On the uniqueness of factors of amalgamated products

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Let $G = A \ast_C B$ be an amalgamated product. We study conditions under which the factor $B$ is determined by $A$ and $C$, i.e. under which the existence of another splitting of $G$ as an amalgamated product $G = A \ast_C D$ implies $B = D$ or $B \cong D$. We also describe the structure of the family of amalgamated products along a given malnormal subgroup.

There are situations where $D \not\cong B$. We describe them and show that they are the only ones after giving a short account of controlled subgroups as introduced in [FRW]. Mary Jones has independently constructed an example of the type below and Ilya Kapovich has provided an insightful example to a related question. This work was motivated by the first author’s work on splittings of surface groups [B].

1 Examples and main results

The two types of examples are similar in that $B$ splits over a subgroup of $C$ and that $D$ is obtained by conjugating a boundary monomorphism with an element of $A$ such that this conjugation cannot be done in $B$. We describe the amalgamated product and the HNN-extension case.

Case 1. Suppose that $B = B_1 \ast_{C_1} B_2$ where $C_1 \leq C \leq B_1$ and $a^{-1}C_1a \leq C$ for some $a \in A$. Then $A \ast_C B = A \ast_C D$ where

$$D = B_1 \ast_{a^{-1}C_1a} a^{-1}B_2a.$$ This holds since $G = A \ast_C B = A \ast_C (B_1 \ast_{C_1} B_2) = (A \ast_C B_1) \ast_{C_1} B_2 = (A \ast_C B_1) \ast_{a^{-1}C_1a} a^{-1}B_2a = A \ast_C (B_1 \ast_{a^{-1}C_1a} a^{-1}B_2a) = A \ast_C D.$$

Case 2. Suppose that $B = \langle B_1, t \mid tC_1t^{-1} = C_1' \rangle$ is an HNN-extension with base group $B_1$ and associated subgroups $C_1$ and $C_1'$ where $C_1 \leq C \leq B_1$ and $a^{-1}C_1a \leq C$ for some $a \in A$. Then $A \ast_C B = A \ast_C D$ where

$$D = \langle B_1, ta \mid (ta) \cdot a^{-1}C_1a \cdot (ta)^{-1} = C_1' \rangle.$$ This holds since $G = A \ast_C B = A \ast_C ((B_1, t \mid tC_1t^{-1} = C_1')) = (\langle A \ast_C B_1 \rangle, t \mid tC_1t^{-1} = C_1') = (\langle A \ast_C B_1 \rangle, ta \mid (ta) \cdot a^{-1}C_1a \cdot (ta)^{-1} = C_1') = A \ast_C (\langle B_1, ta \mid (ta) \cdot a^{-1}C_1a \cdot (ta)^{-1} = C_1' \rangle) = A \ast_C D$

In the first case we say that the splitting $A \ast_C D$ is obtained from the splitting $A \ast_C B$ by a move of type 1, in the second case by a move of type 2.

It is easy to see that there are situations where such moves yield non-isomorphic $B$ and $D$. A simple example for a move of type 1 is the following. Suppose that
A = BS(1, 2) = \langle a, x | a^{-1}xa = x^2 \rangle$, that $C = B_1 = \langle x \rangle$, that $B_2 = \langle y \rangle$ and that $C_1 = \langle x \rangle = \langle y^2 \rangle$. We then have $B = B_1 \ast_{C_1} B_2 = \langle x \rangle *_{\langle x = y^2 \rangle} \langle y \rangle \cong \mathbb{Z}$ but $D = B_1 \ast_{a^{-1}c_1a} a^{-1}B_2a = \langle x \rangle *_{\langle x^2 = a^{-1}y^2a \rangle} \langle a^{-1}ya \rangle$ which is the fundamental group of the Klein bottle.

