Low-rank approximation of integral operators by using the second Green formula and quadrature

Jessica Gördes

joint work with Steffen Börm

CAU Kiel

September 8, 2011

Content

Content

Model Problem

Let $\Omega \subseteq \mathbb{R}^d$. We consider a Fredholm integral operator of the form

$$\mathcal{G}[u](x) = \int_\Omega g(x,y)u(y)\mathrm{d}y\,.$$

Model Problem

Let $\Omega \subseteq \mathbb{R}^d$. We consider a Fredholm integral operator of the form

$$\mathcal{G}[u](x) = \int_\Omega g(x,y)u(y)\mathrm{d}y\,.$$

Model Problem: Singularity function in $\mathbb{R}^2$ is

$$g(x,y) = -\frac{1}{2\pi}\log\|x-y\|\,.$$

Model Problem

Let $\Omega \subseteq \mathbb{R}^d$. We consider a Fredholm integral operator of the form

$$\mathcal{G}[u](x) = \int_\Omega g(x,y)u(y)\mathrm{d}y\,.$$

Model Problem: Singularity function in $\mathbb{R}^2$ is

$$g(x,y) = -\frac{1}{2\pi}\log\|x-y\|\,.$$

Standard Galerkin discretisation of $\mathcal{G}$ yields a matrix G with entries

$$G_{ij} := \int_\Omega \int_\Omega \varphi_i(x)g(x,y)\varphi_j(y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Model Problem

Let $\Omega \subseteq \mathbb{R}^d$. We consider a Fredholm integral operator of the form

$$\mathcal{G}[u](x) = \int_\Omega g(x,y)u(y)\mathrm{d}y\,.$$

Model Problem: Singularity function in $\mathbb{R}^2$ is

$$g(x,y) = -\frac{1}{2\pi}\log\|x-y\|\,.$$

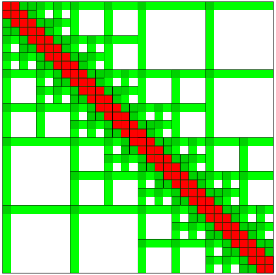
Standard Galerkin discretisation of $\mathcal{G}$ yields a matrix G with entries

$$G_{ij} := \int_\Omega\int_\Omega \varphi_i(x)g(x,y)\varphi_j(y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Since g is in general not local, one expects a **dense matrix** G.

$\mathcal{H}$-matrix

Approximation of G as $\mathcal{H}$-matrix:



$$G|_{t\times s} = \begin{cases} AB^\top & (\text{ low rank matrix }) \quad \text{if } (t,s) \text{ admissible} \\ G & (\text{ full matrix }) \quad \text{if } (t,s) \text{ not admissible} \end{cases}$$

Degenerate kernel function

degenerated kernel function:

$$g(x,y) \approx \tilde{g}(x,y) = \sum_{\nu=1}^{k} a_\nu(x)\, b_\nu(y)$$

Degenerate kernel function

degenerated kernel function: $\rightsquigarrow$ separated variables

$$g(x,y) \approx \tilde{g}(x,y) = \sum_{\nu=1}^{k} a_\nu(x)\, b_\nu(y)$$

Degenerate kernel function

degenerated kernel function: $\rightsquigarrow$ separated variables

$$g(x,y) \approx \tilde{g}(x,y) = \sum_{\nu=1}^{k} a_\nu(x)\, b_\nu(y)$$

$$\widetilde{G}_{ij} := \sum_{\nu=1}^{k} \underbrace{\int_\Omega \varphi_i(x) a_\nu(x) \mathrm{d}x}_{=A_{i\nu}} \underbrace{\int_\Omega \varphi_j(y)\, b_\nu(y)\, \mathrm{d}y}_{=B_{j\nu}}$$

