Chapter 9
The Holomorphic Functional Calculus for Sectorial Operators

9.1 Sectorial Operators

For $0 < \omega \leq \pi$ and $0 < \omega'$ we consider the open sector and the open strip

$$S_\omega := \{ z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \omega \},$$
$$St_{\omega'} := \{ z \in \mathbb{C} \mid |\text{Im} z| < \omega' \}.$$

So, $S_\omega$ is symmetric about the positive real axis with opening angle $2\omega$. For $\omega = \pi/2$ we have $S_{\pi/2} = \mathbb{C}_+$. (Note that $2\omega > \pi$ is allowed here.) The strip $St_{\omega'}$ extends horizontally, symmetric about the real axis. We also define

$$S_0 := (0, \infty) \quad \text{and} \quad St_0 := \mathbb{R}$$

and consider them as a degenerate sector and strip, respectively. For $0 \leq \omega < \pi$, the exponential function is a biholomorphic mapping (conformal equivalence)

$$e^z : St_{\omega} \to S_{\omega} \quad \text{with inverse} \quad \log z : S_{\omega} \to St_{\omega},$$

the (principal branch of the) logarithm. Note that the additive group $\mathbb{R}$ acts by translations on each strip, whereas the multiplicative group $(0, \infty)$ acts by multiplication on each sector.

An operator $A$ on a Banach space $X$ is called sectorial of angle $\omega \in [0, \pi)$ if $\sigma(A) \subseteq S_\omega$ and for each $\alpha \in (\omega, \pi)$

$$M(A, \alpha) := \sup\{ \|\lambda R(\lambda, A)\| \mid \lambda \in \mathbb{C} \setminus S_\alpha \} < \infty.$$ 

An operator $A$ is simply called sectorial if it is sectorial of angle $\omega$ for some $\omega \in [0, \pi)$. In this case,

---

1 In these notes, the meaning of the symbol “$M(A, \alpha)$” is heavily depending on the context, cf. Section 8.1
\[ \omega_{sc}(A) := \min\{\omega \in [0, \pi) \mid A \text{ sectorial of angle } \omega \} \]

is called the \textit{sectoriality angle} of \( A \). Analogously to the half-plane case, we say that a set \( \mathcal{A} \) of operators is \textbf{uniformly sectorial of angle} \( \omega < \pi \) if

\[ \sup_{A \in \mathcal{A}} M(A, \alpha) < \infty \]

for all \( \alpha \in (\omega, \pi) \).

\textbf{Examples 9.1.} (See Exercise 9.1).

1) Let \( \Omega \) be a semi-finite measure space and \( a : \Omega \to \mathbb{C} \) a measurable mapping. The multiplication operator \( M_a \) on \( L^p(\Omega) \), \( 1 \leq p \leq \infty \), is sectorial of angle \( \omega \) if and only if \( a \in S_{\omega} \) almost everywhere.

2) Let \( H \) be a Hilbert space and \( A \) a closed operator on \( H \) with numerical range \( \text{W}(A) \subseteq S_{\omega} \) and such that ran(I + A) = \( H \). Then \( A \) is sectorial of angle \( \omega \). In particular, a positive, self-adjoint operator is sectorial of angle \( 0 \).

3) Suppose that the resolvent of an operator \( A \) satisfies an estimate of the form

\[ \|R(\lambda, A)\| \leq \frac{M}{|\text{Re} \lambda|} \quad \text{for all } \text{Re} \lambda < 0. \]

Then \( A \) is sectorial of angle \( \pi/2 \). In particular, this is the case if \( -A \) generates a bounded \( C_0 \)-semigroup.

4) Suppose that the resolvent of an operator \( A \) satisfies an estimate of the form

\[ \|R(\lambda, A)\| \leq \frac{M}{|\text{Im} \lambda|} \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

Then \( A^2 \) is sectorial of angle \( 0 \). In particular, this is the case if \( -iA \) generates a bounded \( C_0 \)-group.

Let us list some elementary properties of sectorial operators.

\textbf{Theorem 9.2.} An operator \( A \) on a Banach space \( X \) is sectorial if and only if \( (-\infty, 0) \subseteq \rho(A) \) and \( M := \sup_{t \geq 0} \|t(t + A)^{-1}\| < \infty \). Moreover, if \( A \) is sectorial, the following assertions hold:

\begin{enumerate}
\item \( x \in \text{dom}(A) \) if and only if \( \lim_{t \to \infty} t(t + A)^{-1}x = x \);
\item \( x \in \text{ran}(A) \) if and only if \( \lim_{t \to 0} t(t + A)^{-1}x = 0 \).
\end{enumerate}

\textbf{Proof.} Note that, trivially, if \( A \) is sectorial then the stated criterion holds. The converse is proved in Exercise 9.2, as well as the assertions in a) and b). Let \( X \) be reflexive and let \( x \in X \). Then the bounded sequence \( n(n + A)^{-1}x \) has a subsequence which is weakly convergent to some \( y \). Since \( \text{dom}(A) \) is a...
subspace, by Mazur’s theorem we have $y \in \text{dom}(A)$. On the other hand,

$$(1 + A)^{-n}(n + A)^{-1} = \frac{n}{n - 1}((1 + A)^{-1} - (n + A)^{-1}) \to (1 + A)^{-1}$$

in operator norm. It follows that $(1 + A)^{-1}x = (1 + A)^{-1}y$ and hence $x = y \in \text{dom}(A)$.

The proof that $X = \ker(A) \oplus \text{ran}(A)$ is similar. First, for $x \in X$ find a weak limit $y$ of a subsequence of $(A(\frac{1}{n} + A)^{-1}x_{n \in \mathbb{N}}$. Then, by Mazur’s theorem, $y \in \text{ran}(A)$. Finally, observe that $x - y \in \ker(A(1 + A)^{-1}) = \ker(A)$. 

**Remark 9.3.** In dealing with sectorial operators and functions on sectors, the following observations are frequently helpful. Each sector $S_\omega$ is invariant under inversion $z^{-1}$ and under multiplication by positive scalars. Moreover, $C \setminus S_\omega = -S_{\pi - \omega}$.

If $\omega + \omega' \leq \pi$ then

$$S_\omega \cdot S_{\omega'} = S_{\omega+\omega'} \quad \text{and} \quad S_\omega + S_{\omega'} = S_{\max(\omega,\omega')}.$$ 

If $0 < \omega < \alpha < \pi$ then

$$\sup_{z,\lambda} \left\{ \left| \frac{\lambda}{\lambda - z} \right| + \left| \frac{z}{\lambda - z} \right| < \infty, \right.$$ (9.1)

where the supremum runs over all $z \in S_\omega$ and $\lambda \in C \setminus S_\alpha$. (The reason is that for these choices of $z$ and $\lambda$ one has $z/\lambda, \lambda/z \in C \setminus S_{\alpha - \omega}$ and hence these fractions have a uniform distance to 1.)

In accordance with these properties of sectors, sectorial operators enjoy certain permanence properties, see Exercise 9.3.

