Tensor Multigrid

Steffen Börm

Christian-Albrechts-Universität, Kiel

CSE Workshop Plön, 11.3.2015

Overview

Overview

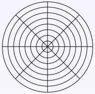
Model problem

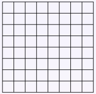
Poisson's equation on the unit disc

$$
\begin{aligned}
-\Delta u(x) &= f(x) && \text{for all } x \in D := \{x \in \mathbb{R}^2 \ : \ \|x\| < 1\},\\
u(x) &= 0 && \text{for all } x \in \partial D = \{x \in \mathbb{R}^2 \ : \ \|x\| = 1\}.
\end{aligned}
$$

Approach: Use angular coordinates

$$
\Phi : [0,1] \times [0, 2\pi] \to D, \quad \widehat{x} = (r, \alpha) \mapsto \begin{pmatrix} r\sin(\alpha) \\ r\cos(\alpha) \end{pmatrix},
$$





with $u \circ \Phi = \widehat{u}$ and $f \circ \Phi = \widehat{f}$ to obtain the equivalent equation

$$
\begin{aligned}
-\operatorname{div} \begin{pmatrix} r & \\ & 1/r \end{pmatrix} \operatorname{grad} \widehat{u}(\widehat{x}) &= r\widehat{f}(\widehat{x}) && \text{for all } \widehat{x} \in \widehat{D} := (0,1) \times (0, 2\pi),\\
\widehat{u}(\widehat{x}) &= 0 && \text{for all } \widehat{x} \in \partial\widehat{D}.
\end{aligned}
$$

Challenge: Unbounded coefficients for $r \to 0$.

Variational formulation

Multiplication by test functions $\widehat{v}$ and partial integration yields

$$\underbrace{\int_{\widehat{D}} \left\langle \begin{pmatrix} r & \\ & 1/r \end{pmatrix} \operatorname{grad} \widehat{v}(\widehat{x}), \operatorname{grad} \widehat{u}(\widehat{x}) \right\rangle d\widehat{x}}_{=:a(\widehat{v},\widehat{u})} = \underbrace{\int_{\widehat{D}} r \widehat{v}(\widehat{x}) f(\widehat{x})\, d\widehat{x}}_{=:f(\widehat{v})}.$$

Bilinear form is

- symmetric,
- positive definite, but
- not H^1-coercive.

Galerkin discretization: Choose hierarchy $V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots$ of finite-dimensional subspaces and solve

$$a(\widehat{v}_\ell, \widehat{u}_\ell) = b(\widehat{v}_\ell) \qquad \text{for all } \widehat{v}_\ell \in V_\ell.$$

Multigrid solver

Discretization leads to very large systems of linear equations.
$\rightarrow$ Apply multigrid solver.

```
procedure MG(ℓ, f_ℓ, var u_ℓ);
begin
  if ℓ > 0 begin
    d_ℓ ← f_ℓ − a(·, u_ℓ);
    Find c_ℓ ∈ V_ℓ with w_ℓ(v_ℓ, c_ℓ) = d_ℓ(v_ℓ) for all v_ℓ ∈ V_ℓ;
    u_ℓ ← u_ℓ + c_ℓ;      d_ℓ ← f_ℓ − a(·, u_ℓ);
    f_{ℓ−1} ← d_ℓ|_{V_{ℓ−1}};      u_{ℓ−1} ← 0;
    MG(ℓ − 1, f_{ℓ−1}, u_{ℓ−1});
    u_ℓ ← u_ℓ + u_{ℓ−1};      d_ℓ ← f_ℓ − a(·, u_ℓ);
    Find c_ℓ ∈ V_ℓ with w_ℓ(v_ℓ, c_ℓ) = d_ℓ(v_ℓ) for all v_ℓ ∈ V_ℓ;
    u_ℓ ← u_ℓ + c_ℓ
  end else
    Find u_0 ∈ V_0 with a_0(v_0, u_0) = f_0(v_0) for all v_0 ∈ V_0
end
```

Multigrid solver

Discretization leads to very large systems of linear equations.
$\rightarrow$ Apply multigrid solver.

