On the rank of quotients of hyperbolic groups

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Abstract

We show that the rank does not decrease if one passes from a torsion-free locally quasi-convex hyperbolic group to the quotient by the normal closure of certain high powered element. An argument provided by Ilya Kapovich further shows that the quasiconvexity assumption cannot be dropped without adding other assumptions.

Introduction

In [D] T. Delzant shows that the quotient of a hyperbolic group by the normal closure of some high powered element is again hyperbolic. This result is at least philosophically related to the celebrated result of W. Thurston [Th] that states that almost all Dehn fillings of a cusped hyperbolic manifolds again yield a hyperbolic manifold. The result of Delzant has since been extended to relatively hyperbolic groups by D. Osin [O] and by D. Groves and J. Manning [GM] thus yielding a true algebraic version of Thurston’s result.

Another interesting result regarding Dehn fillings of hyperbolic 3-manifolds is a result of Y. Moriah and H. Rubinstein [MR] that states that for most Dehn fillings $M(\alpha)$ of a cusped hyperbolic 3-manifold $M$ we have $g(M) = g(M(\alpha))$ where $g(M)$ denotes the Heegaard genus of $M$. In many situations the rank of the fundamental group of a 3-manifold behaves in a similar fashion as its Heegaard genus, however recent work of Abert and Nikolov [AN] (possibly) suggests that it behaves very differently when passing to finite index subgroups. This makes it a natural question whether rank $\pi_1(M) = \pi_1(M(\alpha))$ for almost all $\alpha$. See J. Souto’s interesting paper [Sou] for a discussion of this question.

In this note we investigate a question for hyperbolic groups that is philosophically related to the above question. We investigate whether killing a high power of some element of a hyperbolic group does not decrease its rank. We prove the following result.

Theorem 1 Let $G$ be a torsion-free locally quasi-convex hyperbolic group and $g \in G$. Then there exists some $N \in \mathbb{N}$ such that

$$\text{rank } G/(g^N) = \text{rank } G.$$

At first it seems plausible that the assertion of the theorem should be true for all sufficiently large $N$. This however turns out the be false as the group

$$G = \langle a, b \mid - \rangle * \langle [a, b] = z^2 \rangle \langle z \mid - \rangle$$

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is clearly a torsion-free locally quasi-convex hyperbolic group of rank 3 as quotienting out the normal closure of the amalgam yields the group $\mathbb{Z}^2 \ast \mathbb{Z}_2$ which is of rank 3 by Grushko’s theorem. However adding the relation $z^N$ clearly produces a group that is generated by $a$ and $b$ provided that $N$ is odd. It turns out that besides requiring $N$ to be very large we also need to impose certain arithmetic conditions on $N$.

The hypothesis that $G$ is locally quasi-convex is a very strong one and the theorem probably remains true in many situations where the group $G$ is not locally quasi-convex. It might very well be true that adding a random relator preserves the rank with probability one, this is supported by the result of I. Kapovich and P. Schupp that asserts that the rank of a generic group (their genericity defintion implies that the group is a small cancellation group) is equal to the number of generators in the presentation, thus the introduced relators do not change the rank.

Our theorem however turns out to not be true in the class of hyperbolic groups. As observed by Ilya Kapovich the Rips construction helps as usual. Indeed, take a finitely presented group $H$ of rank at least 3 that contains an element $h$ of infinite order such that the normal closure of any non-trivial power of $h$ coincides with $H$. The Rips construction $[R]$ then gives an exact sequence

$$1 \to N \to G \to H \to 1$$

such that $G$ is hyperbolic and that $N$ is generated by two elements. As $H$ is of rank at least 3 the groups $G$ cannot be 2-generated. Let $g \in G$ be an element that projects to a nontrivial element $h \in H$ of infinite order. Then for every $n \geq 1$ the group $G/\langle \langle g^n \rangle \rangle$ is generated by the image of $N$ and thus has rank at most 2. As pointed out by Mark Sapir such a group $H$ is easily constructed using an amalgamated product of simple groups along a malnormal subgroup and then applying Corollary 1 of [W1].

