Analysis of invariant PDO's on the Heisenberg group

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1 Analysis on the Heisenberg group: Basic Facts

(see e.g.[7], [24])

1.1 The Heisenberg group and its automorphisms

The *Heisenberg group* $\mathbb{H}_n$ is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, endowed with the product

$$(x, y, u) \cdot (x', y', u') := (x + x', y + y', u + u' + \frac{1}{2}(x \cdot y' - y \cdot x')).$$

Observe:

- 0 is the neutral element of $\mathbb{H}_n$
- $(x, y, u)^{-1} = (-x, -y, -u)$
- Writing $z = (x, y) \in \mathbb{R}^{2n}$, and regarding z as a column vector, we may regard $\mathbb{H}_n$ also as $\mathbb{R}^{2n} \times \mathbb{R}$, with product

$$(z, u) \cdot (z', u') = (z + z', u + u' + \frac{1}{2}\langle z, z'\rangle), \tag{1.1}$$

where $\langle\ ,\ \rangle$ denotes the *canonical symplectic form*

$$\langle z, w\rangle := {}^t z \cdot J \cdot w, \qquad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

on $\mathbb{R}^{2n}$.

Exercise: $\mathbb{H}_n$ is isomorphic to the group of upper triangular matrices

$$\begin{pmatrix} 1 & p_1 & \dots & p_n & t \\ & 1 & & 0 & q_1 \\ & & \ddots & & \vdots \\ & & & 1 & q_n \\ & & & & 1 \end{pmatrix}, \qquad p_j, q_j, t \in \mathbb{R}.$$

If ω is any symplectic bilinear form on a finite dimensional vector space V over $\mathbb{R}$ or $\mathbb{C}$, denote by

$$Sp(\omega) := \{T \in L(V, V) :\ \omega(Tz, Tw) = \omega(z, w)\ \forall z, w \in V\} \tag{1.2}$$

the corresponding *symplectic group*. If $\omega = \langle\ ,\ \rangle$, we also write $Sp(n,\mathbb{R})$ resp. $Sp(n,\mathbb{C})$ for these groups. Notice:

$$T \in Sp(n,\mathbb{R}) \iff {}^tT \cdot J \cdot T = J$$

If $t \mapsto T(t)$ is a smooth curve in $Sp(\omega)$ with $T(0) = I$, one finds from (1.2) that $S := \frac{dT}{dt}(0)$ satisfies

$$\omega(Sz, w) + \omega(z, Sw) = 0, \tag{1.3}$$

i.e. S is skew symmetric w.r. to ω.

This shows that the Lie algebra $\mathfrak{sp}(\omega)$ of $Sp(\omega)$ consists of all linear endomorphisms S of V satisfying (1.3). In particular,

$$\mathfrak{sp}(n,\mathbb{R}) := \mathrm{Lie}(Sp(n,\mathbb{R})) = \{S :\ {}^tSJ + JS = 0\}.$$

The Lie bracket in $\mathfrak{sp}(n,\mathbb{R})$ is just the commutator

$$[S_1, S_2] = S_1S_2 - S_2S_1.$$

- If $T \in Sp(n,\mathbb{R})$, we identify T with the automorphism

  $$T(z,u) := (Tz, u)$$

  of $\mathbb{H}_n$, so that $Sp(n,\mathbb{R})$ embeds into the automorphism group $\mathrm{Aut}\,(\mathbb{H}_n)$ of $\mathbb{H}_n$.

- Further automorphisms are the (anisotropic) *dilations*

  $$\delta_r(z,u) := (rz, r^2u),\ \ r > 0,$$

  and the "*Cartan involution*"

  $$\theta(x,y,u) := (x, -y, -u).$$

Proposition 1.2 $\mathrm{Aut}\,(\mathbb{H}_n)$ *is generated by* $Sp(n,\mathbb{R})$*, the dilations* δ_r*, the inner automorphisms and* θ*.*

1.2 Integration on $\mathbb{H}_n$

The Lebesgue measure $dg := dzdu$ is a *bi-invariant Haar measure* on $\mathbb{H}_n$, i.e.

$$\int_{\mathbb{H}_n} f(hg)\,dg = \int_{\mathbb{H}_n} f(gh)\,dg = \int_{\mathbb{H}_n} f(g)\,dg \qquad \forall h \in \mathbb{H}_n.$$

The *convolution* of two suitable functions (or distributions) f_1, f_2 on $\mathbb{H}_n$ is defined by

$$\begin{aligned} f_1 \star f_2(g) &:= \int_{\mathbb{H}_n} f_1(h) f_2(h^{-1}g)\,dh \\ &= \int_{\mathbb{H}_n} f_1(gh^{-1}) f_2(h)\,dh. \end{aligned}$$

Define the *reflection at the origin* and the *involution* of f by

$$\check{f}(g) := f(g^{-1}) \text{ and } f^*(g) := \overline{f(g^{-1})}, \text{ respectively.}$$

Then, for $\sharp = \check{}\,, {}^*$, one has $(f^\sharp)^\sharp = f$, and

$$(f_1 \star f_2)^\sharp = f_2^\sharp \star f_1^\sharp, \qquad ||f^\sharp||_{L^1} = ||f||_{L^1}.$$

Notice that the *group algebra* $L^1(\mathbb{H}_n, +, \star, {}^*)$ is a non-commutative involutive Banach algebra.

Remarks. (a) Identify $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $(z_1, \dots, z_n) := (x_1 + iy_1, \dots, x_n + iy_n) \in \mathbb{C}^n$, and call f *polyradial*, if $f(z, u) = \tilde{f}(|z_1|, \dots, |z_n|, u)$ for some function $\tilde{f}$ on $\mathbb{R}^n_+ \times \mathbb{R}$. Under this identification of the underlying manifold of $\mathbb{H}_n$ with $\mathbb{C}^n$, the n-torus $\mathbb{T}^n = \{(e^{i\varphi_1}, \dots e^{i\varphi_n}) : \varphi_i \in [0, 2\pi[\}$ acts by (symplectic) automorphisms $(z_1, \dots, z_n, u) \mapsto (e^{i\varphi_1} z_1, \dots, e^{i\varphi_n} z_n, u)$ on $\mathbb{H}_n$, and f is polyradial iff $f \circ \tau = f \; \forall \tau \in \mathbb{T}^n$. Consequently, since

$$(f_1 \star f_2) \circ \alpha = (f_1 \circ \alpha) \star (f_2 \circ \alpha)$$

for every $f_1, f_2 \in L^1(\mathbb{H}_n)$ and $\alpha \in \mathrm{Aut}\,(\mathbb{H}_n)$ with $\det D\alpha = 1$,

$$L^1_{\mathrm{pr}}(\mathbb{H}_n) := \{ f \in L^1(\mathbb{H}_n) : f \text{ is polyradial } \}$$

forms a subalgebra of $L^1(\mathbb{H}_n)$. Even more is true:

Proposition 1.2 *$L^1_{\mathrm{pr}}(\mathbb{H}_n)$ is a commutative involutive Banach algebra.*

Proof. If $f \in L^1_{\mathrm{pr}}(\mathbb{H}_n)$, then $\check{f} = f \circ \theta$. Hence, for $f_1, f_2 \in L^1_{\mathrm{pr}}(\mathbb{H}_n)$,

$$\begin{aligned} f_1 \star f_2 &= (\check{f}_2 \star \check{f}_1)\check{} = ((f_2 \circ \theta) \star (f_1 \circ \theta))\check{} \\ &= ((f_2 \star f_1) \circ \theta)\check{} = ((f_2 \star f_1)\check{})\check{} = f_2 \star f_1. \end{aligned}$$

Q.E.D.

(b) If one replaces $\mathbb{T}^n$ by the unitary group $U(n)$ in this discussion, one finds in a similar way that the *radial* L^1-functions f, i.e. functions which depend only on $|z| := (|z_1|^2 + \ldots + |z_n|^2)^{1/2}$ and u, form a commutative subalgebra $L^1_{\mathrm{r}}(\mathbb{H}_n)$ of $L^1(\mathbb{H}_n)$.

For $(z, u) \in \mathbb{H}_n$, define the so-called *Koranyi-norm* by

$$|(z, u)| := (|z|^4 + 16u^2)^{1/4} = ||z|^2 \pm 4iu|^{1/2}. \tag{1.4}$$

It has the following properties (Exercise):

(i) $|\delta_r g| = r|g| \qquad \forall g \in \mathbb{H}_n,\; r > 0.$

(ii) $|g| = 0 \iff g = 0.$

(iii) $|g^{-1}| = |g|.$

(iv) $|g_1 g_2| \le |g_1| + |g_2| \quad \forall g_1, g_2 \in \mathbb{H}_n.$

In particular, $|\cdot|$ is a so-called *homogeneous norm*, and $d_K(g_1, g_2) := |g_1^{-1} g_2|$ is a left-invariant metric on $\mathbb{H}_n$.

