

LIE ALGEBRAS WITH A COALGEBRA SPLITTING

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1. INTRODUCTION AND PRELIMINARIES

In their recent article [5], the authors endow every finite-dimensional simple complex Lie algebra \mathfrak{g} with a coalgebra structure such that the composition $\mu \circ \delta$ of the two structure maps $\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g}$ and $\mu : \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g} \longrightarrow \mathfrak{g}$ coincides with the identity. Moreover, the dual algebra $(\mathfrak{g}^*, \delta^*)$ associated to the Lie coalgebra is isomorphic to (\mathfrak{g}, μ) . The coalgebra map δ is given explicitly for $\mathfrak{sl}(n)$, those for the other types are obtained via embeddings $\mathfrak{g} \hookrightarrow \mathfrak{sl}(n)$.

The purpose of the present short note is to elicit the conceptual sources of [5], starting from the observation that the coalgebra maps defined in [5] are in fact homomorphisms of \mathfrak{g} -modules. For Lie algebras affording non-degenerate symmetric associative forms, such coalgebra maps naturally arise by dualizing the multiplication. This immediately implies the abovementioned duality, and the formulae displayed in [5, §4] can also be subsumed under our general approach.

By demanding that δ be \mathfrak{g} -linear, we depart from the usual compatibility condition of a Lie bialgebra, which requires δ to be a derivation. In that case, $\mu \circ \delta$ is a derivation of \mathfrak{g} . For fields of characteristic zero, only nilpotent Lie algebras afford invertible derivations (cf. [7]), so that non-zero Lie bialgebras over such fields never satisfy $\mu \circ \delta = \text{id}_{\mathfrak{g}}$.

Let k be a field. Given a finite-dimensional k -vector space V , we consider the k -linear maps

$$\tau_V : V \otimes_k V \longrightarrow V \otimes_k V \quad ; \quad u \otimes v \mapsto v \otimes u$$

and

$$\xi_V : V \otimes_k V \otimes_k V \longrightarrow V \otimes_k V \otimes_k V \quad ; \quad u \otimes v \otimes w \mapsto v \otimes w \otimes u.$$

Identifying $(V \otimes_k \cdots \otimes_k V)^*$ with $V^* \otimes_k \cdots \otimes_k V^*$, the transpose maps satisfy

$$\tau_V^* = \tau_{V^*}^{-1} \quad \text{as well as} \quad \xi_V^* = \xi_{V^*}^{-1}.$$

A pair (\mathfrak{g}, δ) consisting of a k -vector space \mathfrak{g} and a k -linear map $\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes_k \mathfrak{g}$ is called a *Lie coalgebra* if

- (a) $(\text{id}_{\mathfrak{g} \otimes_k \mathfrak{g}} + \tau_{\mathfrak{g}}) \circ \delta = 0$, and
- (b) $(\text{id}_{\mathfrak{g} \otimes_k \mathfrak{g} \otimes_k \mathfrak{g}} + \xi_{\mathfrak{g}} + \xi_{\mathfrak{g}}^2) \circ (\text{id}_{\mathfrak{g}} \otimes \delta) \circ \delta = 0$.

In the following, we consider a finite-dimensional Lie algebra \mathfrak{g} , defined over a field k (of arbitrary characteristic), whose bracket will be interpreted as a \mathfrak{g} -linear map $\mu : \mathfrak{g} \otimes_k \mathfrak{g} \longrightarrow \mathfrak{g}$. As usual, the *adjoint representation* $\text{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is defined via

$$(\text{ad } x)(y) = [x, y] \quad \forall x, y \in \mathfrak{g}.$$

Throughout, we assume that \mathfrak{g} possesses a non-degenerate, symmetric, associative form $(,) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, with corresponding isomorphism $\Theta : \mathfrak{g} \longrightarrow \mathfrak{g}^*$; $x \mapsto (x, -)$. We let $\mu^* : \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \otimes_k \mathfrak{g}^*$ be the transpose μ and define a \mathfrak{g} -linear map $\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes_k \mathfrak{g}$ via

$$\delta := (\Theta^{-1} \otimes \Theta^{-1}) \circ \mu^* \circ \Theta.$$

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Using Sweedler notation, this amounts to

$$(*) \quad \delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \Leftrightarrow (x, [y, z]) = \sum_{(x)} (x_{(1)}, y)(x_{(2)}, z) \quad \forall x, y, z \in \mathfrak{g}.$$

Our first subsidiary result shows that Theorems 5.1 and 5.3 of [5] hold in this wider context.