It should be noted that all constants that appear in this article are computable. This follows immediately from the proofs. Whenever proofs are entirely standard we will not give the details but leave details to the reader. We therefore choose constants to be sufficiently large to work with any conceivable proof the reader might have in mind. We firmly beliee that this lack of detail significantly increases the readability of this note.

## 1 Small cancellation theory

Small cancellation theory states that groups that are given by a presentation that satisfies certain technical conditions have the property that an element given by a freely reduced word in the generators represents the non-trivial element if it cannot be replaced by a shorter word by replacing a subword $u$ by a shorter subword $v$ where $uv^{-1}$ is a cyclic conjugate of one of the relations or their inverses. This is equivalent to saying that Dehn’s algorithm can be used to solve the word problem for such groups.

T. Delzant used small cancellation arguments to study quotients of hyperbolic groups. Let now $G = \langle X \rangle$ be a hyperbolic group and $w$ be a geodesic word in $X \cup X^{-1}$. Note that we denote the element of $G$ represented by a word
We say that \( w \) almost contains a geodesic word \( u \) if \( w = v_1 u' v_2 \) and \( u' = xuv \) with \( x \) and \( y \) of length at most \( 10\delta \), see Figure 1.

![Figure 1: A geodesic word \( w \) almost containing a geodesic word \( u \).](image)

The following can easily be extracted from Lemme 2.4 of [D] and the proof of Théorème 2.1. It should be noted that Delzant is not only concerned with 1-relator quotients; this more general setting is responsible for the fact that in his statements he requires the number \( N \) to be a multiple of some other constant. In the case of a single element this is not necessary.

**Proposition 2** Let \( G \) be a hyperbolic group, \( g \in G \) be an element of infinite order and \( \alpha \in (0,1) \). Then there exists some \( n_0 \in \mathbb{N} \) with the following property:

Suppose that \( N \geq n_0 \) and that \( w \) is a \( G \)-geodesic word representing the trivial element of \( G/\langle \langle g^N \rangle \rangle \). Then \( w \) almost contains a subword \( u \) of \( g^N \) or \( g^{-N} \) of length at least \( \alpha |g^N| \).

## 2 Quasi-convex subgroups of hyperbolic groups

Let \( G \) be a hyperbolic group with finite generating set \( X \) such that the Cayley graph \( \Gamma(G,X) \) is \( \delta \)-hyperbolic.

Recall that a finitely generated subgroup \( U \) of a hyperbolic group \( G \) is called \( C \)-quasi-convex if any geodesic in \( \Gamma(G,X) \) joining any two elements \( u, v \in U \) stays in the \( C \)-neighborhood of \( U \). \( U \) is further called quasi-convex if \( U \) is \( C \)-quasi-convex for some \( C \in \mathbb{N} \). Note that quasi-convexity does not depend on the choice of \( X \) but that the quasi-convexity constant \( C \) does.

In the remainder of this section we recall a well-known characterization of quasi-convex subgroups. Recall that the limit set \( \Lambda(U) \) of a subgroups \( U \) of a hyperbolic group \( G \) is the set of all accumulation points of \( U \) in \( \partial G \). Recall that \( \Lambda(U) \) contains at least two points unless \( U \) is finite.

The convex hull \( X_U \) of an infinite subgroup \( U \) is then defined to be the set of all geodesics joining points of \( \Lambda(U) \). It is easily seen that \( X_U \) is \( 4\delta \)-quasiconvex. It is further clear that \( X_U \) is \( U \)-invariant and that \( U \) acts freely on \( X_U \). The action does not need to be cocompact, it is for example possible that \( X_U \) is all of \( \Gamma(G,X) \) although \( U \) is a subgroup of infinite index, examples for this phenomenon are hyperbolic mapping tori of automorphisms of surface groups. We do however have the following well-known fact, its proof follows directly from the definitions.

**Lemma 3** Let \( G, X \) and \( U \) be as above.

1. The action of \( U \) on \( X_U \) is cocompact if and only if \( U \) is quasi-convex.
2. If \( U \) is \( C \)-quasi-convex then \( X_U \subset N_{C+10\delta}(U) \).
In the case of an infinite cyclic subgroup $U = \langle g \rangle$ of $G$ we call $A_g := X_U$ the axis of $U$ or the axis of $g$. Note that if we define the translation length of an element $g$ to be
\[ ||g|| = \lim_{i \to \infty} \frac{1}{i} \inf_{x \in X} d(x, g^i x) \]
then we have the following fact.