Remark 1.3 *$\mathbb{H}_n$, endowed with the Koranyi-metric d_K and the Haar measure, forms a space of homogeneous type in the sense of Coifman and Weiss.*

Denote by

$$B_r(g) := \{ h \in \mathbb{H}_n : |g^{-1} h| < r \}$$

the open ball of radius $r > 0$ centered at $g \in \mathbb{H}_n$. Then, by left-invariance and (i),

$$|B_r(g)| = |B_r(0)| = |\delta_r(B_1(0))| = r^Q |B_1(0)|,$$

where

$$Q = 2n + 2$$

is the so-called *homogeneous dimension* of $\mathbb{H}_n$.

1.3 Left-invariant differential operators on $\mathbb{H}_n$

A linear operator $T : \mathcal{S}(\mathbb{H}_n) \to \mathcal{S}'(\mathbb{H}_n)$ is called *left* respectively *right - invariant*, if

$$T(\lambda_g\varphi) = \lambda_g(T\varphi) \quad \text{respectively} \quad T(\varrho_g\varphi) = \varrho_g(T\varphi)$$

for every $g \in G,\ \varphi \in \mathcal{S}$, where λ and ϱ denote the *left-regular* and *right-regular action*

$$(\lambda_g\varphi)(h) := \varphi(g^{-1}h), \quad (\varrho_g\varphi)(h) := \varphi(hg).$$

T is called *homogeneous of degree* $\alpha \in \mathbb{C}$, if

$$T(\varphi \circ \delta_r) = r^\alpha (T\varphi) \circ \delta_r \qquad \forall r > 0,\ \varphi \in \mathcal{S}.$$

The Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$

Identify the tangent space $T_0\mathbb{H}_n$ with $\mathbb{R}^{2n} \times \mathbb{R}$. For $X \in T_0\mathbb{H}_n$, let L_X denote the Lie-derivative

$$(L_X\varphi)(g) := \frac{d}{dt}\varphi(g \cdot \gamma(t))|_{t=0},$$

where $\gamma : [0,1] \to \mathbb{H}_n$ is any smooth curve with $\gamma(0) = 0,\ \dot\gamma(0) = X$. Then L_X is a left-invariant vector field on $\mathbb{H}_n$, and the mapping $X \to L_X$ is bijective from $T_0\mathbb{H}_n$ onto the space of all left-invariant real vector fields on $\mathbb{H}_n$. In particular, the *Lie bracket* $[\ ,\ ]$ on $T_0\mathbb{H}_n$ can be defined by

$$L_{[X,Y]} = [L_X, L_Y] := L_X L_Y - L_Y L_X.$$

$T_0\mathbb{H}_n$, endowed with $[\ ,\ ]$, forms the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$. As usually, we shall henceforth identify $X \in \mathfrak{h}_n$ with the corresponding Lie derivative L_X.

One computes easily that a basis of $\mathfrak{h}_n$ is given by the vector fields

$$(1.5) \quad X_j := \frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial u}, \qquad Y_j := \frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial u}, \qquad j = 1, \dots, n, \quad \text{and} \quad U := \frac{\partial}{\partial u}.$$

These satisfy the "*Heisenberg commutation relations*"

$$\begin{aligned} [X_j, Y_k] &= \delta_{jk}U, \\ [X_j, X_k] &= [Y_j, Y_k] = 0, \\ [X_j, U] &= [Y_j, U] = 0. \end{aligned}$$

Observe: The X_j, Y_j are homogeneous of degree 1, the "central derivative" U is homogenous of degree 2.

If $n = 1$, we shall often write X, Y in place of X_1, Y_1.

Notice that the exponential mapping $\exp : \mathfrak{h}_n \to \mathbb{H}_n$ is the identity mapping.

Denote by $\mathfrak{u}(\mathfrak{h}_n)$ the associative algebra of all left-invariant differential operators on $\mathbb{H}_n$. $\mathfrak{u}(\mathfrak{h}_n)$ can be identified with the *universal enveloping algebra* of $\mathfrak{h}_n$. In particular, it is generated by the elements of $\mathfrak{h}_n$.

2 Local solvability

(see e.g. [9])

Let $P = P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ be a linear PDO on $\mathbb{R}^d$ of order m, where $D^\alpha = D_1^{\alpha_1} \dots D_d^{\alpha_d}$, $D_j = \frac{1}{2\pi i} \frac{\partial}{\partial x_j}$. Denote by

$$P_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha, \quad (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

its *principal symbol.* Assume that the coefficients a_α are smooth.

P is said to be *locally solvable (l.s.)* at $x^0 \in \mathbb{R}^d$ if there exists an open neighborhood Ω of x^0, such that for every $f \in C_0^\infty(\Omega)$ there exists a distribution $u \in \mathcal{D}'(\Omega)$ solving the equation

$$Pu = f \qquad \text{in } \Omega. \tag{2.1}$$

We call P locally solvable (in $\mathbb{R}^d$), if it is locally solvable at every $x^0 \in \mathbb{R}^d$.

Remark 2.1 *By the theorem of Malgrange/Ehrenpreis, every constant coefficient PDO is locally solvable.*

Example 2.2 Consider the left-invariant complex vector field

$$Z = X + iY \qquad \text{on } \mathbb{H}_1.$$

This is just the famous *Lewy-operator*, historically the first example of a linear PDO which is nowhere locally solvable.

Observe: A left-invariant PDO on a Lie group is l.s. at one point of the group iff it is l.s. at every other point.

Shortly after Lewy's example, Hörmander produced the following

Theorem 2.3 (Hörmander's criterion) *Assume there exists $\xi^0 \in \mathbb{R}^d$ s.t.*

$$P_m(x^0, \xi^0) = 0 \quad \text{and} \quad \{\Re\mathrm{e} P_m, \Im\mathrm{m} P_m\}(x^0, \xi^0) \ne 0, \tag{H}$$

where

$$\{a, b\} := \sum_{j=1}^d \left(\frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial b}{\partial \xi_j} \right)$$

denotes the Poisson bracket of a and b. Then $P(x, D)$ is not locally solvable at x^0.

Recall that ξ^0 is called *characteristic* for P at x^0, if $P_m(x^0, \xi^0) = 0$.

The lengthy proof makes use of the following

Basic Lemma 2.4 *The equation (2.1) can be solved in Ω if and only if the following holds true:*

For every relatively compact open subset $\Lambda \subset \Omega$ (shortly: $\Lambda \Subset \Omega$) there exist constants C and $k \in \mathbb{N}$, s.t. for every $f, v \in C_0^\infty(\Lambda)$,

$$\left| \int f v \, dx \right| \le C \sum_{|\alpha| \le k} ||D^\alpha f||_2 \sum_{|\beta| \le k} ||D^\beta \, {}^tPv||_2 \tag{2.2}$$

Here, ${}^tP$ denotes the formal transposed of P, defined by

$$\int v(Pu) \, dx = \int ({}^tPv) u \, dx.$$

Proof. The sufficiency of (2.2) follows by Hahn-Banach (exercise).

Conversely, if $Pu = f$ can be solved for every $f \in \mathcal{D}(\Omega)$ by some $u \in \mathcal{D}'(\Omega)$, then

$$(*) \qquad \langle f, v\rangle = \int f v dx = \langle u, {}^tPv\rangle \quad \forall v \in \mathcal{D}(\Lambda).$$

Consider $\langle f, v\rangle$ as a bilinear form on $C_0^\infty(\overline{\Lambda}) \times C_0^\infty(\Lambda)$, where $C_0^\infty(\overline{\Lambda})$ is a Frechet space with the topology induced by the semi-norms $||D^\alpha f||_2$, and where $C_0^\infty(\Lambda)$ is endowed with the metrizable topology induced by the semi-norms $||D^\beta\, {}^tPv||_2$.

Obviously, $f \mapsto \langle f, v\rangle$ is continuous for fixed v.

The continuity of $v \mapsto \langle f, v\rangle$, for fixed f, follows on the other hand by $(*)$.

Thus, $(f, v) \mapsto \langle f, v\rangle$ is separately continuous, hence continuous, by Banach-Steinhaus. This proves (2.2).

Q.E.D.

Remark 2.5 *Condition (2.2) is equivalent to*

$$(2.3) \qquad ||v||_{(-k)} \le C||\,{}^tPv||_{(k)},$$

where $||f||_{(\alpha)} = (\int(1 + |\xi|^2)^\alpha |\hat{f}(\xi)|^2 d\xi)^{1/2}$ *denotes the Sobolev-norm of order* α.

Illustration of the proof of Theorem 2.3 in the case of Lewy's operator Z

Assume w.r. that $x^0 = 0$.