procedure *MG*(ℓ, f_ℓ, var u_ℓ);
begin
  if $\ell > 0$ begin
    $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    Find $c_\ell \in V_\ell$ *with* $w_\ell(v_\ell, c_\ell) = d_\ell(v_\ell)$ *for all* $v_\ell \in V_\ell$;
    $u_\ell \leftarrow u_\ell + c_\ell$; $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    $f_{\ell-1} \leftarrow d_\ell|_{V_{\ell-1}}$; $u_{\ell-1} \leftarrow 0$;
    MG($\ell - 1$, $f_{\ell-1}$, $u_{\ell-1}$);
    $u_\ell \leftarrow u_\ell + u_{\ell-1}$; $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    Find $c_\ell \in V_\ell$ *with* $w_\ell(v_\ell, c_\ell) = d_\ell(v_\ell)$ *for all* $v_\ell \in V_\ell$;
    $u_\ell \leftarrow u_\ell + c_\ell$
  end else
    Find $u_0 \in V_0$ *with* $a_0(v_0, u_0) = f_0(v_0)$ *for all* $v_0 \in V_0$
end

Multigrid solver

Discretization leads to very large systems of linear equations.
$\rightarrow$ Apply multigrid solver.

procedure *MG*(ℓ, f_ℓ, var u_ℓ);
begin
  if $\ell > 0$ begin
    $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    Find $c_\ell \in V_\ell$ with $w_\ell(v_\ell, c_\ell) = d_\ell(v_\ell)$ for all $v_\ell \in V_\ell$;
    $u_\ell \leftarrow u_\ell + c_\ell$; $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    $f_{\ell-1} \leftarrow d_\ell|_{V_{\ell-1}}$; $u_{\ell-1} \leftarrow 0$;
    MG($\ell - 1$, $f_{\ell-1}$, $u_{\ell-1}$);
    $u_\ell \leftarrow u_\ell + u_{\ell-1}$; $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    Find $c_\ell \in V_\ell$ with $w_\ell(v_\ell, c_\ell) = d_\ell(v_\ell)$ for all $v_\ell \in V_\ell$;
    $u_\ell \leftarrow u_\ell + c_\ell$
  end else
    Find $u_0 \in V_0$ with $a_0(v_0, u_0) = f_0(v_0)$ for all $v_0 \in V_0$
end

Multigrid solver

Discretization leads to very large systems of linear equations.
$\rightarrow$ Apply multigrid solver.

procedure *MG*(ℓ, f_ℓ, *var* u_ℓ);
begin
  if $\ell > 0$ begin
    $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    Find $c_\ell \in V_\ell$ *with* $w_\ell(v_\ell, c_\ell) = d_\ell(v_\ell)$ *for all* $v_\ell \in V_\ell$;
    $u_\ell \leftarrow u_\ell + c_\ell$; $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    $f_{\ell-1} \leftarrow d_\ell|_{V_{\ell-1}}$; $u_{\ell-1} \leftarrow 0$;
    MG($\ell - 1$, $f_{\ell-1}$, $u_{\ell-1}$);
    $u_\ell \leftarrow u_\ell + u_{\ell-1}$; $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    Find $c_\ell \in V_\ell$ *with* $w_\ell(v_\ell, c_\ell) = d_\ell(v_\ell)$ *for all* $v_\ell \in V_\ell$;
    $u_\ell \leftarrow u_\ell + c_\ell$
  end else
    Find $u_0 \in V_0$ *with* $a_0(v_0, u_0) = f_0(v_0)$ *for all* $v_0 \in V_0$
end

Multigrid solver

Discretization leads to very large systems of linear equations.
$\rightarrow$ Apply multigrid solver.

procedure $MG(\ell, f_\ell,\ var\ u_\ell)$;
begin
  if $\ell > 0$ begin
    $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    Find $c_\ell \in V_\ell$ *with* $w_\ell(v_\ell, c_\ell) = d_\ell(v_\ell)$ *for all* $v_\ell \in V_\ell$;
    $u_\ell \leftarrow u_\ell + c_\ell$; $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    $f_{\ell-1} \leftarrow d_\ell|_{V_{\ell-1}}$; $u_{\ell-1} \leftarrow 0$;
    $MG(\ell-1, f_{\ell-1}, u_{\ell-1})$;
    $u_\ell \leftarrow u_\ell + u_{\ell-1}$; $d_\ell \leftarrow f_\ell - a(\cdot, u_\ell)$;
    Find $c_\ell \in V_\ell$ *with* $w_\ell(v_\ell, c_\ell) = d_\ell(v_\ell)$ *for all* $v_\ell \in V_\ell$;
    $u_\ell \leftarrow u_\ell + c_\ell$
  end else
    Find $u_0 \in V_0$ *with* $a_0(v_0, u_0) = f_0(v_0)$ *for all* $v_0 \in V_0$
end

Convergence

First experiment: Poisson problem on the unit square.