**Lemma 4** $A_g$ is $10\delta$-Hausdorff-close to the set $\{ x \in X | d(x, gx) \leq ||g|| + 10\delta \}$.

### 3 The intersection of a quasi-convex subgroup with an infinite cyclic group

In this section we observe that if the convex hull $X_H$ of some quasi-convex subgroup $H$ of some hyperbolic group $G$ contains a sufficiently large portion of the axis of some element $g$ then $H$ intersects $\langle g \rangle$ non-trivially. This is probably well-known to the experts but we are not aware of any place in the literature where this has appeared explicitly.

**Proposition 5** There exists a constant $c = c(C, n, k, \delta)$ with the following properties.

Let $X = \{ x_1, \ldots, x_n \}$ and $G = \langle X \rangle$ such that $\Gamma(G, X)$ is $\delta$-hyperbolic. Let further $g \in G$ be an element of infinite order with $||g|| \leq k$ and $H \leq G$ be a $C$-quasiconvex subgroup. Then the following hold:

1. If $N_{10\delta}(X_H)$ contains a subset of $A_g$ of diameter greater than $c$ then $H \cap \langle g \rangle \neq \emptyset$.

2. $H$ contains at most $c$ conjugacy classes of maximal cyclic subgroups that are in $G$ conjugate to a subgroup of $\langle g \rangle$.

**Proof** We first show that the first claim holds provided that $c \geq k(2n)^{C+30\delta} + 50\delta$. Suppose that $G$, $X$, $H$ and $g$ are as above and that $N_{10\delta}(X_H)$ contains a subset of $A_g$ of diameter greater than $k(2n)^{C+30\delta} + 50\delta$.

It follows easily that there exists some vertex $x \in \Gamma(G, X)$ such that the vertices $x, gx, g^2x, \ldots, g^{(2n)^{C+30\delta}}$ lie in the $20\delta$-neighborhood of $X_H$ and therefore in the $(C + 30\delta)$-neighborhood of $H$.

![Diagram](image.png)

Figure 2: A large part of $A_g$ is close to $X_H$. Here $N = (2n)^{C+30\delta}$. 
Put \( x_i = g^i x \) for \( 0 \leq i \leq (2n)^{C+30\delta} \). Because of the triangle inequality and the above remarks it follows that for each \( i \) there exists some \( h_i \in H \) such that 
\[ d(x_i, h_i) \leq C + 30\delta. \]
As there are fewer than \((2n)^{C+30\delta}\) elements of length at most \( C + 30\delta \) it follows that there exists \( i < j \) such that 
\[ x_i^{-1}h_i = x_j^{-1}h_j. \]
This clearly implies that 
\[ g^{j-i} = (g^j x)(x^{-1} g^{-1}) = x_j x_i^{-1} = h_j h_i^{-1} \in H \]
which proves the first claim.

To verify the second claim note first that the proof of the first claim implies that whenever \( H \) intersects \( \langle \bar{g} g \rangle \) then \( H \) contains \( \langle \bar{g} g^N \rangle \) with \( 1 \leq N \leq C + 30\delta \). The number of such conjugacy classes is clearly bounded by a constant \( l \) that only depends on \( C, \delta, n \) and \( k \).

Thus the Proposition holds with \( c := \max(k(2n)^{C+30\delta} + 50\delta, l) \). \( \square \)

4 The intersection of the axes of two elements

In this section we study the intersection of the axes of two hyperbolic elements. The observation we need is an immediate consequence of Proposition 5. We make it explicit as it is needed in a different part of the proof of our main theorem.

It is well-known that hyperbolic elements \( g \) and \( h \) either have the same fixed points in the boundary or their axes are far apart outside of some bounded set \( Y \) inside which the axes of \( g \) and \( h \) fellow-travel. In the latter case sufficiently high powers of \( g \) and \( h \) generate a free subgroup of rank 2. Note that the size of \( Y \) depends on \( g \) and \( h \) as the axes might fellow-travel for an arbitrarily long time.

The purpose of this section is to observe that there exists an a priori bound on the size of \( Y \) provided that \( g \) and \( h \) have short translation length.