A first important step is to find, for a given characteristic ξ^0 at 0 satisfying condition (H), a complex phase function of the form

$$(2.4) \qquad w(x) = \xi^0 \cdot x + i\, {}^tx \cdot A \cdot x + O(|x|^3),$$

where $\Re e A$ is a positive-definite matrix, such that, if possible,

$$(2.5) \qquad {}^tP(x, D)e^{2\pi i\lambda w} = 0 \qquad \forall\lambda \gg 1.$$

(This cannot always be achieved in the strict sense, only asymptotically as $\lambda \to \infty$, but a necessary condition is that w satisfies the *"eikonal equation"*

$$P_m(x, \nabla w) = 0.)$$

If $P = Z$ is Lewy's operator, then one computes that the characteristic points at 0 are $(0, 0, \mu^0)$, which satisfy (H) if and only if $\mu^0 \ne 0$.

A suitable phase can here be constructed directly by means of the following observation: Let

$$(2.6) \qquad q_\pm(z, u) := |z|^2 \pm 4iu$$

be the expression appearing implicitely in (1.4). Then one computes that

$$(2.7) \qquad Zq_+ = 0,$$

so that $Z(f \circ q_+) = 0$ for every holomorphic function f. Since ${}^tZ = -Z$, we may thus choose w such that

$$2\pi iw = -q_+ + q_+^2 = -4iu - (|z|^2 + 16u^2) + O((|z| + |u|)^3)$$

in (2.5), with $\mu^0 = -2/\pi$.

Given this phase, put

$$v_\lambda := e^{2\pi i\lambda w}\chi, \quad f_\lambda := \lambda^3\chi(\lambda\cdot),$$

where $\chi \in \mathcal{D}(\mathbb{H}_1)$ is supported where $|z| + |u| < 2\varepsilon$, and $\chi \equiv 1$ in $|z| + |u| \leq \varepsilon$. Then, as $\lambda \to +\infty$,

$$\int_{\mathbb{H}_1} f_\lambda v_\lambda \, dg = \int\int \chi(z,u)\chi(z/\lambda, u/\lambda)e^{2\pi i\lambda w(z/\lambda,u/\lambda)}\, dzdu \to \int\int \chi(z,u)e^{-4iu}dzdu = \hat{\chi}(0,-\mu^0).$$

On the other hand,

$${}^tZv_\lambda = e^{2\pi i\lambda w}\; {}^tZ\chi,$$

where ${}^tZ\chi$ is supported in the region where $|z|+|u| \sim \varepsilon$. If ε is sufficiently small, then, by (2.4), $\Im\mathrm{m}\, w \sim \varepsilon^2$ in this region, hence $|e^{2\pi i\lambda w}| \sim e^{-\delta\lambda}$, for some $\delta > 0$. This easily implies

$$||f_\lambda||_{(k)} \cdot ||\,{}^tZv_\lambda||_{(k)} \to 0 \quad \text{as} \quad \lambda \to +\infty.$$

Thus, if we choose χ s.t. $\hat{\chi}(0,-\mu^0) \neq 0$, we obtain a contradiction to (2.2).

Q.E.D.

Remark: In general, (2.4) cannot be satisfied exactly, and the proof becomes considerably more involved.

For homogeneous left-invariant PDO's on $\mathbb{H}_n$, the following necessary criterion for local solvability has proven extremely useful (analogues hold on general homogeneous groups).

Theorem 2.6 [5], [14]. *Let $P \in \mathfrak{u}(\mathfrak{h}_n)$ be homogeneous. If P is locally solvable, then there exist a Sobolev-norm $||\cdot||_{(k)}$ and a continuous "Schwartz-norm" $||\cdot||_{\mathcal{S}}$ on $\mathcal{S}(\mathbb{H}_n)$, s.t.*

$$|f(0)| \leq ||f||_{\mathcal{S}}^{1/2}\, ||\,{}^tPf||_{(k)}^{1/2} \quad \forall f \in \mathcal{S}(\mathbb{H}_n). \tag{2.8}$$

Corollary 2.7 [5] *Suppose there exists a non-trivial $f \in \mathcal{S}(\mathbb{H}_n)$ s.t.*

(CR) $\qquad {}^tPf = 0.$

Then P is not locally solvable.

Proof. Let Q be an elliptic, right-invariant Laplacian on $\mathbb{H}_n$, and let Ω be an open neighborhood of $0, m \geq 1$.Then, for $\varphi \in \mathcal{D}(\Omega)$, by Poincaré's inequality and standard elliptic regularity theory,

$$|\varphi(0)| \leq C'||Q^m\varphi||_2 \leq C||Q^{m+k/2}\varphi||_{(-k)},$$

provided Ω is chosen sufficiently small. We choose k is as in (2.3), and assume k to be even. Since $Q^{m+k/2}$ commutes with the left-invariant operator ${}^tP$, by (2.3) we have

$$\begin{aligned} ||Q^{m+k/2}\varphi||_{(-k)} &\leq C||Q^{m+k/2}\;{}^tP\varphi||_{(k)} \\ &\leq C'||\,{}^tP\varphi||_{(2m+2k)}, \end{aligned}$$

i.e. there exists a $K \in \mathbb{N},\ C \geq 0$, s.t.

$$|\varphi(0)| \leq C\,||\,{}^tP\varphi||_{(K)} \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{2.9}$$

Rescaling, we may assume w.r. that $\Omega = B_2$, where $B_r := B_r(0)$. Let ${}^tP$ be homogeneous of degree q. Choose $\chi \in \mathcal{D}(B_2)$ s.t. $\chi \equiv 1$ on B_1. Then, for $f \in \mathcal{S}$, by (2.9)

$$|f(0)| \leq C\,||\,{}^tP(\chi(f\circ\delta_r))||_{(K)} \quad \forall r > 0. \tag{2.10}$$

But:

$$\begin{aligned} {}^tP(\chi(f\circ\delta_r)) &= \chi\,{}^tP(f\circ\delta_r)+R(f\circ\delta_r) \\ &= r^q\chi\,({}^tPf)\circ\delta_r+R(f\circ\delta_r), \end{aligned}$$

where $R=[{}^tP,\chi]$ is a PDO whose coefficients are supported in $\{1\leq |x|\leq 2\}$. Thus, for $r\geq 1$,

$$\begin{aligned} &||\,{}^tP(\chi(f\circ\delta_r))||_{(K)} \\ &\leq Cr^A\{||\,{}^tPf||_{(K)}+\sum_{|\alpha|\leq N}(\int_{1<|x|<2}|f^{(\alpha)}(\delta_r x)|^2dx)^{1/2}\}, \end{aligned}$$

for some constants $A>0, N\geq 0$. Now,

$$\begin{aligned} \int_{1<|x|<2}|f^{(\alpha)}(\delta_r x)|^2dx &\leq r^{-B}\int_{1<|x|<2}|\delta_r x|^B|f^{(\alpha)}(\delta_r x)|^2dx \\ &\leq r^{-B-Q}\int |x|^B|f^{(\alpha)}(x)|^2dx. \end{aligned}$$

Choosing B s.t. $A-B-Q=-A$, we find a Schwartz-norm $||\cdot||_{\mathcal{S}}$ s.t.

$$||\,{}^tP(\chi(f\circ\delta_r))||_{(K)}\leq C(r^A||\,{}^tPf||_{(K)}+r^{-A}||f||_{\mathcal{S}}).$$

Combining this with (2.10) and optimizing in r we obtain (2.8) (if we assume w.r. that $|f(0)|\leq ||f||_{\mathcal{S}}$).

Q.E.D.

In order to apply the "(CR)-test" from Corollary 2.7, one needs to construct functions in the kernel of ${}^tP$. Here, representation theory can help.

3 The group Fourier transform

(see e.g. [7], [4], [24])

Let G be a locally compact group and $\mathcal{H}$ a Hilbert space. A *unitary representation* of G on $\mathcal{H}$ is a strongly continuous homomorphism

$$\pi : G\to U(\mathcal{H})$$

of G into the group $U(\mathcal{H})$ of unitary operators on $\mathcal{H}$. We shall also write $\mathcal{H}_\pi$ in place of $\mathcal{H}$, if we want to emphasize that $\mathcal{H}$ is the representation space of π. Two representations π and ρ are called *equivalent*, if there exists a linear isometry T from $\mathcal{H}_\rho$ onto $\mathcal{H}_\pi$ such that $\pi(g)T=T\rho(g)$ for every $g\in G$. π is called *irreducible*, if the only closed and $\pi(G)$-invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$. The unitary dual $\hat{G}$ of G consists of all equivalence classes $[\pi]$ of irreducible unitary representations. Often one identifies $\hat{G}$ also with a system of representatives of representations.

As a consequence of the Stone-von Neumann theorem, such a system is given for the Heisenberg group $\mathbb{H}_n$ by the following irreducible representations:

(i) For $\mu\in\mathbb{R}^\times:=\mathbb{R}\setminus\{0\}$, the *Schrödinger representation* π_μ acts on $L^2(\mathbb{R}^n)$ as follows:

(3.1) $$[\pi_\mu(p,q,u)f](x):=e^{2\pi i\mu(u+q\cdot x+\frac{1}{2}q\cdot p)}f(x+p),\quad f\in L^2(\mathbb{R}^n).$$

(ii) For $\zeta\in\mathbb{R}^{2n}$, the characters

$$\omega_\zeta(z,u):=e^{2\pi i\zeta\cdot z}$$

are 1-dimensional representations of $\mathbb{H}_n$.