9.2 Elementary Functions on Sectors and Strips

Quite analogously to the half-plane case in the previous chapter, we introduce the algebra of elementary functions\footnote{Actually $S_{\pi/2} = C_+ = \mathbb{R}_0$ and hence the symbol $\mathcal{E}(C_+)$ is ambiguous. To avoid confusion, the elementary functions on the sector $C_+$ will be denoted by $\mathcal{E}(S_{\pi/2})$, while the elementary functions on the half-plane $C_+$ are denoted by $\mathcal{E}(\mathbb{R}_0)$.} on $S_\omega$, $0 < \omega \leq \pi$, by

$$\mathcal{E}(S_\omega) := \left\{ f \in H^\infty(S_\omega) \mid \int_0^\infty |f(re^{i\alpha})| \frac{dr}{r} < \infty \text{ for all } |\alpha| < \omega \right\}.$$ 

So, $f : S_\omega \to C$ is elementary if $f$ is holomorphic and bounded and, for each $0 \leq \alpha < \omega$, $f$ is integrable over $\partial S_\alpha$ with respect to the measure $\frac{|dz|}{|z|^\alpha}$. We
shall write \( f \in L^1(\partial S_\alpha) \) for this. The boundary \( \partial S_\alpha \) shall be so oriented that the points of \( S_\alpha \) are to its left, i.e.

\[
\int_{\partial S_\alpha} f(z) \frac{dz}{z} = \int_0^\infty f(xe^{-i\alpha}) \frac{dx}{x} - \int_0^\infty f(xe^{i\alpha}) \frac{dx}{x} = \int_{\partial S_\alpha} f(e^z) dz,
\]

formally.

For technical reasons here, and for later use, we also introduce the algebra of \textbf{elementary functions} on the strip \( S_\omega, \omega > 0, \) by

\[
E(S_\omega) := \{ f \in H^\infty(S_\omega) \mid \int_{-\infty}^{\infty} |f(r + i\alpha)| dr < \infty \text{ for all } |\alpha| < \omega \}.
\]

In other words, a function \( f : S_\omega \to \mathbb{C} \) is elementary if \( f \) is holomorphic and bounded and, for each \( 0 \leq \alpha < \omega, \) \( f \) is integrable over \( \partial S_\alpha \) with respect to arclength measure. We shall write \( f \in L^1(\partial S_\alpha) \) for this. The boundary \( \partial S_\alpha \) shall be so oriented that the points of \( S_\alpha \) are to its left, i.e.

\[
\int_{\partial S_\alpha} f(z) \frac{dz}{z} - f(a) = \int_\mathbb{R} f(x - i\alpha) dx - \int_\mathbb{R} f(x + i\alpha) dx,
\]

formally. Note that if \( 0 < \omega < \pi, \)

\[
f \in E(S_\omega) \iff f(e^z) \in E(S_\omega).
\]

The following is the analogue of Lemma 8.2 for elementary functions on strips and sectors.

\textbf{Lemma 9.4.} \ a) \ Let \( 0 < \delta < \omega, \ f \in E(S_\omega), \) and \( a \in \mathbb{C} \setminus \partial S_\delta. \) Then

\[
\frac{1}{2\pi i} \int_{\partial S_\delta} \frac{f(z)}{z - a} \frac{dz}{z} = \begin{cases} 
0 & \text{if } |\text{Im } a| < \delta, \\
f(a) & \text{if } |\text{Im } a| > \delta.
\end{cases}
\]

Moreover, \( f \in C_0(S_\delta) \) and \( \int_{\partial S_\delta} f(z) \frac{dz}{z} = 0. \)

b) \ Let \( 0 < \delta < \omega \leq \pi, \ f \in E(S_\omega) \) and \( a \in \mathbb{C} \setminus \partial S_\delta. \) Then

\[
\frac{1}{2\pi i} \int_{\partial S_\delta} \frac{f(z)}{z - a} \frac{dz}{z} = \begin{cases} 
0 & \text{if } |\arg a| < \delta, \\
f(a) & \text{if } |\arg a| > \delta.
\end{cases}
\]

Moreover, \( f \in C_0(S_\delta \setminus \{0\}) \) and \( \int_{\partial S_\delta} f(z) \frac{dz}{z} = 0. \)

\textbf{Proof.} \ a) \ For \( R > 0 \) let \( \gamma_R \) be the positively oriented boundary of the rectangle \([-R, R] \times [-\delta, \delta]. \) Then

\[
\lim_{R \to \infty} \int_{\gamma_R} \frac{f(z)}{z - a} \frac{dz}{z} = \int_{\partial S_\delta} \frac{f(z)}{z - a} \frac{dz}{z}.
\]
since \( f \) is uniformly bounded on \( \overline{S_{\delta}} \). If \( a \notin S_{\delta} \) then
\[
\int_{\gamma_R} \frac{f(z)}{z-a} \, dz = 0
\]
for all \( R > 0 \), by Cauchy’s theorem. And if \( a \in S_{\delta} \) then \( a \in \text{Int}(\gamma_R) \) for all sufficiently large \( R > 0 \) and hence
\[
\int_{\gamma_R} \frac{f(z)}{z-a} \, dz = 2\pi i f(a)
\]
again by Cauchy’s theorem.

For the next assertion, fix \( 0 < \delta' < \delta \). By the dominated convergence theorem and the first, already proved, assertion, it follows that \( f(a) \to 0 \) as \( a \to \infty \) within \( \overline{S_{\delta'}} \). As \( 0 < \delta < \omega \) is arbitrary, the claim is proved. For the last claim we take \( \gamma_R \) as above and observe that, since \( f \in C_0(S_{\delta'}) \),
\[
0 = \int_{\gamma_R} f(z) \, dz \to \int_{\partial S_{\delta'}} f(z) \, dz \quad (R \to \infty).
\]

b) We let \( b := \log a \) and \( g := f(e^x) \) and note that
\[
h := \frac{e^x}{e^x - e^b} - \frac{1}{z-b} \in H^\infty(S_{\omega}).
\]
In particular, \( gh \in \mathcal{E}(S_{\omega}) \) and hence, by the last assertion of a),
\[
\int_{\partial S_{\delta}} \frac{f(z)}{z-a} \, dz = \int_{\partial S_{\delta'}} \frac{e^x g(z)}{e^x - e^b} \, dz = \int_{\partial S_{\delta'}} g(z) \, dz.
\]
Now all claims in b) follow from a).

\[\Box\]

\textbf{Remark 9.5}. Fix \( 0 < \omega < \pi \) and let \( f \in \text{Hol}(S_{\omega}) \) be such that for some \( s > 0 \) and some \( C \geq 0 \) one has
\[
|f(z)| \leq C \min\{|z|^s, |z|^{-s}\} \quad (z \in S_{\omega}). \tag{9.2}
\]
Then \( f \in \mathcal{E}(S_{\omega}) \). Even more, one has
\[
\sup_{|\alpha| \leq \delta} \int_0^\infty |f(re^{i\alpha})| \frac{dr}{r} < \infty \quad (0 \leq \delta < \omega). \tag{9.3}
\]
One can show that (9.3) holds actually for each \( f \in \mathcal{E}(S_{\omega}) \), see Exercise 9.11.
9.3 The Sectorial Functional Calculus

Let $A$ be a sectorial operator and let $\omega_{se}(A) < \omega < \pi$. Then, for $f \in \mathcal{E}(S_\omega)$ we define

$$\Phi_A(f) := \frac{1}{2\pi i} \int_{\partial S_\delta} f(z) R(z, A) \, dz,$$

(9.4)

where $\omega_{se}(A) < \delta < \omega$. The integral does not depend on $\delta$ (by Cauchy’s theorem) because $f \in C_0(S_\omega')$ for each $0 < \omega' < \omega$.

