Problem 1. Suppose that $L$ is a finite-dimensional Lie algebra over an algebraically closed field $F$. Let $D : L \rightarrow L$ be a derivation. For $\alpha \in F$, we let $L_\alpha$ be the generalized eigenspace of $D$ with eigenvalue $\alpha$.

(1) Show that $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.
(2) Let $D = S + N$ be the Jordan decomposition of $D$, with $S$ being diagonalizable and $N$ being nilpotent. Show that $S$ and $N$ are derivations of $L$.

Problem 2. Let $F$ be an algebraically closed field of characteristic 0. Show that $M \cong M^*$ for every finite-dimensional $\mathfrak{sl}(2, F)$-module $M$.

Problem 3. Let $R \subseteq E$ be a root system in a Euclidean vector space $E$. We put $\gamma^\vee := \frac{2\gamma}{(\gamma, \gamma)}$ for all $\gamma \in E \setminus \{0\}$.

(1) Show that $R^\vee := \{ \alpha^\vee : \alpha \in R \}$ is a root system.
(2) Show that the Weyl group $W^\vee$ of $R^\vee$ is isomorphic to the Weyl group $W$ of $R$.

Problem 4. Let $L$ be a finite-dimensional Lie algebra over an algebraically closed field $F$ such that every finite-dimensional $L$-module is semi-simple. Show that $L$ is a semi-simple Lie algebra.