The characters are the irreducible representations which act trivially on the center

$$Z_n := \{(0,0,u) : u \in \mathbb{R}\}$$

of $\mathbb{H}_n$, and they will play no role in the discussions to follow.

If π is a unitary representation of G, and if $f \in L^1(G, dg)$ (dg= left-invariant Haar measure), one defines $\pi(f) \in \mathcal{B}(\mathcal{H})$ by

$$\pi(f)\xi := \int_G f(g)\pi(g)\xi \, dg, \qquad \xi \in \mathcal{H}.$$

One checks that the operator norm of $\pi(f)$ satisfies $||\pi(f)|| \leq ||f||_{L^1}$, and that the following holds true:

The "integrated" representation π is a continuous homomorphism

$$\pi : (L^1(G), +, \star, ^*) \to (B(\mathcal{H}), +, \circ, ^*)$$

of involutive Banach algebras.

For $f \in L^1(G)$, we define the *(group-) Fourier transform* of f as the mapping $\hat{f} : \hat{G} \to \dot{\bigcup}_{\pi \in \hat{G}} \mathcal{B}(\mathcal{H}_\pi)$, given by

$$\hat{f}(\pi) := \int f(g)\pi(g)^* \, dg = \int f(g)\pi(g^{-1}) \, dg.$$

Observe that for instance for $G = \mathbb{H}_n$,

$$\hat{f}(\pi) = \pi(\check{f}),$$

which implies

$$(f_1 \star f_2)^\wedge(\pi) = \hat{f}_2(\pi) \circ \hat{f}_1(\pi). \tag{3.2}$$

On $\mathbb{H}_n$, one has the following explicit *Fourier-inversion formula* for "nice" functions, such as for example Schwartz-functions:

$$f(g) = \int_{\mathbb{R}^\times} \operatorname{tr}(\hat{f}(\pi_\mu)\pi_\mu(g)) \, |\mu|^n \, d\mu, \quad g \in \mathbb{H}_n. \tag{3.3}$$

The corresponding *Plancherel-formula* reads as follows:

$$\int_{\mathbb{H}_n} |f(g)|^2 \, dg = \int_{\mathbb{R}^\times} ||\hat{f}(\pi_\mu)||^2_{\mathcal{HS}} |\mu|^n \, d\mu. \tag{3.4}$$

Here, $\operatorname{tr} A$ denotes the trace of the operator A, and $||A||_{\mathcal{HS}} := (\operatorname{tr} A^* A)^{1/2}$ its Hilbert-Schmidt norm.

This holds for $f \in L^2(G)$ in a similar sense as in the Euclidean case. For $f \in L^1 \cap L^2(G)$, where $\hat{f}(\pi_\mu)$ is well-defined for every $\mu \neq 0$, part of the statement is that $\hat{f}(\pi_\mu)$ is a Hilbert-Schmidt-operator for a.e. $\mu \in \mathbb{R}^\times$.

Notice that the characters ω_ζ do not enter in these formulas.

Formulas (3.3) and (3.4) can be deduced from the Euclidean Fourier inversion formula as follows:

Direct computations, based on formula (3.1), show that $\hat{f}(\pi_\mu)$ can be represented as a kernel operator

$$(\hat{f}(\pi_\mu)\varphi)(x) = \int_{\mathbb{R}^n} K_f^\mu(x,y)\varphi(y) \, dy, \qquad \varphi \in L^2(\mathbb{R}^n), \tag{3.5}$$

with integral kernel

$$K_f^\mu(x,y) = \int\int f(x-y,q,u)e^{-2\pi i\mu(u+\frac{q}{2}(x+y))}\,dqdu, \tag{3.6}$$
$$= f(x-y, \widehat{\frac{\mu}{2}(x+y)}, \hat{\mu}).$$

Since $\mathrm{tr}\hat{f}(\pi_\mu) = \int K_f^\mu(x,x)\,dx$, (3.3) follows easily (Exercise).

The Fourier transform of a differential operator

If $P \in \mathfrak{u}(\mathfrak{h}_n)$, then

$$P\varphi = P(\varphi \star \delta) = \varphi \star (P\delta), \quad \varphi \in \mathcal{S},$$

i.e. P can be represented by convolution from the right with the compactly supported distribution $P\delta$. But from (3.6), one sees that K_f^μ is well-defined as a tempered distribution kernel $K_f^\mu \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ supported near the diagonal $x=y$, for every distribution $f \in \mathcal{E}'(\mathbb{H}_n)$ with compact support. This implies that the integral operator (3.5), defined in the Schwartz-sense of distributions, is well-defined on $\mathcal{S}(\mathbb{R}^n)$, and

$$\hat{f}(\pi_\mu) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is continuous for every $f \in \mathcal{E}'(\mathbb{H}_n)$.

For $P \in \mathfrak{u}(\mathfrak{h}_n)$, we now define its Fourier transform by

$$\hat{P}(\pi_\mu) := \widehat{P\delta}(\pi_\mu) := \pi_\mu((P\delta)\check{\ }).$$

Approximating $P\delta$ by $P\delta \star \varphi_\varepsilon \in \mathcal{D}$, where $\{\varphi_\varepsilon\}_{\varepsilon>0}$ denotes a Dirac sequence in $\mathcal{D}$, one finds from (3.2) that

$$\widehat{P\varphi}(\pi_\mu) = \hat{P}(\pi_\mu) \circ \hat{\varphi}(\pi_\mu), \qquad \varphi \in \mathcal{S}, \tag{3.7}$$

and

$$\widehat{AB}(\pi_\mu) = \hat{A}(\pi_\mu) \circ \hat{B}(\pi_\mu), \qquad \forall A, B \in \mathfrak{u}(\mathfrak{h}_n), \tag{3.8}$$

since $(AB)\delta = A(B\delta \star \delta) = B\delta \star A\delta$.

Since $X_j\delta = \frac{\partial}{\partial x_j}\delta$, $Y_j\delta = \frac{\partial}{\partial y_j}\delta$, $U\delta = \frac{\partial}{\partial u}\delta$, we find from (3.6) that

$$\hat{X}_j(\pi_\mu) = \frac{\partial}{\partial x_j}, \quad \hat{Y}_j(\pi_\mu) = 2\pi i\mu x_j, \quad \hat{U}(\pi_\mu) = 2\pi i\mu. \tag{3.9}$$

Also, from (3.6), one sees that

$$K^{r^2\mu}_{f\circ\delta_r}(x,y) = r^{-n-2}K_f^\mu(rx,ry), \qquad r>0. \tag{3.10}$$

If $P \in \mathfrak{u}(\mathfrak{h}_n)$ is homogeneous of degree q, then $f = P\delta$ satisfies $f \circ \delta_r = r^{-Q-q}f$, hence from (3.10) we get

$$K^{r^2\mu}_{P\delta}(x,y) = r^{q+n}K^\mu_{P\delta}(rx,ry). \tag{3.11}$$

From Corollary 2.7, we can now deduce

Corollary 3.1 *Let $P \in \mathfrak{u}(\mathfrak{h}_n)$ be homogeneous, and assume there exist $\mu^0 \in \mathbb{R}^\times$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\phi \neq 0$, s.t. $\widehat{{}^tP}(\pi_{\mu^0})\phi = 0$. Then P is not locally solvable.*

Proof. Assume for instance $\mu^0 > 0$. For $\mu > 0$, put

$$\phi^\mu(x) := \phi\left(\left(\frac{\mu}{\mu^0}\right)^{1/2} x\right).$$

Then, by (3.11), $\widehat{{}^tP}(\pi_\mu)\phi^\mu = 0 \qquad \forall \mu > 0$. Let $\chi \in C_0^\infty(\mathbb{R}^+)$, and put

$$K^\mu(x,y) := \chi(\mu)\phi^\mu(x)\phi^\mu(y).$$

From (3.6), it follows that $K^\mu = K_f^\mu$ for some unique function $f \in \mathcal{S}(\mathbb{H}_n)$. And,

$$({}^tPf)^\wedge(\pi_\mu) = \widehat{{}^tP}(\pi_\mu)\hat{f}(\pi_\mu) = 0,$$

since $\hat{f}(\pi_\mu)$ is represented by the kernel K^μ. Thus, by Fourier inversion, ${}^tPf = 0$.

Q.E.D.

Example 3.2. By (3.9), for the Lewy operator $Z = X + iY$ on $\mathbb{H}_1$, one has

$$\widehat{{}^tZ}(\pi_{\mu^0}) = -\left(\frac{d}{dx} + x\right), \quad \text{if } \mu^0 = -1/2\pi.$$

Thus, the Gaussian $e^{-x^2/2}$ lies in the kernel of $\widehat{{}^tZ}(\pi_{\mu^0})$.