Lemma 1.1. *The following statements hold:*

- (1) (\mathfrak{g}, δ) is a Lie coalgebra.
- (2) The Lie algebras (\mathfrak{g}, μ) and $(\mathfrak{g}^*, \delta^*)$ are isomorphic.
- (3) We have $\text{ad} = \eta \circ (\Theta \otimes \text{id}_{\mathfrak{g}}) \circ \delta$, where $\eta : \mathfrak{g}^* \otimes_k \mathfrak{g} \rightarrow \text{Hom}_k(\mathfrak{g}, \mathfrak{g})$ is given by $\eta(f \otimes y)(x) = f(x)y$.

Proof. (1) In view of the above observations, we have

$$\begin{aligned} \tau_{\mathfrak{g}} \circ \delta &= \tau_{\mathfrak{g}} \circ (\Theta^{-1} \otimes \Theta^{-1}) \circ \mu^* \circ \Theta = (\Theta^{-1} \otimes \Theta^{-1}) \circ \tau_{\mathfrak{g}^*} \circ \mu^* \circ \Theta \\ &= (\Theta^{-1} \otimes \Theta^{-1}) \circ (\mu \circ \tau_{\mathfrak{g}}^{-1})^* \circ \Theta = -\delta. \end{aligned}$$

Owing to

$$\begin{aligned} (\text{id}_{\mathfrak{g}} \otimes \delta) \circ \delta &= (\Theta^{-1} \otimes \Theta^{-1} \otimes \Theta^{-1}) \circ (\text{id}_{\mathfrak{g}^*} \otimes \mu^*) \circ \mu^* \circ \Theta \\ &= (\Theta^{-1} \otimes \Theta^{-1} \otimes \Theta^{-1}) \circ (\mu \circ (\text{id}_{\mathfrak{g}} \otimes \mu))^* \circ \Theta, \end{aligned}$$

the Jacobi identity implies $(\text{id}_{\mathfrak{g} \otimes_k \mathfrak{g} \otimes_k \mathfrak{g}} + \xi_{\mathfrak{g}} + \xi_{\mathfrak{g}}^2) \circ (\text{id}_{\mathfrak{g}} \otimes \delta) \circ \delta = 0$.

- (2) By dualizing the identity $(\Theta \otimes \Theta) \circ \delta = \mu^* \circ \Theta$ and identifying \mathfrak{g} with $(\mathfrak{g}^*)^*$, we obtain

$$\delta^* \circ (\Theta^* \otimes \Theta^*) = \Theta^* \circ \mu,$$

so that $\Theta^* : \mathfrak{g} \rightarrow \mathfrak{g}^*$ is an isomorphism between the Lie algebras (\mathfrak{g}, μ) and $(\mathfrak{g}^*, \delta^*)$.

- (3) Using (*), we have $((\text{ad } x)(y), z) = \sum_{(x)} (x_{(1)}, y)(x_{(2)}, z) = \sum_{(x)} (\Theta(x_{(1)})(y)x_{(2)}, z)$ for every $z \in \mathfrak{g}$, whence $\text{ad } x = \eta \circ (\Theta \otimes \text{id}_{\mathfrak{g}}) \circ \delta(x)$. \square

We fix a basis $\{x_1, \dots, x_n\}$ of \mathfrak{g} along with its dual basis $\{y_1, \dots, y_n\}$ relative to our form $(,)$. Then

$$c_{\text{ad}} : \mathfrak{g} \rightarrow \mathfrak{g} ; \quad x \mapsto \sum_{j=1}^n (\text{ad } x_j) \circ (\text{ad } y_j)(x)$$

is the *Casimir operator* of the adjoint representation $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ of \mathfrak{g} (cf. [8, p.77f]). It is well-known that c_{ad} does not depend on the choice of the basis $\{x_1, \dots, x_n\}$. Moreover, if κ is another non-degenerate, symmetric, associative form, then there exists a \mathfrak{g} -linear automorphism $f : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $\kappa(x, y) = (x, f(y))$ for every $x, y \in \mathfrak{g}$. Accordingly,

$$\tilde{c}_{\text{ad}} := f^{-1} \circ c_{\text{ad}}$$

is the Casimir operator associated to κ .

