Derivative Pricing Based on Local Utility Maximization

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Abstract

This paper discusses a new approach to contingent claim valuation in general incomplete market models. We determine the *neutral derivative price* which occurs if investors maximize their *local utility* and if derivative demand and supply are balanced. We also introduce the *sensitivity process* of a contingent claim. This process quantifies the reliability of the neutral derivative price and it can be used to construct price bounds. Moreover, it allows to calibrate market models in order to be consistent with initially observed derivative quotations.

Key words: option pricing, incomplete markets, local utility, neutral derivative price, sensitivity process, local sensitivity

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1 Introduction

Consider a frictionless securities market where underlyings and derivatives are traded. Suppose that you have a good statistical model for the underlyings. This paper deals with the following question: What are reasonable derivative prices, or more precisely, how can one extend the model for the underlyings to a reasonable probabilistic model including both underlyings and derivatives?

From a practical point of view, such a model extension serves at least three needs. Firstly, it suggests a reasonable price to the *issuer* of a not yet traded contingent claim. Secondly, it provides the *risk manager* with a probability distribution on which she can base the risk assessment of a portfolio containing underlyings *and* derivatives. Thirdly, some approaches to contingent claim *hedging* rely on a model for the whole market including derivatives (cf. e.g. Kallsen (1999)).

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The literature addressing derivative pricing is far too extensive to be listed here. Very roughly, one may distinguish arbitrage reasoning in complete models (e.g. Black and Scholes (1973), Cox et al. (1979), Harrison and Pliska (1981), Kallsen and Taqqu (1998), Hobson and Rogers (1998)), equilibrium-type approaches (e.g. Rubinstein (1976), Naik and Lee (1990), Duan (1995), Aase (1997)), L^2- and similar projection methods (e.g. Föllmer and Schweizer (1991), Schweizer (1991,1996), Keller (1997), Grandits (1999a,b), Chan (1999), Miyahara (1999), Frittelli (2000), Goll and Rüschendorf (2001)), pricing by superhedging arguments (e.g. El Karoui and Quenez (1995)), and approaches that are more closely linked to specific models.

Our aim is to provide a theoretical framework that focuses on the demands of practitioners. More specifically, we want to compromise between three partly contradictory goals: Firstly, the methodology shall be applicable to a large class of semimartingale models for the underlyings. This rules out e.g. a purely arbitrage-based pricing which only works in the narrow set of complete markets. Nevertheless, our reasoning has to be based on economically meaningful assumptions. But at the same time, we want the resulting formulas to be simple enough for use in practice.

The approach in this paper is composed of two basic ingredients, namely *local utility maximization* and *neutral derivative prices*. Local utility optimization can be interpreted as expected utility maximization of the gains over infinitesimal time intervals (cf. Kallsen (1999)). It is remotely related to maximization of utility from consumption and to local risk minimization in the sense of Föllmer & Schweizer (1989, 1991), Schweizer (1991). We give a formal definition in Section 2.

A derivative price will be called *neutral* if the optimal portfolio contains no contingent claim. Intuitively, neutral prices are stable in the sense that they do not lead to unmatched supply of or demand for derivatives. To an economist, this valuation principle may sound quite natural or even familiar. Moreover, it is not restricted to the use of local utility. Nevertheless, we can produce almost no reference where such an approach has been taken. Maybe the most explicit ones are He & Pearson (1991a,b) and Davis (1997) (cf. also Kallsen (2001)). Section 3 contains an introduction to neutral derivative pricing in the context of local utility optimization. Mathematically, this approach amounts to choosing one particular equivalent martingale measure.

Neutral prices are based on stronger assumptions than purely arbitrage-based values. Therefore it is important to assess their reliability. We express the robustness of neutral derivative prices in terms of a *sensitivity process*. The idea is quite simple. Neutral prices are based on the assumption that the net demand for derivatives is 0. We measure the sensitivity of contingent claim prices to violation of this assumption by computing how much the price reacts to small demand perturbations (i.e. small positive or negative demand). Attainable claims are characterized by the fact that their prices do not depend on supply and demand. The sensitivity process also allows to calibrate the neutral pricing model to initially observed market quotations. These topics are addressed in Section 4. Finally, Section 5 contains some examples illustrating the new methodology.

Throughout, we use the notation of Jacod and Shiryaev (1987) (henceforth JS) and Jacod (1979,1980). In particular, we write stochastic and Stieltjes integrals as $\int_0^t H_s dS_s = H \cdot S_t$. The transposed of a vector or matrix x is denoted by $x^\top$ and its components by superscripts. Increasing processes are identified with their corresponding Lebesgue-Stieltjes measure. All proofs are relegated to the end of the respective section.

2 Local utility maximization

Our general mathematical framework for a frictionless market model with a finite number of traded securities is as follows: Fix a terminal time $T \in \mathbb{R}_+$. We work with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ in the sense of JS, I.1.2. *Securities* $0, \dots, d$ are modelled by their respective price processes $S^0, \dots, S^d$. Security 0 is assumed to be positive and plays a special role. As a *numeraire* by which all other assets are discounted it can be interpreted as the benchmark for risklessness. From now on we consider only the *discounted price process* $\widehat{S} := (\frac{1}{S^0}S^1, \dots, \frac{1}{S^0}S^d)$. We assume that $\widehat{S}$ is a $\mathbb{R}^d$-valued special semimartingale with characteristics (B, C, ν) (cf. JS, II.2.6). By JS, II.2.9 and II.2.29, one can write (B, C, ν) in the form

$$B_t + (x - h(x)) * \nu_t = \int_0^t b_s dA_s, \;\; C_t = \int_0^t c_s dA_s, \;\; \nu = A \otimes F, \tag{2.1}$$

where $A \in \mathscr{A}^+_{\text{loc}}$ is a predictable process, b is a predictable $\mathbb{R}^d$-valued process, c is a predictable $\mathbb{R}^{d\times d}$-valued process whose values are non-negative, symmetric matrices, and F is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$. Note that $B + (x - h(x)) * \nu$ is the predictable part of finite variation (i.e. the *drift*) in the canonical decomposition of the special semimartingale $\widehat{S}$ (cf. JS, II.2.29). Typical choices for A are $A_t := t$ (e.g. for Lévy processes, diffusions, Itô processes etc.) and $A_t := \sum_{s\leq t} 1_{\mathbb{N}\setminus\{0\}}(s)$ (discrete-time processes). Especially for $A_t = t$, one can interpret b_t as a drift rate, c_t as a diffusion coefficient, and F_t as a local jump measure.

Trading strategies are modelled by $\mathbb{R}^d$-valued, predictable stochastic processes $\varphi = (\varphi^1, \dots, \varphi^d)$, where φ^i_t denotes the number of shares of security i in your portfolio at time t. If the (vector) stochastic integral exists (in the sense of Jacod (1980)), we can define the real-valued *discounted gain process* $G(\varphi)$ by $G_t(\varphi) := \int_0^t \varphi_s^\top d\widehat{S}_s$. In order for gain processes and other expressions to exist, we will restrict our attention to $\mathscr{L}^1(\widehat{S})$, which denotes the set of all trading strategies φ satisfying

$$\int_0^T \Big(|\varphi_t^\top b_t| + \varphi_t^\top c_t \varphi_t + \int ((\varphi_t^\top x)^2 \wedge |\varphi_t^\top x|) F_t(dx)\Big) dA_t \in L^1(\Omega, \mathcal{F}, P). \tag{2.2}$$

As an investor, you may want to choose your trading strategy in some optimal way. Our notion of optimality is based on maximization of expected local utility.

Definition 2.1 1. We call a function $u : \mathbb{R} \to \mathbb{R}$ *utility function* if

(a) u is two times continuously differentiable.

(b) The derivatives u', u'' are bounded and $\lim_{x\to\infty} u'(x) = 0$.

(c) $u(0) = 0$, $u'(0) = 1$

(d) $u'(x) > 0$ for any $x \in \mathbb{R}$.

(e) $u''(x) < 0$ for any $x \in \mathbb{R}$.

2. For any $\psi \in \mathbb{R}^d$, $t \in \mathbb{R}_+$ the random variable

$$\gamma_t(\psi) := \psi^\top b_t + \frac{u''(0)}{2}\psi^\top c_t \psi + \int (u(\psi^\top x) - \psi^\top x) F_t(dx)$$

is termed *local utility* of ψ in t.

3. We call a strategy $\varphi \in \mathscr{L}^1(\widehat{S})$ *u-optimal* if

$$E\Big(\int_0^T \gamma_t(\varphi_t) dA_t\Big) \geq E\Big(\int_0^T \gamma_t(\widetilde{\varphi}_t) dA_t\Big)$$

for any $\widetilde{\varphi} \in \mathscr{L}^1(\widehat{S})$.

For motivation of u-optimality we refer the reader to Kallsen (1999). Intuitively, a u-optimal strategy maximizes the expected utility of the gains over infinitesimal time intervals, or put differently, the expected utility of aggregate consumption among all strategies whose financial gains are immediately consumed.

Remarks.

1. A strategy $\varphi \in \mathscr{L}^1(\widehat{S})$ is u-optimal if and only if, for any $\widetilde{\varphi} \in \mathscr{L}^1(\widehat{S})$, we have $\gamma_t(\varphi_t) \geq \gamma_t(\widetilde{\varphi}_t)$ $(P \otimes A)$-almost everywhere on $\Omega \times [0, T]$.

2. A typical example for a utility function is $u_\kappa : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{1}{\kappa}(1 + \kappa x - \sqrt{1 + \kappa^2 x^2})$, where the parameter $\kappa = -u''(0) \in (0, \infty)$ can be interpreted as the investor's *risk aversion*. Since the mappings u_κ are of a simple analytic form, we call them *standard utility functions*.

3. In Kallsen (1999) it is assumed that the process $A \in \mathscr{A}^+_{\text{loc}}$ is deterministic. A careful inspection of the proofs reveals that all statements in that paper remain true in this slightly more general setting. Note that the local utility depends on the chosen process A. However, the definition of u-optimality and the statements in Kallsen (1999) and in this paper do not depend on the particular choice of A.

Theorem 2.2 *A trading strategy $\varphi \in \mathscr{L}^1(\widehat{S})$ is u-optimal if and only if*

$$b_t^i + u''(0) c_t^{i\cdot} \varphi_t + \int x^i (u'(\varphi_t^\top x) - 1) F_t(dx) = 0 \tag{2.3}$$

$(P \otimes A)$-almost everywhere for $i = 1, \ldots, d$.

The proof to Theorem 2.2 can be found in Kallsen (1999), Corollary 3.6.

3 Neutral derivative pricing

In this section we turn to derivative pricing. More exactly, we propose a way to extend a market model for the underlyings to a model for both underlyings and derivatives. In a sense, the approach mimics the reasoning in complete models, but under stronger preference assumptions.

In complete models there exist unique arbitrage-free derivative values. The assertion that real market prices have to coincide with these values can be easily justified. It suffices to assume the existence of traders (from now on called *derivative speculators*) who exploit favourable market conditions once they detect them. The existence of derivative speculators explains why the market price cannot deviate too strongly from the right value: If it did, the huge demand for (resp. supply of) the mispriced security would push its price immediately closer to the rational value. The only assumption on the preferences of the speculators is that they do not reject riskless profits – which most people may agree on. The elegance of this approach comes at a price. It only works in complete models, or more exactly, for attainable claims.

We want to extend this reasoning to incomplete markets by imposing stronger assumptions on the preferences of derivative speculators. We suppose that they trade by maximizing a certain kind of utility. The role of the unique arbitrage-free price will now be played by the *neutral* derivative value. This is the unique price such that the speculators' optimal portfolio contains no contingent claim. Similarly as in the complete case we argue that the speculators' presence should prevent the market price from deviating too strongly from the neutral value.