Remark 3.3. For a representation theoretic pendant to Theorem 2.6, see [14].

4 Second order PDO's on $\mathbb{H}_n$ with real coefficients and the metaplectic group

In the remaining part of these lectures, we shall discuss the following (still largely open)

PROBLEM. Classify all second order left-invariant PDO's on $\mathbb{H}_n$ which are locally solvable.

Let me remark that local solvability has also been studied for operators of higher order, and on more general Lie groups, in particular for bi-invariant PDO's and for "transversally elliptic" operators. Some reference to the vast literature on the subject can be found in [1] and [22].

We shall concentrate here on the case of homogeneous operators of degree 2, which are of the form

(4.1) $$L = \sum_{j,k=1}^{2n} a_{jk} W_j W_k + i\alpha U, \qquad a_{jk}, \alpha \in \mathbb{C},$$

where $W_j := X_j,\ W_{n+j} := Y_j,\ j = 1, \ldots, n.$

Throughout this section, the a_{jk} will be real; the case of complex coefficients will be discussed in the last section. For results in the non-homogeneous case, see e.g. [21], [17].

Let us put $A := (a_{jk})_{j,k=1,\ldots,2n}$ and

(4.2) $$S := -AJ.$$

Observe that A is real and symmetric if and only if $S \in \mathfrak{sp}(n, \mathbb{R})$. Since, as it turns out, solvability of the operator L is very much ruled by the spectral properties of S, we shall put

$$\Delta_S := \sum_{j,k=1}^{2n} a_{jk} W_j W_k, \quad S \in \mathfrak{sp}(n,\mathbb{R}),$$

where A is related to S by (4.2).

The following theorem gives a complete answer for operators of the form (4.1) and A real (for a generalization to arbitrary 2-step nilpotent groups, see [20]).

Theorem 4.1 [19] *The operator $L_\alpha := \Delta_S + i\alpha U$ is not locally solvable if and only if all of the following three conditions hold:*

(i) $\alpha \in \mathbb{R}$;

(ii) S is semisimple and has purely imaginary spectrum $\sigma(S)$; in this case, there exists some $T \in \mathrm{Sp}(n,\mathbb{R})$ such that $S' := TST^{-1}$ takes on the normal form

$$S' = \left(\begin{array}{ccc|ccc} & & & \lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \\ \hline -\lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & -\lambda_n & & & \end{array}\right), \tag{4.3}$$

with "frequencies" $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

(iii) There are no constants $C, N > 0$, s.t.

$$\left| \sum_{j=1}^{n} (2k_j+1)\lambda_j \pm \alpha \right| \geq C\ (1 + k_1 + \ldots + k_n)^{-N} \tag{4.4}$$

for all $k_1, \ldots, k_n \in \mathbb{N}$.

Before we discuss some of the methods employed in its proof, let us consider some examples:

Example 1. Assume S is given by (4.3). Then

$$\Delta_S = -\sum_{j=1}^{n} \lambda_j (X_j^2 + Y_j^2).$$

If all λ_j are of the same sign, Δ_S is a so-called *sub-Laplacian.* In this case, condition (4.4) is equivalent to $\alpha \notin \mathcal{C}$, where $\mathcal{C}$ is the "*critical set*"

$$\mathcal{C} := \{\pm \sum_{j=1}^{n} (2k_j+1)\lambda_j \ :\ k_j \in \mathbb{N}\}.$$

Observe that local non-solvability for these operators does not only depend on the principal part of order 2, but in fact in a crucial way on the first order part $i\alpha U$. This phenomenon, which is in sharp contrast to the behaviour of so-called "principal type" operators (see e.g.[22]), had first been observed in the fundamental work [8] on the so-called Kohn-Laplacian $\Delta_K = \sum_{j=1}^n (X_j^2 + Y_j^2)$. For general sub-Laplacians, see also [2].

It is interesting to remark that the approach by Folland/Stein in [8] avoids representation theory. It is based on the explicit formula

$$(\Delta_K + i\alpha U)\Phi_\alpha = \gamma_\alpha \delta, \tag{4.5}$$

where

$$\Phi_\alpha := q_+^{-\frac{n-\alpha}{2}} q_-^{-\frac{n+\alpha}{2}},$$

with $q_\pm(z,u) = |z|^2 \pm 4iu$ given by (2.6), and

$$\gamma_\alpha := \frac{c_n}{\Gamma\left(\frac{n+\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)}.$$

Clearly, for $\alpha \notin \mathcal{C}$, $\gamma_\alpha \neq 0$, hence $\frac{1}{\gamma_\alpha}\Phi_\alpha$ is a fundamental solution of $\Delta_K + i\alpha U$, which implies its local solvability.

This approach, however, is restricted to rather particular operators (compare also [6]).

Example 2. $X_1^2 + Y_1^2 - \lambda(X_2^2 + Y_2^2)$ on $\mathbb{H}_2$ is locally solvable if and only if there are constants $C, N > 0$ s.t.

$$|\lambda - p/q| > Cq^{-N} \qquad \text{(compare (4.4))},$$

for all odd $p, q \in \mathbb{N}$, i.e. if and only if λ is neither a rational number p/q with odd p and q, nor a Liouville number of "odd type".

Example 3. $X_1^2 - Y_1^2 + i\alpha U$ is locally solvable on $\mathbb{H}_1$ for every $\alpha \in \mathbb{C}$.

In fact, here $S = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, hence $\sigma(S) = \{-1, 1\}$ is real.

Basic tools in the proof of Theorem 4.1

4.1 "Symplectic" changes of coordinates

If $T \in Sp(n,\mathbb{R}) \hookrightarrow \text{Aut}\,(\mathbb{H}_n)$, then, since exp = id for $\mathbb{H}_n$, $X(f\circ T)(g) = \frac{d}{dt}f(T(g\exp tX))|_{t=0} = \frac{d}{dt}f(T(g)\exp tT(X))|_{t=0} = (T(X)f)(Tg)$ for every $X \in \mathfrak{h}_n$. This implies (Exercise)

$$\Delta_S(f\circ T) = (\Delta_{TST^{-1}}f)\circ T. \tag{4.6}$$

Since $U(f\circ T) = (Uf)\circ T$, this shows that *solvability of $\Delta_S + i\alpha U$ depends only on the conjugacy class of $S \in \mathfrak{sp}(n,\mathbb{R})$ under the real symplectic group* Sp$(n,\mathbb{R})$.

4.2 Application of the group Fourier transform

Whereas Hörmander's criterion cannot be used here to prove non-solvability, since L_α has a real principal symbol, Theorem 2.6 does apply in a very similar way as in Example 3.2 .

Let us illustrate this in the case of the operators

$$L_\alpha = X^2 + Y^2 + i\alpha U \quad \text{ on } \mathbb{H}_1. \tag{4.7}$$

By (3.9), we have

$$\widehat{L_\alpha}(\pi_\mu) = \frac{d^2}{dx^2} - (2\pi\mu x)^2 - 2\pi\alpha\mu.$$

But, $\frac{d^2}{dx^2} - (2\pi\mu x)^2$ is just a re-scaled *Hermite operator*, with eigenfunctions

$$h_k^\mu(x) := (2\pi|\mu|)^{1/4} h_k((2\pi|\mu|)^{1/2}x)$$

and associated eigenvalues

$$-2\pi|\mu|\ (2k+1), \qquad k \in \mathbb{N}.$$

Here,

$$h_k(x) = c_k(-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2}$$

denotes the L^2-normalized Hermite function of order k.

Consequently,

$$\widehat{L_\alpha}(\pi_\mu) h_k^\mu = -2\pi|\mu|(2k+1+(\text{sign}\mu)\alpha) h_k^\mu, \tag{4.8}$$

i.e. there exist μ and k with $\widehat{L_\alpha}(\pi_\mu)h_k^\mu = 0$ iff $\alpha \in \mathcal{C} = \{\pm(2k+1) : k \in \mathbb{N}\}$.

So, by Corollary 3.1, L_α is not l.s., if $\alpha \in \mathcal{C}$.

In general, if S satisfies (i), (ii) and (iii) in Theorem 4.1, then, by standard symplectic linear algebra, one finds $T \in \text{Sp}(n, \mathbb{R})$, which conjugates S into the form (4.3) (see e.g. [19, Lemma 3.1], and assuming that $S = S'$, one can argue in a similar way as above, keeping Remark 3.3 in mind.

On the other hand, if conditions (i) and (ii) in Theorem 4.1 do apply, but the diophantine condition (4.4) fails, one can prove local solvability by means of the Fourier inversion formula (3.3).