![Fig. 9.1](image)

Fig. 9.1 The integration contour $\partial S_\delta$ lies within the domain $S_\omega$ of the function $f$ and outside the sector $S_{\tilde{\omega}}$, where $\tilde{\omega} = \omega_{se}(A) > \pi/2$ is the sectoriality angle of $A$.

**Theorem 9.6.** The so-defined mapping $\Phi_A : \mathcal{E}(S_\omega) \to \mathcal{L}(X)$ has the following properties ($f \in \mathcal{E}(S_\omega)$):

a) $\Phi_A$ is a homomorphism of algebras.

b) If $T \in \mathcal{L}(X)$ satisfies $TA \subseteq AT$, then $\Phi_A(f)T = T\Phi_A(f)$.

c) $\Phi_A(f \cdot (\lambda - z)^{-1}) = \Phi_A(f)R(\lambda, A)$ whenever $\lambda \in \mathbb{C} \setminus S_\omega$.

d) $\Phi_A(z(\lambda - z)^{-1}(\mu - z)^{-1}) = AR(\lambda, A)R(\mu, A)$ whenever $\lambda, \mu \in \mathbb{C} \setminus S_\omega$.

e) $\sup_{r > 0} \|\Phi_A(f(rz))\| \leq \inf \left\{ \frac{M(A, \delta)}{2\pi} \|f\|_{L_1(\partial S_\delta)} \mid \omega_{se}(A) < \delta < \omega \right\} < \infty$.

**Proof.** The proof of a)–c) is analogous to the proof of Theorem 8.3.
9.3 The Sectorial Functional Calculus

Fig. 9.2 An example where $\omega_{se}(A) < \delta < \omega < \frac{\pi}{2}$.

d) Fix $\omega_{se}(A) < \delta < \omega$ and for $R > 0$ let $\gamma_R$ be the positively oriented boundary of the “cake piece” $B[0, R] \setminus S_\delta$. If $R$ is large enough, the points $\lambda$ and $\mu$ are contained in the interior of $\gamma_R$. Since $zR(z, A)$ is bounded on $\gamma_R$ uniformly in $R > 0$, it follows that

$$\frac{1}{2\pi i} \int_{\partial S_\delta} \frac{z}{(z - \lambda)(z - \mu)} R(z, A) \, dz = \lim_{R \to \infty} \frac{-1}{2\pi i} \int_{\gamma_R} \frac{z}{(z - \lambda)(z - \mu)} R(z, A) \, dz.$$ 

By the residue theorem, the right-hand side equals

$$-\frac{\lambda}{\lambda - \mu} R(\lambda, A) - \frac{\mu}{\mu - \lambda} R(\mu, A)$$

which is equal to $AR(\mu, A)R(\lambda, A)$ by an elementary computation.

e) We use the definition of $\Phi_A$ to estimate

$$\|\Phi_A(f(rz))\| \leq \frac{1}{2\pi} \int_{\partial S_\delta} |f(rz)| \|R(z, A)\| |dz| \leq \frac{M(A, \delta)}{2\pi} \|f\|_{L^1(\partial S_\delta)}.$$ 

This yields the claim.

Corollary 9.7. In the situation from before, $\Phi_A$ is non-degenerate if and only if $A$ is injective.

Proof. If $A$ is injective, then so is $AR(\lambda, A)R(\mu, A)$. On the other hand, for $x \in \ker(A)$ and $e \in E(S_\omega)$ we have
The Holomorphic Functional Calculus for Sectorial Operators

\[ \Phi_A(e) x = \frac{1}{2\pi i} \int_{\partial S_\omega} e(z) R(z, A) x \, dz = \left( \frac{1}{2\pi i} \int_{\partial S_\omega} e(z) \frac{dz}{z} \right) x = 0 \]

by Lemma \[9.4\]. Hence, \( \ker(A) \subseteq \ker(\Phi(e)) \).

So, if \( A \) is not injective one has to extend the calculus \( \Phi_A \) in order to render it non-degenerate. The extension to \( \mathcal{E}(S_\omega) \oplus \mathbb{C}1 \) as described in Section \[7.1\] would do, but it has the unpleasant feature that no “resolvent function” \( (\lambda - z)^{-1} \) is anchored in that algebra. (Look at limits at 0 and at \( \infty \).) Therefore, one rather takes the larger algebra

\[ \mathcal{E}_e(S_\omega) := \mathcal{E}(S_\omega) \oplus \mathbb{C}1 \oplus \mathbb{C}(1 + z)^{-1}. \]

It follows from Theorem \[9.6\].c) that this extension, which is again denoted by \( \Phi_A \), is a non-degenerate algebra homomorphism, see Exercise \[9.12\]. Note that for \( \lambda \in \mathbb{C} \setminus S_\omega \) one has \( (\lambda - z)^{-1} \in \mathcal{E}_e(S_\omega) \), see Exercise \[9.6\].

The domain within \( \text{Mer}(S_\omega) \) of the canonically extended calculus is denoted by \( \text{Mer}_A(S_\omega) \). As the operator \( A \) is clearly the generator of this calculus, we write \( f(A) \) in place of \( \Phi_A(f) \) and say that \( f(A) \) is defined by the sectorial calculus for \( A \).

### Remark 9.8 (Compatibility with the Hille–Phillips Calculus).

Suppose that \( -A \) generates a bounded \( C_0 \)-semigroup \( (T_t)_{t \geq 0} \) on a Banach space \( X \). Then \( A \) is sectorial of angle \( \pi/2 \) (Example \[9.1.3\]). By essentially the same argument as in the proof of Theorem \[8.20\] one can show that each \( f \in \mathcal{E}(S_\omega) \), \( \omega > \pi/2 \), is the Laplace transform \( f = \mathcal{L}\varphi \) of a function \( \varphi \in L^1(\mathbb{R}_+) \), and that

\[ f(A) = \int_0^\infty \varphi(t) T_t \, dt, \]

see [1], Lemma 3.3.1. Hence, the Hille–Phillips calculus, which takes the right-hand side as the definition of “\( f(A) \)” is an extension of the sectorial calculus.

### Injective vs. Non-Injective Sectorial Operators

If \( A \) is not injective, then a function \( f \in H^\infty(S_\omega) \) is anchored in \( \mathcal{E}(S_\omega) \) if and only \( f(0) := \lim_{z \searrow 0} f(z) \) exists and \( (1 + z)^{-1}(f - f(0)) \in \mathcal{E}(S_\omega) \). In this case, \( (1 + z)^{-1} \) is an anchor element for \( f \) (Exercise \[9.7\]). In particular, imaginary powers \( (z^s)(A) \) of \( A \) for \( s \neq 0 \) are not defined.

If \( A \) is injective, then the \( \mathcal{E}(S_\omega) \)-calculus is already non-degenerate and every \( f \in H^\infty(S_\omega) \) is anchored in \( \mathcal{E}(S_\omega) \) (e.g., by \( e = z(1+z)^{-2} \)). In particular, the function \( (1 + z)^{-1} \) is anchored, and hence \( \mathcal{E}_e \subseteq \langle \mathcal{E} \rangle_{\Phi_A} \). By Exercise \[7.4\], this means that \( \langle \mathcal{E} \rangle_{\Phi_A} = \langle \mathcal{E}_e \rangle_{\Phi_A} \), and one does not need the algebra \( \mathcal{E}_e \) in this case.