Lemma 1.2. *The following statements hold:*

- (1) $\delta(x) = \sum_{j=1}^n [y_j, x] \otimes x_j$ for all $x \in \mathfrak{g}$.
- (2) If $\mathfrak{n} \triangleleft \mathfrak{g}$ is an ideal, then $\delta(\mathfrak{n}) \subseteq \mathfrak{n} \otimes_k \mathfrak{n}$.
- (3) $\mu \circ \delta = -c_{\text{ad}}$.

Proof. (1) Let x be an element of \mathfrak{g} . Writing

$$(\operatorname{ad} x)(x_j) = \sum_{i=1}^n \alpha_{ij}(x)x_i \quad \text{as well as} \quad (\operatorname{ad} x)(y_j) = \sum_{i=1}^n \beta_{ij}(x)y_i,$$

we obtain $\beta_{ij} = -\alpha_{ji}$ for $1 \leq i, j \leq n$ (cf. [6, p.27]). We put $\delta(x) = \sum_{r,s=1}^n \gamma_{rs}(x)y_r \otimes x_s$ and apply (1.1(3)) to see that

$$\begin{aligned} \sum_{i=1}^n \alpha_{ij}(x)x_i &= (\operatorname{ad} x)(x_j) = [\eta \circ (\Theta \otimes \operatorname{id}_{\mathfrak{g}}) \circ \delta(x)](x_j) = [\eta(\sum_{r,s=1}^n \gamma_{rs}(x)\Theta(y_r) \otimes x_s)](x_j) \\ &= \sum_{r,s=1}^n \gamma_{rs}(x)(y_r, x_j)x_s = \sum_{s=1}^n \gamma_{js}(x)x_s, \end{aligned}$$

whence $\gamma_{ij}(x) = \alpha_{ji}(x)$. Consequently,

$$\delta(x) = \sum_{i,j=1}^n \alpha_{ji}(x)y_i \otimes x_j = \sum_{i,j=1}^n -\beta_{ij}(x)y_i \otimes x_j = \sum_{j=1}^n [y_j, x] \otimes x_j,$$

as desired.

(2) Let $\mathfrak{n} \triangleleft \mathfrak{g}$ be an ideal. Owing to (1), we have $\delta(\mathfrak{n}) \subseteq \mathfrak{n} \otimes_k \mathfrak{g}$. Since $\tau \circ \delta = -\delta$, this implies

$$\delta(\mathfrak{n}) \subseteq (\mathfrak{n} \otimes_k \mathfrak{g}) \cap (\mathfrak{g} \otimes_k \mathfrak{n}) = \mathfrak{n} \otimes_k \mathfrak{n}.$$

(3) Application of μ to (1) yields

$$\mu(\delta(x)) = \sum_{j=1}^n [[y_j, x], x_j] = -\sum_{j=1}^n [x_j, [y_j, x]] = -c_{\operatorname{ad}}(x),$$

as claimed. \square

Example. For $n \geq 2$ we put $\mathbb{N}_n := \{1, \dots, n\}$ and consider the general linear Lie algebra $\mathfrak{g} := \mathfrak{gl}(n)$. Let $\{E_{r,s} ; (r,s) \in \mathbb{N}_n \times \mathbb{N}_n\}$ be the standard basis, given by the matrices $E_{r,s} := (\delta_{ir}\delta_{sj})_{1 \leq i,j \leq n}$. The trace form

$$(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \longrightarrow k \quad ; \quad (x, y) \mapsto \operatorname{tr}(xy)$$

is non-degenerate with $\{E_{s,r} ; (r,s) \in \mathbb{N}_n \times \mathbb{N}_n\}$ being the basis dual to the standard basis. Consequently, (1.2(1)) implies

$$\delta(x) = \sum_{r,s=1}^n [E_{s,r}, x] \otimes E_{r,s} \quad \forall x \in \mathfrak{gl}(n).$$