On a formal level, and grossly oversimplifying compared to the general case, the argument, which we shall again demonstrate in the case of the operator (4.7), is as follows:

Suppose $\alpha \notin \mathcal{C}$, i.e. that (4.4) fails. Then, by (4.8), the operator $\widehat{L_\alpha}(\pi_\mu)$ is invertible, with

$$||(\widehat{L_\alpha}(\pi_\mu))^{-1}|| \le C|\mu|^{-1}. \tag{4.9}$$

Now, given $f \in \mathcal{D}(\mathbb{H}_1)$, try to define a function w on $\mathbb{H}_1$ by putting

$$w(g) := \int_{\mathbb{R}^\times} \text{tr}(\widehat{L_\alpha}(\pi_\mu)^{-1} \hat{f}(\pi_\mu)\pi_\mu(g))\ |\mu|\, d\mu. \tag{4.10}$$

Since then $\hat{w}(\pi_\mu) = \widehat{L_\alpha}(\pi_\mu)^{-1}\hat{f}(\pi_\mu)$, one finds that $(L_\alpha w)^\wedge(\pi_\mu) = \widehat{L_\alpha}(\pi_\mu)\hat{w}(\pi_\mu) = \hat{f}(\pi_\mu)$, hence $L_\alpha w = f$ (at least on a formal level).

To make this argument rigorous, the main problem is that (4.10) will in general not converge, because of the blow-up of estimate (4.9) as $\mu \to 0$. This can be overcome as follows:

Define v as w by (4.10), only with $\widehat{L_\alpha}(\pi_\mu)^{-1}$ replaced by $2\pi i\mu\ \widehat{L_\alpha}(\pi_\mu)^{-1}$. Then v turns out to be well-defined, and one finds that

$$L_\alpha v = Uf. \tag{4.11}$$

But, since U is locally solvable, given any $\varphi \in \mathcal{D}$, there is some $f \in \mathcal{D}$ s.t. $Uf = \varphi$ on the support of φ. But then

$$L_\alpha v = \varphi \qquad \text{on supp}\,\varphi,$$

hence L_α is locally solvable.

4.3 Twisted convolution and the metaplectic group

For generic $S \in \mathrm{Sp}(n, \mathbb{R})$, the operator $\widehat{\Delta_S}(\pi_\mu)$ will no longer have a discrete spectrum, and the approach described above breaks down.

What saves the day is the following

Lemma 4.2 *For $S_1, S_2 \in \mathfrak{sp}(n, \mathbb{R})$, we have*

$$[\Delta_{S_1}, \Delta_{S_2}] = -2U\Delta_{[S_1,S_2]}.$$

Proof. Exercise.

Denote by f^μ the partial Fourier transform of f "along the center" of $\mathbb{H}_n$, i.e.

$$f^\mu(z) := \int_{\mathbb{R}} f(z,u) e^{-2\pi i\mu u} du, \qquad \mu \in \mathbb{R}.$$

Moreover, for suitable functions or distributions φ, ψ on $\mathbb{R}^{2n}$, define the *μ-twisted convolution* of φ and ψ by

$$\varphi \times_\mu \psi(z) = \int_{\mathbb{R}^{2n}} \varphi(z - z')\psi(z') e^{\pi i\mu\langle z-z',z'\rangle}\, dz'.$$

One easily verifies that, for suitable distributions f_1, f_2 on $\mathbb{H}_n$,

$$\begin{aligned} (f_1 \star f_2)^\mu &= f_1^\mu \times_\mu f_2^\mu, \\ (f^*)^\mu &= (f^\mu)^*. \end{aligned} \tag{4.12}$$

One also easily sees that $L^1(\mathbb{R}^{2n}, +, \times_\mu, *)$ is a (non-commutative) involutive Banach algebra, and (4.12) shows that $f \mapsto f^\mu$ is a *-homomorphism of $L^1(\mathbb{H}_n, +, \star, ^*)$ onto it (another way to verifying these facts is by passage through the "reduced" Heisenberg group; compare [7]).

If $\mu = 1$, we just speak of the *twisted convolution*, and write $\varphi \times \psi$ in place of $\varphi \times_1 \psi$.

Remark 4.3 Twisted convolution shares many features of ordinary convolution. For example, one has Young's inequality

$$||\varphi \times \psi||_{L^r} \le ||\varphi||_{L^p} ||\psi||_{L^q},$$

if $1/p + 1/q = 1 + 1/r$. More surprising is the following fact (see [7]): If $\varphi, \psi \in L^2(\mathbb{R}^{2n})$, then also $\varphi \times \psi \in L^2(\mathbb{R}^{2n})$, and

$$||\varphi \times \psi||_{L^2} \le ||\varphi||_{L^2} ||\psi||_{L^2}.$$

Now, if $P \in \mathfrak{u}(\mathfrak{h}_n)$, then from (4.12) we get

$$(Pf)^\mu = (f \star P\delta)^\mu = f^\mu \times_\mu (P\delta)^\mu,$$

where clearly $(P\delta)^\mu$ is a distribution supported at $0 \in \mathbb{R}^{2n}$. This shows that there exists a PDO P^μ on $\mathbb{R}^{2n}$ such that

$$(Pf)^\mu = P^\mu f^\mu, \quad f \in \mathcal{S}(\mathbb{R}^{2n}). \tag{4.13}$$

For instance, by (1.5),

$$X_j^\mu = \frac{\partial}{\partial x_j} - i\pi\mu y_j, \quad Y_j^\mu = \frac{\partial}{\partial y_j} + i\pi\mu x_j, \quad U^\mu = 2\pi i\mu. \tag{4.14}$$

In particular, from Lemma 4.2, we get

$$[\Delta^\mu_{S_1}, \Delta^\mu_{S_2}] = -4\pi i\mu\, \Delta^\mu_{[S_1,S_2]}.$$

Moreover, Δ_S^μ is formally self-adjoint, hence the mapping

$$S \mapsto \frac{i}{4\pi\mu}\Delta_S^\mu \tag{4.15}$$

is a representation of $\mathfrak{sp}(n,\mathbb{R})$ by (formally) skew-adjoint operators on $L^2(\mathbb{R}^{2n})$.

Let us consider the case $\mu = 1$. In [11], R. Howe has proved for this case that the map (4.15) can be exponentiated to a unitary representation of the *metaplectic group* $M_p(n,\mathbb{R})$, a two-fold covering of the symplectic group. $M_p(n,\mathbb{R})$ can in fact be represented by twisted convolution operators of the form $f \mapsto f \times \gamma$, where the γ's are suitable measures which, generically, are multiples of purely imaginary Gaussians

$$e_A(z) := e^{-i\pi\, {}^t z\cdot A\cdot z}, \tag{4.16}$$

with real, symmetric $2n \times 2n$ matrices A. In particular, one has

$$e^{i\frac{t}{4\pi}\Delta_S^1} f = f \times \gamma_{t,S}, \quad t \in \mathbb{R}. \tag{4.17}$$

The measures $\gamma_{t,S}$ have been determined explicitly in [18]. To indicate how this can be accomplished, let us argue on a completely formal basis:

If e_A, e_B are two Gaussians (4.16) such that $\det(A+B) \neq 0$, one computes that

$$e_A \times e_B = [\det(A+B)]^{-1/2} e_{A-(A-J/2)(A+B)^{-1}(A+J/2)},$$

where a suitable determination of the root has to be chosen. Choosing $A = \frac{1}{2}JS_1,\ B = \frac{1}{2}JS_2$, with $S_1, S_2 \in \mathfrak{sp}(n,\mathbb{R})$, and assuming that S_1 and S_2 commute, one finds that

$$e_{\frac{1}{2}JS_1} \times e_{\frac{1}{2}JS_2} = 2^n(\det(S_1+S_2))^{-1/2}\, e_{\frac{1}{2}J[S_1S_2+I)(S_1+S_2)^{-1}]}.$$

This reminds of the addition law for the hyperbolic cotangent, namely

$$\coth(x+y) = \frac{\coth x \coth y + 1}{\coth x + \coth y}.$$

We are thus led to define, for non-singular S,

$$A(t) := \frac{1}{2}J\coth(tS/2), \tag{4.18}$$

which is well-defined at least for $|t| > 0$ small.

Then

$$e_{A(t_1)} \times e_{A(t_2)} = 2^n(\det(A(t_1)+A(t_2)))^{-1/2}\, e_{A(t_1+t_2)}.$$

And, from

$$\coth x + \coth y = \frac{\sinh(x+y)}{\sinh x \sinh y},$$

we obtain (ignoring again the determination of roots)

$$\begin{aligned}[]
[\det\sinh((t_1+t_2)S/2)]^{1/2}\, [\det(A(t_1)+A(t_2))]^{-1/2} \\
= [\det\sinh(t_1S/2)]^{1/2}\, [\det\sinh(t_2S/2)]^{1/2}.
\end{aligned}$$

Together this shows that

$$\gamma_{t,S} := 2^{-n}[\det\sinh(tS/2)]^{-1/2} e_{A(t)} \tag{4.19}$$

forms a (local) 1-parameter group under twisted convolution, and it is not hard to check that its infinitesimal generator is $\frac{i}{4\pi}\Delta_S^1$.

Warning: Formula (4.18) only holds true for "generic" $S \in \mathfrak{sp}(n,\mathbb{R})$ and $t \in \mathbb{R}$.