---

\[3\] or: within
The sectorial calculus is much nicer for injective operators than for non-injective ones. On reflexive spaces $X$ one has the decomposition $X = \ker(A) \oplus \operatorname{ran}(A)$, by which one can reduce problems for general sectorial operators to injective ones.

### Convergence Theorems

Similar to the half-plane case, one can coin the notion of a sectorial approximation $(A_n)_{n \in \mathbb{N}}$ of a sectorial operator $A$ and prove analogues of Lemma 8.6 and Theorem 8.8 for the sectorial calculus. However, we shall not do this here, but rather turn to the approximations of functions. We say that a subset $F \subseteq \mathcal{E}(S_\omega)$ is dominated if for each $0 < \delta < \omega$ sufficiently close to $\omega$ the set $\{f|_{S_\delta} \mid f \in F\}$ is dominated in $L_1^*(\partial S_\delta)$. Clearly, if $F \subseteq \mathcal{E}(S_\omega)$ is dominated then

$$\sup_{f \in F} \|f(A)\| < \infty$$

for each sectorial operator $A$ with $\omega_{se}(A) < \omega$.

**Lemma 9.9.** Let $A$ be sectorial of angle $\omega_{se}(A) < \omega$ and $(e_n)_{n \in \mathbb{N}}$ a sequence in $\mathcal{E}(S_\omega)$ converging to 0 pointwise and boundedly on $S_\omega$. Then the following assertions hold:

a) If $\{e_n \mid n \in \mathbb{N}\}$ is dominated, then $e_n(A) \rightarrow 0$ in norm.

b) If $e_n(A) \rightarrow T$ strongly for some bounded operator $T$, then $T = 0$.

**Proof.** a) This is a consequence of Lebesgue’s theorem like in Lemma 8.9.

b) Let $e := \frac{z}{(1+z)^2}$. By a), $(ee_n)(A) \rightarrow 0$ in norm. On the other hand,

$$(ee_n)(A) = e(A)e_n(A) \rightarrow e(A)T = A(1+A)^{-2}T$$

strongly. Hence, $\text{ran}((1+A)^{-2}T) \subseteq \ker(A)$. On the other hand, $(1+z)^{-2}e_n \in \mathcal{E}(S_\omega)$ and hence, by Exercise 9.5,

$$(1+A)^{-2}Tx = \lim_{n \rightarrow \infty} ((1+z)^{-2}e_n)(A)x \in \operatorname{ran}(A) \quad (x \in X).$$

Since $\ker(A) \cap \operatorname{ran}(A) = \{0\}$ (Theorem 9.2), it follows that $(1+A)^{-2}T = 0$, which implies that $T = 0$.

**Theorem 9.10 (“Convergence Lemma”).** Let $0 < \omega < \pi$ and let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $H^\infty(S_\omega)$ converging pointwise and boundedly on $S_\omega$ to some $f \in H^\infty(S_\omega)$. Let, furthermore, $A$ be an injective sectorial operator on a Banach space $X$ with $\omega_{se}(A) < \omega$. Suppose that $f_n(A) \in \mathcal{L}(X)$ for all $n \in \mathbb{N}$. Then the following assertions hold:

a) If $f_n(A) \rightarrow T \in \mathcal{L}(X)$ strongly, then $f(A) = T$. 


b) If \( \text{dom}(A) \cap \text{ran}(A) \) is dense in \( X \) and \( \sup_n \|f_n(A)\| < \infty \) then \( f(A) \in \mathcal{L}(X) \), \( f_n(A) \to f(A) \) strongly, and \( \|f(A)\| \leq \liminf_{n \to \infty} \|f_n(A)\| \).

**Proof.** Let \( e = z(1 + z)^{-2} \). For the proof of a), apply Lemma 9.9.b) with \( e_n := e(f_n - f) \). This yields \( e(A)T = (ef)(A) \). Hence,

\[
 f(A) = (e^{-1})(A)(ef)(A) = e(A)^{-1}e(A)T = T
\]
as claimed.

For the proof of b), apply Lemma 9.9.a) to conclude that \( \lim_{n \to \infty} f_n(A)x \) exists for all \( x \in \text{ran}(e(A)) = \text{ran}(A) \cap \text{dom}(A) \). Uniform boundedness in combination with the density yields that \( (f_n(A))_{n \in \mathbb{N}} \) converges strongly, hence to \( f(A) \) by a). The rest is standard.

An analogue for non-injective sectorial operators is treated in the supplementary Section 9.5 below.

### 9.4 Holomorphic Semigroups

In this section we shall see the sectorial calculus “at work”.

Observe that the function \( e^{-z} \) is an element of \( \mathcal{E}_e(S_{\pi/2}) \) since the function

\[
 f := e^{-z} - \frac{1}{1 + z} \in \mathcal{E}(S_{\pi/2})
\]

satisfies (9.2) with \( s = 1 \) for each \( \omega < \pi/2 \). Consequently, for each \( \lambda \in S_{\pi/2} \) the function

\[
 e^{-\lambda z} = f(\lambda z) + \frac{1}{1 + \lambda z}
\]
is contained in \( \mathcal{E}_e(S_{\omega}) \), where \( \omega = \pi - |\arg \lambda| \). In particular,

\[
 e^{-\lambda A} := (e^{-\lambda z})(A)
\]
is defined for each sectorial operator \( A \) of angle \( \omega_{sc}(A) < \pi - |\arg \lambda| \).

Let us abbreviate \( \omega_A := \omega_{sc}(A) \) and \( \theta_A := \pi/2 - \omega_A \). Then \( e^{-\lambda A} \) is defined for each \( \lambda \in S_{\theta_A} \). Moreover, functional calculus rules imply that

\[
 e^{-\lambda A}e^{-\mu A} = e^{-(\lambda + \mu)A} \quad (\lambda, \mu \in S_{\theta_A}).
\]
The operator family \( (e^{-\lambda A})_{\lambda \in S_{\theta_A}} \) is called the **holomorphic semigroup generated by** \(-A\). The reason for this terminology is the following classical result, see [2, Section II.4].

**Theorem 9.11.** Let \( A \) be a sectorial operator of angle \( \omega_A < \pi/2 \) and let \( \theta_A := \pi/2 - \omega_A \). Then the mapping

\[
 e^{-\lambda A}e^{-\mu A} = e^{-(\lambda + \mu)A} \quad (\lambda, \mu \in S_{\theta_A}).
\]
9.4 Holomorphic Semigroups

\[ S_{\theta_A} \to \mathcal{L}(X), \quad \lambda \mapsto e^{-\lambda A} \]

is holomorphic with

\[ \sup_{\lambda \in S_{\theta}} \| e^{-\lambda A} \| < \infty \quad \text{for each } 0 < \theta < \theta_A. \] (9.6)

The derivatives are given by

\[ \frac{d^n}{d\lambda^n} e^{-\lambda A} = (-A)^n e^{-\lambda A} \in \mathcal{L}(X) \quad (n \in \mathbb{N}). \] (9.7)

For each \( \mu \in S_{\pi/2} \)

\[ (\mu + A)^{-1} = \int_0^\infty e^{-\mu t} e^{-tA} \, dt. \] (9.8)

Finally, \( x \in \overline{\text{dom}(A)} \) if and only if \( e^{-\lambda A}x \to x \) as \( \lambda \to 0 \) within \( S_\theta \) for one/each \( 0 < \theta < \theta_A \).