Degenerate kernel function

degenerated kernel function: $\rightsquigarrow$ separated variables

$$g(x,y) \approx \tilde{g}(x,y) = \sum_{\nu=1}^{k} a_\nu(x)\, b_\nu(y)$$

$$\widetilde{G}_{ij} := \sum_{\nu=1}^{k} \underbrace{\int_\Omega \varphi_i(x) a_\nu(x) \mathrm{d}x}_{=A_{i\nu}} \underbrace{\int_\Omega \varphi_j(y)\, b_\nu(y)\, \mathrm{d}y}_{=B_{j\nu}}$$

low rank matrix: $G|_{t\times s} \approx AB^\top$ with $A \in \mathbb{R}^{t\times k}$ and $B \in \mathbb{R}^{s\times k}$

Degenerate kernel function

degenerated kernel function: $\rightsquigarrow$ separated variables

$$g(x,y) \approx \tilde{g}(x,y) = \sum_{\nu=1}^{k} a_\nu(x)\, b_\nu(y)$$

$$\widetilde{G}_{ij} := \sum_{\nu=1}^{k} \underbrace{\int_\Omega \varphi_i(x) a_\nu(x) \mathrm{d}x}_{=A_{i\nu}} \underbrace{\int_\Omega \varphi_j(y)\, b_\nu(y)\, \mathrm{d}y}_{=B_{j\nu}}$$

low rank matrix: $G|_{t\times s} \approx AB^\top$ with $A \in \mathbb{R}^{t\times k}$ and $B \in \mathbb{R}^{s\times k}$

$\rightsquigarrow$ $k(\#t + \#s)$ matrix entries instead of $\#t \cdot \#s$

Interpolation

Standard technique: via **Lagrange interpolation**

Let $t \times s \subseteq I \times I$ and corresponding domains

$$\tau := \bigcup_{i \in t} \mathrm{supp}(\varphi_i), \quad \sigma := \bigcup_{j \in s} \mathrm{supp}(\varphi_j)$$

and (minimal) axially parallel boxes B_τ, B_σ containing τ and σ.

Interpolation

Standard technique: via **Lagrange interpolation**

Let $t \times s \subseteq I \times I$ and corresponding domains

$$\tau := \bigcup_{i \in t} \operatorname{supp}(\varphi_i), \quad \sigma := \bigcup_{j \in s} \operatorname{supp}(\varphi_j)$$

and (minimal) axially parallel boxes B_τ, B_σ containing τ and σ.

Standard admissibility condition

$$\min\{\operatorname{diam}(B_\tau), \operatorname{diam}(B_\sigma)\} \leq \operatorname{dist}(B_\tau, B_\sigma)$$

Interpolation

Standard technique: via **Lagrange interpolation**

Let $t \times s \subseteq I \times I$ and corresponding domains

$$\tau := \bigcup_{i \in t} \operatorname{supp}(\varphi_i), \quad \sigma := \bigcup_{j \in s} \operatorname{supp}(\varphi_j)$$

and (minimal) axially parallel boxes B_τ, B_σ containing τ and σ.

Standard admissibility condition

$$\min\{\operatorname{diam}(B_\tau), \operatorname{diam}(B_\sigma)\} \leq \operatorname{dist}(B_\tau, B_\sigma)$$

$\rightsquigarrow$ interpolation in the bounding box with the smaller diameter

Interpolation

Interpolation in B_τ:

$$g(x,y) \approx \tilde{g}(x,y) = \sum_{\nu \in K} g(\xi_{\tau,\nu}, y)\mathcal{L}_{\tau,\nu}(x)$$

with

- $K := \{0, \dots, m-1\}^d$
- $\xi_{\tau,\nu}$ multidimensional transformed interpolation points
- $\mathcal{L}_{\tau,\nu}(x)$ multidimensional Lagrange polynomials