If one defines the *symplectic Fourier transform* of f on $\mathbb{R}^{2n}$ by

$$\overset{\triangle}{f}(\zeta) := \int_{\mathbb{R}^{2n}} f(z) e^{-i\pi\langle\zeta,z\rangle} dz = \hat{f}(\frac{1}{2}J\zeta),$$

one obtains a formula analogous to (4.19) for $\overset{\triangle}{\gamma}_{t,S}$:

$$\overset{\triangle}{\gamma}_{t,S} = [\det\cosh(tS/2)]^{-1/2} e_{B(t)}, \tag{4.20}$$

where

$$B(t) := \frac{1}{2}J\tanh(tS/2). \tag{4.21}$$

4.4 Solvability of L_α if $\sigma(S) \subset \mathbb{C}\setminus(i\mathbb{R})$ and $\alpha \in \mathbb{R}$.

In this case, formulas (4.19), (4.20) do apply in the strict sense. Observing that $\langle\gamma_{t,S},\varphi\rangle = 2^{-2n}\langle\overset{\triangle}{\gamma}_{t,S},\overset{\triangle}{\varphi}\rangle$, they easily imply that there are a Schwartz norm $||\cdot||_{\mathcal{S}}$ and a constant $\beta \neq 0$, s.t.

$$|\langle\gamma_{t,S},\varphi\rangle| \le \frac{1}{\cosh\beta t}||\varphi||_{\mathcal{S}}. \tag{4.22}$$

For arbitrary $\mu \neq 0$, put

$$\gamma^\mu_{t,S}(z) := \begin{cases} \mu^n\gamma_{t,S}(\mu^{1/2}z), & \mu > 0, \\ |\mu|^n\overline{\gamma_{t,S}(|\mu|^{1/2}z)}, & \mu < 0. \end{cases} \tag{4.23}$$

Then one verifies (see [18]) that

$$e^{i\frac{t}{4\pi\mu}\Delta^\mu_S} f = f \ \times_\mu \ \gamma^\mu_{t,S}. \tag{4.24}$$

Now, the idea to solve the equation

$$L_\alpha F = (\Delta_S + i\alpha U)F = f \tag{4.25}$$

is as follows: By taking a partial Fourier transformation, (4.25) is equivalent to

$$\left(\frac{i}{4\pi\mu}\Delta^\mu_S - \frac{i\alpha}{2}\right)F^\mu = \frac{i}{4\pi\mu}f^\mu \quad \forall\mu \in \mathbb{R}^\times.$$

Formally, we then obtain F^μ by

$$F^\mu = -\int_0^\infty e^{\frac{it}{4\pi\mu}\Delta^\mu_S - \frac{i\alpha}{2}t}\left(\frac{i}{4\pi\mu}f^\mu\right)dt,$$

hence

$$F(z,u) = -\int_{\mathbb{R}^\times}\int_0^\infty e^{-i\frac{\alpha}{2}t} f^\mu \times_\mu \gamma^\mu_{t,S}(z)\,dt\,\frac{e^{2\pi i\mu u}}{4\pi i\mu}\,d\mu = f \star K(z,u),$$

where K is the distribution, formally defined by

$$K(z,u) = -\int_{\mathbb{R}^\times}\int_0^\infty e^{-i\frac{\alpha}{2}t}\gamma^\mu_{t,S}(z)\,dt\frac{e^{2\pi i\mu u}}{4\pi i\mu}\,d\mu.$$

This suggests to *define* K by

$$\langle K,\varphi\rangle = -\int_0^\infty\int_{\mathbb{R}^\times}\left\langle \gamma^\mu_{t,S},\varphi^{-\mu}\right\rangle e^{-i\frac{\alpha}{2}t}\,\frac{d\mu}{4\pi i\mu}\,dt. \tag{4.26}$$

Now from (4.22) one derives that there are constants $N, M \in \mathbb{N}$ s.t.

$$\left|\left\langle \gamma^\mu_{t,S},\varphi^{-\mu}\right\rangle\right| \leq C\frac{(1+|\mu|)^N}{(\cosh\beta t)|\mu|^M}||\varphi^{-\mu}||_{\mathcal{S}}.$$

This estimate implies that the distribution $\tilde{K}$, defined in the same way as K, only with $d\mu$ replaced by $(2\pi i\mu)^{M+1}d\mu$, is in fact well-defined. Moreover, we then have

$$L_\alpha(f\star\tilde{K}) = U^{M+1}f.$$

From here on, one can argue in a similar way as in §4.2 to show that L_α is locally solvable.

Remark. The discussion of the remaining cases in Theorem 4.1 requires considerably more care (see [19]).

5 Second order PDO's on $\mathbb{H}_n$ with complex coefficients

The classification of locally solvable PDO's on $\mathbb{H}_n$ of the form

$$L = \sum_{j,k=1}^{2n} a_{jk}W_jW_k + \text{ lower order terms} \tag{5.1}$$

with complex coefficients a_{jk} appears to be a challenging problem, which as of yet has only been answered for a few classes of operators (see [6], [16], [17] and [12]). Let us briefly survey those results.

We write the principal part of L again as Δ_S, however, now with $S \in \mathfrak{sp}(n,\mathbb{C})$, i.e. $S = S_1 + iS_2$, with $S_1, S_2 \in \mathfrak{sp}(n,\mathbb{R})$. The operators studied in [6], [16], [17] can be described as follows:

Assume $\mathbb{R}^{2n}$ decomposes into symplectic subspaces

$$\mathbb{R}^{2n} = V_1\oplus\cdots\oplus V_r, \tag{5.2}$$

where each (V_j,ω_j), with $\omega_j := \langle\ ,\ \rangle|_{V_j\times V_j}$, is a symplectic vector space, and where the V_j's are pairwise orthogonal w.r. to $\langle\ ,\ \rangle$.

Moreover, assume that each V_j is *S-invariant*, i.e. that $S_i(V_j) \subset V_j$ for $i = 1,2$. Recall that a basis $e_1,\dots,e_m,\ f_1,\dots,f_m$ of a symplectic vector space (V,ω) is called *canonical* or *symplectic*, if

$$\omega(e_j,e_k) = \omega(f_j,f_k) = 0,\quad \omega(e_j,f_k) = \delta_{jk}.$$

Then, choosing such a basis for each subspace V_j, we assume that S can be written as a block diagonal matrix

$$S = \begin{pmatrix} \gamma_1S_{(1)} & & & \\ & \gamma_2S_{(2)} & & \\ & & \ddots & \\ & & & \gamma_rS_{(r)}\end{pmatrix}, \tag{5.3}$$

with $\gamma_j \in \mathbb{C}^\times$ and

$$S_{(j)}^2 = -I, \qquad j = 1, \ldots, r. \tag{5.4}$$

Observe that (5.3) generalizes the case (ii), formula (4.3), in Theorem 4.1, which appears to be of particular interest, to the complex setting.

We may and shall assume that each of the symplectic subspaces V_j in (5.2) is *minimal* in the sense that it does not contain any proper S-invariant symplectic subspace.

Theorem 5.1 [17] *If at least one of the minimal subspace V_j has dimension > 2, then $\Delta_S + P$ is not locally solvable for all first order (not necessarily invariant) differential operators P with smooth coefficients.*

This result is proved by means of Hörmander's criterion Theorem 2.3: If we put again $S = -AJ$, then it follows from (1.5) that the principal symbol of Δ_S is given by

$$\sigma_S((z,u),(\zeta,\mu)) := -\,{}^t(\zeta - \pi\mu Jz) \cdot A \cdot (\zeta - \pi\mu Jz). \tag{5.5}$$

And, a straight-forward computation yields (compare Lemma 4.2)

$$\{\sigma_S, \sigma_{S'}\} = 4\pi\mu\sigma_{[S,S']} \qquad \forall S, S' \in \mathfrak{sp}(n, \mathbb{C}). \tag{5.6}$$

Thus, Hörmander's criterion, applied to $\Delta_S + P$, just reads as follows:

There is some $\zeta \in \mathbb{R}^{2n}$ such that

$$ {}^t\zeta A_1 \zeta = {}^t\zeta_{A_2} \zeta = 0 \text{ and } {}^t\zeta A_3 \zeta \neq 0, \tag{H$'$}$$

where

$$A_1 := S_1 J, \quad A_2 := S_2 J \;\text{ and }\; A_3 := [S_1, S_2]J.$$

Open Problem. Classify all $S = S_1 + iS_2 \in \mathfrak{sp}(n, \mathbb{C})$ for which (H$'$) applies.