**Proof.** Fix \( 0 < \theta < \theta_A \), let \( \omega := \pi/2 - \theta \) and \( f := e^{-z} - (1 + z)^{-1} \in \mathcal{E}(S_{\pi/2}) \).

Since

\[ (1 + \lambda z)^{-1}(A) = -\lambda^{-1} R(-\lambda^{-1}, A), \]

and this is holomorphic in \( \lambda \) and uniformly bounded for \( \lambda \in S_\theta \) (even for \( \lambda \in S_{\pi/2} \)), it suffices to consider the function \( \lambda \mapsto f(\lambda A) \). Fix \( \omega_A < \delta < \omega \) and define

\[ F_n(\lambda) := \frac{1}{2\pi i} \int_{\Gamma_n} f(\lambda z) R(z, A) \, dz \quad (n \in \mathbb{N}, \ \lambda \in S_{\theta}), \]

where \( \Gamma_n \) is just \( \partial S_{\delta} \), but restricted to the region \( \frac{1}{n} \leq |z| \leq n \). By standard arguments, \( F_n \) is holomorphic on \( S_{\theta} \). Moreover, \( F_n(\lambda) \to f(\lambda A) \) pointwise and boundedly on \( S_{\theta} \) since

\[ \|F_n(\lambda)\| \lesssim \int_{\partial S_{\delta}} |f(\lambda z)| \frac{|dz|}{|z|} \leq \sup_{|\omega| \leq \delta + \theta} \int_0^\infty |f(re^i\alpha)| \frac{dr}{r} \]

for all \( \lambda \in S_{\theta} \) (cf. Remark 9.5). The uniform bound (9.6) follows readily.

Since \( z^n e^{-\lambda z} \in \mathcal{E}(S_{\pi/2}) \), it follows that \( A^n e^{-\lambda A} \in \mathcal{L}(X) \) for all \( \lambda \in S_{\theta} \). In particular, \( e^{-\lambda A} \) maps into \( \text{dom}(A^n) \) for each \( n \in \mathbb{N} \). Hence, if \( e^{-\lambda A}x \to x \) as \( \lambda \to 0 \), then \( x \in \text{dom}(A) \).

For the converse implication note that it follows from Theorem 9.2.a) and the uniform boundedness of \( (t(t + A)^{-1})_{t > 0} \) that

\[ \overline{\text{dom}(A)} = \overline{\text{dom}(A^2)}. \]

So by (9.6) it suffices to suppose that \( x \in \text{dom}(A^2) \). For \( \lambda \in S_{\theta} \) consider the function
\[ f_\lambda := \frac{1}{(1+z)^2}(1-e^{-\lambda z}) = \lambda \frac{1-e^{-\lambda z}}{\lambda z} \frac{z}{(1+z)^2} \in \mathcal{E}(S_\omega). \]

Since the function \((1-e^{-z})/z\) is bounded on \(S_{\pi/2}\), the functions \(f_\lambda/\lambda\) are dominated in \(\mathcal{E}(S_\omega)\). It follows that
\[ (1-e^{-\lambda A})(1+A)^{-2} = f_\lambda(A) \to 0 \]
in norm as \(0 \leftarrow \lambda \in S_\theta\).

For the computation of the derivatives it suffices to compute right derivatives since we already know that \(\lambda \mapsto e^{-\lambda A}\) is holomorphic. Fix \(\lambda \in S_\theta\) and note that
\[ \frac{1}{t}(e^{-\lambda A} - e^{-(\lambda+t)A}) - Ae^{-\lambda A} = \left[ \frac{1-e^{-tz}}{tz} - 1 \right] te^{-\lambda z}(A) =: g_t(A). \]

Since the function \(ze^{-\lambda z}\) is contained in \(\mathcal{E}(S_\omega)\) and the term in big round brackets converges to 0 pointwise and boundedly on \(S_\omega\), Lemma 9.9.a) yields that \(g_t(A) \to 0\) in operator norm as \(t \downarrow 0\). The claim about higher derivatives follows easily.

Finally, fix \(\mu \in S_{\pi/2}\) and compute
\[ (1+A)^{-2} \left( \mu \int_0^{\infty} e^{-\mu t} e^{-tA} dt - 1 \right) = \int_0^{\infty} \frac{e^{-tz} - 1}{(1+z)^2} \mu e^{-\mu t} dt \]
\[ = \frac{1}{2\pi i} \int_{\partial S_\delta} \frac{e^{-tz} - 1}{tz} R(z,A) \frac{dz}{(1+z)^2} \mu e^{-\mu t} dt \]
\[ = \frac{1}{2\pi i} \int_{\partial S_\delta} \frac{-z}{\mu(\mu+z)} \frac{R(z,A)}{(1+z)^2} \mu e^{-\mu t} dt \]
\[ = \frac{1}{2\pi i} \int_{\partial S_\delta} \frac{-z}{\mu(\mu+z)} \frac{R(z,A)}{(1+z)^2} \mu e^{-\mu t} dt \]
\[ = -(1+A)^{-2} A(\mu + A)^{-1}. \]

(We have used that \((e^{-tz} - 1)/(1+z)^2 \in \mathcal{E}(S_{\pi/2})\) and that Fubini’s theorem is applicable since \(te^{-\mu t} \in L^1(\mathbb{R}^+)\).) It follows that
\[ \mu \int_0^{\infty} e^{-\mu t} e^{-tA} dt = 1 - A(\mu + A)^{-1} = \mu(\mu + A)^{-1} \]
and hence (9.8).

Suppose that \(A\) is as above and, in addition, densely defined. Then, by Theorem 9.11 \((e^{-tA})_{t \geq 0}\) is a bounded \(C_0\)-semigroup with generator \(-A\). Moreover, this semigroup has a holomorphic extension to the sector \(S_{\theta_A}\) and is uniformly bounded on each smaller sector.

Conversely, suppose that an operator \(-A\) generates a \(C_0\)-semigroup \((T_t)_{t \geq 0}\) which for some \(0 < \theta_0 \leq \frac{\pi}{2}\) has a holomorphic extension to \(S_{\theta_0}\), uniformly
bounded on each smaller sector. Then $A$ is sectorial of angle $\pi/2 - \theta_0$. (The proof of this claim is Exercise 9.8.)

9.5 Supplement: A Topological Extension of the Sectorial Calculus

Let $A$ be a non-injective sectorial operator and $\omega \in (\omega_{sr}(A), \pi)$. As we have observed above, there are bounded and holomorphic functions $f$ such that $f(A)$ is not defined. This is unavoidable, since if $f \in \mathcal{H}^\infty(S_\omega)$ is such that $f(A)$ is defined, the limit $f(0) := \lim_{z \to 0, z \in S_\omega} f(z)$ must exist.