On the other hand there is a situation where we can guarantee that $B$ and $D$ are isomorphic, namely in the case that the conjugation by the element could have been done by an element of $B_1$, i.e. if there exists an element $b \in B_1$ such that $b^{-1}c_1b = a^{-1}c_1a$ for all $c_1 \in C_1$. This is clear since then $D = B_1 \ast_{a^{-1}c_1a} a^{-1}B_2a \cong B_1 \ast_{b^{-1}c_1b} b^{-1}B_2b = B$. It is further clear that in this case there exists an isomorphism $\phi : B \to D$ such that $\phi|_{B_1} = \text{Id}_{B_1}$, namely the extension of the map $\phi|_{B_1} = \text{Id}_{B_1}$ and the map $\phi|_{b^{-1}B_2b}$ that maps $b^{-1}b_2b$ to $(a^{-1}b)b^{-1}b_2b(b^{-1}a) = a^{-1}b_2a$. These two maps extend to a homomorphism $\phi : B \to D$ since by assumption $a^{-1}c_1a = b^{-1}c_1b$ for all $c_1 \in C_1$, i.e. they coincide when restricted to the amalgam. This extension is clearly an isomorphism.

Our main result is the following corollary.

**Theorem 1** Suppose that $G = A \ast_C B = A \ast_C D$ where $G$ and $C$ are finitely generated. Then the splitting $A \ast_C D$ can be obtained from the splitting $A \ast_C B$ by a finite number of moves of type 1 or 2.

If any conjugation of subgroups of $C$ in $A$ can already be done in $C$, then our observation above immediately yields the following.

**Corollary** Let $G = A \ast_C B$ where $G$ and $C$ are finitely generated and suppose that for any $C_1 \leq C$ and $a \in A$ with $aC_1a^{-1} \leq C$ there exists an element $c \in C$ such that $aC_1a^{-1} = cc_1c^{-1}$ for all $c_1 \in C_1$.

Then $G = A \ast_C D$ implies that there exists an isomorphism $\phi : B \to D$ such that $\phi|_C = \text{Id}_C$.

The hypothesis of the Corollary is clearly fulfilled if $C$ is a malnormal subgroup of $A$, i.e. if $aC_1a^{-1} \cap C = 1$ for all $a \in A - C$. In the case of a malnormal subgroup we are able to give the following stronger result.

**Theorem 2** Let $G$ be a finitely generated group and $C \neq 1$ be a malnormal subgroup that does not lie in a proper free factor of $G$. Then there exists a unique decomposition of type $G = \ast_{i=1}^n G_i$ such that for any splitting $G = A \ast_C B$ we have that $A = \ast_{i \in I_1} G_i$ and $B = \ast_{i \in I_2} G_i$ where $I_1 \cup I_2 = \{1, \ldots, n\}$ and $I_1 \cap I_2 = \emptyset$.

### 2 Controlled subgroups

We will assume familiarity of the reader with the Bass-Serre theory. Details can be found in [Sr]. Suppose that $G$ acts minimally, simplicially and without inversion on a simplicial tree $T$. Let $U$ be a subgroup of $G$. We define $T_U$ to be a vertex fixed under the action of $U$ if $U$ acts with a global fixed point and to be the minimal $U$-invariant subtree of $T$ otherwise. The induced splitting of $U$ is the splitting corresponding to the action of $U$ on $T_U$. For a graph $\Gamma$ we denote the set of vertices of $\Gamma$ by $VT$ and the set of edges by $ET$. For $x, y \in VT$ we denote by $[x, y]$ the geodesic segment joining $x$ and $y$ and by $(x, y)$ the segment without its boundary.
We follow [FRW] and describe a situation where the induced splitting can be read off information that comes with \( U \). In the definition below the tree \( T_1 \) corresponds to a lift of a maximal subtree of \( U \setminus T_U \) to \( T_U \), the tree \( T_2 \) corresponds to a subtree in \( T_U \) whose edge set projects bijectively onto the edge set of \( U \setminus T_U \) and the elements \( t_v \) to the stable letters. 

We say that the subgroup \( U \) of \( G \) is **controlled** by the tuple 

\[
(T_1,T_2,\{G_v|v \in VT_2\},\{t_v|v \in VT_2 - VT_1\})
\]