In general, this seems to be a hard "semi-algebraic" problem. The proof of Theorem 5.1 makes use of a classification of normal forms of matrices $S \in \mathfrak{sp}(n, \mathbb{C})$ satisfying $S^2 = -I$, with respect to conjugation by real symplectic matrices $T \in \mathrm{Sp}(n, \mathbb{R})$. Such a classification has been given in [23]. There remains the

The case where all of the "blocks" $\gamma_j S_{(j)}$ are of size 2×2

According to the classification of normal forms in [23], the $S_{(j)}$ can then be assumed to be either of the form

$$S_{(j)} = \begin{pmatrix} i\varepsilon_j\lambda_j & \lambda_j^2 - 1 \\ 1 & -i\varepsilon_j\lambda_j \end{pmatrix} \qquad \text{"Type 1"}, \tag{5.7}$$

with $\lambda_j \in \{-1\} \cup [0, \infty[$ and

$$\varepsilon_j = \begin{cases} 1, & \text{if } |\lambda_j| \leq 1, \\ \pm 1, & \text{if } \lambda_j > 1, \end{cases}$$

or of the form

$$S_{(j)} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \text{"Type 3"}. \tag{5.8}$$

The corresponding operators $\Delta_{S_{(j)}}$ are given by

$$L_{\lambda_j,\varepsilon_j} := (1-\lambda_j^2)X_j^2 + Y_j^2 - i\varepsilon_j\lambda_j(X_jY_j + Y_jX_j) \qquad \text{“Type 1”} \tag{5.9}$$

and

$$-i(X_j^2 - Y_j^2) \qquad \text{“Type 3”}. \tag{5.10}$$

The case $n = 1$

Let us briefly discuss operators

$$L_\lambda := (1-\lambda^2)X^2 + Y^2 + i\lambda(XY + YX) \tag{5.11}$$

on $\mathbb{H}_1$. We call L_λ a *generalized sub-Laplacian*, if $0 \le \lambda < 1$, i.e. if $\Re e L_\lambda$ is a sub-Laplacian. In the case $\lambda = 1$, i.e.

$$L_1 = Y^2 + i(XY + YX), \tag{5.12}$$

we speak of a *degenerate generalized sub-Laplacian.* If $\lambda > 1$, then $\Re e L_\lambda$ is more of “hyperbolic type”.

For $\mathbb{H}_1$, local solvability of left-invariant operators (5.1) can be discussed in a complete way ([17]). To indicate the flavour of these results, let me highlight a few examples:

Example 1. If L_λ is a generalized sub-Laplacian, then $L_\alpha + i\alpha U$ is l.s. if and only if

$$\alpha \notin \mathcal{C} := \{\pm(2k+1) : k \in \mathbb{N}\}.$$

This extends the result for sub-Laplacians.

Example 2. If $L_1 = Y^2 + i(XY + YX)$ is a degenerate generalized sub-Laplacian, then $L_1 + i\alpha U$ is locally solvable if and only if

$$\alpha \notin \mathcal{C}^+ := \{(2k+1) : k \in \mathbb{N}\}.$$

For instance, for $\alpha = -1$ and $\alpha = 1$, respectively, putting $\tilde{Z} := Y + 2iX$, one has

$$\begin{aligned} L_1 - iU &= Y\tilde{Z}, \\ L_1 + iU &= \tilde{Z}Y. \end{aligned}$$

Since $\tilde{Z}$ is of “Lewy-type”, hence non-solvable, clearly $L_1 + iU$ cannot be solvable. The fact that $Y\tilde{Z}$ is locally solvable is more of a surprise (see[16]).

Example 3. If $\lambda > 1$, then $L_\lambda + P$ is not locally solvable for every $P \in \mathfrak{u}(\mathfrak{h}_1)$ of order 1. This result cannot be obtained from Hörmander's criterion, since this fails to apply for arbitrary operators Δ_S on $\mathbb{H}_1$ (Exercise). It is proved in [17] by means of a variant of Corollary 3.1, which applies even to non-homogeneous operators.

The case $n \ge 2$

In this case, “most” of the operators $\Delta_S + P$ are locally non-solvable, as can be shown, with some effort, by means of Hörmander's criterion. The “exceptional” operators Δ_S, to which (H$'$) does not apply, are listed in [17, §6.1]. There are five such exceptional classes, of which I want to mention two here:

(i) On $\mathbb{H}_n$, $n \geq 2$, "positive combinations of generalized sub-Laplacians and of degenerate generalized sub-Laplacians", more precisely

$$\Delta_S = \sum_{j=1}^{m} \gamma_j[(1-\lambda_j^2)X_j^2 + Y_j^2 + i\lambda_j(X_jY_j + Y_jX_j)] + i \sum_{j=m+1}^{n} \beta_j(X_j^2 - Y_j^2),$$

where $|\lambda_j| \leq 1$, $\gamma_j \in \mathbb{C}^\times$, $\beta_j > 0$, and where all of the quadratic forms

$$\Re\mathrm{e}(\gamma_j[(1-\lambda_j^2)\xi_j^2 + \eta_j^2 + 2i\lambda_j\xi_i\eta_j])$$

are positive-semidefinite.

(ii) On $\mathbb{H}_2$, for $\lambda > 1$,

$$\begin{aligned}\Delta_S = &(1-\lambda^2)X_1^2 + Y_1^2 + i\lambda(X_1Y_1 + Y_1X_1)\\ &+(1-\lambda^2)X_1^2 + Y_1^2 - i\lambda(X_2Y_2 + Y_2X_2)\end{aligned}$$

ad (i). Observe that here the matrix $A = SJ$ satisfies $\Re\mathrm{e}\, A \geq 0$. Defining $B(t)$ as in (4.21), one then finds that $\Re\mathrm{e}(iB(-it))$ is positive semidefinite for every $t \geq 0$, so that $\overset{\triangle}{\gamma}_{-it,S}$, defined by (4.20), still remains a "good" Gaussian. As has been shown in [17], this can be used to treat these operators by means of suitable modifications of the approach outlined in §4.4. In particular, one finds that $\Delta_S S + i\alpha U$ is locally solvable for every α not contained in the exceptional set

$$E := \{\pm \sum_{j=1}^{n} \gamma_j(2k_j + 1) : \; k_1, \ldots, k_n \in \mathbb{N}\},$$

provided $m = n$.

If $\Re\mathrm{e}A$ is positive definite, this result follows also from the general theory of "transversally elliptic" partial differential operators; see e.g. [10], [3].

Open Problem. Will $\Delta_S + i\alpha U$ be locally solvable for generic $\alpha \in \mathbb{C}$, if $S \in \mathfrak{sp}(n, \mathbb{C})$ and $\Re\mathrm{e}(SJ)$ is positive semidefinite, but not definite?

One can show that Hörmander's criterion fails in this case (Exercise).

ad(ii). As has been proved recently in [12], the operator $\Delta_S + P$ is locally solvable for arbitrary left-invariant lower order terms P.

In fact, the symplectic change of basis

$$\begin{aligned}\tilde{X}_1 &:= Y_1 - \sqrt{\lambda^2-1}X_2, \qquad & \tilde{Y}_1 &:= Y_2 + \sqrt{\lambda^2-1}X_1,\\ \tilde{X}_2 &:= Y_2 - \sqrt{\lambda^2-1}X_1, \qquad & \tilde{Y}_2 &:= Y_1 + \sqrt{\lambda^2-1}X_2,\end{aligned}$$

transforms Δ_S into the operator

$$Q_\lambda := \left(1 + \frac{\lambda}{2\sqrt{\lambda^2-1}}\right) DE - \frac{\lambda}{2\sqrt{\lambda^2-1}} \overline{D}\,\overline{E},$$

where

$$D := \tilde{X}_1 - i\tilde{X}_2, \quad E := \tilde{Y}_2 + i\tilde{Y}_1.$$

We may thus reduce ourselves to the study of operators in $\mathfrak{u}(\mathfrak{h}_n)$, whose leading terms are of the form

$$L_{\mathcal{A}} := \sum_{j,k} \alpha_{jk} X_j Y_k,$$

where $\mathcal{A} = (\alpha_{jk})_{jk}$ is a complex 2×2-matrix. Now,

$$\widehat{L_{\mathcal{A}}}(\pi_\mu) = 2\pi i\mu \sum_{j,k=1}^{2} \alpha_{jk} \frac{\partial}{\partial x_j} \circ x_k$$

is homogeneous of degree 0. The second important property of Q_λ is that $\widehat{Q_\lambda}(\pi_\mu)$ is elliptic away from the origin, since its principal symbol is given by

$$-4\pi\mu \left[(1 + \frac{\lambda}{2\sqrt{\lambda^2 - 1}})(x_2 + ix_1)(\xi_1 - i\xi_2) - \frac{\lambda}{2\sqrt{\lambda^2 - 1}} \overline{(x_2 + ix_2)(\xi_1 - i\xi_2)} \right].$$

It has been proved in [12] that a left-invariant operator on $\mathbb{H}_2$ with leading term $L_{\mathcal{A}}$ is locally solvable, whenever $\widehat{L_{\mathcal{A}}}(\pi_\mu)$ is elliptic away from 0 and $\det \mathcal{A} \neq 0$.

In higher dimensions, such an ellipticity property of the operators $L_{\mathcal{A}}$ can never hold, which seems to explain why the exceptional operators of type (ii) only arize on $\mathbb{H}_2$.

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