However, there is a more serious shortcoming. Suppose that $\mu \in M(0, \infty)$ is a bounded complex measure on $(0, \infty)$ and $f$ is given by

$$f(z) = \int_0^\infty \frac{t}{t + z} \mu(dt) \quad \text{for} \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

Then $f \in \mathcal{H}^\infty(S_\omega)$ for each $0 < \omega < \pi$ and $\lim_{z \to 0, z \in S_\omega} f(z)$ exists. Moreover, one clearly expects the formula

$$f(A) = \int_0^\infty t(t + A)^{-1} \mu(dt) \in \mathcal{L}(X). \quad (9.9)$$

However, there are examples of measures $\mu$ such that $f$ is not anchored in $\mathcal{E}_e(S_\omega)$ [3, Example 5.2]. That is, $f(A)$ is not defined within the sectorial calculus for $A$ even if the right-hand side of (9.9) is a perfectly well-defined expression.

The best one can say is that if $f(A)$ is defined in the sectorial calculus for $A$, then (9.9) holds. This is a consequence of the following analogue of Theorem 9.10.a), see also Exercise 9.13.

**Theorem 9.12.** Let $A$ be a non-injective sectorial operator on a Banach space $X$, let $\omega \in (\omega_{sr}(A), \pi)$ and let $(f_n)_n$ be a sequence in $\mathcal{H}^\infty(S_\omega)$ and $f \in \mathcal{H}^\infty(S_\omega)$ such that all $f_n(A)$ and $f(A)$ are defined within the sectorial calculus for $A$.

Suppose in addition that $f_n \to f$ pointwise and boundedly on $S_\omega \cup \{0\}$, that $f_n(A) \in \mathcal{L}(X)$ for all $n \in \mathbb{N}$ and that $f_n(A) \to T \in \mathcal{L}(X)$ strongly. Then $f(A) = T$.

**Proof.** Let $e := (1+z)^{-1}$, so that $e(A) = (1+A)^{-1}$. By passing to $f_n - f_n(0)e$ and $f - f(0)e$ we may suppose that $f_n(0) = f(0) = 0$. By Exercise 9.7, $ef_n, ef \in \mathcal{E}(S_\omega)$. Now apply Lemma 9.9(b) to $e_n := e(f_n - f)$. This yields $e(A)T = (ef)(A)$ and hence

$$f(A) = e(A)^{-1}(ef)(A) = e(A)^{-1}e(A)T = T.$$
The Holomorphic Functional Calculus for Sectorial Operators

In order to cover all instances of (9.9) it is necessary to extend the sectorial calculus again, but now in a topological way. There is an abstract framework for this.

Abstract Functional Calculus (V) — Topological Extensions

Let $F$ be an algebra. A sequential convergence structure on $F$ is a mapping

$$\tau : F^N \supseteq \text{dom}(\tau) \to F$$

with the following properties:

1) $\text{dom}(\tau)$ is a subalgebra of $F^N$ and $\tau$ is an algebra homomorphism.

2) If $(f_n)_n \in F^N$ and $f_n = 0$ for all but at most finitely many $n \in \mathbb{N}$, then $(f_n)_n \in \text{dom}(\tau)$ and $\tau((f_n)_n) = 0$.

One writes $f_n \tau \to f$ in place of $f = \tau((f_n)_n)$.

From now on we suppose that $F$ is endowed with a fixed sequential convergence structure $\tau$.

Let $E$ be a subalgebra of $F$ and $\Phi : E \to \mathcal{L}(X)$ a representation. Then the set

$$E^\tau := \{ f \in F | \exists (e_n)_n \in E, T \in \mathcal{L}(X) : e_n \tau \to f, \Phi(e_n) \to T \text{ strongly} \}$$

is a subalgebra of $F$ containing $E$. Suppose in addition that $\Phi$ is closable with respect to $\tau$, which means that

$$(e_n)_n \in E^N, T \in \mathcal{L}(X), e_n \tau \to 0, \Phi(e_n) \to T \text{ strongly} \implies T = 0. \quad (9.10)$$

Then one can define the $\tau$-extension $\Phi^\tau : E^\tau \to \mathcal{L}(X)$ of $\Phi$ by

$$\Phi^\tau(f) := \lim_{n \to \infty} \Phi(e_n)$$

whenever $(e_n)_n \in E^N$, $e_n \tau \to f$, and $\lim_{n \to \infty} \Phi(e_n)$ exists (strongly) in $\mathcal{L}(X)$.

The following theorem is straightforward.

**Theorem 9.13.** The so-defined mapping $\Phi^\tau : E^\tau \to \mathcal{L}(X)$ is an algebra homomorphism which extends $\Phi$.

Now suppose in addition that $\Phi : E \to \mathcal{L}(X)$ is non-degenerate, so that we can speak of its canonical (algebraic) extension. Since, as easily seen, $\mathcal{Z}(E) \subseteq \mathcal{Z}(E^\tau)$, Theorem 8.19 yields

$$\langle E \rangle_{\Phi} \subseteq \langle E^\tau \rangle_{\Phi^\tau} \quad \text{and} \quad \hat{\Phi}^\tau|_{\langle E \rangle_{\Phi}} = \hat{\Phi}$$
for the canonical (algebraic) extensions of $\Phi$ and $\Phi^\tau$ within $\mathcal{F}$.

**Topological Extension of the Sectorial Calculus**

Let $A$ be a sectorial operator on a Banach space $X$, let $\omega_{sc}(A) < \omega < \pi$ and

$$\Phi_A : \mathcal{E}_c(S_\omega) \to \mathcal{L}(X)$$

the sectorial calculus for $A$. As an immediate consequence of Theorems 9.10 and Theorem 9.12 (with $f_n \in \mathcal{E}_c(S_\omega)$ and $T = 0$) we obtain:

**Theorem 9.14.** The sectorial calculus $\Phi_A$ on $\mathcal{E}_c(S_\omega)$ is closable in $H^\infty(S_\omega)$ with respect to bounded and pointwise convergence on $S_\omega \cup \{0\}$.

Applying Theorem 9.13 yields the bp-extension $\Phi_A^{bp}$ of the sectorial calculus $\Phi_A$ for $A$. Note that if $A$ is injective then, by Theorem 9.10.a), $\Phi_A$ on $\mathcal{E}_c(S_\omega)$ is even closed (and not just closable) in $H^\infty(S_\omega)$ with respect to bp-convergence on $S_\omega$. Hence, the bp-extension of the sectorial calculus does not lead to a larger calculus in this case.

This is different for non-injective sectorial operators, as the following example shows.

**Example 9.15 (Hirsch Functional Calculus).** Suppose that $\mu \in M(0, \infty)$ and

$$f(z) := \int_0^\infty \frac{t}{t + z} \mu(dt) \quad (z \in \mathbb{C} \setminus (-\infty, 0]). \quad (9.11)$$

Functions of this form are the core of the so-called Hirsch calculus, see [4, Chap.4]. It is easy to see that $f$ is bounded and holomorphic on $S_\omega$ with limits

$$f(0) = \int_0^\infty 1 \, d\mu \quad \text{and} \quad f(\infty) = 0$$

for each $\omega \in (0, \pi)$. The approximants

$$f_n(z) = \int_{[n/n, n]} \frac{t}{t + z} \mu(dt)$$

converge to $f$ uniformly on each such sector $S_\omega$. Moreover, one can show that $f_n \in \mathcal{E}_c(S_\omega)$ and, by an application of Fubini’s theorem, that

$$f_n(A) = \int_{[n/n, n]} t(t + A)^{-1} \mu(dt)$$

for every sectorial operator $A$. It follows that $f(A)$ is defined in the bp-extension of the sectorial calculus for $A$ and

$$f(A) = \int_0^\infty t(t + A)^{-1} \mu(dt)$$
as expected. (Cf. also Exercise [9.13]).