**Proposition 6** For any \( \delta, l \geq 0 \) and \( n \geq 1 \) there exists a constant \( c = c(\delta, l, n) \) such that the following hold:

Let \( X = \{x_1, \ldots, x_n\} \) and \( G = \langle X \rangle \) be a \( \delta \)-hyperbolic group. Suppose that \( g, h \in G \) are elements of infinite order such that \( ||g||, ||h|| \leq l \) and that a segment \([p, q]\) of length \( c \) lies in the \( 10\delta \)-neighborhood of both \( A_g \) and \( A_h \). Then the following hold:

1. \( A_g = A_h \)
2. \( \langle g, h \rangle \) is an elementary subgroup of \( G \).
3. If \( G \) is torsion-free then \( \langle g, h \rangle = \langle \bar{g} \rangle \) for some \( \bar{g} \in G \).

**Proof** Proposition 6 easily follows from Proposition 5. Indeed put \( H = \langle h \rangle \). It is easily verified that \( H \) is \( C \)-quasiconvex with \( C = ||h|| + 20\delta \leq l + 20\delta \). It then follows from Proposition 5 that for \( c \) large enough the hypothesis of Proposition 6 implies that \( \langle g \rangle \cap \langle h \rangle \) is non-empty. This implies that \( g \) and \( h \) have common powers which implies that \( g \) and \( h \) have the same fixed points \( x_1, x_2 \) in the boundary. It follows that \( A_g = A_h \). It further follows that \( \langle g, h \rangle \) fixes \( x_1 \) and \( x_2 \). Thus \( \langle g, h \rangle \) is an elementary subgroup of \( G \). Part (3) is trivial as elementary subgroups of a torsion-free hyperbolic groups are infinite cyclic. \( \square \)
5 Representing subgroups by $G$-graphs

In this section we review how subgroups of a torsion-free locally quasiconvex hyperbolic group $G$ can be efficiently represented by a folded $G$-graph. The main result stated is proven in [KW2], the formulation however is in the language of [KW4].

**Definition 7** Let $G$ be a group. A $G$-graph $B$ is a graph of groups $B$ with trivial edge groups such that the following hold:

1. Every vertex group $B_v$ is a subgroup of $G$.
2. Every edge $e \in EB$ is labelled by an element $g_e$ and $g_{e^{-1}} = g_e^{-1}$.

We further call a $G$-graph marked if to any vertex $v \in VB$ there is an associated generating tuple $T_v$ of the vertex group $B_v$. We call the sum of the word lengths of the edge elements the volume of $B$.

A $G$-graph $B$ in particular encodes a morphism from the fundamental group of its underlying graph of groups $B$ to $G$.

**Lemma 8** For any $G$-graph $B$ with a base-vertex $v_0$ we have that the map $\nu : \pi_1(B, v_0) \to G$ given by

$$[b_0, e_1, b_1, \ldots, b_k-1, e_k, b_k] \mapsto b_0 g_{e_1} b_1 \cdots b_{k-1} g_e b_k$$

is a homomorphism. We call the subgroup $\nu(\pi_1(B, v_0))$ of $G$ the subgroup represented by $B$.

A marked $G$-graph with base vertex $v_0$ further represents a Nielsen equivalence class of generating tuples of the subgroup it represents. Indeed choose a maximal subtree $Y$ of $B$ and let for any $v \in VB$

$$\gamma_v := 1, e_{v,1}, 1, \ldots, 1, e_{v,k_v}, 1$$

be the $B$-path whose underlying path $e_{v,1}, \ldots, e_{v,k_v}$ is the unique reduced path that connects $v_0$ to $v$ in $Y$. If $E$ is now a set of edges of $EB$ that contains one edge out of every edge pair of $EB - EY$ then a generating set $T^B_Y$ of $\pi_1(B, v_0)$ is given by the following elements:

1. For ever edge $e \in E$ take $[\gamma_{\alpha(e)}, e, \gamma_{\omega(e)}^{-1}]$.
2. For every element $g$ of some generating tuple $T_v$ take $[\gamma_v \cdot g \cdot \gamma_v^{-1}]$.