if

1. \( T_1 \) and \( T_2 \) are subtrees of \( T \), \( T_1 \subset T_2 \).
2. For any \( v \in VT_2 - VT_1 \) there exists an edge \( e_v \in ET_2 \) with initial vertex \( v \) and terminal vertex in \( VT_1 \).
3. \( G_v \subset G \) and \( G_v v = v \) for all \( v \in VT_2 \).
4. \( U \) is generated by the \( G_v \) and the \( t_v \).
5. \( G_v \cap Stab (e) = G_w \cap Stab (e) \) for all edges \( e \in ET_2 \) with initial vertex \( v \) and terminal vertex \( w \).
6. For every \( v \in VT_2 - VT_1 \) there exists a \( x_v \in VT_1 \) such that \( t_v v = x_v \) and \( G_v = t_v^{-1}G_x t_v \).
7. \( t_v e_v \neq t_w e_w \), \( t_v e_v \notin T_2 \) and \( t_w e_w \notin T_2 \) for all \( v,w \in VT_2 - VT_1 \) and \( v \neq w \).
8. All edges of \( T_3 := T_2 \cup \{t_v e_v|v \in VT_2 - VT_1\} \) emanating at a vertex \( v \in VT_1 \) are \( G_v \)-inequivalent.
9. There exists no vertex \( x \in VT_1 \) such that a component \( C \) of \( T_1 - \{x\} \) is also a component of \( T_3 - \{x\} \) and that \( G_v \subset G_x \) for all \( v \in VC \).

There is a simple way to read the induced splitting of \( U \) off the above tuple. This graph of groups has as the underlying graph the graph obtained from \( T_2 \) by identifying \( v \) and \( x_v \) for every \( v \in VT_2 - VT_1 \). The vertex and edge groups are simply the groups \( G_v \) and \( G_x \) given above. The number of edges of the induced splitting coincides with the number of edges of \( T_2 \) and the groups \( G_v \) coincide with \( Stab(v) \cap U \) for all \( v \); see [FRW] for details. Conversely for every subgroup \( U \) of \( G \) there exists such a tuple that makes \( U \) controlled.

We say that a subtree \( T^U \) of \( T \) is a **generating tree** for \( U \) if there exists a generating set \( M \) of \( U \) such that \( mT^U \cap T^U \neq \emptyset \) for all \( m \in M \). It is clear that \( UT^U \) is connected and \( U \)-invariant. In particular \( UT^U \) contains the minimal \( U \)-invariant subtree \( T_U \), it follows that \( UT^U \) has at least as many \( U \)-equivalence classes of edges as \( T_U \). It is further clear that \( gT^U \) is a generating tree of \( gUg^{-1} \) iff \( T^U \) is a generating tree for \( U \). We have shown the following:

**Lemma 1** Let \( T^U \) be a generating tree for \( U \). Then the induced splitting of \( U \) has at most as many edges as \( T^U \).
3 The proofs

Proof of Theorem 1 We study the action of $D$ on the Bass–Serre tree $T$ associated to the splitting $G = A * C B$. We assume that the number of edges of the induced splitting of $D$ with respect to the action on $T$ is minimal among all groups $D'$ such that the splitting $A * D D'$ can be obtained from $A * C D$ by a finite numbers of moves of type 1 or 2. Such $D'$ must exist since any splitting of a finitely generated group contains only finitely many edges and $D$ is finitely generated since $G$ and $C$ are finitely generated.

Let $x$ be the vertex fixed under the action of $A$, $y$ be the vertex fixed under the action of $B$ and $e = [x, y]$ be the edge fixed under the action of $C$. Let further $T_D \subset T$ be as in section 2. We distinguish the cases that $T_D$ lies in the component of $T - \{x\}$ containing $y$, that $T_D$ lies in a component of $T - \{x\}$ not containing $y$ and that $T_D$ contains $x$ where the last case has the subcases that $T_D$ contains $y$ and that it doesn’t.

Case 1: $T_D$ lies in the component of $T - \{x\}$ containing $y$. Choose the vertex $z$ of $T_D$ that has minimal distance to $x$, possibly we have $y = z$, but by assumption $z \neq x$. Choose a tuple $(T_1, T_2, \{G_v|v \in VT_2\}, \{t_v|v \in VT_2 - VT_1\})$ that makes $D$ controlled such that $z \in VT_1$. It is clear that $C \leq G_z$, otherwise $T_D \cap T_D = \emptyset$ for any $c \in C - G_z$ as $c$ fixes $x$ but not $z$. This contradicts the $D$-invariance of $T_D$.