Interpolation

Interpolation in B_τ:

$$g(x,y) \approx \tilde{g}(x,y) = \sum_{\nu \in K} g(\xi_{\tau,\nu}, y)\mathcal{L}_{\tau,\nu}(x)$$

with

- $K := \{0, \ldots, m-1\}^d$
- $\xi_{\tau,\nu}$ multidimensional transformed interpolation points
- $\mathcal{L}_{\tau,\nu}(x)$ multidimensional Lagrange polynomials

$$\widetilde{G}_{ij} := \sum_{\nu \in K} \underbrace{\int_\tau \varphi_i(x)\mathcal{L}_{\tau,\nu}(x)\,\mathrm{d}x}_{=A_{i\nu}} \underbrace{\int_\sigma \varphi_j(y) g(\xi_{\tau,\nu}, y)\,\mathrm{d}y}_{=B_{j\nu}}$$

Interpolation

Interpolation in B_τ:

$$g(x,y) \approx \tilde{g}(x,y) = \sum_{\nu \in K} g(\xi_{\tau,\nu}, y)\mathcal{L}_{\tau,\nu}(x)$$

with

- $K := \{0, \dots, m-1\}^d$
- $\xi_{\tau,\nu}$ multidimensional transformed interpolation points
- $\mathcal{L}_{\tau,\nu}(x)$ multidimensional Lagrange polynomials

$$\widetilde{G}_{ij} := \sum_{\nu \in K} \underbrace{\int_\tau \varphi_i(x)\mathcal{L}_{\tau,\nu}(x)\,\mathrm{d}x}_{=A_{i\nu}} \underbrace{\int_\sigma \varphi_j(y) g(\xi_{\tau,\nu}, y)\,\mathrm{d}y}_{=B_{j\nu}}$$

$\rightsquigarrow \quad \widetilde{G} = AB^\top$

Interpolation

Interpolation in B_τ:

$$g(x,y) \approx \tilde{g}(x,y) = \sum_{\nu \in K} g(\xi_{\tau,\nu}, y)\mathcal{L}_{\tau,\nu}(x)$$

with

- $K := \{0, \ldots, m-1\}^d$
- $\xi_{\tau,\nu}$ multidimensional transformed interpolation points
- $\mathcal{L}_{\tau,\nu}(x)$ multidimensional Lagrange polynomials

$$\widetilde{G}_{ij} := \sum_{\nu \in K} \underbrace{\int_\tau \varphi_i(x)\mathcal{L}_{\tau,\nu}(x)\,\mathrm{d}x}_{=A_{i\nu}} \underbrace{\int_\sigma \varphi_j(y) g(\xi_{\tau,\nu}, y)\,\mathrm{d}y}_{=B_{j\nu}}$$

$\leadsto \quad \widetilde{G} = AB^\top \qquad \leadsto \quad \operatorname{rank} \widetilde{G} \leq \#K = m^d$

Content

Green formula for the kernel function

Theorem (Second Green formula)

Let $\Theta \subseteq \mathbb{R}^d$ be a normal domain, $\Gamma = \partial\Theta$ and let $n : \Gamma \to \mathbb{R}^d$ be the outer normal direction. If $u \in C^2(\overline{\Theta})$ satisfies the potential equation $\Delta u = 0$, we have

$$u(x) = \int_\Gamma g(x,z)\frac{\partial u}{\partial n}(z)\,\mathrm{d}z - \int_\Gamma \frac{\partial g}{\partial n_y}(x,z)u(z)\,\mathrm{d}z$$

for all $x \in \Theta$.

Green formula for the kernel function

Theorem (Second Green formula)

Let $\Theta \subseteq \mathbb{R}^d$ be a normal domain, $\Gamma = \partial\Theta$ and let $n : \Gamma \to \mathbb{R}^d$ be the outer normal direction. If $u \in C^2(\overline{\Theta})$ satisfies the potential equation $\Delta u = 0$, we have

$$u(x) = \int_\Gamma g(x,z)\frac{\partial u}{\partial n}(z)\,\mathrm{d}z - \int_\Gamma \frac{\partial g}{\partial n_y}(x,z)u(z)\,\mathrm{d}z$$

for all $x \in \Theta$.