Although the tuple $T^B_Y$ does clearly depend on the choice of $Y$ and $E$, it is easy to see that the Nielsen equivalence class does not, i.e. the marked $G$-graph $B$ defines a Nielsen equivalence class of generating tuples of $\pi_1(B, v_0)$ and via the homomorphism $\nu$ a Nielsen equivalence class of generating tuples of the represented subgroup $\nu(\pi_1(B, v_0))$, the proofs of the above claims are identical to those in [W2].

To any $\mathcal{A}$-path $p = b_0, e_1, b_1, \ldots, b_{k-1}, e_k, b_k$ we have an associated path in $\Gamma(G, X)$ connecting the identity and $\nu([b_0, e_1, b_1, \ldots, b_{k-1}, e_k, b_k])$, namely the path

$$\gamma_p = [v_0, u_0] \cup [u_0, v_1] \cup [v_1, u_1] \cup \ldots \cup [u_k, v_k] \cup [v_k, u_{k+1}]$$

where
1. \( u_i = b_0e_i, b_1 \cdot \cdot \cdot b_{i-1}e_i, b_i \) for \( 0 \leq i \leq k \) and

2. \( v_0 = 1 \) and \( v_i = b_0e_i, b_1 \cdot \cdot \cdot b_{i-1}e_i \) for \( 1 \leq i \leq k \).

This path is not quite unique as the geodesic segments \([v_i, u_{i+1}]\) and \([v_i, u_i]\) are not unique, we will however ignore this issue as usual.

\[
\begin{align*}
\text{Figure 3: The path } \gamma_p \text{ corresponding to an } \Lambda \text{-path of type } p &= b_0, e_1, b_1, e_2, b_2. \\
\end{align*}
\]

It is clear that in general the path \( \gamma_p \) may contain lots of backtracking and the path might be closed even if \( p \) is reduced. However in the case of a subgroup \( U \) of a finitely generated torsion-free locally quasiconvex hyperbolic group \( G \) with a given finite generating tuple \( T \) it is always possible to find an efficient marked \( G \)-graph \( B \) such that \( \nu(\pi_1(B, v_0)) = U \), that \( B \) represents the Nielsen equivalence class of \( T \) and that no substantial backtracking of \( \gamma_p \) occurs for reduced paths \( p \).

**Theorem 9 ([KW2])** Suppose that \( G = \langle X \rangle \) is a locally quasiconvex torsion-free hyperbolic group and that \( m \in \mathbb{N} \). Then there exist integers \( k_1 = k_1(m, G, X), k_2 = k_2(m, G, X), \ldots, k_m = k_m(m, G, X) \) with the following property:

Let \( T = (g_1, \ldots, g_m) \in G^m \). Then the Nielsen equivalence class of \( T \) is represented by a marked \( G \)-graph \( B \) such that the following hold:

1. The elements of the generating tuples of the vertex groups are of length at most \( k_i \) for some \( i \leq m \).

2. For any reduced \( B \)-path \( p = b_0, e_1, b_1, \ldots, b_{k-1}, e_k, b_k \) any geodesic segment \([1, \nu(p)]\) lies in the \( 10\delta \)-neighborhood of \( \gamma_p \).

3. The length of the element represented by any closed loop of \( B \) and any simple path connecting two vertices with non-trivial vertices is of length at least \( \frac{1}{2}k_{i+1} \) and at least half of the corresponding subpath of \( \gamma_p \) lies in the \( 10\delta \)-neighborhood of \([1, \nu(p)]\).

4. The map \( \nu : \pi_1(B) \to G \) maps \( \pi_1(B) \) isomorphically to \( \langle T \rangle \).

5. Any valence 1 or valence 2 vertex except possibly the base vertex has non-trivial vertex group.
We can further require that \( k_{i+1} \geq f(k_i) \) for \( 1 \leq i \leq m - 1 \) where \( f : \mathbb{N} \to \mathbb{N} \) is an arbitrary map possibly dependent on \( m \).

It should be noted that the function \( f \) never occurs in the discussion in [KW2] however an analysis of the proofs show that it never hurts to choose the constants bigger. In fact the whole work done in [KW2] is to find an upper bound for these constants. There is no problem in choosing the constants as in the above Theorem as \( k_i \) is only computed after all \( k_j \) with \( j < i \) are already known.