Level	1	2	3	4	5	6	7
Rate	0.100	0.153	0.167	0.168	0.170	0.170	0.170

Second experiment: Model problem with unbounded coefficients.

Level	1	2	3	4	5	6	7
Rate	0.301	0.685	0.903	0.974	0.994	0.998	0.999

Convergence

First experiment: Poisson problem on the unit square.

Level	1	2	3	4	5	6	7
Rate	0.100	0.153	0.167	0.168	0.170	0.170	0.170

Second experiment: Model problem with unbounded coefficients.

Level	1	2	3	4	5	6	7
Rate	0.301	0.685	0.903	0.974	0.994	0.998	0.999

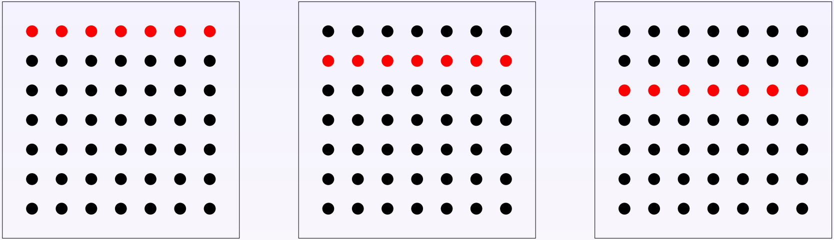
Third experiment: Model problem with tensor multigrid solver.

Level	1	2	3	4	5	6	7
Rate	0.175	0.197	0.198	0.199	0.199	0.200	0.200

Overview

Block smoother

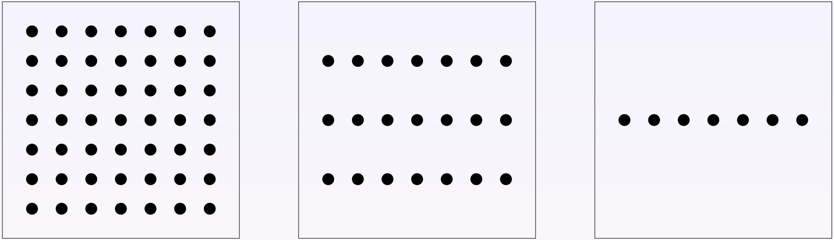
Idea: Treat rows of grid points simultaneously.



Implementation: Where the standard Jacobi and Gauß-Seidel iterations divide by the diagonal element, the block versions solve a tridiagonal system of equations.
$\rightarrow$ Only moderately increased runtime.

Semicoarsened mesh hierarchy

Idea: Keep the mesh resolution fixed in r direction,
apply coarsening only in α direction.



Advantage: No smoothing in r direction required.

Cost: 50% increase in grid points ($2n$ instead of $\frac{4}{3}n$).

Tensor spaces

We discretize by tensor spaces

$$V_\ell = V_r \otimes V_{\alpha,\ell} = \mathrm{span}\{v_r \otimes v_\ell \ : \ v_r \in V_r,\ v_\ell \in V_{\alpha,\ell}\},$$

where $V_r \subseteq H_0^1[0,1]$ and $V_{\alpha,0} \subseteq V_{\alpha,1} \subseteq V_{\alpha,2} \ldots \subseteq H_0^1[0,2\pi]$.

Bilinear form for tensor functions:

$$a(v_r \otimes v_\alpha, u_r \otimes u_\alpha) = a_r(v_r,u_r)m_\alpha(v_\alpha,u_\alpha) + m_r(v_r,u_r)a_\alpha(v_\alpha,u_\alpha)$$

with the one-dimensional bilinear forms

$$a_r(v_r,u_r) = \int_0^1 r v_r'(r)u_r'(r)\,dr, \qquad m_r(v_r,u_r) = \int_0^1 \frac{1}{r} v_r(r)u_r(r)\,dr,$$

$$a_\alpha(v_\alpha,u_\alpha) = \int_0^{2\pi} v_\alpha'(\alpha)u_\alpha'(\alpha)\,d\alpha, \quad m_\alpha(v_\alpha,u_\alpha) = \int_0^{2\pi} v_\alpha(\alpha)u_\alpha(\alpha)\,d\alpha.$$

Tensor smoothers

Bilinear form satisfies

$$a(v_r \otimes v_\alpha, u_r \otimes u_\alpha) = a_r(v_r, u_r) m_\alpha(v_\alpha, u_\alpha) + m_r(v_r, u_r) a_\alpha(v_\alpha, u_\alpha).$$

Block smoothers are given by

$$w_\ell(v_r \otimes v_\alpha, u_r \otimes u_\alpha) = a_r(v_r, u_r) w_{m,\ell}(v_\alpha, u_\alpha) + m_r(v_r, u_r) w_{a,\ell}(v_\alpha, u_\alpha),$$

where $w_{m,\ell}$ and $w_{a,\ell}$ correspond to one-dimensional smoothers for the bilinear forms m_α and a_α, respectively.