For fixed $y \in \mathbb{R}^2$ the potential equation in $\mathbb{R}^2 \setminus \{y\}$ is solved by $g(x,y)$.

Green formula for the kernel function

Theorem (Second Green formula)

Let $\Theta \subseteq \mathbb{R}^d$ be a normal domain, $\Gamma = \partial\Theta$ and let $n : \Gamma \to \mathbb{R}^d$ be the outer normal direction. If $u \in C^2(\overline{\Theta})$ satisfies the potential equation $\Delta u = 0$, we have

$$u(x) = \int_\Gamma g(x,z)\frac{\partial u}{\partial n}(z)\,\mathrm{d}z - \int_\Gamma \frac{\partial g}{\partial n_y}(x,z)u(z)\,\mathrm{d}z$$

for all $x \in \Theta$.

For fixed $y \in \mathbb{R}^2$ the potential equation in $\mathbb{R}^2 \setminus \{y\}$ is solved by $g(x,y)$.

So we can apply the second Green formula to $u(x) := g(x,y)$ and get

$$g(x,y) = \int_\Gamma g(x,z)\frac{\partial g}{\partial n_x}(z,y)\,\mathrm{d}z - \int_\Gamma \frac{\partial g}{\partial n_y}(x,z)g(z,y)\,\mathrm{d}z\,.$$

Idea of the algorithm

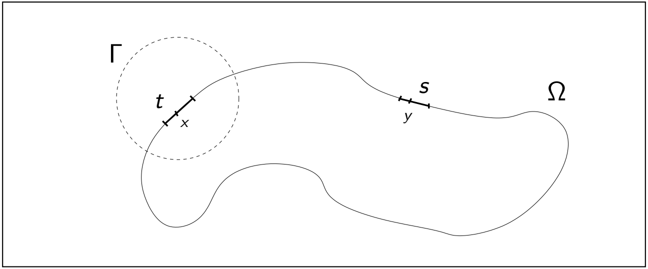
Green formula:

$$g(x,y) = \int_\Gamma g(x,z)\frac{\partial g}{\partial n_x}(z,y)\,\mathrm{d}z - \int_\Gamma \frac{\partial g}{\partial n_y}(x,z)g(z,y)\,\mathrm{d}z$$

Idea of the algorithm

Green formula:

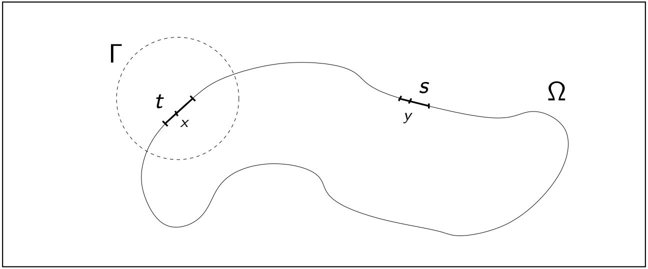
$$g(x,y) = \int_\Gamma g(x,z)\frac{\partial g}{\partial n_x}(z,y)\,\mathrm{d}z - \int_\Gamma \frac{\partial g}{\partial n_y}(x,z)g(z,y)\,\mathrm{d}z$$



Idea of the algorithm

Green formula:

$$g(x,y) = \int_\Gamma g(x,z)\frac{\partial g}{\partial n_x}(z,y)\,\mathrm{d}z - \int_\Gamma \frac{\partial g}{\partial n_y}(x,z)g(z,y)\,\mathrm{d}z$$



Quadrature:

$$g(x,y) \approx \sum w_\nu g(x,z_\nu)\frac{\partial g}{\partial n_x}(z_\nu,y) - \sum w_\nu \frac{\partial g}{\partial n_y}(x,z_\nu)g(z_\nu,y)$$