Note further that part 5 of the above proposition implies that \( B \) has at most \( 3m - 1 \) edges as rank \( \pi_1(B) \leq \text{rank} \pi_1(B) \) and a graph of rank \( m \) without valence 1 or 2 vertices has at most \( 3m - 3 \) edges. Adding the base vertex increases this number by at most 2. We will sometimes introduce valence 2 vertices, i.e. we will replace an edge of \( B \) by two edges \( e_1 \) and \( e_2 \) with \( \alpha(e_1) = \alpha(e_2) \) and \( \omega(e_1) = \omega(e_2) \) such that for the edge elements we have \( g_{e_1} = g_{e_2} \) and \( |g_{e_1}| + |g_{e_2}| \leq |g_{e_2}| + 100\delta \). We call this modification a subdivision. It clearly increases the volume of the \( G \)-graph by at most \( 100\delta \).

We also need the following remark, it is an immediate consequence of the proofs in [KW2].

**Remark 10** If in Theorem 9 the tuple \( T \) is represented by some marked \( G \)-graph \( B' \) then \( B \) can be chosen to be of at most the same volume as \( B' \).

6 The proof of the theorem

We now fix the locally quasi-convex word-hyperbolic group \( G \) and the finite generating set \( X = \{x_1, \ldots, x_n\} \). Let \( \delta \) be the quasi-convexity constant of \( \Gamma(G, X) \). We further fix the element \( g \) for which we want to study quotients of type \( G/\langle \langle g^n \rangle \rangle \); without loss of generality we can assume that \( g \) is primitive, i.e. that \( g \) is not a proper power. All quasiconvexity constants below are with respect to this setting. Let \( C = 10 \cdot c(\delta, ||g||, n) \) be the constant provided by Proposition 6.

We now define the function \( f : \mathbb{N} \to \mathbb{N} \) from the last part of the statement of Theorem 9. For \( k \in \mathbb{N} \) let \( \mathcal{V}_k \) be the set of all subgroups of \( G \) that are generated by at most \( n \) elements of \( G \) of word length at most \( k \). \( \mathcal{V}_k \) is clearly a finite set of subgroups of \( G \). We further define \( q(k) \) to the maximum of the quasiconvexity constants of the elements of \( \mathcal{V}_k \). Finally we put \( f(k) = 100 \cdot n \cdot q(k) + 10 \cdot n \cdot C + 1000\delta \).

Note that rank \( G/\langle \langle g^n \rangle \rangle \leq \text{rank} G \leq n \). Let \( \mathcal{V} = \bigcup_{k=1}^n \mathcal{V}_k \) be the set of groups that occur as vertex groups of marked \( G \)-graphs as in the conclusion of Theorem 9 for \( m := n \) and \( f \) as above. We now choose \( N_0 \) such that the following hold:

1. For any \( V \in \mathcal{V} \) and \( h \in G \) such that \( G \cap \langle hgh^{-1} \rangle = \langle hgh^{-1} \rangle \neq 1 \) the constant \( N_0 \) is a multiple of \( q \).
2. \(|g^{N_0}| \geq 100 \cdot n \cdot M \) where \( M \) is the maximum over all the constants provided by Proposition 5 for the different subgroups \( V \in \mathcal{V} \).
3. Such that $N_0$ is greater than the number $n_0$ provided by Proposition 2 for $\alpha = 1 - \frac{1}{100}$. 

4. $|g^{N_i}| \geq 100 \cdot n \cdot k_i$ for $1 \leq i \leq n$ with the $k_i = k_i(n, G, X)$ from Proposition 9 (and $f$ as above).

Such a $N_0$ exists as $\mathcal{V}$ is a finite set and by Proposition 5 any $V \in \mathcal{V}$ only contains finitely many conjugacy classes (in $V$) of maximal of subgroups conjugate (in $G$) to subgroups of $(g)$. From now on we choose $N$ to be a non-zero multiple of $N_0$. Our Theorem now clearly follows from the following:

**Proposition 11** With $N$ as above $\text{rank } G/\langle\langle g^N \rangle\rangle = \text{rank } G$.

**Proof** In order to show that $\text{rank } G/\langle\langle g^N \rangle\rangle = \text{rank } G$ we study minimal generating tuples of rank $G/\langle\langle g^N \rangle\rangle$ and observe that there exists a generating tuple of the same cardinality of $G$. We will see that we can lift the generating tuple of rank $G/\langle\langle g^N \rangle\rangle$ to $G$ provided me make the appropriate minimality assumptions.