Overview

General multigrid theory

If we can prove the smoothing property

$$a(v_\ell, v_\ell) \leq w_\ell(v_\ell, v_\ell) \qquad \text{for all } v_\ell \in V_\ell$$

and the approximation property

$$w_\ell(v_\ell, v_\ell) \leq C_A a(v_\ell, v_\ell) \qquad \text{for all } v_\ell \in V_\ell \text{ with } a(v_\ell, V_{\ell-1}) = 0,$$

the convergence rate of the multigrid solver is bounded by

$$\varrho \leq \frac{C_A}{C_A + 2}.$$

☞ D. N. Arnold, R. S. Falk, R. Winther: Preconditioning in $H(\mathrm{div})$ and applications, Math. Comp. 66(219):957–984 (1997)

Eigenvector basis

Approach: For tensor functions, we have

$$a(v_r \otimes v_\alpha, v_r \otimes v_\alpha) = a_r(v_r, v_r) m_\alpha(v_\alpha, v_\alpha) \quad + m_r(v_r, v_r) a_\alpha(v_\alpha, v_\alpha)$$
$$w_\ell(v_r \otimes v_\alpha, v_r \otimes v_\alpha) = a_r(v_r, v_r) w_{m,\ell}(v_\alpha, v_\alpha) + m_r(v_r, v_r) w_{a,\ell}(v_\alpha, v_\alpha)$$

Eigenvector basis

Approach: For tensor functions, we have

$$a(v_r \otimes v_\alpha, v_r \otimes v_\alpha) = a_r(v_r, v_r) m_\alpha(v_\alpha, v_\alpha) + m_r(v_r, v_r) a_\alpha(v_\alpha, v_\alpha)$$
$$w_\ell(v_r \otimes v_\alpha, v_r \otimes v_\alpha) = a_r(v_r, v_r) w_{m,\ell}(v_\alpha, v_\alpha) + m_r(v_r, v_r) w_{a,\ell}(v_\alpha, v_\alpha)$$

Eigenvector basis

Approach: For tensor functions, we have

$$a(v_r \otimes v_\alpha, v_r \otimes v_\alpha) = a_r(v_r, v_r) m_\alpha(v_\alpha, v_\alpha) \;\; + m_r(v_r, v_r) a_\alpha(v_\alpha, v_\alpha)$$
$$w_\ell(v_r \otimes v_\alpha, v_r \otimes v_\alpha) = a_r(v_r, v_r) w_{m,\ell}(v_\alpha, v_\alpha) + m_r(v_r, v_r) w_{a,\ell}(v_\alpha, v_\alpha)$$

Idea: Use m_r-orthonormal eigenvector basis $(e_\nu)_{\nu \in \mathcal{I}}$ given by

$$a_r(e_\nu, e_\mu) = \lambda_\nu \delta_{\nu\mu}, \qquad m_r(e_\nu, e_\mu) = \delta_{\nu\mu} \qquad \text{for all } \nu, \mu \in \mathcal{I}.$$

Result: For each $v_\ell \in V_\ell$, we can find $(v_\nu)_{\nu \in \mathcal{I}}$ such that

$$v_\ell = \sum_{\nu \in \mathcal{I}} e_\nu \otimes v_\nu, \qquad a(v_\ell, v_\ell) = \sum_{\nu \in \mathcal{I}} \lambda_\nu m_\alpha(v_\alpha, v_\alpha) + a_\alpha(v_\alpha, v_\alpha),$$
$$w_\ell(v_\ell, v_\ell) = \sum_{\nu \in \mathcal{I}} \lambda_\nu w_{m,\ell}(v_\alpha, v_\alpha) + w_{a,\ell}(v_\alpha, v_\alpha).$$

Smoothing property

Let $v_\ell \in V_\ell$. Let $(v_\nu)_{\nu\in\mathcal{I}}$ be such that

$$v_\ell = \sum_{\nu\in\mathcal{I}} e_\nu \otimes v_\nu.$$

Since a_α and m_α are well-behaved, we can prove

$$m_\alpha(v_\alpha, v_\alpha) \leq w_{m,\alpha}(v_\alpha, v_\alpha), \qquad a_\alpha(v_\alpha, v_\alpha) \leq w_{a,\alpha}(v_\alpha, v_\alpha).$$