Approximation by quadrature

Admissibility condition for our method

$$\operatorname{diam}_{\mathsf{max}}(B_\tau) \leq \operatorname{dist}_{\mathsf{max}}(B_\tau, B_\sigma)$$

Approximation by quadrature

Admissibility condition for our method

$$\mathrm{diam}_{\mathsf{max}}(B_\tau) \leq \mathrm{dist}_{\mathsf{max}}(B_\tau, B_\sigma)$$

For the construction of a parametrisation we define

$$\delta := \frac{1}{2}\mathrm{diam}_{\mathsf{max}}(B_\tau) \qquad \text{and build} \qquad \Gamma = \bigcup_{\iota=1}^{4} \Gamma_\iota \qquad \text{with}$$

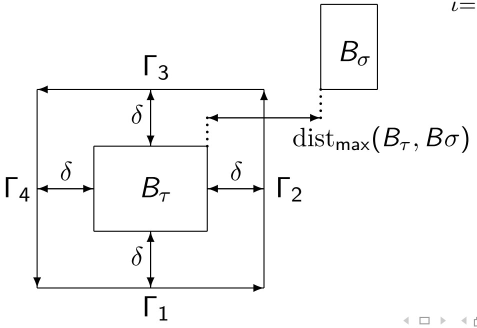
Approximation by quadrature

Admissibility condition for our method

$$\mathrm{diam}_{\mathsf{max}}(B_\tau) \leq \mathrm{dist}_{\mathsf{max}}(B_\tau, B_\sigma)$$

For the construction of a parametrisation we define

$$\delta := \frac{1}{2}\,\mathrm{diam}_{\mathsf{max}}(B_\tau) \qquad \text{and build} \qquad \Gamma = \bigcup_{\iota=1}^{4} \Gamma_\iota \qquad \text{with}$$



Approximation by quadrature

We use composite quadrature, e.g., Gaussian quadrature, with ℓ subsections with m quadrature points each.

Approximation by quadrature

We use composite quadrature, e.g., Gaussian quadrature, with ℓ subsections with m quadrature points each.

$$g(x,y) \approx \tilde{g}(x,y) = \sum_{\iota=1}^{4} \sum_{\nu=1}^{m\ell} \left(\|\gamma_\iota'(t_\nu)\| w_\nu \, g(x, \gamma_\iota(t_\nu)) \, \frac{\partial g}{\partial n_x}(\gamma_\iota(t_\nu), y) - \|\gamma_\iota'(t_\nu)\| w_\nu \, \frac{\partial g}{\partial n_y}(x, \gamma_\iota(t_\nu)) \, g(\gamma_\iota(t_\nu), y) \right)$$

where $t_1, \ldots, t_{m\ell} \in [-1,1]$ and $w_1, \ldots, w_{m\ell} \in \mathbb{R}$ are the whole of the quadrature points and weights

Low rank approximation of the matrix block

For $i \in t$, $j \in s$ and $\nu \in \{1, \ldots, m\ell\}$ and defining

$$(A_\iota)_{i\nu} = \|\gamma_\iota'(t_\nu)\| w_\nu \int_\Omega g(x, \gamma_\iota(t_\nu))\varphi_i(x)\,\mathrm{d}x$$

$$(\widehat{A}_\iota)_{i\nu} = \|\gamma_\iota'(t_\nu)\| w_\nu \int_\Omega \frac{\partial g}{\partial n_y}(x, \gamma_\iota(t_\nu))\varphi_i(x)\,\mathrm{d}x$$

$$(B_\iota)_{j\nu} = \int_\Omega \frac{\partial g}{\partial n_x}(\gamma_\iota(t_\nu), y)\varphi_j(y)\,\mathrm{d}y$$

$$(\widehat{B}_\iota)_{j\nu} = \int_\Omega g(\gamma_\iota(t_\nu), y)\varphi_j(y)\,\mathrm{d}y$$