It is easy to check that the subgroup generated by $A$ and $D$ is controlled by the tuple $(T'_1, T'_2, \{G_v|v \in VT'_2\}, \{t_v|v \in VT'_2 - VT'_1\})$ where $T'_i$ is the tree spanned by $T_i$ and $x$, the $t_v$ are as before, $G_x = A$, $G_v = C$ for all $v \in VT'_2 - VT_2$ with $v \neq x$ and all other $G_v$ are as before. This subgroup however is $G$ by assumption. It follows that the induced splitting has two vertices and therefore $VT'_1 = VT'_2 = \{x, y\}$ which implies that $D$ fixes $y$, i.e. $D \leq B$, and therefore $D = B$ since otherwise $A$ and $D$ do not generate $G$.

Case 2: $T_D$ lies in a component of $T - \{x\}$ not containing $y$. Choose $z$ and $(T_1, T_2, \{G_v|v \in VT_2\}, \{t_v|v \in VT_2 - VT_1\})$ as in the first case.

We argue as in the first case to show that $D$ fixes a vertex $z$ adjacent to $x$. Clearly $y \neq z$. So, there is an $a \in A$ such that $ae = [x, z]$. We have $D \leq \text{Stab}(z) = aBa^{-1}$ and $C \leq \text{Stab}(ae) = aCa^{-1}$. Now, $G = A * C D = (A * C) * C D = (A * aCa^{-1} aCa^{-1}) * C D = A * aCa^{-1} (aCa^{-1} * C D)$. We also clearly have $G = A * C B = A * aCa^{-1} aBa^{-1}$. Note, that the equality $X *z Y = X *z Y_1$ with $Y \leq Y_1$ implies $Y = Y_1$ for any amalgamated product. Therefore $aCa^{-1} * C D = aBa^{-1}$. Hence $B = C *a^{-1}Ca a^{-1}Da$.

This implies that the splitting $A * C D$ is obtained from the splitting $A * C B$ by a move of type 1.

Case 3: $T_D$ contains $x$ and $y$. Note that in this case $D$ is not elliptic with respect to the action on $T$, i.e. $T_D$ is by definition the minimal $D$-invariant subtree. Since minimal subtrees cannot contain valence 1 vertices it follows that $T_D$ must contain an edge $ae = a[x, y] = [ax, ay] = [x, z]$ different from $e$. It follows that $D$ splits over the subgroup $C_1 := \text{Stab}(ae) \cap D = \text{Stab}(ae) \cap (D \cap \text{Stab}(x)) = aCa^{-1} \cap C$. Since $D \cap \text{Stab}(x) = D \cap C$ we further get that $ae$ is not $(D \cap \text{Stab}(x))$-equivalent to $e$.

In the case that $D$ splits as an amalgamated product $D = D_1 * C_1 D_2$ we can choose a tuple $(T_1, T_2, \{G_v|v \in VT_2\}, \{t_v|v \in VT_2 - VT_1\})$ such that $ae \in ET_1$
and $e \in ET_2$ and that the two components $T^1$ and $T^2$ of $T_2-(x, z)$ are generating trees for $D_1$ and $D_2$, where we choose $D_1$ to be the group corresponding to the component containing $x$ and therefore also $e$. This implies in particular that $C \leq D_1$. We define $D_2 := a^{-1}D_2a$ and $\bar{D} = \langle D_1, D_2 \rangle$. The subtree $\bar{T} := T^1 \cup a^{-1}T^2$ is connected since $a^{-1}$ maps $z \in T^2$ to $y \in T^1$, it follows that $\bar{T}$ is a generating tree for $\bar{D}$. Since $\bar{T}$ has less edges than $T_1$ it follows from Lemma 1 that the induced splitting of $\bar{D}$ has fewer edges than the induced splitting of $D$. In order to get a contradiction to the minimality assumption we have to show that $G = A \ast_C D$ and that the splitting $G = A \ast_C D$ can be obtained from the splitting $G = A \ast_C D$ by a move of type 1.

We have $G = A \ast_C D = A \ast_C \langle D_1 \ast_{C_1} D_2 \rangle = (A \ast_C D_1) \ast_{C_1} D_2 = (A \ast_C D_1) \ast_{a^{-1}C_1 a} \ast_{a^{-1}D_2a} = (A \ast_C D_1) \ast_{C_2} D_2 = A \ast_C (D_1 \ast_{C_2} D_2) = A \ast_C \bar{D}$ where $C_2 = a^{-1}C_1 a = a^{-1}(C \cap aCa^{-1})a = a^{-1}C \cap C \leq C$. This calculation shows in particular that $\bar{D} = D_1 \ast_{C_2} D_2$.