As mentioned in the beginning of this section, there are examples of measures $\mu$ such that the function $f$ defined by (9.11) is not covered by the sectorial calculus for non-injective operators. Hence, in this case, the bp-extension of the sectorial calculus is strictly larger than the sectorial calculus.

Exercises

9.1 (Examples of Sectorial Operators).

a) Let $\Omega$ be a semi-finite measure space and $a : \Omega \to \mathbb{C}$ a measurable mapping. Show that the multiplication operator $M_a$ on $L^p(\Omega)$, $1 \leq p \leq \infty$, is sectorial of angle $\omega$ if and only if $a \in \mathcal{S}_{\omega}$ almost everywhere.

b) Let $H$ be a Hilbert space and $A$ a closed operator on $H$ with numerical range $W(A) \subseteq \overline{\mathcal{S}_{\omega}}$ and such that $\text{ran}(I+A) = H$. Show that $A$ is sectorial of angle $\omega$.

c) Suppose that the resolvent of an operator $A$ satisfies an estimate of the form

$$\|R(\lambda, A)\| \leq \frac{M}{|\text{Re} \lambda|}$$

for all $\text{Re} \lambda < 0$.

Show that $A$ is sectorial of angle $\pi/2$.

d) Suppose that the resolvent of an operator $A$ satisfies an estimate of the form

$$\|R(\lambda, A)\| \leq \frac{M}{|\text{Im} \lambda|}$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Show that $A^2$ is sectorial of angle $0$.

[Hints: For a) observe that $\|R(\lambda, A)\| = 1/\text{dist}(\lambda, \mathcal{S}_{\omega})$ and use (9.11); for b) use Theorem A.23 and the same argument as in a); for d) prove first that $\|\lambda R(A, A^2)\| \leq M/|\text{Im} \lambda|$.

9.2. Let $A$ be an operator on a Banach space $X$ such that $(-\infty, 0) \subseteq \rho(A)$ and $M := \sup_{t>0} \|t(t+A)^{-1}\| < \infty$. Show that $M \geq 1$ and that $A$ is sectorial of angle $\omega_{\text{sc}}(A) \geq \pi - \arcsin(\frac{1}{M})$. Moreover, show for $x \in X$ the equivalences

$$x \in \overline{\text{dom}(A)} \quad \text{if and only if} \quad \lim_{t \to \infty} t(t+A)^{-1}x = x$$

$$x \in \overline{\text{ran}(A)} \quad \text{if and only if} \quad \lim_{t \to 0} t(t+A)^{-1}x = 0.$$

Conclude that $\ker(A) \cap \overline{\text{ran}(A)} = \{0\}$.

[Hint: For the first statement, fix $\omega > \pi - \arcsin(\frac{1}{M})$ and $\lambda \in \mathbb{C} \setminus \overline{\mathcal{S}_{\omega}}$ and pick $a < 0$ such that that the triangle with vertices $0$, $a$ and $\lambda$ has a right angle at $\lambda$. Then use the Taylor expansion of $R(z, A)$ at $z = a$ to estimate $\lambda R(\lambda, A)$.]
9.3. Let $A$ be a sectorial operator of angle $\omega \in (0, \pi)$. Show that

a) for each $r > 0$ the operator $rA$ is sectorial of angle $\omega$ with

$$M(rA, \alpha) = M(A, \alpha) \quad (\omega < \alpha < \pi);$$

b) if $A$ is injective then $A^{-1}$ is sectorial of angle $\omega$ with

$$M(A^{-1}, \alpha) \leq 1 + M(A, \alpha) \quad (\omega < \alpha < \pi);$$

c) for each $|\theta| < \pi - \omega$ the operator $e^{i\theta}A$ is sectorial of angle $\omega + |\theta|$ with

$$M(e^{i\theta}A, \alpha') \leq M(A, \alpha' - |\theta|) \quad (\omega + |\theta| < \alpha' < \pi);$$

d) for each $\mu \in \mathbb{C} \setminus \{0\}$ with $|\arg \mu| =: \omega' < \pi - \omega$ the operator $A + \mu$ is sectorial of angle $\max(\omega, \omega')$ with

$$M(A + \mu, \alpha') \leq \frac{1}{\sin(\min(\alpha' - \omega', \pi/2))} \cdot M(A, \alpha),$$

where $\alpha = \min(\alpha', \pi - \omega')$ and $\max(\omega, \omega') < \alpha' < \pi$.

[Hint for b) cf. Exercise 2.4.]

9.4. Show that for a closed operator $A$ on a Banach space $X$ and a number $0 < \theta_0 \leq \pi/2$ the following assertions are equivalent:

a) $e^{\pm i\theta_0}A$ are sectorial of angle $\pi/2$.

b) $A$ is sectorial of angle $\pi/2 - \theta_0$.

[Hint: For the implication b) $\Rightarrow$ a) note Exercise 9.3.c.]

9.5. Let $A$ be a sectorial operator and let $f \in \mathcal{E}(S_{\omega})$ for some $\omega \in (\omega_{se}(A), \pi)$. Show that

$$\text{ran}(f(A)) \subseteq \overline{\text{dom}(A)} \cap \text{ran}(A).$$

Moreover, show that the following assertions are equivalent for $x \in X$:

(i) $x \in \overline{\text{dom}(A)} \cap \text{ran}(A)$;

(ii) $x \in \text{dom}(A) \cap \text{ran}(A)$;

(iii) $n^2(1 + nA)^{-1}(n + A)^{-1}x \to x$ as $n \to \infty$.

9.6 (Functions with polynomial limits). Let $0 < \omega < \pi$. Show that

$$(\lambda - z)^{-1} \in \mathcal{E}_c(S_{\omega}) \quad (\lambda \in \mathbb{C} \setminus \overline{S_{\omega}}).$$

More generally, let $f \in \mathcal{H}^\infty(S_{\omega})$ such that

$$f(z) - c = O(|z|^s) \quad (z \to 0) \quad \text{and} \quad f(z) - d = O(|z|^{-s'}) \quad (z \to \infty)$$

for some $c, d \in \mathbb{C}$ and $s, s' > 0$. Show that
\[ f - d1 - \frac{c - d}{1 + z} \in \mathcal{E}(S_\omega) \]

and hence \( f \in \mathcal{E}_c(S_\omega) \). (One says that \( f \) has polynomial limit \( c \) at 0 and \( d \) at \( \infty \).) Finally, show that \( f \) has a polynomial limit at 0 if \( f \) has a holomorphic extension to a neighborhood of 0.

**9.7.** Let \( A \) be a non-injective sectorial operator of angle \( \omega \) on a Banach space \( X \) and let \( f \in H^\infty(S_\omega) \), \( \omega \in (\omega_{sc}(A), \pi) \). Show that the following assertions are equivalent:

(i) \( f(A) \) is defined in the sectorial calculus for \( A \);

(ii) \( f(0) := \lim_{z \to 0} f(z) \) exists and \((1 + z)^{-1}(f - f(0)) \in \mathcal{E}(S_\omega)\);

(iii) \((1 + z)^{-1}\) is an anchor element for \( f \) w.r.t. the sectorial calculus for \( A \).