In particular,

$$\delta(E_{i,j}) = \sum_{s=1}^n E_{s,j} \otimes E_{i,s} - E_{i,s} \otimes E_{s,j}.$$

Owing to (1.2(2)), the restriction of δ to the ideal $\mathfrak{n} := \mathfrak{sl}(n)$ defines a comultiplication, which is proportional to the one given in [5, §4] by the factor $-2n$. Note that the factor $2n$ just accounts for the passage from the trace form to the Killing form (cf. [6, p.31]). Letting $I_n \in \mathfrak{gl}(n)$ be the identity matrix, we have

$$(\mu \circ \delta)(E_{i,j}) = 2(\delta_{ij}I_n - nE_{i,j}),$$

so that $\mu \circ \delta|_{\mathfrak{sl}(n)} = -2n \operatorname{id}_{\mathfrak{sl}(n)}$.

The Lie algebra $\mathfrak{sl}(n)$ thus affords an $\mathfrak{sl}(n)$ -linear comultiplication. However, if $2 < p := \operatorname{char}(k) \mid n$, then $\mathfrak{sl}(n)$ has center kI_n , which, by virtue of $\mathfrak{sl}(n) = [\mathfrak{sl}(n), \mathfrak{sl}(n)]$, is contained in the radical of

any associative form on $\mathfrak{sl}(n)$. If $\{f_{i,j} ; (i,j) \in \mathbb{N}_n \times \mathbb{N}_n\} \subseteq \mathfrak{gl}(n)^*$ is the basis dual to the standard basis, then

$$\gamma : \mathfrak{gl}(n) \longrightarrow \mathfrak{gl}(n)^* \quad ; \quad E_{i,j} \mapsto f_{j,i}$$

defines an isomorphism $\mathfrak{gl}(n) \xrightarrow{\sim} \mathfrak{gl}(n)^*$ that identifies $\mathfrak{sl}(n)$ with the ideal of those linear forms that annihilate I_n . (By (1.2(1)), the center $C(\mathfrak{g})$ of \mathfrak{g} is contained in $\ker \delta$.)

2. LIE ALGEBRAS WITH NON-SINGULAR CASIMIR OPERATORS

Retaining the notation of Section 1, we turn to those Lie algebras $\mathfrak{g} \neq (0)$ for which the map $\mu \circ \delta$ is invertible. According to (1.2(3)) this amounts to studying Lie algebras with non-singular Casimir operators. Our remarks concerning Casimir operators show that this property does not depend on the choice of the non-degenerate associative form.

Recall that a *restricted Lie algebra* $(\mathfrak{g}, [p])$ over a field of characteristic $\text{char}(k) = p > 0$ is a Lie algebra \mathfrak{g} together with a map $[p] : \mathfrak{g} \longrightarrow \mathfrak{g} ; x \mapsto x^{[p]}$ that satisfies the formal properties of an associative p -th power operator. The reader is referred to [12, Chap.II] for basic properties of restricted Lie algebras.

As before, we are working over a field k of arbitrary characteristic. An early result of Dieudonné [3] states that every semisimple Lie algebra with a non-degenerate symmetric associative bilinear form is a direct sum of simple ideals. In our case, the semisimplicity of \mathfrak{g} is a consequence of invertibility of c_{ad} .