The neutral pricing approach can be applied to various types of expected utility maximization (cf. Kallsen (2001)). In fact, the first to suggest this kind of valuation seems to be Davis (1997) (cf. also Karatzas and Kou (1996)), who considered expected utility of terminal wealth. Davis' suggestion for a reasonable derivative price is such that among all strategies that buy an infinitesimal number of contingent claims and hold it till maturity, a portfolio containing no derivative is optimal. Although he does not claim that this remains true if we compare among all portfolios trading *arbitrarily* with the contingent claim, this follows in an Itô-process setting from duality results by He and Pearson (1991b) and Karatzas et al. (1991). The first to notice the duality between portfolio optimization and *least favourable market completion* seem to be He and Pearson (1991a,b). They extended related earlier work by Pliska (1986), Karatzas et al. (1987), Cox and Huang (1989) and even Bismut (1975) on complete models, where, of course, selecting a pricing measure is not an issue. Recently, many papers addressed and applied the duality between portfolio optimization and the choice of martingale measures (e.g. Cvitanić and Karatzas (1992), Kramkov and Schachermayer (1999), Schachermayer (2001), Cvitanić et al. (2001), Frittelli (2000), Bellini and Frittelli (2000), Kallsen (2000, 2001), Goll and Kallsen (2000), Goll and Rüschendorf (2001), Xia and Yan (2000), Delbaen et al. (2000)).

Here, we assume that derivative speculators maximize their local utility in the sense of the previous section. In contrast to more common forms of utility, this leads to relatively explicit results for diverse models (cf. Section 5). From a theoretical point of view one may critisize that neutral derivative values depend on the utility function. However, the numerical differences are often small in practice. In models with continuous paths, the neutral prices do not depend on the utility function at all.

The general setting is as in the previous section. Fix a utility function u. We distinguish two kinds of securities: *underlyings* $1,\dots,m$ and *derivatives* $m+1,\dots,m+n$. The underlyings are given in terms of their discounted price process $\widehat{S}=(\widehat{S}^1,\dots,\widehat{S}^m)$. At this stage, the only information on the derivatives is their discounted terminal payoffs $R^{m+1},\dots,R^{m+n}$ at time T, which are supposed to be $\mathcal{F}_T$-measurable random variables. As explained above, our goal is to determine *neutral* price processes in the sense of the following

Definition 3.1 Special semimartingales $\widehat{S}^{m+1},\dots,\widehat{S}^{m+n}$ are called *neutral derivative price processes* if

1. $\widehat{S}^{m+i}_T=R^{m+i}$ P-almost surely for $i=1,\dots,n$,
2. there exists a u-optimal portfolio $\overline{\varphi}$ in the extended market $(\widehat{S}^1,\dots,\widehat{S}^{m+n})$ with $\overline{\varphi}^i=0$ for $i=m+1,\dots,m+n$.

For the following, we need some

Assumptions 3.2 1. There exists a u-optimal strategy $\varphi\in\mathscr{L}^1(\widehat{S})$ for the market $\widehat{S}=(\widehat{S}^1,\dots,\widehat{S}^m)$.

2. The local martingale $\mathscr{E}(N)$ is a martingale, where

$$N:=u''(0)\int_0^\cdot \varphi_t^\top d\widehat{S}^c_t+\frac{u'(\varphi^\top x)-1}{1+V}*(\mu^{\widehat{S}}-\nu)$$

and $V_t:=\int(u'(\varphi_t^\top x)-1)\nu(\{t\}\times dx)$ for $t\in[0,T]$. We define the probability measure $P^\star\sim P$ by $\frac{dP^\star}{dP}=\mathscr{E}(N)_T$.

3. The $P^\star$-local martingales $\widehat{S}^1,\dots,\widehat{S}^m$ are $P^\star$-martingales.

4. We assume that $R^{m+1},\dots,R^{m+n}$ can be *superhedged* by simple trading strategies, i.e. for $i=1,\dots,n$ there exist $M\in\mathbb{R}$ and simple strategies ξ,χ such that

$$-M+\int_0^T\xi_t^\top d\widehat{S}_t\le R^{m+i}\le M+\int_0^T\chi_t^\top d\widehat{S}_t.$$

By *simple strategy* we refer to a predictable $\mathbb{R}^m$-valued process of the form $\sum_{i=1}^k\psi_i 1_{]T_{i-1},T_i]}$ where $k\in\mathbb{N}$, $0\le T_1\le\dots\le T_k\le T$ are stopping times, and ψ_i is a bounded $\mathcal{F}_{T_{i-1}}$-measurable, $\mathbb{R}^m$-valued random variable for $i=1,\dots,k$.

Remark. The density process of $P^\star$ can also be written as

$$\mathscr{E}(N)=\exp(X-K^X),$$

where $X:=u''(0)\int_0^\cdot\varphi_t^\top d\widehat{S}_t+\sum_{t\le\cdot}(\log(u'(\varphi_t^\top\Delta\widehat{S}_t))-u''(0)\varphi_t^\top\Delta\widehat{S}_t)$ and K^X denotes the *modified Laplace cumulant process* introduced in Kallsen and Shiryaev (2000). In that paper one can also find sufficient conditions ensuring that $\mathscr{E}(N)$ is actually a martingale.

Definition 3.3 We call the above probability measure $P^\star$ *neutral pricing measure*.

The following theorem treats existence and uniqueness of neutral derivative prices. Moreover, it shows that these prices are obtained via conditional expectation relative to some equivalent martingale measure. This implies that the corresponding securities market allows no arbitrage opportunity.

Theorem 3.4 *Suppose that Assumptions 3.2 hold. Then the semimartingales $\widehat{S}^{m+1}, \dots, \widehat{S}^{m+n}$ defined by*

$$\widehat{S}_t^{m+i} := E^\star(R^{m+i}|\mathcal{F}_t)$$

for $t \in [0, T]$, $i = 1, \dots, n$ are neutral derivative price processes, where $E^\star$ denotes (conditional) expectation relative to $P^\star$. These are up to indistinguishability the only neutral derivative price processes that do not *lead to simple arbitrage opportunities (cf. Remark 1 below).*

Remarks.

1. By *simple arbitrage* in the previous theorem, we refer to a $\mathbb{R}^{m+n}$-valued simple strategy $\xi = \sum_{i=1}^k \psi_i 1_{]T_{i-1}, T_i]}$ with stopping times $0 \leq T_1 \leq \dots \leq T_k \leq T$ and bounded $\mathcal{F}_{T_{i-1}}$-measurable random variables ψ_i such that $G_T(\xi) = \int_0^T \xi_t^\top d(\widehat{S}^1, \dots, \widehat{S}^{m+n})_t \geq 0$ a.s. and > 0 with positive probability.

   It may seem counterintuitive that simple arbitrages are not automatically excluded if derivatives are neutrally priced. On the mathematical side, this phenomenon corresponds to the fact that local martingales are not necessarily martingales. Put differently, some games as e.g. the doubling or the suicide strategy are locally fair but turn out to be unfair on a global level.

2. Note that $P^\star$ is an equivalent martingale measure (EMM) for the extended market $(\widehat{S}^1, \dots, \widehat{S}^{m+n})$. In particular, neutral derivative prices coincide with the unique arbitrage-based prices in complete models.

3. The choice of T does not affect neutral derivative prices because the density process $\mathscr{E}(N)$ of $P^\star$ does not depend on T. Of course, T has to be chosen so large that the terminal payoffs $R^{m+1}, \dots, R^{m+n}$ are $\mathcal{F}_T$-measurable.

   A simple calculation shows that N and hence neutral derivative prices do not depend on the risk aversion parameter κ if standard utility functions in the sense of Remark 2 following Definition 2.1 are chosen. For contiuous processes (i.e. $F = 0$), $P^\star$ does not depend on the utility function u at all.

For explicit calculations it is often helpful to know the dynamics of the securities price process $\widehat{S}$ under the pricing measure, as stated in the following

Lemma 3.5 *The $P^\star$-characteristics of $\widehat{S} = (\widehat{S}^1, \dots, \widehat{S}^m)$ are of the form (2.1) with $(b^\star, c^\star, F^\star)$ instead of (b, c, F), where $b^\star_t = 0$, $c^\star_t = c_t$, and*

$$F^\star_t(G) = \int_G \frac{u'(\varphi_t^\top x)}{1 + \int (u'(\varphi_t^\top \widetilde{x}) - 1)\nu(\{t\} \times d\widetilde{x})} F_t(dx)$$

for $t \in [0, T]$ and $G \in \mathcal{B}(\mathbb{R}^m)$.

In discrete-time markets, the above results can be formulated in a simpler fashion. By *discrete-time market* we refer to the case that T is an integer and that $\widehat{S} = (\widehat{S}^1, \dots, \widehat{S}^m)$ and $(\mathcal{F}_t)_{t\in[0,T]}$ are constant on the open intervals between neighbouring integers.

Lemma 3.6 *Suppose that the market $\widehat{S} = (\widehat{S}^1, \dots, \widehat{S}^m)$ is discrete in time and that the first of Assumptions 3.2 holds. Then the second of Assumptions 3.2 holds as well. If, moreover, the price processes $\widehat{S}^1, \dots, \widehat{S}^m$ are non-negative, then the third assumption also holds, i.e. $P^\star$ is an equivalent martingale measure. Its density is of the form*

$$\frac{dP^\star}{dP} = \prod_{t=1,\dots,T} \frac{u'(\varphi_t^\top \Delta\widehat{S}_t)}{E(u'(\varphi_t^\top \Delta\widehat{S}_t)|\mathcal{F}_{t-1})},$$

where φ_t solves

$$E(u'(\varphi_t^\top \Delta\widehat{S}_t)\Delta\widehat{S}_t|\mathcal{F}_{t-1}) = 0$$

for $t = 1, \dots, T$.

Proofs

PROOF OF THE STATEMENTS IN ASSUMPTIONS 3.2. In the first two steps we prove that $\mathscr{E}(N)$ is a well-defined local martingale and that it can be written as in the remark following Assumptions 3.2.

Step 1: In the proof of Kallsen (1999), Theorem 3.1 it is shown that $\varphi \in L(\widehat{S})$, which implies that $u''(0)\varphi^\top \cdot \widehat{S}$ is a well-defined semimartingale. Note that $\log(u'(x)) - u''(0)x \leq Mx^2$ for $x \in [-1, 1]$ and some $M \in \mathbb{R}$ that is independent of x. Since $\sum_{t\leq\cdot}(\Delta X_t)^2 1_{\{|\Delta X_t|\leq 1\}} \in \mathscr{V}$ for any semimartingale X, it follows that $X := u''(0)\varphi^\top \cdot \widehat{S} + \sum_{t\leq\cdot}(\log(u'(\varphi_t^\top \Delta\widehat{S}_t)) - u''(0)\varphi_t^\top \Delta S_t)$ is a well-defined semimartingale as well. The fact that $e^{\Delta X}$ is bounded by some constant implies that e^X is a special semimartingale (cf. Kallsen and Shiryaev (2000), Lemma 2.13).

Step 2: Let $\widehat{W}_t := \int (e^x - 1)\nu^X(\{t\} \times dx)$ for $t \in [0, T]$. From Kallsen and Shiryaev (2000), Theorem 2.19 it follows that the local martingale $\exp(X - K^X)$ equals $\mathscr{E}(N)$ with $N := X^c + \frac{e^x-1}{1+\widehat{W}} * (\mu^X - \nu^X)$. It remains to be shown that N can be written as in Assumptions 3.2. Obviously, we have $X^c = u''(0)\varphi^\top \cdot \widehat{S}^c$ (cf. e.g. Goll and Kallsen (2000), Proposition A.2). Moreover, $\Delta X = \log(u'(\varphi^\top \Delta\widehat{S}))$ implies that $\widehat{W}_t = \int (u'(\varphi_t^\top x) - 1)\nu(\{t\} \times dx) = V_t$ and $\frac{e^x-1}{1+\widehat{W}} * (\mu^X - \nu^X) = \frac{u'(\varphi^\top x)-1}{1+V} * (\mu^{\widehat{S}} - \nu)$.