In the case that $D$ splits as a HNN-extension $D = D_1 \ast_{C_1}$ we can choose a tuple $(T_1, T_2, \{G_i, v \in VT_2\}, \{t_v | v \in VT_2 - VT_1\})$ such that $ae = [x, z] \in ET_2 - ET_1$ and $e \in ET_2$. In particular $T^1 = T_2-(x, z)$ is connected and is a generating tree for $D_1$. We have $D = \langle D_1, t_z | t_zc_1 t_z^{-1} = \psi(c_1) \rangle$ for all $c_1 \in C_1$ where $\psi$ maps $C_1$ isomorphically to a subgroup of $D_1$. We define $\bar{D} = \langle D_1, t_z \rangle$. Since $t_z$ maps $z$ to a vertex $x_z$ of $T^1$ it follows $t_z(a)$ maps $y$ to $x_z$ which implies that $T^1$ is a generating tree for $\bar{D}$ which implies that the induced splitting of $\bar{D}$ has fewer edges than the induced splitting of $D$. In order to get a contradiction to the minimality assumption we have to check that $G = A \ast_C \bar{D}$ and that the splitting $G = A \ast_C \bar{D}$ can be obtained from the splitting $G = A \ast_C D$ by a move of type 2. The proof is analogous to the case of an amalgamated product.

We conclude with the proof of Theorem 2.

Proof of Theorem 2 Corollary 1 of [W] states that rank $(G) - 1$ is an upper bound on the number of factors of a decomposition of $G$ as an amalgamated product of type $\ast_{C_i} G_i$ provided that $C \neq 1$ and that $C \leq G$ is malnormal. This guarantees the existence of a maximal decomposition $G = \ast_{C_i} G_i$. This also follows from Z.Sela’s acylindrical accessibility result [Sl]. Suppose $G = A \ast_C B$ is an arbitrary splitting. In order to prove Theorem 1 it clearly suffices to show that for any $i \in \{1, \ldots, n\}$ either $G_i \leq A$ or $G_i \leq B$.

Choose $T$, $x$, $y$ and $e$ be as in the proof of Theorem 1. We have to show that $G_i$ fixes either $x$ or $y$. Define $T_i := T_{G_i}$. Let $z$ be the vertex of $T_i$ that is in minimal distance to $e$. As in the proof of Theorem 1 we see that $C \leq G_i \cap \text{Stab}(z)$. Since the action of $G$ on $T$ is 1-acylindrical, i.e. does not
fix a segment of length 2, this implies that $z = x$ or that $z = y$. W.l.o.g. we can assume that $z = x$. We have to show that $G_i$ fixes $x$. As $C \leq \text{Stab } (x) \cap G_i$, it follows that $C$ is a subgroup of a vertex group of the induced splitting of $G_i$. This implies that the induced splitting of $G_i$ does not contain a trivial edge group since otherwise $C$ is contained in a free factor of $G_i$ and therefore in a free factor of $G$ which contradicts the assumption.

We show that every edge of $T_i$ is $G_i$-equivalent to $e$. If $T_i$ contains an edge this implies that $G_i$ splits as an amalgamated product over $C$ which contradicts the maximality of the splitting $\bigast_{i=1}^{n} G_i$. Note that in this case $G_i$ cannot split as a HNN-extension since $x$ and $y$ are not $G$-equivalent. If $T_i$ contains no edge then $T_i$ consists of the vertex $x$ and therefore fixes $x$ which finishes the proof.

Let $f$ be an edge of $T_i$ and choose $w \in G$ such that $w e = f$. Since all edge stabilizers are non-trivial there exists an element $g \in G_i$ such that $g f = f$, in particular $g \in wCw^{-1}$, i.e. $g = wcw^{-1}$ for some $c \in C$. The malnormality of $C$ implies the malnormality of $G_i$, i.e. $G_i \cap wG_iw^{-1} = 1$ for all $w \in G - G_i$. Since $g \in G_i$ and $c \in C \leq G_i$ it follows that $w \in G_i$. Thus $e$ and $f$ are $G_i$-equivalent.

References