Proposition 2.1. *Suppose that c_{ad} is non-singular. Then the following statements hold:*

- (1) *The adjoint representation is a direct summand of $\mathfrak{g} \otimes_k \mathfrak{g}$.*
- (2) *Every derivation of \mathfrak{g} is inner.*
- (3) *$\mathfrak{g} = \mathfrak{g}_1 \perp \mathfrak{g}_2 \perp \cdots \perp \mathfrak{g}_n$ is an orthogonal direct sum of simple Lie algebras, with each summand having a non-degenerate symmetric associative form with non-singular Casimir operator.*
- (4) *There exist a \mathfrak{g} -linear comultiplication $\delta' : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes_k \mathfrak{g}$ such that $\mu \circ \delta' = \text{id}_{\mathfrak{g}}$.*
- (5) *If $\text{char}(k) = p > 0$, then \mathfrak{g} is a restricted Lie algebra with unique p -map and with each \mathfrak{g}_i being a p -subalgebra.*

Proof. (1) Since μ is a split-surjective homomorphism of \mathfrak{g} -modules, our assertion follows.

(2) This is a direct consequence of Whitehead's Lemma, [8, p.77].

(3) Let $\mathfrak{n} \triangleleft \mathfrak{g}$ be an ideal. Directly from the definition we obtain $c_{\text{ad}}(\mathfrak{n}) \subseteq \mathfrak{n}$, and our current assumption implies that this inclusion is in fact an equality. Now let \mathfrak{n} be abelian. Thanks to (1.2(2),(3)), we have

$$\mathfrak{n} = c_{\text{ad}}(\mathfrak{n}) = (\mu \circ \delta)(\mathfrak{n}) \subseteq \mu(\mathfrak{n} \otimes_k \mathfrak{n}) = (0).$$

On the other hand, the orthogonal complement \mathfrak{n}^\perp of any ideal $\mathfrak{n} \triangleleft \mathfrak{g}$ is an ideal, and for $x, y \in \mathfrak{n} \cap \mathfrak{n}^\perp$ and $z \in \mathfrak{g}$, we have

$$([x, y], z) = (x, [y, z]) \in (\mathfrak{n}, \mathfrak{n}^\perp) = (0).$$

Thus, $\mathfrak{n} \cap \mathfrak{n}^\perp$ is abelian, so that $\mathfrak{n} \cap \mathfrak{n}^\perp = (0)$. As a result, any ideal \mathfrak{n} of \mathfrak{g} defines a direct sum decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{n}^\perp$$

of \mathfrak{g} -modules. This means that the adjoint representation endows \mathfrak{g} with the structure of a semi-simple \mathfrak{g} -module, whose simple constituents are simple Lie algebras. Consider two constituents $\mathfrak{g}_i \neq \mathfrak{g}_j$ of \mathfrak{g} , so that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_i \cap \mathfrak{g}_j = (0)$. Since $[\mathfrak{g}_j, \mathfrak{g}_j] = \mathfrak{g}_j$, we obtain

$$(\mathfrak{g}_i, \mathfrak{g}_j) = (\mathfrak{g}_i, [\mathfrak{g}_j, \mathfrak{g}_j]) \subseteq ([\mathfrak{g}_i, \mathfrak{g}_j], \mathfrak{g}_j) = (0).$$

Consequently, the above decomposition is an orthogonal decomposition, and the restriction $(,)|_{\mathfrak{g}_i \times \mathfrak{g}_i}$ is non-degenerate. By choosing dual bases $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_m\}$ that are compatible with the orthogonal decomposition, we see that $c_{\text{ad}}|_{\mathfrak{g}_i}$ is the Casimir operator of the form $(,)|_{\mathfrak{g}_i \times \mathfrak{g}_i}$. Thus, each \mathfrak{g}_i also affords a non-singular Casimir operator.

(4) Let $f : \mathfrak{g} \rightarrow \mathfrak{g}$ be \mathfrak{g} -linear. Directly from (1.2(1)) we obtain $\delta \circ f = (f \otimes \text{id}_{\mathfrak{g}}) \circ \delta$, whence $(\text{id}_{\mathfrak{g}} \otimes f) \circ \delta = \tau_{\mathfrak{g}} \circ (f \otimes \text{id}_{\mathfrak{g}}) \circ \delta = \tau_{\mathfrak{g}} \circ \delta \circ f = \delta \circ (-f)$. This implies

$$(\text{id}_{\mathfrak{g}} \otimes (\delta \circ f)) \circ \delta \circ f = (\text{id}_{\mathfrak{g}} \otimes \delta) \circ (\text{id}_{\mathfrak{g}} \otimes f) \circ \delta \circ f = (\text{id}_{\mathfrak{g}} \otimes \delta) \circ \delta \circ (-f^2).$$

As a result, $\delta \circ f$ is also a comultiplication.