Step 3: Let $Z := \mathscr{E}(N)$. From $\Delta N = \frac{u'(\varphi^\top \Delta\widehat{S})-1}{1+V} - \frac{V}{1+V} = \frac{u'(\varphi^\top \Delta\widehat{S})}{1+\widehat{W}} - 1$ it follows that $Z = Z_-(1+\Delta N) = Z_- \frac{u'(\varphi^\top \Delta\widehat{S})}{1+\widehat{W}}$. Define $\beta := u''(0)\varphi$ and $Y : \Omega \times [0,T] \times \mathbb{R}^m \to \mathbb{R}_+$ by $Y(t,x) := \frac{u'(\varphi_t^\top x)}{1+\widehat{W}_t}$. Since $x = \Delta\widehat{S}_t(\omega)$ for $M^P_{\mu^{\widehat{S}}}$-almost all $(\omega,t,x) \in \Omega \times [0,T] \times \mathbb{R}^m$, we have $UZ = UZ_- \frac{u'(\varphi^\top \Delta\widehat{S})}{1+\widehat{W}} = UZ_- Y$ $M^P_{\mu^{\widehat{S}}}$-almost everywhere for any $\widetilde{\mathcal{P}}$-measurable function U. Moreover, $Z^c = Z_- \cdot (u''(0)\varphi^\top \cdot \widehat{S}^c) = (Z_-\beta)^\top \cdot \widehat{S}^c$ implies that $\langle Z^c, \widehat{S}^{i,c}\rangle = (Z_- c^{i\cdot}\beta) \cdot A$ for $i = 1,\dots,m$. From Girsanov's theorem for semimartingales (cf. JS, III.3.24), it follows that the $P^\star$-characteristics $(B^\star, C^\star, \nu^\star)$ of $\widehat{S}$ are given by $C^\star = C$, $\nu^\star = Y \cdot \nu$, and $B^{\star,i} = B^i + (u''(0)c^{i\cdot}\varphi) \cdot A + h^i(x)(Y-1) * \nu$ for $i = 1,\dots,m$.

Step 4: Fix $i \in \{1,\dots,m\}$. Note that Condition (2.3) implies $b^i_t + \int (h^i(x) - x^i) F_t(dx) = -u''(0)c^{i\cdot}_t\varphi_t - \int (x^i u'(\varphi_t^\top x) - h^i(x)) F_t(dx)$. On the set $\{\widehat{W}_t = 0\}$ this equals $-u''(0)c^{i\cdot}_t\varphi_t - \int (x^i Y(t,x) - h^i(x)) F_t(dx)$. On $\{\widehat{W}_t \neq 0\}$ we have $\Delta A_t \neq 0$, $c_t = 0$, and $\int h^i(x) F_t(dx)$ $\Delta A_t = \int h^i(x) \nu(\{t\} \times dx) = \Delta B^i_t = (b^i_t + \int (h^i(x) - x^i) F_t(dx)) \Delta A_t$, which implies $b^i_t + \int (h^i(x) - x^i) F_t(dx) = \int h^i(x) F_t(dx)$. Therefore $\int x^i u'(\varphi_t^\top x) F_t(dx) = 0$ and hence $\int x^i Y(t,x) F_t(dx) = 0$. This in turn implies that $b^i_t + \int (h^i(x) - x^i) F_t(dx) = -u''(0)c^{i\cdot}_t\varphi_t - \int (x^i Y(t,x) - h^i(x)) F_t(dx)$ holds on the set $\{\widehat{W}_t \neq 0\}$ as well. Since $B^i = (b^i_t + \int (h^i(x) - x^i) F_t(dx)) \cdot A$, it follows from the previous step that $B^{\star,i} = -(x^i - h^i(x)) * \nu^\star$. By JS, II.2.29 this means that $\widehat{S}^i$ is a $P^\star$-local martingale. □

Proposition 3.7 *Let U, V be special semimartingales. If X is a semimartingale with $U \leq X \leq V$, then X is a special semimartingale as well.*

PROOF. Since $X = (X - U) + U$, it suffices to consider the case $U = 0$. Let $B := \sum_{t\leq\cdot} \Delta X_t 1_{\{|\Delta X_t|>1\}}$ and $\widetilde{X} := X - B$. By JS, I.4.24, $\widetilde{X}$ is a special semimartingale. Moreover, B has pathwise only finitely many jumps on any finite interval. Since V is a special semimartingale, we have $\sup_{t\leq\cdot} |V_t - V_0| \in \mathscr{A}^+_{\mathrm{loc}}$ (cf. JS, I.4.23). Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of stopping times with $T_n \uparrow \infty$ P-almost surely such that $|\{t \leq T_n : |\Delta X_t| > 1\}| \leq n$ and $E(\sup_{t\leq T_n} |V_t - V_0|) < \infty$ and $V_0 \leq n$ on $\{T_n > 0\}$. From $|\Delta X| \leq V + V_-$ we can now conclude $\mathrm{Var}(B)_{T_n} = \sum_{t\leq T_n} |\Delta X_t| 1_{\{|\Delta X_t|>1\}} \leq 2n \sup_{t\leq T_n} V_t$ for the variation process of B, which implies that $E(\mathrm{Var}(B)_{T_n}) < \infty$ for any $n \in \mathbb{N}$. Therefore, $B \in \mathscr{A}_{\mathrm{loc}}$ and hence it is a special semimartingale (cf. JS, I.4.23). □

PROOF OF THEOREM 3.4. *Step 1:* From JS, I.1.47, it follows that $\xi^\top \cdot \widehat{S}$ is a $P^\star$-martingale for any simple strategy ξ. This implies $-M + \xi^\top \cdot \widehat{S} \leq \widehat{S}^{m+i} \leq M + \chi \cdot \widehat{S}$ if M, ξ, χ correspond to R^{m+i} as in the fourth part of Assumptions 3.2. In view of Jacod (1979), (2.51) and Proposition 3.7, we conclude that $\widehat{S}^{m+i}$ is a P-special semimartingale for $i = 1,\dots,n$.

Step 2: W.l.o.g. the characteristics of $\overline{S} := (\widehat{S}^1, \dots, \widehat{S}^{m+n})$ are given in the form (2.1), but with $(\overline{b}, \overline{c}, \overline{F})$ instead of (b, c, F). Let us repeat the steps leading to the measure $P^\star$ with the $\mathbb{R}^{m+n}$-valued processes $\overline{S}$ and $\overline{\varphi} := (\varphi, 0, \dots, 0) \in \mathscr{L}^1(\overline{S})$ instead of $\widehat{S}$ and φ. Obviously, the definition of N and $P^\star$ is not affected by this alternative choice. In Step 4 of the next to last proof, we obtained the $P^\star$-local martingale property of $\widehat{S}^i$ for $i = 1,\dots,m$ from the equation $b^i_t + u''(0)c^{i\cdot}_t\varphi_t + \int x^i(u'(\varphi_t^\top x) - 1) F_t(dx) = 0$ or equivalently

$\overline{b}_t^i + u''(0)\overline{c}_t^{i\cdot}\varphi_t + \int x^i(u'(\overline{\varphi}_t^\top x) - 1)\overline{F}_t(dx) = 0$. By reversing the argumentation in that step, we obtain the corresponding equation for $i = m+1, \dots, m+n$ from the $P^\star$-local martingale property of $\widehat{S}^{m+1}, \dots, \widehat{S}^{m+n}$. In view of Theorem 2.2, $\overline{\varphi}$ is a u-optimal strategy for $\overline{S}$, which implies that $\widehat{S}^{m+1}, \dots, \widehat{S}^{m+n}$ are neutral derivative price processes.

Step 3: By JS, I.1.47, $G(\xi)$ is a $P^\star$-martingale for any simple strategy in the market $\overline{S}$. Hence there exists no simple arbitrage in this extended market.

Step 4: For the uniqueness part assume that $\widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n}$ are neutral derivative prices corresponding to some u-optimal portfolio $\widetilde{\varphi} = (\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^m, 0, \dots, 0)$ in the extended market $(\widehat{S}^1, \dots, \widehat{S}^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n})$. Since $\widetilde{\varphi}$ does not contain any derivative, we have that $(\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^m)$ is an optimal strategy for the market $\widehat{S}$ with the same local utility. Similarly, the local utility of φ in the market $\widehat{S}$ and of $\overline{\varphi} := (\varphi, 0, \dots, 0)$ in the market $(\widehat{S}^1, \dots, \widehat{S}^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n})$ tally. Since φ is u-optimal in the market $\widehat{S}$, it follows that $\overline{\varphi}$ is optimal for $(\widehat{S}^1, \dots, \widehat{S}^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n})$. Hence we may w.l.o.g. assume $\widetilde{\varphi} = \overline{\varphi}$.

Step 5: Similar as above we repeat the steps leading to the measure $P^\star$ with the $\mathbb{R}^{m+n}$-valued processes $(\widehat{S}^1, \dots, \widehat{S}^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n})$ and $\overline{\varphi}$ instead of $\widehat{S}$ and φ. As before, the resulting measure $P^\star$ remains the same. As in Step 4 of the next to last proof, we conclude that $\widehat{S}^1, \dots, \widehat{S}^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n}$ are $P^\star$-local martingales.

Step 6: Fix $i \in \{1, \dots, n\}$ and let M, ξ, χ be chosen for derivative $m+i$ as in the fourth part of Assumptions 3.2. The absence of simple arbitrage implies that $-M + \xi^\top \cdot \widehat{S} \leq \widetilde{S}^{m+i} \leq M + \chi^\top \cdot \widehat{S}$. From JS, I.1.47 it follows that $\psi^\top \cdot \widehat{S}$ is a $P^\star$-martingale for any simple strategy ψ. Therefore, $\widetilde{S}^{m+i}$ is bounded from below and above by $P^\star$-martingales, which implies that it is of class (D) relative to $P^\star$ (cf. JS, I.1.47). Hence $\widetilde{S}^{m+i}$ is a $P^\star$-martingale with the same terminal value R^{m+i} as $\widehat{S}^{m+i}$, which yields the uniqueness. $\square$

PROOF OF LEMMA 3.5. The assertion for $c^\star$ and $F^\star$ has been shown in Step 3 of the first proof in this section. Since $\widehat{S}$ is a $P^\star$-martingale, its predictable part of finite variation $b^\star \cdot A$ vanishes. Hence the statement on $b^\star$ follows as well. $\square$

PROOF OF LEMMA 3.6. *Step 1:* For discrete-time processes the canonical choice for A in representation (2.1) of the characteristics is $A := \sum_{t\leq\cdot} 1_{\mathbb{N}\setminus\{0\}}(t)$. Moreover, we have $F_t(G) = P(\Delta\widehat{S}_t \in G \setminus \{0\}|\mathcal{F}_{t-1})$, $b_t = E(\Delta\widehat{S}_t|\mathcal{F}_{t-1})$, and $c_t = 0$ for $t \in [0, T]$, $G \in \mathcal{B}^m$ (cf. JS II.3.14). Equation (2.3) then reads as $0 = E(\Delta\widehat{S}_t^i + \Delta\widehat{S}_t^i(u'(\varphi_t^\top\Delta\widehat{S}_t) - 1)|\mathcal{F}_{t-1}) = E(u'(\varphi_t^\top\Delta\widehat{S}_t)\Delta\widehat{S}_t|\mathcal{F}_{t-1})$. The equation for $\frac{dP^\star}{dP}$ follows from $\Delta\frac{u'(\varphi^\top x)-1}{1+V} * (\mu^{\widehat{S}} - \nu)_t = \frac{u'(\varphi_t^\top\Delta\widehat{S}_t)-1-E(u'(\varphi_t^\top\Delta\widehat{S}_t)-1|\mathcal{F}_{t-1})}{1+E(u'(\varphi_t^\top\Delta\widehat{S}_t)-1|\mathcal{F}_{t-1})} = \frac{u'(\varphi_t^\top\Delta\widehat{S}_t)}{E(u'(\varphi_t^\top\Delta\widehat{S}_t)|\mathcal{F}_{t-1})} - 1$ and from the explicit formula for the stochastic exponential (cf. JS, I.4.63).