Low rank approximation of the matrix block

For $i \in t$, $j \in s$ and $\nu \in \{1, \dots, m\ell\}$ and defining

$$
\begin{aligned}
(A_\iota)_{i\nu} &= \|\gamma_\iota'(t_\nu)\| w_\nu \int_\Omega g(x, \gamma_\iota(t_\nu))\varphi_i(x)\, \mathrm{d}x \\
(\widehat{A}_\iota)_{i\nu} &= \|\gamma_\iota'(t_\nu)\| w_\nu \int_\Omega \frac{\partial g}{\partial n_y}(x, \gamma_\iota(t_\nu))\varphi_i(x)\, \mathrm{d}x \\
(B_\iota)_{j\nu} &= \int_\Omega \frac{\partial g}{\partial n_x}(\gamma_\iota(t_\nu), y)\varphi_j(y)\, \mathrm{d}y \\
(\widehat{B}_\iota)_{j\nu} &= \int_\Omega g(\gamma_\iota(t_\nu), y)\varphi_j(y)\, \mathrm{d}y
\end{aligned}
$$

we get the representation

$$
\widetilde{G}|_{t\times s} = \sum_{\iota=1}^{4} A_\iota B_\iota^\top - \widehat{A}_\iota \widehat{B}_\iota^\top
$$

Low rank approximation of the matrix block

For $i \in t$, $j \in s$ and $\nu \in \{1, \dots, m\ell\}$ and defining

$$(A_\iota)_{i\nu} = \|\gamma_\iota'(t_\nu)\| w_\nu \int_\Omega g(x, \gamma_\iota(t_\nu))\varphi_i(x)\,\mathrm{d}x$$

$$(\widehat{A}_\iota)_{i\nu} = \|\gamma_\iota'(t_\nu)\| w_\nu \int_\Omega \frac{\partial g}{\partial n_y}(x, \gamma_\iota(t_\nu))\varphi_i(x)\,\mathrm{d}x$$

$$(B_\iota)_{j\nu} = \int_\Omega \frac{\partial g}{\partial n_x}(\gamma_\iota(t_\nu), y)\varphi_j(y)\,\mathrm{d}y$$

$$(\widehat{B}_\iota)_{j\nu} = \int_\Omega g(\gamma_\iota(t_\nu), y)\varphi_j(y)\,\mathrm{d}y$$

we get the representation

$$\widetilde{G}|_{t\times s} = \sum_{\iota=1}^{4} A_\iota B_\iota^\top - \widehat{A}_\iota \widehat{B}_\iota^\top$$

$\rightsquigarrow$ $\mathrm{rank}\, \widetilde{G}|_{t\times s} \leq 8m\ell \sim m^{d-1}$ (rank $m\ell$ for each $A_\iota B_\iota^\top$ and $\widehat{A}_\iota \widehat{B}_\iota^\top$).

Content

Error analysis

If we have

- an **asymptotically smooth kernel function**:
  $\exists C_{\mathrm{as}} > 0, c_0 \geq 1, \sigma \in \mathbb{N}_0 : \forall n \in \mathbb{N}$, $x, y \in \mathbb{R}^d$ with $x \neq y$ and all directions $p \in \mathbb{R}^d \times \mathbb{R}^d$ holds

$$|\partial_p^n g(x,y)| \leq C_{\mathrm{as}} \frac{(\sigma - 1 + n)! c_0^n \|p\|^n}{\|x-y\|^{\sigma+n}}$$