By specializing $f = -c_{\text{ad}}^{-1}$, we conclude from (1.2(3)) that $\delta' := \delta \circ f$ has the requisite property.

(5) By (2), the p -th power $(\text{ad } x)^p$ of any inner derivation is inner. Thus, [12, (II.2.3)] provides a p -map on \mathfrak{g} . According to (3), the algebra \mathfrak{g} has trivial center, so that [12, (II.2.2)] yields the unicity of the p -map. Clearly, each \mathfrak{g}_i is a p -subalgebra. \square

As a first consequence, we record the following generalization of [5, Thm.6.1]:

Theorem 2.2. *Let \mathfrak{g} be a semisimple Lie algebra over a field k of characteristic 0. Then there exists a comultiplication $\delta' : \mathfrak{g} \rightarrow \mathfrak{g} \otimes_k \mathfrak{g}$ such that $\mu \circ \delta' = \text{id}_{\mathfrak{g}}$.*

Proof. By Cartan's solvability criterion [12, (I.7.8)], the Lie algebra \mathfrak{g} is a direct sum of simple Lie algebras, whose constituents possess non-degenerate Killing forms. Thus, the restriction $c_{\text{ad}}|_{\mathfrak{g}_i}$ of the Casimir operator to each of these ideals is either invertible or zero. Since $\text{tr}(c_{\text{ad}}|_{\mathfrak{g}_i}) = \dim_k \mathfrak{g}_i \cdot 1 \in k \setminus \{0\}$, the former alternative applies, and our result follows from (2.1(4)). \square

The foregoing result reduces us to the consideration of simple Lie algebras over fields of positive characteristic. If the base field k is algebraically closed, the Block-Wilson classification theorem [2] asserts that these are either classical or of Cartan type. The classical simple Lie algebras are described in [11, Chap.II], those of Cartan type can be found in [12, Chap.IV].

Thanks to [1, 9, 10], all classical simple Lie algebras of characteristic $p \geq 7$ possess a non-degenerate symmetric associative form. The following Lemma implies that certain simple Lie algebras do not afford invertible Casimir operators.

Lemma 2.3. *Let \mathfrak{g} be a simple Lie algebra.*

- (1) *If $\mathfrak{g} = \bigoplus_{i=-r}^s \mathfrak{g}_i$ ($r, s \geq 0$) is \mathbb{Z} -graded and c_{ad} is non-singular, then $r = s$.*
- (2) *If $\text{char}(k) = p > 0$, then $\mathfrak{g} = \mathfrak{psl}(pn)$ does not possess a non-singular Casimir operator.*

Proof. (1) According to [4, (3.1)], we have

$$(\mathfrak{g}_i, \mathfrak{g}_j) = 0 \quad \text{for } i + j \neq s - r.$$

Consequently, the linear map c_{ad} is homogeneous of degree $s - r$. Being non-singular, c_{ad} must have degree 0, whence $s = r$.

(2) Since $\mathfrak{psl}(pn)$ is the derived algebra of $\mathfrak{gl}(pn)/kI_{pn}$, not all derivations of $\mathfrak{psl}(pn)$ are inner. Thus, (2.1(2)) implies the assertion. \square

We now consider the classical simple Lie algebras of types B_ℓ, C_ℓ and D_ℓ over an algebraically field k of arbitrary characteristic. By definition, these algebras are Lie algebras of linear transformations on an n -dimensional vector space V , with $n = 2\ell + 1$ for type B_ℓ and $n = 2\ell$, otherwise. If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is one of these algebras, then the trace form

$$(x, y) = \text{tr}(xy) \quad \forall x, y \in \mathfrak{g}$$

is non-degenerate.