Step 2: Any non-negative local martingale is a supermartingale and in particular integrable (cf. Jacod (1979), (5.17)). Since any integrable discrete-time local martingale is a martingale (cf. JS, I.1.64 and the following remark), it follows that any non-negative discrete-time local martingale is actually a martingale. Hence we obtain the second of Assumptions 3.2. If $\widehat{S}^1, \dots, \widehat{S}^m$ are non-negative, the same argument yields that these processes are $P^\star$-martingales. $\square$

4 Sensitivity processes

In complete models there is no uncertainty about derivative prices as long as one believes in the model for the underlyings and in absence of arbitrage. In incomplete models derivative prices are subject to supply and demand. Contingent claim valuation approaches can only produce a suggestion based on more or less reasonable assumptions. Therefore it is desirable to quantify how much the price actually depends on supply and demand. Very intuitively, the situation may be compared to parameter estimation: One does not only produce an estimate but one also assesses its accuracy e.g. in terms of its variance or by confidence regions.

Below, we express the robustness of neutral prices in terms of a *sensitivity process*, which can be interpreted as a kind of derivative (in the mathematical sense) of the contingent claim price with respect to demand for the claim. By multiplying this process with some fixed number one obtains price bounds which serve as a counterpart to confidence intervals in statistics. Our approach is related to *consistent derivative pricing* in Kallsen (1998, 2000), but it allows to obtain more explicit results. An alternative concept is proposed in Cochrane and Saá-Requejo (2000), Černý and Hodges (1999) under the name *no-good-deal pricing* (cf. the end of this section).

If all participants in a derivative market are speculators in the sense of the preceding section, then derivative prices must coincide with neutral values: If they did not tally, then all derivative speculators would trade in the same direction (either buy or sell) and nobody would be there to take the counterposition. Let us now consider the more realistic situation that some traders (from now on called *other investors*) do not purely speculate (in the above sense) but trade for different reasons (e.g. insurance purposes). How are market prices affected if the aggregate demand for derivatives by these other investors does not sum up to 0? Since the speculators have to take the counterposition, market prices must be such that the number of derivatives in the speculators' aggregate portfolio exactly offsets the demand from the other traders. Intuitively, a positive demand by the other investors should lead to market prices which are higher then the neutral value in order to prompt speculators to sell contingent claims. Below we compute how robust prices are against small supply or demand from the other investors.

In the following definition, δ stands for a kind of derivative (in the mathematical sense) of the optimal portfolio relative to some price perturbation D: If the price process differs from a given model $\widehat{S}$ by a small amount εD, then one should adjust the optimal portfolio for $\widehat{S}$ by $\varepsilon\delta$.

Definition 4.1 Let $\widehat{S} = (\widehat{S}^1, \dots, \widehat{S}^d)$ be a securities market as in Section 2 and φ a corresponding u-optimal portfolio. Moreover, let $D = (D^1, \dots, D^d)$ denote a $\mathbb{R}^d$-valued special semimartingale and δ a $\mathbb{R}^d$-valued process. We call the family of strategies $(\varphi + \varepsilon\delta : \varepsilon \in \mathbb{R})$ *asymptotically u-optimal* for the family of markets $(\widehat{S} + \varepsilon D : \varepsilon \in \mathbb{R})$ if, outside some $(P \otimes A)$-null set, we have for any $\widetilde{\delta} \in \mathbb{R}^d$:

$$\gamma_t^{\widehat{S}+\varepsilon D}(\varphi_t + \varepsilon\delta_t) \geq \gamma_t^{\widehat{S}+\varepsilon D}(\varphi_t + \varepsilon\widetilde{\delta}) + o(\varepsilon^2),$$

where $\frac{o(\varepsilon^2)}{\varepsilon^2} \to 0$ for $\varepsilon \to 0$. (Here, $\gamma_t^{\widehat{S}+\varepsilon D}$ denotes the local utility of the securities price process $\widehat{S}+\varepsilon D$. It is w.l.o.g. assumed that the characteristics of both $\widehat{S}$ and D are absolutly continuous with respect to the same $A \in \mathscr{A}^+_{\text{loc}}$.) We call δ a *derivative of the optimal strategy relative to D in $\widehat{S}$* and we write $\delta = \frac{\partial \varphi}{\partial D}(\widehat{S})$.

From now on, let $\widehat{S} = (\widehat{S}^1, \dots, \widehat{S}^m)$ be a securities price process as in Section 3 (including Assumptions 3.2). Moreover, let $\widehat{S}^{m+1}, \dots \widehat{S}^{m+n}$ denote neutral derivative prices of contingent claims with discounted payoffs $R^{m+1}, \dots, R^{m+n}$ and let $\overline{\varphi} = (\varphi^1, \dots, \varphi^m, 0, \dots, 0)$ be a u-optimal strategy in the extended market $\overline{S} := (\widehat{S}^1, \dots, \widehat{S}^{m+n})$. The previous definition considers how the optimal portfolio is affected by changing the price process. But our principal question is rather the opposite: If the optimal portfolio of the speculators contains a small non-zero number $\varepsilon\delta^{(2)}$ of derivatives (caused by supply or demand from the other investors), what does this mean for the derivative price process?

Definition 4.2 A $\mathbb{R}^{n\times n}$-valued locally square-integrable special semimartingale $\partial\widehat{S}$ with $\partial\widehat{S}_T = 0$ is called *sensitivity process* if the following holds:

1. Under the neutral pricing measure $P^\star$, we have that $\partial\widehat{S}$ is a special semimartingale whose local martingale part is a square-integrable martingale.

2. For any $\delta^{(2)} \in \mathbb{R}^n$ there exists some $\mathbb{R}^m$-valued process $\delta^{(1)}$ such that $(\delta^{(1)}, \delta^{(2)})$ is a derivative of the optimal strategy relative to $(0, \dots, 0, \partial\widehat{S}^{1\cdot}\delta^{(2)}, \dots, \partial\widehat{S}^{n\cdot}\delta^{(2)})$ in $\overline{S} := (\widehat{S}^1, \dots \widehat{S}^{m+n})$.

Remarks.

1. The first property is a moderate integrability condition. We include it in the above definition to avoid pathological cases.

2. The real number $\partial\widehat{S}^{ij}_t\delta$ indicates how the price of claim $m+i$ at time t is affected by a small constant demand $\delta \in \mathbb{R}$ for derivative $m+j$. Let us paraphrase the previous definition for one contingent claim (i.e. $n = 1$): In a market where the derivative price process equals $\widehat{S}^{m+1} + \partial\widehat{S}\delta^{(2)}$, it is approximately optimal to hold $\delta^{(2)}$ contingent claims.

The sensitivity process will later be expressed in terms of *local sensitivity* in the sense of

Definition 4.3 Suppose that $\overline{S} = (\widehat{S}^1, \dots, \widehat{S}^{m+n})$ meets the integrability condition

$$\int_0^T \Big(|\overline{c}_t| + \frac{1}{1+V_t}\int |x|^2 \overline{F}_t(dx)\Big) dA_t \in L^2(\Omega, \mathcal{F}, P^\star), \tag{4.1}$$

where $\overline{b}, \overline{c}, \overline{F}$ are chosen as in (2.1) for $\overline{S}$ instead of $\widehat{S}$. Define the $\mathbb{R}^{(m+n)\times(m+n)}$-valued stochastic process H by

$$H^{ij}_t := \Big(-u''(0)\overline{c}^{ij}_t - \frac{1}{1+V_t}\int u''(\overline{\varphi}_t^\top x) x^i x^j \overline{F}_t(dx)\Big)$$

for $i, j = 1, \ldots, m+n$ and $t \in [0,T]$. We denote some sub-matrices of H by $H^{(uu)} := (H^{ij})_{i=1,\ldots,m}^{j=1,\ldots,m}$, $H^{(ud)} := (H^{ij})_{i=1,\ldots,m}^{j=m+1,\ldots,m+n}$, $H^{(du)} := (H^{ij})_{i=m+1,\ldots,m+n}^{j=1,\ldots,m} = (H^{(ud)})^\top$, $H^{(dd)} := (H^{ij})_{i=m+1,\ldots,m+n}^{j=m+1,\ldots,m+n}$. We call the $\mathbb{R}^{n\times n}$-valued process

$$E := H^{(dd)} - H^{(du)}(H^{(uu)})^- H^{(ud)}$$

local sensitivity, where $(H^{(uu)})^-$ denotes the Moore-Penrose pseudo inverse of the matrix $H^{(uu)}$ (cf. Albert (1972)).

Proposition 4.4 *The values of H and of the local sensitivity E are non-negative, symmetric matrices.*

The following theorem characterizes sensitivity processes in terms of local sensitivity.

Theorem 4.5 *Suppose that Condition (4.1) holds and that $(E^\star(\int_0^T E_s dA_s|\mathcal{F}_t))_{t\in[0,T]}$ is a locally square-integrable special semimartingale, where E^* denotes expectation relative to $P^\star$. Then there exists an up to indistinguishability unique sensitivity process. It is given by*

$$\partial\widehat{S}_t := E^*\Big(-\int_{t+}^T E_s dA_s \Big| \mathcal{F}_t\Big)$$

for $t \in [0,T]$, where $\int_{t+}^T E_s dA_s := \int_0^T E_s dA_s - \int_0^t E_s dA_s$.

Remark. The (infinitesimal) number $E_s^{ij} dA_s$ indicates how the price of claim $m+i$ is affected by a small demand shock for derivative $m+j$ which lasts for an (infinitesimal) period ds.

Roughly speaking, a contingent claim can be replicated if and only if its price process is insensitive to small supply and demand:

Theorem 4.6 *Suppose that the assumptions in the previous theorem hold and denote by $\partial\widehat{S}$ the sensitivity process. Let $i \in \{1,\ldots,n\}$. Then we have equivalence between:*

1. $\partial\widehat{S}_0^{ii} = 0$
2. $\partial\widehat{S}^{ij} = 0$ *for* $j = 1,\ldots,n$
3. *There exists some $\vartheta \in L(\widehat{S})$ such that $\widehat{S}^{m+i} = \widehat{S}_0^{m+i} + \int_0^\cdot \vartheta_s^\top d\widehat{S}_s$.*

Remark. Statement 3 in the previous theorem essentially means that the derivative price processes $\widehat{S}^{m+i}$ and in particular its terminal payoff R^{m+i} can be replicated by using the trading strategy ϑ. The restriction *essentially* refers to the fact that ϑ generally belongs to the set $L(\widehat{S})$ of all $\widehat{S}$-integrable processes and not necessarily to some particular smaller class of admissible trading strategies.

Sensitivity processes can be used to construct price bounds as a valuation counterpart to confidence regions in statistics. Suppose for simplicity that only one derivative is given (i.e.

$n = 1$) and consider time $t = 0$. Fix a demand/supply parameter $\delta^{(2)} \in \mathbb{R}$, e.g. $\delta^{(2)} = 1$. The interval $[\widehat{S}_0^{m+1} + \partial\widehat{S}_0\delta^{(2)}, \widehat{S}_0^{m+1} - \partial\widehat{S}_0\delta^{(2)}]$ can be interpreted as the set of initial prices which correspond to at most moderate external demand resp. supply of contingent claims. In the general case (i.e. for arbitrary n and t), one obtains time-varying *price regions*. The previous theorem shows that these price regions reduce to a single point for attainable claims.