Error analysis

If we have

- an **asymptotically smooth kernel function**:
  $\exists C_{\mathrm{as}} > 0, c_0 \geq 1, \sigma \in \mathbb{N}_0 : \forall n \in \mathbb{N},\ x, y \in \mathbb{R}^d$ with $x \neq y$ and all directions $p \in \mathbb{R}^d \times \mathbb{R}^d$ holds
  $$|\partial_p^n g(x,y)| \leq C_{\mathrm{as}} \frac{(\sigma - 1 + n)! c_0^n \|p\|^n}{\|x-y\|^{\sigma+n}}$$
- a parametrisation where the **second derivative is zero**, so that
  $$\frac{d^n}{dt^n} g(x, \gamma_\iota(t)) = \partial^n_{\gamma_\iota'(t)} g(x, \gamma_\iota(t))\,.$$

Error analysis

If we have

- an **asymptotically smooth kernel function**:
  $\exists C_{\mathrm{as}} > 0, c_0 \geq 1, \sigma \in \mathbb{N}_0 : \forall n \in \mathbb{N}$, $x, y \in \mathbb{R}^d$ with $x \neq y$ and all directions $p \in \mathbb{R}^d \times \mathbb{R}^d$ holds

$$|\partial_p^n g(x,y)| \leq C_{\mathrm{as}} \frac{(\sigma - 1 + n)! c_0^n \|p\|^n}{\|x - y\|^{\sigma+n}}$$

- a parametrisation where the **second derivative is zero**, so that

$$\frac{d^n}{dt^n} g(x, \gamma_\iota(t)) = \partial^n_{\gamma'_\iota(t)} g(x, \gamma_\iota(t)) \, .$$

we can prove the following error estimate:

Error analysis

Theorem (Quadrature error)

For the approximation error for composite quadrature of n-th degree with ℓ subsections with m quadrature points each we have

$$|g(x,y) - \tilde{g}(x,y)| \le p(n+1)\left(\frac{c_0}{\ell}\right)^{n+1}$$

for all $x \in B_\tau$ and $y \in B_\sigma$ with $c_0 \ge 1$ and a polynomial p.

Error analysis

Theorem (Quadrature error)

For the approximation error for composite quadrature of n-th degree with ℓ subsections with m quadrature points each we have

$$|g(x,y) - \tilde{g}(x,y)| \leq p(n+1)\left(\frac{c_0}{\ell}\right)^{n+1}$$

for all $x \in B_\tau$ and $y \in B_\sigma$ with $c_0 \geq 1$ and a polynomial p.

$\rightsquigarrow$ choose ℓ with $\ell > c_0$

Error analysis

Theorem (Quadrature error)

For the approximation error for composite quadrature of n-th degree with ℓ subsections with m quadrature points each we have

$$|g(x,y) - \tilde{g}(x,y)| \leq p(n+1)\left(\frac{c_0}{\ell}\right)^{n+1}$$

for all $x \in B_\tau$ and $y \in B_\sigma$ with $c_0 \geq 1$ and a polynomial p.

- $\rightsquigarrow$ choose ℓ with $\ell > c_0$
- $\rightsquigarrow$ exponential decay against polynomial growth

Error analysis

Theorem (Quadrature error)

For the approximation error for composite quadrature of n-th degree with ℓ subsections with m quadrature points each we have

$$|g(x,y) - \tilde{g}(x,y)| \le p(n+1)\left(\frac{c_0}{\ell}\right)^{n+1}$$

for all $x \in B_\tau$ and $y \in B_\sigma$ with $c_0 \ge 1$ and a polynomial p.

- $\rightsquigarrow$ choose ℓ with $\ell > c_0$
- $\rightsquigarrow$ exponential decay against polynomial growth