Let $m = \text{rank } G/\langle\langle g^N \rangle\rangle$, clearly $m \leq n$. We consider Nielsen equivalence classes of $m$-tuples of elements of $G$ that project onto generating tuples of $G/\langle\langle g^N \rangle\rangle$. Let $B$ be a minimal volume marked $G$-graph that represents one of these Nielsen equivalence classes and satisfies the hypothesis of Proposition 9. We claim that the elements read off this marked $G$-graph form a generating tuple $T$ of $G$. The minimality of $m$ therefore implies that $\text{rank } G = \text{rank } G/\langle\langle g^N \rangle\rangle$ which we are out to prove.

It clearly suffices to show that any generator of $G$ lies in the subgroup generated by $T$. Let $x$ be one such generator. Choose a reduced $\mathcal{B}$-path $p = b_0, e_1, b_1, \ldots, b_{k-1}, e_k, b_k$ such that $\nu(p)$ is of minimal length such that $\pi(\nu(p)) = \pi(x)$ where $\pi : G \to G/\langle\langle g^N \rangle\rangle$ is the canonical projection. Such a path exist as we assume that $T$ projects onto a generating tuple of $G/\langle\langle g^N \rangle\rangle = \pi(G)$. We assume that all notations, in particular the $u_i$ and $v_i$ are as in the previous section, thus we have Figure 3 in mind.

After choosing a geodesic word for $\nu(p)$ it follows that that the word $\nu(p)x^{-1}$ can be transformed to the trivial word by applying small cancellation replacements and relations from $G$ as $\pi(\nu(p)x^{-1}) = 1$. If $\nu(p)x^{-1}$ admits no small cancellation replacement then the identity $\nu(p) = x$ holds in $G$ already and we are done. Thus we can assume that $\nu(p)x^{-1}$ admits a small cancellation replacement, in particular $\nu(p)$ almost contains almost the whole relation $g^N$ or $g^{-N}$ (recall that we have chosen $\alpha = 1 - \frac{1}{100 \alpha}$). Denote the corresponding subword of $\nu(p)$ by $w$.

There are different ways how the subword $w$ can be picked up when walking along the path $\gamma_p$. Recall that the subpaths of $\gamma_p$ do partially cancel but that Theorem 9 still guarantees that $w$ lies close to the union of the subpaths. We need to distinguish a number of different cases, in each of them we will either find a contradiction to the minimality assumption for the $G$-graph $B$ or to the minimality assumption for the element $\nu(p)$. The cases below clearly cover all possibilities.

**Case 1:** A subsegment of $\hat{w}$ of length at least $2M$ (see 2 in the above choice of $N_0$) of $w$ is picked up at a vertex group, i.e. lies in a small neighborhood of one of the subpaths $[v_i, u_i]$ of $\gamma(p)$. Recall that $b_i = u_i^{-1}v_i \in B_{\alpha(u_i)}$. The element $b_i$ can be represented by an $(1, 1008)$-quasi-geodesic word $uwv$. Note that a portion
of length greater than $M$ of the axis of $u^g g^{-1}$ lies in $N_{10M}(X_{B_{\alpha(e_i)}})$ where $\tilde{g}$ is some cyclic permutation of $g$ or $g^{-1}$ (the segment $[u, u\tilde{w}]$ does). It follows from the definition of $M$ and Lemma 5 that some power $u\tilde{g}^j u^{-1}$ of $u^g g^{-1}$ lies in $B_{\alpha(e_i)}$. We can choose $L$ to be minimal, in particular we have $L|N_6|N$ because of part 1 in the choice of $N_6$. It follows in particular that $u\tilde{g}^{N} u^{-1} \in B_{\alpha(e_i)}$.