In the sequel, we let κ be the Killing form of \mathfrak{g} . Since k is algebraically closed, there exists an element $m_{\mathfrak{g}} \in k$ such that

$$\kappa = m_{\mathfrak{g}}(\cdot, \cdot).$$

Suppose that $m_{\mathfrak{g}} \neq 0$. Letting c_{κ} and c_{tr} be the Casimir operators of \mathfrak{g} relative to these two forms, our remarks preceding (1.2) yield

$$c_{\text{tr}} = m_{\mathfrak{g}} c_{\kappa}.$$

If $\text{char}(k) = 0$, then $c_{\kappa} = \text{id}_{\mathfrak{g}}$, so that we have

$$c_{\text{tr}} = m_{\mathfrak{g}} \text{id}_{\mathfrak{g}}$$

in that case.

Proposition 2.4. *Suppose that $p := \text{char}(k) > 2$. Let \mathfrak{g} be a classical simple Lie algebra of type B_{ℓ}, C_{ℓ} , or D_{ℓ} . Then \mathfrak{g} affords an invertible Casimir operator if and only if:*

- (1) \mathfrak{g} is of type B_{ℓ} , and $p \nmid 2\ell - 1$, or
- (2) \mathfrak{g} is of type C_{ℓ} , and $p \nmid \ell + 1$, or
- (3) \mathfrak{g} is of type D_{ℓ} , and $p \nmid \ell - 1$.

Proof. Let $\mathcal{C}(\mathfrak{g}) = (c_{ij}(\mathfrak{g}))$ be the Cartan matrix of \mathfrak{g} . Then we have

$$\det(\mathcal{C}(\mathfrak{g})) = \begin{cases} 2 & \text{if } \mathfrak{g} \text{ has type } B_{\ell} \text{ or } C_{\ell} \\ 4 & \text{if } \mathfrak{g} \text{ has type } D_{\ell}, \end{cases}$$

see [6, p.63].

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}$ be a root space decomposition relative to some Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Given $h \in \mathfrak{h}$, we have

$$\kappa(h, h) = \text{tr}((\text{ad } h)^2) = 2 \sum_{\alpha > 0} \alpha(h)^2.$$

Accordingly, the constant $m_{\mathfrak{g}}$ is determined by

$$2 \sum_{\alpha > 0} \alpha(h)^2 = m_{\mathfrak{g}}(h, h).$$

Using the Chevalley bases given on pages 139–141 of [8], we obtain

$$m_{\mathfrak{g}} = \begin{cases} 2\ell - 1 & \text{if } \mathfrak{g} \text{ has type } B_{\ell} \\ 2(\ell + 1) & \text{if } \mathfrak{g} \text{ has type } C_{\ell} \\ 2(\ell - 1) & \text{if } \mathfrak{g} \text{ has type } D_{\ell}. \end{cases}$$

We denote the aforementioned Chevalley basis by $\{h_1, \dots, h_{\ell}\} \cup \{e_{\alpha} ; \alpha \in R\}$. Direct computation shows $(e_{\alpha}, e_{-\alpha}) \in \{1, 2\}$. Let $\{h'_1, \dots, h'_{\ell}\}$ be the basis of \mathfrak{h} that is dual to $\{h_1, \dots, h_{\ell}\}$ relative to the non-degenerate form $(\cdot, \cdot)|_{\mathfrak{h} \times \mathfrak{h}}$. Writing $h'_j = \sum_{i=1}^{\ell} s_{ij} h_i$, we obtain

$$\delta_{ij} = (h_i, h'_j) = \sum_{n=1}^{\ell} s_{nj} (h_i, h_n) = \sum_{n=1}^{\ell} \frac{1}{2} c_{in}(\mathfrak{g})(h_i, h_i) s_{nj},$$

so that the base change is described by the inverse of the matrix $(\frac{1}{2} c_{ij}(\mathfrak{g})(h_i, h_i))_{i,j} = \mathcal{C}(\mathfrak{g})$.