Let us relate this approach to the concept of *no-good-deal pricing* in the sense of Černý and Hodges (1999). Among all model extensions, neutral price processes correspond to the lowest expected utility for the derivative speculators: They cannot increase their utility by investing in contingent claims at all. Instead of focusing on demand for derivatives, the concept of no-good-deal pricing considers the rise of expected utility if derivative prices deviate from neutral values. If one assumes that utility does not exceed its lower bound by some given number (i.e. there are no attractive derivative investments (*good deals*) in the market), then one obtains price regions similarly as above (*no-good-deal bounds*).

For risk management purposes one does not need a contingent claim valuation approach to determine derivative prices because current quotations are observable at the market. In this case, the concept of neutral valuation is used to obtain a reasonable *distribution of future price changes*, which in turn is needed to determine the risk involved in a portfolio of underlyings and derivatives. If the neutral values deviate from observed market quotations, one can use the sensitivity process to calibrate the model. The idea is to determine the demand vector $\delta^{(2)} \in \mathbb{R}^n$ in such a way that the calibrated model prices $\widehat{S}_0^{m+i} + \partial\widehat{S}_0^{i\cdot}\delta^{(2)}$ for $i = 1, \dots, n$ coincide with the observed quotations. The distribution of $(\widehat{S}^1, \dots, \widehat{S}^m, \widehat{S}^{m+1} + \partial\widehat{S}^{1\cdot}\delta^{(2)}, \dots, \widehat{S}^{m+n} + \partial\widehat{S}^{n\cdot}\delta^{(2)})$ serves as an improved model for risk management purposes. This approach may be compared to the use of *implicit volatilities* to make theoretical and observed derivative prices tally. But in contrast to the latter, the calibration via *implicit demand* $\delta^{(2)}$ does not affect the model for the underlyings. Moreover, the vector $\delta^{(2)}$ can be interpreted economically in the sense that a large value $\delta^{(2),j}$ indicates a strong demand for derivative $m + j$.

Proofs

PROOF OF PROPOSITION 4.4. The symmetry of H and E is obvious. $\overline{c}$ is non-negative definite by JS, II.2.9. Set $M := (-\int u''(\overline{\varphi}_t^\top x)x^i x^j \overline{F}_t(dx))_{i=1,\dots,m+n}^{j=1,\dots,m+n} \in \mathbb{R}^{(m+n)\times(m+n)}$. For $\lambda \in \mathbb{R}^{m+n}$ we have $\lambda^\top M\lambda = -\int(\sum_{i,j=1}^{m+n} \lambda^i x^i \lambda^j x^j)u''(\overline{\varphi}_t^\top x)\overline{F}_t(dx)$. Since $\sum_{i,j=1}^{m+n} \lambda^i x^i \lambda^j x^j = (\lambda^\top x)^2 \geq 0$, it follows that M is non-negative definite. Together, we have that the sum $H = -u''(0)\overline{c} + (1 + V)^{-1}M$ is non-negative definite. By Albert (1972), Theorem 9.1.6, E is non-negative definite as well. □

PROOF OF THEOREM 4.5. *Step 1:* In this proof, superscripts at the local utility γ or at the characteristics b, c, F in the sense of (2.1) refer to the process under consideration. Let $D = (D^1, \dots, D^{m+n})$ be a $\mathbb{R}^{m\times n}$-valued locally square-integrable special semimartingale with $D^1 = \dots = D^m = 0$. Suppose that, relative to $P^\star$, the process D is a special semimartingale whose local martingale part is square-integrable. Let δ be a $\mathbb{R}^{m+n}$-valued process and $\varepsilon \in (0,\infty)$. By definition, we have that the local utility of $\overline{S}+\varepsilon D$ in $\overline{\varphi}+\varepsilon\delta$ equals $\gamma^{\overline{S}+\varepsilon D}(\overline{\varphi}+\varepsilon\delta) = (\overline{\varphi}+\varepsilon\delta)^\top b^{\overline{S}+\varepsilon D} + \frac{u''(0)}{2}(\overline{\varphi}+\varepsilon\delta)^\top c^{(\overline{S}+\varepsilon D,\overline{S}+\varepsilon D)}(\overline{\varphi}+\varepsilon\delta) + \int (u((\overline{\varphi}+\varepsilon\delta)^\top x) - (\overline{\varphi}+\varepsilon\delta)^\top x) F^{\overline{S}+\varepsilon D}(dx)$. We will consider the three terms on the right-hand side seperately. Note that the first part can be written as $(\overline{\varphi}+\varepsilon\delta)^\top b^{\overline{S}+\varepsilon D} = \overline{\varphi}^\top b^{\overline{S}} + \varepsilon(\delta^\top b^{\overline{S}} + \overline{\varphi}^\top b^D) + \varepsilon^2 \delta^\top b^D$. Moreover, the second term without the factor $\frac{u''(0)}{2}$ equals

$$\begin{aligned}(\overline{\varphi}+\varepsilon\delta)^\top c^{(\overline{S}+\varepsilon D,\overline{S}+\varepsilon D)}(\overline{\varphi}+\varepsilon\delta) &= \overline{\varphi}^\top c^{(\overline{S},\overline{S})}\overline{\varphi} + \varepsilon\Big(2\delta^\top c^{(\overline{S},\overline{S})}\overline{\varphi} + 2\overline{\varphi}^\top c^{(\overline{S},D)}\overline{\varphi}\Big)\\ &\quad + \varepsilon^2\Big(\delta^\top c^{(\overline{S},\overline{S})}\delta + 2\delta^\top c^{(D,\overline{S})}\overline{\varphi} + 2\delta^\top c^{(\overline{S},D)}\overline{\varphi} + \overline{\varphi}^\top c^{(D,D)}\overline{\varphi}\Big)\\ &\quad + o(\varepsilon^2),\end{aligned}$$

where $\frac{o(\varepsilon^2)}{\varepsilon^2} \to 0$ for $\varepsilon \to 0$. For the third term, we need the second order representation $u(x) = u(x_0) + u'(x_0)(x-x_0) + \frac{1}{2}u''(x_0)(x-x_0)^2 + \int_0^1 (u''(x_0+t(x-x_0)) - u''(x_0))(1-t)dt(x-x_0)^2$ for $x_0, x \in \mathbb{R}$. Note that $\overline{\varphi}_t^\top \Delta D_t = 0$ and hence $\overline{\varphi}_t^\top y = 0$ for $F_t^{(\overline{S},D)}$-almost all $(x,y) \in \mathbb{R}^{2(m+n)}$. Moreover, the local square-integrability of D and Condition (4.1) imply that $\int |(x,y)|^2 F^{(\overline{S},D)}(d(x,y)) < \infty$ (cf. JS, II.2.29). Therefore, we have

$$\begin{aligned}\int (u((\overline{\varphi}+\varepsilon\delta)^\top x) - (\overline{\varphi}+\varepsilon\delta)^\top x) F^{\overline{S}+\varepsilon D}(dx) &= \int (u(\overline{\varphi}^\top x) - \overline{\varphi}^\top x) F^{\overline{S}}(dx)\\ &\quad + \varepsilon \int \delta^\top x (u'(\overline{\varphi}^\top x) - 1) F^{\overline{S}}(dx)\\ &\quad + \varepsilon^2 \Big(\int \delta^\top y (u'(\overline{\varphi}^\top x) - 1) F^{(\overline{S},D)}(d(x,y)) + \frac{1}{2}\int u''(\overline{\varphi}^\top x)(\delta^\top x)^2 F^{\overline{S}}(dx)\Big)\\ &\quad + o(\varepsilon^2).\end{aligned}$$

Since $\overline{\varphi}^\top b^D = 0$, $c^{(\overline{S},D)}\overline{\varphi} = 0$ and $c^{(D,D)}\overline{\varphi} = 0$, we obtain

$$\begin{aligned}\gamma^{\overline{S}+\varepsilon D}(\overline{\varphi}+\varepsilon\delta) &= \gamma^{\overline{S}}(\overline{\varphi}) + \varepsilon\delta^\top\Big(b^{\overline{S}} + u''(0)c^{(\overline{S},\overline{S})}\overline{\varphi} + \int x(u'(\overline{\varphi}^\top x) - 1)F^{\overline{S}}(dx)\Big)\\ &\quad + \varepsilon^2\Big(\delta^\top\Big(b^D + u''(0)c^{(D,\overline{S})}\overline{\varphi} + \int y(u'(\overline{\varphi}^\top x) - 1)F^{(\overline{S},D)}(d(x,y))\Big)\\ &\qquad + \frac{1}{2}\delta^\top\Big(u''(0)c^{(\overline{S},\overline{S})} + \int u''(\overline{\varphi}^\top x)xx^\top F^{\overline{S}}(dx)\Big)\delta\Big)\\ &\quad + o(\varepsilon^2). \qquad (4.2)\end{aligned}$$

Note that $b^{\overline{S}} + u''(0)c^{(\overline{S},\overline{S})}\overline{\varphi} + \int x(u'(\overline{\varphi}^\top x) - 1)F^{\overline{S}}(dx) = 0$ because of Equation (2.3). Similarly as in Step 3 of the first proof in Section 3, it follows that

$$\begin{aligned}b^{D,\star} &:= b^D + u''(0)c^{(D,\overline{S})}\overline{\varphi} + \int y\Big(\frac{u'(\overline{\varphi}^\top x)}{1+V} - 1\Big)F^{(\overline{S},D)}(d(x,y))\\ &= \frac{1}{1+V}\Big(b^D + u''(0)c^{(D,\overline{S})}\overline{\varphi} + \int y(u'(\overline{\varphi}^\top x) - 1)F^{(\overline{S},D)}(d(x,y))\Big)\end{aligned}$$

is the $P^\star$-drift of D, in the sense that the predictable part of finite variation in the canonical decomposition of the $P^\star$-special martingale D equals $b^{D,\star} \cdot A$. The second equality follows from the fact that $\Delta A_t \neq 0$, $c_t^{(D,\overline{S})} = 0$ and $b_t^D = \int y F^{(\overline{S},D)}(d(x,y))$ on the set $\{V \neq 0\}$ (cf. JS, II.2.9 and II.2.29). Hence, Equation (4.2) can be rewritten as

$$\gamma^{\overline{S}+\varepsilon D}(\overline{\varphi}+\varepsilon\delta) = \gamma^{\overline{S}}(\overline{\varphi}) + \varepsilon^2(1+V)\Big(\delta^\top b^{D,\star} - \frac{1}{2}\delta^\top H\delta\Big) + o(\varepsilon^2).$$

Consequently, δ is a derivative of the optimal strategy relative to D if and only if $\widetilde{\delta} \mapsto \widetilde{\delta}^\top b^{D,\star} - \frac{1}{2}\widetilde{\delta}^\top H\widetilde{\delta}$ attains its maximum in δ. Since H is non-negative, differentiation yields that this is the case if and only if $0 = b^{D,\star} - H\delta$. In view of $D^1 = \ldots = D^m = 0$, we can rewrite this condition as

$$\begin{aligned} 0 &= H^{(uu)}\delta^{(1)} + H^{(ud)}\delta^{(2)} && (4.3)\\ b^{(D^{m+1},\ldots,D^{m+n}),\star} &= H^{(du)}\delta^{(1)} + H^{(dd)}\delta^{(2)}, && (4.4)\end{aligned}$$

where we use the notation $\delta = (\delta^{(1)}, \delta^{(2)})$ with $\mathbb{R}^m$-valued $\delta^{(1)}$ and $\mathbb{R}^n$-valued $\delta^{(2)}$. The first equation implies that $H^{(du)}\delta^{(1)} + H^{(dd)}\delta^{(2)} = H^{(du)}(H^{(uu)})^- H^{(uu)}\delta^{(1)} + H^{(dd)}\delta^{(2)} = (-H^{(du)}(H^{(uu)})^- H^{(ud)} + H^{(dd)})\delta^{(2)} = E\delta^{(2)}$ (cf. Albert (1972), Theorem 9.1.6). Therefore, Equations (4.3) and (4.4) are equivalent to

$$\begin{aligned} 0 &= H^{(uu)}\delta^{(1)} + H^{(ud)}\delta^{(2)} && (4.5)\\ b^{(D^{m+1},\ldots,D^{m+n}),\star} &= E\delta^{(2)}. && (4.6)\end{aligned}$$