Let now $p'$ be the $B$-path obtained from $p$ by replacing $b_i$ by $b'_i := u\tilde{g}^{N} u^{-1} b_i$. It is clear that $\pi(\nu(p)) = \pi(\nu(p'))$ as $u\tilde{g}^{N} u^{-1}$ lies in the kernel of $\pi$; it is further clear that $|\nu(p')| < |\nu(p)|$ as the change has been an application of a small cancellation relations that shortens the length. This contradicts our minimality assumption for $\nu(p)$.

Case 2: No subsegment of $w$ of length at least $2M$ is picked up at a vertex group, i.e. by part 3 of Proposition 9 significant parts of $w$ are picked up by edge elements, i.e. by subpaths of type $[u_l, v_{i+1}]$ of $\gamma(p)$. Possibly after performing at most two subdivisions of $\mathcal{B}$ and adjust $p$ accordingly we can assume that $w$ is picked up by a sub-$\mathcal{B}$-path $q = b_i, e_{i+1}, b_{i+1}, \ldots, b_{j-1}, e_j, b_j$; thus $\nu(q) = w$.

We distinguish three subcases which are easily seen to cover all possibilities.

Case 2A The $\mathcal{B}$-path $q$ projects onto a simple path in $B$. As the new graph has at most $3n + 1$ edges it follows that at least one of the edges $e$ visited by $q$ has an edge element of length more than $|g^{N}| > 2/10M$. We now modify $\mathcal{B}$ by removing $e$ and adding an edge $f$ with initial vertex $\alpha(f) = \alpha(q)$, terminal element $\omega(f) = \omega(q)$ and edge element the element obtained from $w = \nu(q)$ by applying the small cancellation relation. Clearly $|\nu(q)| < |g^{N}| > 1/10M$ as $w = \pi(q)$ is at least $\alpha = (1 - 1/10M)$ of the relation $|g^{N}|$ and therefore in $G/\langle \langle g^{N}\rangle \rangle$ identical to a word of length at most $|g^{N}| > 1/10M$. The Nielsen equivalence class represented by the new $G$-graph projects onto the same Nielsen equivalence class as the old one but is of smaller volume contradiction our minimality assumption on the $G$-graph $\mathcal{B}$. Note that this modification was first used in [AO], the fact that it does not affect the Nielsen equivalence class of the represented tuple was observed in [KS].

Case 2B The path $q$ contains an edge $e$ with edge element $g_e$ of length at least $2C$ twice with the same orientation and is injective otherwise, after a slight modification of the $G$-graph we can assume that $g_e$ is a power of $g$. Thus $q$ contains a subpath $s = b_k, e_{k+1}, b_{k+1}, \ldots, b_{l-1}, e_l, b_l$ with $e_{k+1} = e_l = e$. This implies that the axis of the conjugate of $g_e$ (which is a power of $g$) by the element $h = \nu(b_{k+1}, \ldots, b_{l-1}, e_l, 1)$ has a segment of length greater than $C$ that lies in the $10\delta$-neighborhood to the axis $A_g = A_{g_\delta}$. It therefore follows from Proposition 5 that this conjugate is also a power of $g$. Thus $h$ normalizes $\langle g \rangle$ which implies that $h$ is a power of $g$ as we are in a torsion-free word hyperbolic group. Thus the element $h$ corresponding to the loop is an element conjugate to some (high) power of $g$. We can then remove this loop by removing a long edge contained in this loop and adding a short element, namely the primitive root of the element represented by this loop. This again yields a contradiction to the minimality of the volume of the $G$-graph $\mathcal{B}$.

Case 2C: The path $q$ contains an edge $e$ with edge element $g_e$ of length at least $2C$ twice with different orientations and is injective otherwise, after a slight modification of the $G$-graph we can assume that $g_e$ is a positive (or negative) power of $g$. Thus $q$ contains a subpath $s = b_k, e_{k+1}, b_{k+1}, \ldots, b_{l-1}, e_l, b_l$ with $e_{k+1} = e_l = e$. The same argument as in case 2B shows that the conjugate of $g_e^{-1}$ by the element $h = \nu(b_{k+1}, \ldots, b_{l-1}, e_l, 1)$ represented by the closed $\mathcal{B}$-
sub-path is a positive (or negative) power of $g$. This however implies that $g$ is conjugate to $g^{-1}$ which is not possible in a torsion-free word-hyperbolic group. □

References


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