Let $\mathfrak{g}_{\mathbb{C}}$ be the complex Lie algebra of the same type, $\mathfrak{g}_{\mathbb{Z}} \subseteq \mathfrak{g}_{\mathbb{C}}$ be the integral form given by the Chevalley basis. By the above, the operator

$$4c_{\text{tr}} = \sum_{i=1}^{\ell} (\text{ad } h_i) \circ (\text{ad}(4h'_i)) + \sum_{\alpha \in R} (\text{ad } e_{\alpha}) \circ (\text{ad}(\frac{4}{(e_{\alpha}, e_{-\alpha})} e_{-\alpha}))$$

sends $\mathfrak{g}_{\mathbb{Z}}$ to $\mathfrak{g}_{\mathbb{Z}}$. Since $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$, the Casimir operator $4c_{\text{tr}}$ on \mathfrak{g} can be computed from $4c_{\text{tr},\mathbb{Z}}$ via

$$4c_{\text{tr}} = 4c_{\text{tr},\mathbb{Z}} \otimes \text{id}_k.$$

In view of $4c_{\text{tr},\mathbb{Z}} = 4m_{\mathfrak{g}} \text{id}_{\mathfrak{g}_{\mathbb{Z}}}$, we obtain

$$c_{\text{tr}} = m_{\mathfrak{g}} \text{id}_{\mathfrak{g}},$$

so that c_{tr} is invertible if and only if $m_{\mathfrak{g}} \not\equiv 0 \pmod{p}$. \square

According to [10] the exceptional Lie algebras of types G_2, F_4, E_6, E_7, E_8 have non-degenerate Killing forms for $p \geq 7$. The dimension formulas for these algebras provide a rough sufficient condition for the invertibility of the corresponding Casimir operators.

Let \mathfrak{g} be a classical Lie algebra. A prime number $p > 7$ is called *suitable* for \mathfrak{g} provided

$$\begin{aligned} p \nmid \ell + 1 & \quad \text{if } \mathfrak{g} \text{ has type } A_{\ell} \text{ or } C_{\ell} \\ p \nmid 2\ell - 1 & \quad \text{if } \mathfrak{g} \text{ has type } B_{\ell} \\ p \nmid \ell - 1 & \quad \text{if } \mathfrak{g} \text{ has type } D_{\ell} \\ p \neq 13 & \quad \text{if } \mathfrak{g} \text{ has type } F_4 \text{ or } E_6 \\ p \neq 19 & \quad \text{if } \mathfrak{g} \text{ has type } E_7 \\ p \neq 31 & \quad \text{if } \mathfrak{g} \text{ has type } E_8. \end{aligned}$$

Summing up, we obtain the following result:

Theorem 2.5. *Let k be an algebraically closed field of characteristic $p > 7$, \mathfrak{g} be a finite-dimensional Lie algebra over k .*

(1) *If \mathfrak{g} possesses a non-singular Casimir operator, then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n$ is a direct sum of classical simple Lie algebras. Moreover, for each constituent of type A, B, C or D , the prime p is suitable.*

(2) *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n$ is a direct sum of classical simple Lie algebras, with p being suitable for each constituent, then \mathfrak{g} possesses a non-singular Casimir operator and there exists a comultiplication $\delta' : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes_k \mathfrak{g}$ with $\mu \circ \delta' = \text{id}_{\mathfrak{g}}$.*

Proof. (1) By (2.1(3),(5)) we may decompose \mathfrak{g} into a direct sum of restricted simple Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, with each summand having a non-singular Casimir operator. By the Block-Wilson theorem [2], each \mathfrak{g}_i is classical or of Cartan type. If L is a restricted Lie algebra of Cartan type, then $L = \bigoplus_{i=-r}^s L_i$ is \mathbb{Z} -graded, with $r \neq s$ (cf. [12, (IV.2-IV.5)]). Thanks to (2.3(1)), these algebras do not afford non-singular Casimir operators. A consecutive application of (2.3(2)) and (2.4) implies that p is suitable for the simple constituents of the given types.

(2) This is a consequence of the above remarks concerning the exceptional types, the example following (1.2), (2.4), as well as (2.1(4)). \square

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