Step 2: Define $\partial\widehat{S}_t = E^*(-E\cdot A_T|\mathcal{F}_t) + E\cdot A_t$ for $t\in[0,T]$. Note that $0 \leq E \leq H^{(dd)}$ by Albert (1972), Theorem 9.1.6. By Huppert (1990), A II.7.7 this implies $\|E\| \leq \|H^{(dd)}\|$ if we set $\|E\| := \sup\{|Ex| : |x| \leq 1\}$ and likewise for $H^{(dd)}$. Since all norms on finite-dimensional spaces are equivalent, it follows that $|E| \leq \alpha|H^{(dd)}|$ for some $\alpha \in \mathbb{R}_+$. Therefore $|E\cdot A_T| \leq |E|\cdot A_T \leq \alpha|H^{(dd)}|\cdot A_T \in L^2(\Omega,\mathcal{F},P^\star)$ by Condition 4.1. In particular, $\partial\widehat{S}$ is well-defined. Moreover, it is a $P^\star$-special semimartingale with drift part $b^{\partial\widehat{S},\star}\cdot A = E\cdot A$ and square-integrable martingale part $E^\star(-E\cdot A_T|\mathcal{F}_\cdot)$ (cf. JS, I.1.42). For $\delta^{(2)} \in \mathbb{R}^n$ choose $\delta^{(1)} := -(H^{(uu)})^- H^{(ud)}\delta^{(2)}$ and let $D := (0, \partial\widehat{S}\delta^{(2)})$. Since Equations (4.5) and (4.6) are satisfied, we have that $\partial\widehat{S}$ is a sensitivity process.

Step 3: Conversely, let $\partial\widehat{S}$ be any sensitivity process. For $\delta^{(2)} \in \mathbb{R}^n$ let the $\mathbb{R}^m$-valued process $\delta^{(1)}$ be as in Definition 4.2. Since $(\delta^{(1)}, \delta^{(2)})$ is a derivative of the optimal strategy relative to $D := (0, \partial\widehat{S}\delta^{(2)})$, Equation (4.6) yields that $0 = b^{(D^{m+1},\ldots,D^{m+n}),\star} - E\delta^{(2)} = (b^{\partial\widehat{S},\star} - E)\delta^{(2)}$. Since this holds for any $\delta^{(2)}$, we have that $\partial\widehat{S} - E\cdot A$ is a $P^\star$-local martingale. From Condition 1 in Definition 4.2 it follows that $\partial\widehat{S} - E\cdot A$ is a square-integrable $P^\star$-martingale. Therefore, $\partial\widehat{S}_t - E\cdot A_t = E^\star(\partial\widehat{S}_T - E\cdot A_T|\mathcal{F}_t) = -E^\star(E\cdot A_T|\mathcal{F}_t)$ for $t\in[0,T]$, which yields the uniqueness.

Observe that in the uniqueness part, we have not used that $E^\star(E\cdot A_T|\mathcal{F}_\cdot)$ is a locally square-integrable special semimartingale. Since this follows automatically from the last equation, we conclude that this assumption is in fact a necessary condition for the existence of a sensitivity process. □

Proposition 4.7 *Let X be a $\mathbb{R}^d$-valued locally square-integrable martingale and ϑ an $\mathbb{R}^d$-valued predictable process. Suppose that $\langle X^i, X^j\rangle = a^{ij} \cdot A$ for $i,j=1,\dots,d$ where $A \in \mathscr{A}^+_{\mathrm{loc}}$ and a is a predictable process whose values are symmetric, non-negative matrices. Then the following statements are equivalent:*

1. $\vartheta \in L(X)$ *and* $\vartheta^\top \cdot X = 0$
2. $(\vartheta^\top a \vartheta) \cdot A = 0$

PROOF. *1⇒2:* By Jacod (1980), Proposition 2 we have that $\vartheta \in L^1_{\mathrm{loc}}(X)$ and hence $0 = [\vartheta^\top \cdot X, \vartheta^\top \cdot X] = (\vartheta^\top \widetilde{a} \vartheta) \cdot \widetilde{A}$, where $\widetilde{A} \in \mathscr{V}^+$ and the adapted $\mathbb{R}^{d\times d}$-valued process $\widetilde{a}$ are chosen such that $[X^i, X^j] = \widetilde{a}^{ij} \cdot \widetilde{A}$ for $i,j=1,\dots,d$. But this implies $\vartheta \in L^2(X)$ and hence $0 = \langle \vartheta^\top \cdot X, \vartheta^\top \cdot X\rangle = (\vartheta^\top a \vartheta) \cdot A$ by Jacod (1979), (4.59), (4.37).

2⇒1: From Jacod (1979), (4.33), (4.37), (4.61) it follows that $\vartheta \in L^2(X) \subset L(X)$ and $\langle \vartheta^\top \cdot X, \vartheta^\top \cdot X\rangle = (\vartheta^\top a \vartheta) \cdot A = 0$. This implies $\vartheta^\top \cdot X = 0$ (cf. JS, I.4.13). □

PROOF OF OF THEOREM 4.6. *Step 1:* The equivalence 1⇔2 follows from Albert (1972), Theorem 9.1.6. Since $\partial \widehat{S}^{ii}$ does not depend on $\widehat{S}^{m+j}$ for $j \neq i$, it suffices to consider the case $n=1$ for the rest of the proof.

Step 2: Define the $\mathbb{R}^{(m+1)\times(m+1)}$-valued process $\widetilde{H}$ by $\widetilde{H}^{ij}_t = \overline{c}^{ij}_t + (1+V)^{-1} \int x^i x^j u'(\overline{\varphi}_t^\top x) \overline{F}_t(dx)$ for $i,j=1,\dots,m+1$ and $t \in [0,T]$. Moreover, define $\widetilde{H}^{(uu)}$, $\widetilde{H}^{(ud)}$, $\widetilde{H}^{(du)}$, $\widetilde{H}^{(dd)}$, $\widetilde{E}$ similarly as in Definition 4.3. As in the proof of Proposition 4.4 one shows that the values of $\widetilde{H}$ and $\widetilde{E}$ are non-negative, symmetric matrices. Let $\lambda \in \mathbb{R}^{m+1}$. It follows from the proof of Proposition 4.4 that $\lambda^\top H \lambda = 0$ holds if and only if $\lambda^\top \overline{c} \lambda = 0$ and $\lambda^\top x = 0$ for $\overline{F}_t$-almost all $x \in \mathbb{R}^{m+1}$. Since the same is true for $\widetilde{H}$ instead of H, we have that $\lambda^\top H \lambda = 0$ holds if and only if $\lambda^\top \widetilde{H} \lambda = 0$.

Step 3: Let $\vartheta \in \mathbb{R}^m$. A straightforward calculation yields that $\binom{-\vartheta}{1}^\top H \binom{-\vartheta}{1} = \vartheta^\top H^{(uu)} \vartheta - 2\vartheta^\top H^{(ud)} + H^{(dd)} = (\vartheta - (H^{(uu)})^- H^{(ud)})^\top H^{(uu)} (\vartheta - (H^{(uu)})^- H^{(ud)}) + E$. Since $H^{(uu)}$ and E are non-negative, we have $\binom{-\vartheta}{1}^\top H \binom{-\vartheta}{1} = 0$ for some $\vartheta \in \mathbb{R}^m$ if and only if $E = 0$ if and only if $\binom{-\psi}{1}^\top H \binom{-\psi}{1} = 0$ for $\psi := (H^{(uu)})^- H^{(ud)}$.

Step 4: Suppose that Statement 3 holds. Observe that $\langle \widehat{S}^i, \widehat{S}^j\rangle^* = \widetilde{H}^{ij} \cdot A$ for $i,j = 1,\dots,m+1$, where $\langle \cdot,\cdot\rangle^*$ refers to the predictable covariation relative to the measure $P^\star$ (cf. JS, II.2.29). From $\binom{-\vartheta}{1}^\top \cdot \overline{S} = 0$ and Proposition 4.7 we conclude that $0 = (\binom{-\vartheta}{1}^\top \widetilde{H} \binom{-\vartheta}{1}) \cdot A$. Using Step 2, this implies $\binom{-\vartheta}{1}^\top H \binom{-\vartheta}{1} = 0$ $(P \otimes A)$-almost everywhere and hence $E = 0$ by Step 3.

Step 5: Conversely, assume that $\partial \widehat{S}^{ii}_0 = 0$ and hence $E = 0$ $(P \otimes A)$-almost everywhere. By Steps 3 and 2 this implies $\binom{-\vartheta}{1}^\top H \binom{-\vartheta}{1} = 0$ and hence $\binom{-\vartheta}{1}^\top \widetilde{H} \binom{-\vartheta}{1} = 0$ for $\vartheta := (H^{(uu)})^- H^{(ud)}$. The predictability of ϑ can be shown using the definition of the Moore-Penrose pseudoinverse in Albert (1972), Theorem 3.4. From Proposition 4.7 it follows that $\binom{-\vartheta}{1} \in L(\overline{S})$ and $\binom{-\vartheta}{1}^\top \cdot \overline{S} = 0$. This in turn implies $\vartheta \in L(\widehat{S})$ and $0 = \binom{-\vartheta}{1}^\top \cdot \overline{S} = -\vartheta \cdot \widehat{S} + \widehat{S}^{m+1}$. □

5 Examples

In this section, we illustrate neutral derivative pricing and sensitivity processes by considering particular cases. Since we do not want to confuse the reader with technicalities, we omit the conditions for existence of the neutral pricing measure etc. In all examples we suppose that there is only one underlying with price process $\widehat{S}^1$ besides the numeraire.

5.1 Markets with continuous paths

We assume that the price process $\widehat{S}^1$ has characteristics of the form (2.1) with $F = 0$ (no jumps). In this case the density process Z of the neutral pricing measure $P^\star$ is given by

$$Z_t = \exp\Big(-\int_0^t \frac{b_s}{c_s} d\widehat{S}^{1,c} - \frac{1}{2}\int_0^t \Big(\frac{b_s}{c_s}\Big)^2 d\langle \widehat{S}^{1,c}, \widehat{S}^{1,c}\rangle_s\Big), \tag{5.1}$$

where $\widehat{S}^{1,c} = \widehat{S}^1 - \int_0^\cdot b_s dA_s$ denotes the P-local martingale part of $\widehat{S}^1$. Observe that Z does not depend on the utility function u. Moreover, $P^\star$ coincides with the *minimal martingale measure* in the sense of Föllmer and Schweizer (1991), Theorem 3.5. The latter is used to determine hedging strategies that are optimal in a locally quadratic sense. But note that this equality holds only for continuous processes.

Now let us introduce a contigent claim R^2 into the market whose discounted price process $\widehat{S}^2$ is neutral and continuous. If we write the predictable covariation in the form $\langle \widehat{S}^{i,c}, \widehat{S}^{j,c}\rangle = c^{ij} \cdot A$ for $i, j = 1, 2$, where A is some predictable increasing process and c is a predictable, $\mathbb{R}^{2\times 2}$-valued process, then the local sensitivity of the derivative equals

$$E_t = -u''(0)\Big(c_t^{22} - \frac{(c_t^{12})^2}{c_t^{11}}\Big) \tag{5.2}$$

on the set $\{c^{11} \neq 0\}$. We will illustrate this equation in the context of bivariate diffusion models in the next subsection.