Remark: $n = 2m - 1$ in case of Gaussian quadrature

Content

Numerical experiments

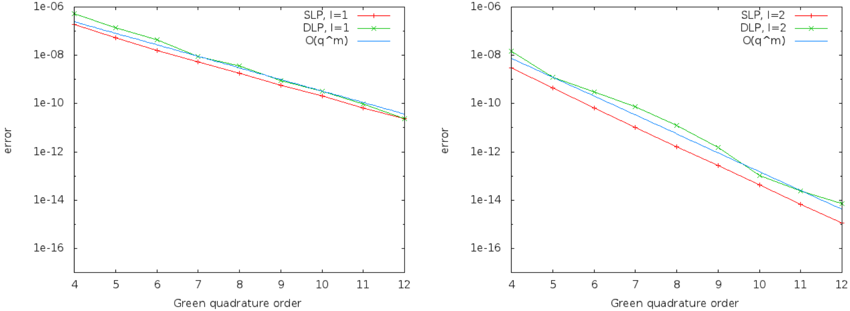


Figure: Error for SLP and DLP: $n = 32768$, $\ell = 1$ (left), $\ell = 2$ (right)

Numerical experiments

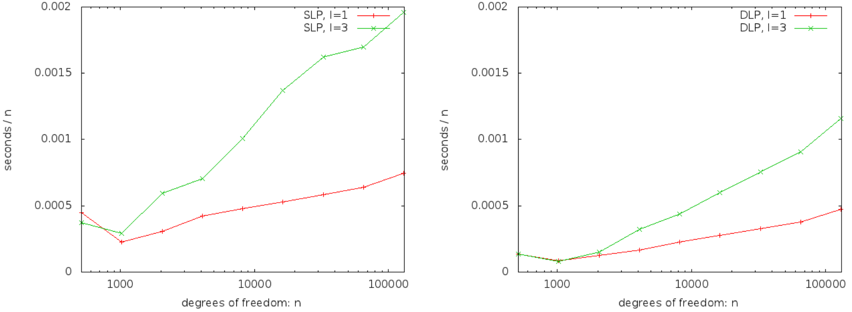


Figure: Building the $\mathcal{H}$-matrix in seconds$/n$

Numerical experiments

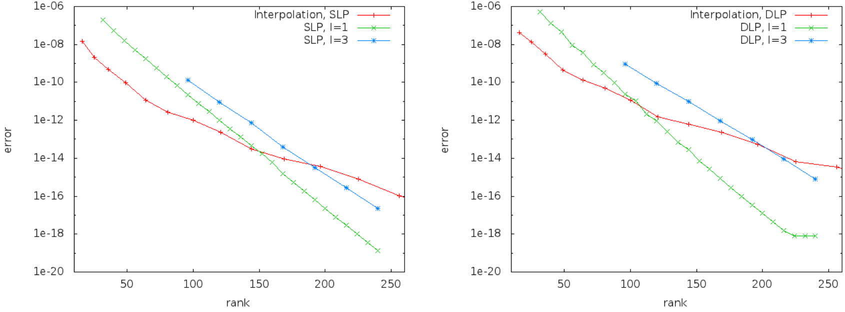


Figure: Error $\|G - \widetilde{G}\|_2$ for interpolation against our approach

Content

Conclusions and future work

We have an algorithm which

- can compete with typical techniques like the interpolation method in matters of error and timing

Conclusions and future work

We have an algorithm which

- can compete with typical techniques like the interpolation method in matters of error and timing
- can be used for various kernel functions satisfying the Green formula

Conclusions and future work

We have an algorithm which

- can compete with typical techniques like the interpolation method in matters of error and timing
- can be used for various kernel functions satisfying the Green formula

Future work includes:

- 3D (which leads to rank $12m^2\ell^2$)

Conclusions and future work

We have an algorithm which

- can compete with typical techniques like the interpolation method in matters of error and timing
- can be used for various kernel functions satisfying the Green formula

Future work includes:

- 3D (which leads to rank $12m^2\ell^2$)
- rank reduction by $1/2$ or via SVD

Conclusions and future work

We have an algorithm which

- can compete with typical techniques like the interpolation method in matters of error and timing
- can be used for various kernel functions satisfying the Green formula

Future work includes:

- 3D (which leads to rank $12m^2\ell^2$)
- rank reduction by 1/2 or via SVD
- $\mathcal{H}^2$-matrix format

Thank you for your attention!