Result: Using the eigenvector expansion, this implies

$$\begin{aligned} a(v_\ell, v_\ell) &= \sum_{\nu\in\mathcal{I}} \lambda_\nu m_\alpha(v_\nu, v_\nu) + a_\alpha(v_\nu, v_\nu) \\ &\leq \sum_{\nu\in\mathcal{I}} \lambda_\nu w_{m,\alpha}(v_\nu, v_\nu) + w_{a,\alpha}(v_\nu, v_\nu) = w_\ell(v_\ell, v_\ell). \end{aligned}$$

Approximation property

Let $v_\ell \in V_\ell$ with $a(v_\ell, V_{\ell-1}) = 0$, let $(v_\nu)_{\nu\in\mathcal{I}}$ be as before.

Semicoarsening: $e_\nu \otimes u_{\ell-1} \in V_{\ell-1}$ for all $u_{\ell-1} \in V_{\alpha,\ell-1}$, so we get

$$\begin{aligned} 0 &= a(v_\ell, e_\nu \otimes u_{\ell-1}) \\ &= \lambda_\nu m_\alpha(v_\nu, u_{\ell-1}) + a_\alpha(v_\nu, u_{\ell-1}) \qquad \text{for all } \nu \in \mathcal{I},\ u_{\ell-1} \in V_{\alpha,\ell-1}. \end{aligned}$$

Standard multigrid theory can be applied to prove

$$\lambda_\nu w_{m,\alpha}(v_\nu, v_\nu) + w_{a,\alpha}(v_\nu, v_\nu) \le C_A(\lambda_\nu m_\alpha(v_\nu, v_\nu) + a_\alpha(v_\nu, v_\nu))$$

with an independent constant C_A (Fourier: $C_A = 4$ for model problem).

Using the eigenvalue decomposition again, we find

$$w_\ell(v_\ell, v_\ell) \le C_A a(v_\ell, v_\ell).$$

Robustness

Goal: Investigate performance for problems of the form

$$-\operatorname{div}\begin{pmatrix}\sigma(x) & \\ & \tau(x)\end{pmatrix}\operatorname{grad} u(x,y) = f(x,y).$$

σ	τ	5	6	7	8	9	10	
1	1	0.35	0.35	0.35	0.35	0.35	0.35	
x	$1/x$	0.40	0.40	0.40	0.39	0.36	0.35	Jacobi
10^8	1	0.56	0.56	0.56	0.56	0.56	0.56	
10^{-8}	1	0.40	0.40	0.40	0.40	0.40	0.40	
1	1	0.09	0.09	0.09	0.09	0.09	0.09	
x	$1/x$	0.09	0.09	0.09	0.09	0.09	0.09	Gauß-Seidel
10^8	1	0.09	0.09	0.09	0.09	0.09	0.09	
10^{-8}	1	0.00	0.00	0.00	0.00	0.00	0.01	

Result: Very robust, particularly with red-black Gauß-Seidel.

Overview

Summary

Tensor multigrid: Use block smoothing and semicoarsening to hide the effects of unbounded coefficients.

Convergence proof: Use eigenvector basis to reduce to family of standard problems that can be treated by standard theory.

Requirements: Both method and proof work for all problems of the form

$$-\operatorname{div}\begin{pmatrix}\sigma(x) & \\ & \tau(x)\end{pmatrix}\operatorname{grad} u(x,y) = f(x,y) \qquad \text{for all } x \in D_x,\ y \in D_y$$

as long as $\sigma, \tau > 0$ almost everywhere.

Generalization

Maxwell's equations lead to

$$\operatorname{curl}^* \mu(x) \operatorname{curl} u(x,y) + \tau(x) u(x,y) = f(x,y) \quad \text{for all } x \in D_x,\ y \in D_y.$$

We consider again the case that $\mu, \tau > 0$ are unbounded.

Discretization: Nédélec's edge basis functions on a tensor grid.

Approach: Use representation operator

$$R_{e,\epsilon} : H_0^1(D_y) \to H_0(\operatorname{curl}, D), \qquad v \mapsto \begin{pmatrix} \epsilon \otimes v \\ 0 \end{pmatrix} + \operatorname{grad}(e \otimes v),$$

with $e \in H_0^1(D_x)$, $\epsilon \in L^2(D_x)$.

Eigenvector decomposition can be constructed by solving a coupled eigenvector problem involving e and ϵ.
$\to$ Reduction to one-dimensional Poisson problems.