5.2 Bivariate diffusion models

A closer look at stock return data reveals that periods of violent price changes alternate with relatively calm intervals. This behaviour led to the introduction of ARCH and GARCH models on the one hand and bivariate diffusion settings on the other (cf. Frey (1997) for a survey in view of derivative pricing). For the latter, the volatility is modelled by a stochastic process following its own dynamic. More specifically, the price process $\widehat{S}^1$ is assumed to satisfy the stochastic differential equations (SDE's)

$$\begin{aligned} d\widehat{S}_t^1 &= \mu(\sigma_t)\widehat{S}_t^1 dt + \sigma_t \widehat{S}_t^1 dW_t \\ d\sigma_t &= \alpha(\sigma_t)dt + \beta(\sigma_t)d\widetilde{W}_t, \end{aligned} \tag{5.3}$$

where $\mu, \alpha : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R} \to \mathbb{R}_+$ are given continuous functions and $W, \widetilde{W}$ denote standard Wiener processes with correlation $\varrho \in [-1, 1]$ (i.e. $\langle W, \widetilde{W}\rangle_t = \varrho t$ for any $t \in \mathbb{R}_+$).

The second SDE describes the dynamic of the stochastic volatility σ. The characteristics of $\widehat{S}^1$ are of the form (2.1) with $b_t^1 = \mu(\sigma_t)\widehat{S}_t^1$, $c_t^{11} = (\sigma_t \widehat{S}_t^1)^2$, $F_t = 0$, and $A_t = t$ for $t \in [0, T]$. By Equation (5.1) the density process Z of the neutral pricing measure $P^\star$ equals

$$Z_t = \exp\Big(- \int_0^t \frac{\mu(\sigma_s)}{\sigma_s} dW_s - \frac{1}{2}\int_0^t \Big(\frac{\mu(\sigma_s)}{\sigma_s}\Big)^2 ds\Big).$$

Girsanov's theorem (cf. e.g. JS, III.3.11, II.4.4) yields that $W^\star := W + \int_0^\cdot \frac{\mu(\sigma_s)}{\sigma_s} ds$ and $\widetilde{W}^\star := \widetilde{W} + \varrho \int_0^\cdot \frac{\mu(\sigma_s)}{\sigma_s} ds$ are $P^\star$-Wiener processes with correlation ϱ. Therefore, the $P^\star$-dynamics of $\widehat{S}^1$ and σ can be better seen from the equations

$$d\widehat{S}_t^1 = \sigma_t \widehat{S}_t^1 dW_t^\star \tag{5.4}$$

$$d\sigma_t = \Big(\alpha(\sigma_t) - \varrho \frac{\mu(\sigma_t)\beta(\sigma_t)}{\sigma_t}\Big) dt + \beta(\sigma_t) d\widetilde{W}_t^\star. \tag{5.5}$$

Now suppose that we want to price a contingent claim $R^2 = g(\widehat{S}_T^1)$ for some measurable mapping $g : \mathbb{R} \to \mathbb{R}$. We set $C_{BS}(x, \Sigma) := \int g(x \exp(\sqrt{\Sigma} y - \frac{\Sigma}{2}))\phi(y)dy$ for $x \in \mathbb{R}$, $\Sigma \in \mathbb{R}_+$, where ϕ denotes the density of the standard normal distribution. Let us assume that the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is generated by $(W, \widetilde{W})$. Moreover, suppose that the SDE (5.5) has a unique strong solution and define the function $C_{bd} : (0,\infty) \times (0,\infty) \times [0,T] \to \mathbb{R}$ by

$$C_{bd}(x, \varsigma, t) := E\Bigg(C_{BS}\Bigg(x \exp\Big(\varrho \int_0^{T-t} \overline{\sigma}_s d\overline{W}_s - \frac{\varrho^2}{2}\int_0^{T-t} \overline{\sigma}_s^2 ds\Big), \sqrt{1-\varrho^2} \int_0^{T-t} \overline{\sigma}_s^2 ds\Bigg)\Bigg),$$

where $(\overline{W}, \overline{\sigma})$ denotes a solution to SDE (5.5) starting in $\overline{\sigma}_0 = \varsigma$ at time 0. Below it is shown that

$$\widehat{S}_t^2 = E^\star(g(\widehat{S}_T^1)|\mathcal{F}_t) = C_{bd}(\widehat{S}_t^1, \sigma_t, t) \tag{5.6}$$

holds for the neutral price process $\widehat{S}^2$ from Theorem 3.4.

If the mapping C_{bd} is of class C^2, Itô's formula and the $P^\star$-martingale property of $\widehat{S}^2$ yield

$$d\widehat{S}_t^2 = D_1 C_{bd}(\widehat{S}_t^1, \sigma_t, t)\sigma_t \widehat{S}_t^1 dW_t^\star + D_2 C_{bd}(\widehat{S}_t^1, \sigma_t, t)\beta(\sigma_t) d\widetilde{W}_t^\star.$$

Therefore the joint characteristics $(\overline{B}, \overline{C}, \overline{\nu})$ of $(\widehat{S}^1, \widehat{S}^2)$ are of the form (2.1) with drift vector

$$\begin{aligned} b_t^1 &= \mu(\sigma_t)\widehat{S}_t^1, \\ b_t^2 &= D_1 C_{bd}(\widehat{S}_t^1, \sigma_t, t)\mu(\sigma_t)\widehat{S}_t^1 + \varrho D_2 C_{bd}(\widehat{S}_t^1, \sigma_t, t)\beta(\sigma_t)\frac{\mu(\sigma_t)}{\sigma_t}, \end{aligned}$$

diffusion matrix

$$\begin{aligned} c_t^{11} &= (\sigma_t \widehat{S}_t^1)^2, \\ c_t^{12} &= c_t^{21} = D_1 C_{bd}(\widehat{S}_t^1, \sigma_t, t)(\sigma_t \widehat{S}_t^1)^2 + \varrho D_2 C_{bd}(\widehat{S}_t^1, \sigma_t, t)\beta(\sigma_t)\sigma_t \widehat{S}_t^1, \\ c_t^{22} &= (D_1 C_{bd}(\widehat{S}_t^1, \sigma_t, t)\sigma_t \widehat{S}_t^1)^2 + (D_2 C_{bd}(\widehat{S}_t^1, \sigma_t, t)\beta(\sigma_t))^2 \\ &\quad + 2\varrho D_1 C_{bd}(\widehat{S}_t^1, \sigma_t, t) D_2 C_{bd}(\widehat{S}_t^1, \sigma_t, t)\beta(\sigma_t)\sigma_t \widehat{S}_t^1, \end{aligned}$$

jump measure $F_t = 0$, and $A_t = t$. This allows to compute the *optimal hedging strategy* for $\widehat{S}^2$ in the sense of Kallsen (1999). If we have sold one contingent claim, the optimal number of shares of the underlying in the hedge portfolio is given by

$$\begin{aligned}\varphi_t^1 &= \frac{c_t^{12}}{c_t^{11}} + \frac{1}{-u''(0)}\frac{b_t^1}{c_t^{11}} \\ &= D_1 C_{bd}(\widehat{S}_t^1, \sigma_t, t) + \varrho \frac{\beta(\sigma_t)}{\sigma_t \widehat{S}_t^1} D_2 C_{bd}(\widehat{S}_t^1, \sigma_t, t) + \frac{1}{-u''(0)} \frac{\mu(\sigma_t)}{\widehat{S}_t^1 \sigma_t^2}.\end{aligned}$$

The third term represents the optimal investment in the underlying when no derivative is present. It can be neglected for large values of the *risk aversion* parameter $-u''(0)$. The dominating part consists of two terms. The first one corresponds to the classical *delta* of the option (i.e. its partial derivative with respect to the underlying) and it provides essentially the optimal hedge for $\varrho = 0$. If, however, the price movements of the underlying are correlated with volatility, one should take care of this dependence by adjusting the hedge portfolio with the second term.

Finally, let us compute the local sensitivity of the option. From Equation (5.2) we obtain

$$E_t = -u''(0)(1-\varrho^2)(\beta(\sigma_t) D_2 C_{bd}(\widehat{S}_t^1, \sigma_t, t))^2. \tag{5.7}$$

As one may expect, the sensitivity depends on the diffusion coefficient of the volatility process and on the partial derivative of the option price with respect to volatility. The latter is called *vega* of the option in the financial literature (cf. Hull (1997), 14.9). Equation (5.7) explains why risk managers try to construct *vega-neutral* portfolios, where this derivative is close to 0.

Note that for $\alpha = \beta = 0$ the volatility is deterministic and constant. Therefore, we recover the Black-Scholes model. As we know already from Theorem 4.6, the local sensitivity vanishes in this case.

Proofs

The following simple result is shown e.g. in Kallsen (1998), Proposition 4.21.

Proposition 5.1 *Let W be a real-valued standard Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, and let $\mathcal{C}$ be a sub-σ-field of $\mathcal{F}$ that is independent of W. Moreover, denote by Y a continuous adapted process that is $\mathcal{C}$-measurable. Then we have*

$$P^{\int_0^T Y_t dW_t | \mathcal{C}} = N\Big(0, \int_0^T Y_t^2 dt\Big) \quad P\text{-almost surely.}$$

PROOF OF EQUATION (5.6). Suppose that $|\varrho| \neq 1$. (The statement for $|\varrho| = 1$ follows similarly.) Define the $P^\star$-Brownian motion $\overline{W}^\star := (1-\varrho^2)^{-\frac{1}{2}}(W^\star - \varrho \widetilde{W}^\star)$ and let $Y^1 := \widehat{S}_0^1\, \mathscr{E}(\varrho \int_0^\cdot \sigma_t d\widetilde{W}_t^\star)$, $Y^2 := \mathscr{E}(\sqrt{1-\varrho^2} \int_0^\cdot \sigma_t d\overline{W}_t^\star)$. Then $\widehat{S}^1 = Y^1 Y^2$. Define a system of

SDE's

$$
\begin{aligned}
dX_t^1 &= \Big(\alpha(X_t^1) - \varrho\frac{\mu(X_t^1)\beta(X_t^1)}{X_t^1}\Big)dt + \beta(X_t^1)dW_t^1 && (5.8)\\
dX_t^2 &= \varrho X_t^1 X_t^2 dW_t^1 && (5.9)\\
dX_t^3 &= \sqrt{1-\varrho^2}X_t^1 X_t^3 dW_t^2, && (5.10)
\end{aligned}
$$

where (W^1, W^2) denotes a standard Wiener process in $\mathbb{R}^2$. If the first SDE has a unique strong solution for any starting value $X_0^1 > 0$, then the system (5.8) – (5.10) has a unique strong solution for any $(X_0^1, X_0^2, X_0^3) \in (0,\infty) \times \mathbb{R}^2$ because the last two equations are uniquely solved by $X^2 = X_0^2\, \mathscr{E}(\varrho \int_0^\cdot X_t^1 dW_t^1)$ and $X^3 = X_0^3\, \mathscr{E}(\sqrt{1-\varrho^2}\int_0^\cdot X_t^1 dW_t^2)$. Note that, relative to $P^\star$, the pair $((W^1, W^2), (X^1, X^2, X^3)) = ((\widetilde{W}^\star, \overline{W}^\star)), (\sigma, Y^1, Y^2))$ is a solution to Equations (5.8) – (5.10) starting in (σ_0, Y_0^1, Y_0^2). If we denote by $P_{(x^1,x^2,x^3)}$ the solution measure to the above system corresponding to $(X_0^1, X_0^2, X_0^3) = (x^1, x^2, x^3)$ and if $(\mathcal{B}_t)_{t\in\mathbb{R}_+}$ denotes the filtration generated by the Wiener process (W^1, W^2), then we have

$$
\begin{aligned}
\widehat{S}_t^2 &= E^*(g(Y_T^1 Y_T^2)|\mathcal{F}_t)\\
&= E_{(\sigma_0, Y_0^1, Y_0^2)}(g(X_T^2 X_T^3)|\mathcal{B}_t)
\end{aligned}
$$

by Theorem 3.4. From the Markov property of Itô diffusions (cf. e.g. Øksendal (1998), Theorem 7.1.2), we conclude that

$$
\begin{aligned}
\widehat{S}_t^2 &= E_{(\sigma_t, Y_t^1, Y_t^2)}(g(X_{T-t}^2 X_{T-t}^3))\\
&= E_{(\sigma_t, Y_t^1, Y_t^2)}(E_{(\sigma_t, Y_t^1, Y_t^2)}(g(X_{T-t}^2 X_{T-t}^3)|\sigma(W^1))). && (5.11)
\end{aligned}
$$

