Problem 1. Let \( L \) be a Lie algebra over \( F \), \( \varrho : L \rightarrow \mathfrak{gl}(V) \) be a finite-dimensional representation such that \( \varrho(x) \) is nilpotent for all \( x \in L \). Show that \( \kappa_{\varrho} = 0 \).

Problem 2. Let \( A \) be an associative \( F \)-algebra. An \( F \)-linear map \( D : A \rightarrow A \) is called a derivation, provided
\[
D(ab) = aD(b) + D(a)b \quad \forall a, b \in A.
\]
(1) Suppose that \( A \) is generated (as an algebra) by the subset \( S \subseteq A \). If \( D_1, D_2 : A \rightarrow A \) are derivations such that \( D_1|_S = D_2|_S \), then \( D_1 = D_2 \).

(2) Let \( A = F[X_1, \ldots, X_n] \) be a polynomial ring in \( n \) indeterminates over \( F \). Given \( i \in \{1, \ldots, n\} \), there exists a unique derivation \( \frac{\partial}{\partial X_i} : A \rightarrow A \) such that \( \frac{\partial}{\partial X_i}(X_j) = \delta_{ij} \) for \( 1 \leq j \leq n \).

Problem 3. Let \( V \) be an \( F \)-vector space. We denote by \( V^\otimes n := V \otimes_F V \otimes \cdots \otimes_F V \) \( (n \geq 2) \) the \( n \)-fold tensor product of \( V \) and write \( V^\otimes 1 := V \) and \( V^\otimes 0 := F \).

(1) The space \( T(V) := \bigoplus_{n \geq 0} V^\otimes n \) obtains the structure of an associative \( F \)-algebra such that
\[
(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m \quad v_i, w_j \in V.
\]

(2) Let \( t : V \rightarrow T(V) ; v \mapsto v \). Show that the pair \( (T(V), t) \) enjoys the following universal property: For any associative \( F \)-algebra \( A \) and any linear map \( f : V \rightarrow A \), there exists a unique homomorphism \( \bar{f} : T(V) \rightarrow A \) of associative \( F \)-algebras such that \( \bar{f} \circ t = f \).

Problem 4. Suppose that \( \text{char}(F) = p > 0 \).

(1) Show that \( W(1) := \bigoplus_{i=1}^{p^2-2} F e_i \) obtains the structure of a Lie algebra via
\[
[e_i, e_j] := (j-i)e_{i+j}.
\]
Here the product is understood to be zero if \( i+j \not\in \{-1, \ldots, p-2\} \). (The algebra \( W(1) \) is called the Witt algebra.)

(2) If \( p \geq 3 \), then \( W(1) \) is simple.

(3) The Killing form \( \kappa_{W(1)} \) is non-degenerate for \( p = 3 \) and identically zero for \( p \geq 5 \).