**9.8.** Let \( -A \) be the generator of a bounded \( C_0 \)-semigroup \( T : \mathbb{R}_+ \to \mathcal{L}(X) \) and suppose that \( T \) has a holomorphic extension (again denoted by \( T \)) to \( S_{\theta_0} \) for some \( 0 < \theta_0 \leq \pi/2 \) such that \( T \) is uniformly bounded on each sector \( S_\theta \), \( 0 < \theta < \theta_0 \). For \( |\theta| < \theta_0 \) define

\[ T^\theta(t) := T(te^{i\theta}), \quad t \geq 0. \]

(a) Show that \( T(z + w) = T(z)T(w) \) for all \( z, w \in S_{\theta_0} \).

(b) Show that

\[ \int_0^\infty e^{-\lambda t}T^\theta(t) \, dt = (\lambda + e^{i\theta}A)^{-1} \quad (\lambda \in S_{\pi/2-\theta}, \ |\theta| < \theta_0). \]

(c) Conclude from (b) that \( T^\theta \) is a bounded \( C_0 \)-semigroup and \(-e^{i\theta}A\) is its generator (\(|\theta| < \theta_0\)).

(d) Show that \( A \) is sectorial of angle \( \pi/2 - \theta_0 \) and that

\[ T(\lambda) = e^{-\lambda A} \quad (\lambda \in S_{\theta_0}). \]

[Hints: For (b) consider \( \int_{e^{i\theta} \mathbb{R}_+} T(z) e^{(\lambda+e^{i\theta}A)z} \, dz \) and apply Cauchy’s theorem. For the strong continuity in (c) note that it suffices to consider elements of the form \( x = (\lambda + e^{i\theta}A)^{-1}y \) for \( y \in X \) and \( \lambda > 0 \). For the first part of (d) see Exercise [9.4] for the second use the uniqueness of the Laplace transform.]

**9.9.** Let \( -iA \) be the generator of a bounded \( C_0 \)-group \( U = (U_s)_{s \in \mathbb{R}} \) on a Banach space \( X \). Show that \( A^2 \) is sectorial of angle 0 and the holomorphic semigroup generated by \( -A^2 \) is given by

\[ e^{-\lambda A^2} = \frac{1}{\sqrt{4\pi \lambda}} \int_{\mathbb{R}} e^{-\frac{\pi^2}{4\lambda} U_s} \, ds \quad (\text{Re} \lambda > 0). \]

[See also Exercise [9.1] (d) and Exercise [9.8]]
9.10. Let $A$ be a normal operator on a Hilbert space $H$ and let $\omega \in (0, \pi)$.

a) Show that $A$ is sectorial of angle $\omega$ if and only if $\sigma(A) \subseteq \overline{S_\omega}$.

b) Suppose that $A$ is sectorial of angle $\omega$ and $f \in \text{Mer}(S_\omega)$ is such that $f(A)$ is defined in the sectorial calculus for $A$. Show that the set $\{ f = \infty \}$ of poles of $f$ is an $A$-null set and that $f(A) = \Psi(g)$, where $\Psi$ is the Borel calculus for $A$ and $g$ is any Borel function on $\mathbb{C}$ such that $[g \neq f]$ is an $A$-null set.

[Hint for a): Exercise 9.1.a.]

Supplementary Exercises

9.11 (Cauchy–Gauss Representation). Let $\omega > 0$. Show that

$$e^{-(a-z)^2} \in \mathcal{E}(\text{St}_\omega) \quad \text{for each } a \in \mathbb{C}.$$

Conclude that for each $f \in H^\infty(\text{St}_\omega)$ one has the Cauchy–Gauss representation

$$f(a) = \frac{1}{2\pi i} \int_{\partial \text{St}_\delta} f(z) \frac{e^{-(z-a)^2}}{z-a} \, dz \quad (a \in \text{St}_\delta, \ 0 < \delta < \omega). \quad (9.12)$$

Use this to show that for each $f \in \mathcal{E}(\text{St}_\omega)$

$$\sup_{|\alpha| \leq \delta} \int_{\mathbb{R}} |f(r + i\alpha)| \, dr < \infty \quad (0 < \delta < \omega).$$

In the case $\omega \leq \pi$ conclude that

$$\sup_{|\alpha| \leq \delta} \int_{0}^{\infty} \left| f(re^{i\alpha}) \right| \frac{dr}{r} < \infty \quad (0 < \delta < \omega)$$

for each $f \in \mathcal{E}(S_\omega)$.

9.12. Let $A$ be a sectorial operator on a Banach space $X$, let $\omega_{se}(A) < \omega < \pi$ and $\Phi_A : \mathcal{E}(S_\omega) \to \mathcal{L}(X)$ the associated sectorial calculus. Show that

$$\mathcal{E}_c(S_\omega) := \mathcal{E}(S_\omega) \oplus \mathbb{C}1 \oplus \mathbb{C} \frac{1}{1+z}$$

is an algebra and that $\Psi : \mathcal{E}_c(S_\omega) \to \mathcal{L}(X)$ given by $(c, d \in \mathbb{C}, e \in \mathcal{E}(S_\omega))$

$$\Psi(c + d1 + d(1 + z)^{-1}) := \Phi_A(e) + c1 + d(1 + A)^{-1}$$

is a well-defined algebra homomorphism. Show further that

$$\sup_{r > 0} \| \Psi(f(rz)) \| < \infty$$
for each \( f \in \mathcal{E}_e(S_\omega) \) and that b) of Theorem 9.6 holds for \( \Psi \) instead of \( \Phi_A \).

(We usually write again \( \Phi_A \) instead of \( \Psi \).)

**9.13.** Let \( \mu \in M(0, \infty) \) and define

\[
f(z) := \int_0^\infty \frac{t}{t + z} \mu(dt) \quad (z \in \mathbb{C} \setminus (-\infty, 0]).
\]

Moreover, for \( n \in \mathbb{N} \) define

\[
f_n(z) := \int_{\lfloor 1/n, n \rfloor} \frac{t}{t + z} \mu(dt) \quad (z \in \mathbb{C} \setminus (-\infty, 0]).
\]

Fix \( 0 < \omega < \pi \) and a sectorial operator \( A \) of angle \( \omega_{set}(A) < \omega \) on a Banach space \( X \). Prove the following assertions:

a) \( f_n = f_n(0) + h_n \), where \( z \cdot h_n \) and \( z^{-1} \cdot h_n \) are bounded on \( S_\omega \). Conclude that \( f_n \in \mathcal{E}_e(S_\omega) \).

b) \( f_n(A) = \int_{\lfloor 1/n, n \rfloor} t(t + A)^{-1} dt \rightarrow \int_0^\infty t(t + A)^{-1} dt \) in operator norm.

c) \( f_n \rightarrow f \) uniformly on \( S_\omega \).

d) If \( f(A) \) is defined within the sectorial calculus for \( A \), then

\[
f(A) = \int_0^\infty t(t + A)^{-1} \mu(dt). \quad (9.13)
\]

e) \( f(A) \) is defined in the bp-extension of the sectorial calculus for \( A \), and [9.13] holds.

[Hint: d) is a direct consequence of Theorem 9.12, but also of e).]

**References**