By Proposition 5.1 the conditional law of $\sqrt{1-\varrho^2}\int_0^{T-t} X_s^1 dW_s^2$ relative to $\sigma(W^1)$ equals $N(0, \sqrt{1-\varrho^2}\int_0^{T-t}(X_s^1)^2 ds)$. Therefore the conditional expectation in (5.11) equals $C_{BS}($ $X_{T-t}^2 X_0^3, \sqrt{1-\varrho^2}\int_0^{T-t}(X_s^1)^2 ds)$. Since $X_{T-t}^2 X_0^3 = X_0^2 X_0^3\, \mathscr{E}(\varrho\int_0^{T-t} X_s^1 dW_s^1)$, this in turn implies that $\widehat{S}_t^2 = C_{bd}(\widehat{S}_t^1, \sigma_t, t)$. □

5.3 Exponential Lévy processes

In the last couple of years, *exponential Lévy processes* have become popular for securities models, since they are mathematically tractable and provide a good fit to real data (cf. Eberlein and Keller (1995), Eberlein et al. (1998), Madan and Senata (1990), Barndorff-Nielsen (1998)). By this notion we refer to the case that the discounted price process $\widehat{S}^1$ is positive and of the form

$$\widehat{S}^1 = \widehat{S}^1\, \mathscr{E}(L), \tag{5.12}$$

where L is some Lévy process with characteristic triplet (b, c, F) relative to the truncation function $h : x \mapsto x$ (i.e. L is a PIIS in the sense of JS, II.4.1 and II.4.19). By Goll and Kallsen (2000), Lemma A.8 these processes coincide with those of the form $\widehat{S}^1 = \widehat{S}^1 \exp(\widetilde{L})$ for real-valued Lévy processes $\widetilde{L}$. In the case $(b, c, F) = (\mu - r, \sigma^2, 0)$ we recover the standard Osborne-Samuelson model with geometric Brownian motion.

How does the underlying price process $\widehat{S}^1$ behave under the neutral pricing measure $P^\star$? Note that the characteristics of $\widehat{S}^1$ are of the form (2.1) with $b_t = \widehat{S}^1_{t-}b$, $c_t = (\widehat{S}^1_{t-})^2 c$, $F_t(G) = F(\frac{1}{\widehat{S}^1_{t-}}G)$, and $A_t = t$ for $t \in [0,T]$, $G \in \mathcal{B}$. From Equation (2.3) we obtain that φ in Assumptions (3.2) is of the form $\varphi_t = \frac{\psi}{\widehat{S}^1_{t-}}$, where the real number ψ solves

$$b + u''(0)c\psi + \int x(u'(\psi x) - 1)F(dx) = 0.$$

Lemma 3.5 yields that the $P^\star$-characteristics of $\widehat{S}^1$ are as above, but with $(b^\star, c^\star, F^\star)$ given by $b^\star = 0$, $c^\star = c$, and

$$F^\star(G) = \int_G u'(\psi x)F(dx) \text{ for } G \in \mathcal{B}$$

instead of (b, c, F). Put differently, the P-Lévy process L in Equation (5.12) remains a process with independent, stationary increments under $P^\star$, but with $P^\star$-characteristic triplet $(b^\star, c^\star, F^\star)$.

5.4 Discrete-time markets

In this final example we consider the case that $\widehat{S}^1$ is a discrete-time process of the form

$$\widehat{S}^1_t = \widehat{S}^1_{t-1}(1 + Y_t),$$

where $Y_1, Y_2, \ldots, Y_T$ are identically distributed random variables (with some distribution Q on $(\mathbb{R}, \mathcal{B})$) such that Y_t is independent of $\mathcal{F}_{t-1}$ for $t = 1, 2, \ldots, T$. In this case the characteristics of $\widehat{S}^1$ are of the form (2.1) with $A_t = \sum_{s \leq t} 1_{\mathbb{N}\setminus\{0\}}(s)$, $b_t = \int x F_t(dx)$, $c_t = 0$, $F_t(G) = Q(\frac{1}{\widehat{S}^1_{t-1}}(G \setminus \{0\}))$ for $t = 1, \ldots, T$, $G \in \mathcal{B}$ (cf. JS, II.3.14). By Equation (2.3) we have that φ in Assumptions 3.2 is of the form $\varphi_t = \frac{\psi}{\widehat{S}^1_{t-1}}$, where ψ solves the equation

$$\int x u'(\psi x) Q(dx) = 0.$$

Lemma 3.5 yields that the $P^\star$-dynamics of $\widehat{S}_1$ are of the same form as above, but with $Q^\star$ instead of Q where $\frac{dQ^\star}{dQ}(x) := \frac{u'(\psi x)}{\int u'(\psi \tilde{x})Q(d\tilde{x})}$. In other words, the random variables $Y_1, \ldots, Y_T$ are independent and identically distributed under $P^\star$ as well. Note that this corresponds to the similar statement on L in the previous subsection.

As a concrete example let us consider a discretized version of the Osborne-Samuelson setting underlying the Black-Scholes model: If Q is a lognormal distribution with parameters $-\mu + r + \frac{\sigma^2}{2}$, $\frac{1}{\sigma}$, -1 (i.e. the law of $\log(1 + Y_t)$ is normal with mean $\mu - r - \frac{\sigma^2}{2}$ and variance σ^2), then the joint law of $\widehat{S}^1_0, \widehat{S}^1_1, \ldots, \widehat{S}^1_T$ is the same as for the geometric Brownian motion underlying the Black-Scholes formula, which satisfies $d\widehat{S}^1_t = (\mu - r)\widehat{S}^1_t dt + \sigma \widehat{S}^1_t dW_t$. The chosen parameters are $r = 1.05/250$, $\mu = 1.09/250$, $\sigma = 0.25/\sqrt{250}$. Note that this discrete-time model is incomplete and hence does not allow derivative pricing solely based on the absence of arbitrage.

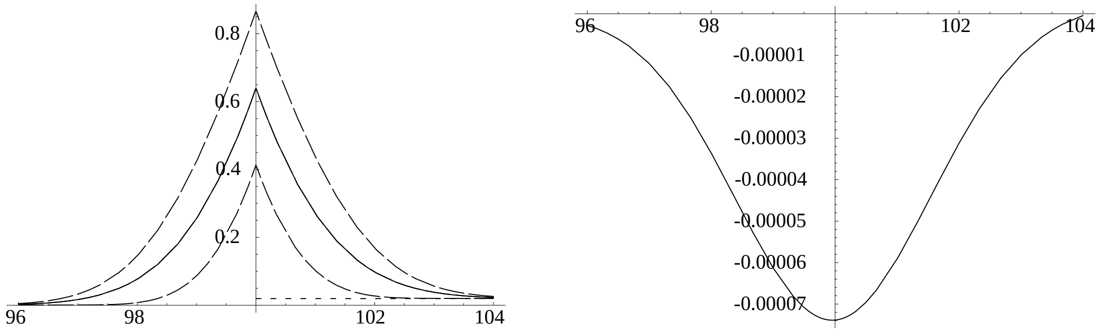

Figure 1: Time value and difference to Black-Scholes 1 day to maturity

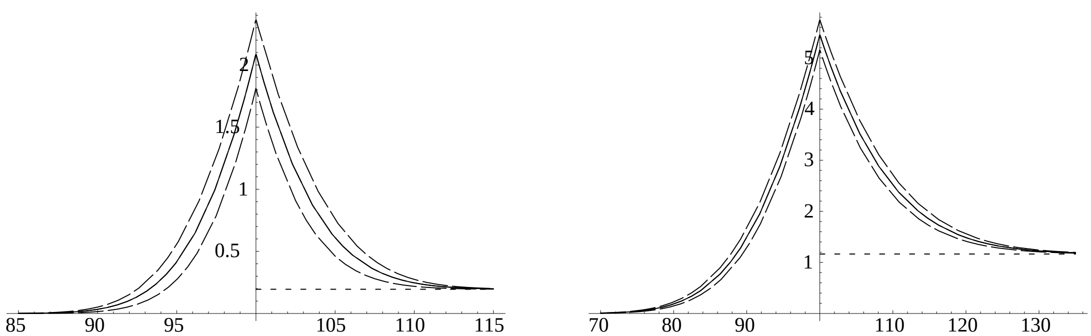

Figure 2: Time value 10 and 60 trading days to maturity

One may wonder how strongly the discretization of the Black-Scholes model affects option prices. Consider a European call option with strike price $K = 100$ expiring in 1, 10, 60 trading days, respectively. We define the *time value* of the option as the difference of its current price S_0^2 and the payoff $(S_0^1 - K)^+$ if it were to expire immediately. Note that the time value of a European call option is non-negative because $(S_0^1 - Ke^{-rT})^+ \geq (S_0^1 - K)^+$ is a lower arbitrage bound. The solid line in the left diagram of Figure 1 shows the time value of our European call one day before expiration as a function of the current stock price S_0^1. The dotted horizontal line represents the lower arbitrage bound. In fact, the solid line in the left diagram consists of two curves, firstly the time value in the discrete-time setting and secondly in the continuous-time Black-Scholes model. We use standard utility functions in the sense of Remark 2 following Definition 2.1. The tiny difference between the two curves is plotted on the right, i.e. the Black-Scholes value is slightly greater than the price in the discrete model. In Figure 2 we repeat the calculations for an option ten and sixty days before expiration.

Having seen that the effect of discretization to neutral call prices in negligible, let us now turn to price bounds based on the sensitivity process. For numerical computations it is useful to note that the coefficients of the matrix H in Definition 4.3 are of the form

$$H_t^{ij} = \frac{E(-u''(\varphi_t \Delta \widehat{S}_t^1)\Delta \widehat{S}_t^i \Delta \widehat{S}_t^j | \mathcal{F}_{t-1})}{E(u'(\varphi_t \Delta \widehat{S}_t^1)|\mathcal{F}_{t-1})} = \frac{E(-u''(\psi Y_t)\Delta \widehat{S}_t^i \Delta \widehat{S}_t^j | \mathcal{F}_{t-1})}{E(u'(\psi Y_1))}$$

for $t = 1, \ldots, T$ and $i, j = 1, 2$. The dashed lines in the left diagram of Figure 1 and in Figure 2 indicate the price interval in the sense of Section 4 relative to $\delta^{(2)} = \pm 1$ and standard utility functions with $\kappa = 1$. As one may expect, the sensitivity of an option in absolute numbers is highest at the money and its gets slightly larger with increasing time to maturity. However, compared to the time value of the option, we observe an entirely different behaviour. Since the value of an option increases rapidly with time to expiration, the price of long-lived options is relatively more robust against supply and demand (cf. the narrow bounds in the right diagram of Figure 2). Conversely, the price interval is comparatively large e.g. for options that are far out of the money.

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