Problem 1. Let $G$ be a group acting on a set $S$. A subset $D \subseteq S$ is referred to as a fundamental domain if $|G.x \cap D| = 1$ for every $x \in S$.

1. Given any group operation $G \times S \to S$, show that there exists a fundamental domain $D \subseteq S$.
2. Let $E$ be a Euclidean vector space with root system $R$ and base $\Delta \subseteq R$. Show that the fundamental Weyl chamber $E(\Delta)$ is a fundamental domain for the canonical action of the Weyl group on the set of regular elements of $E$.

Problem 2. Let $R \subseteq E$ be a root system with Weyl group $W$.

1. Show that $\ell(w) \leq |R^+|$ for every $w \in W$. (Here $\ell(w)$ is the length of $w$ relative to an arbitrary base $\Delta \subseteq R$.)
2. Show that there exists a unique element $w_0 \in W$ such that $\ell(w_0) = |R^+|$. The element $w_0$ is referred to as the longest element of $W$.

Problem 3. Let $W \subseteq \text{GL}(E)$ be the Weyl group of the root system $R \subseteq E$. Show that $w_0^2 = \text{id}_E$.

Problem 4. Consider the root system $R$ of type $A_2$. Given a base $\Delta = \{\alpha, \beta\} \subseteq R$, compute the length of the reflection $s_{\alpha+\beta}$.