# Directional Compression of Helmholtz Potentials

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# Overview

# Helmholtz equation

Goal: Solve the Helmholtz equation

$$\Delta u(x) + \kappa^2 u(x) = 0 \qquad \text{for all } x \in \Omega,$$
$$u(x) = f(x) \qquad \text{for all } x \in \partial\Omega.$$

Challenge: Indefinite operator, pollution effect.

Approach: Boundary integral formulation

$$u(x) = \int_{\partial\Omega} g(x,y) \frac{\partial u}{\partial n}(y)\,dy - \int_{\partial\Omega} \frac{\partial g}{\partial n(y)}(x,y) u(y)\,dy \quad \text{for all } x \in \Omega$$

with fundamental solution

$$g(x,y) = \frac{\exp(\iota\kappa\|x-y\|)}{4\pi\|x-y\|}.$$

Challenge: Standard discretization leads to dense matrix.

# Fast summation

Standard approach: Approximate $g$ by degenerate kernel,

$$g(x,y) \approx \sum_{\nu=1}^{k} v_\nu(x) \sum_{\mu=1}^{k} s_{\nu\mu} \overline{w_\mu(y)}$$

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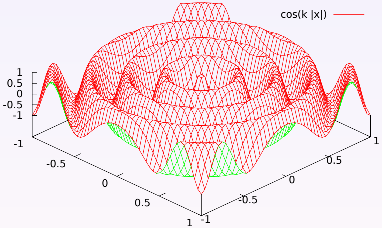
Challenge: Helmholtz kernel oscillates rapidly if $\kappa$ large.




$$g(x,y) = \frac{\exp(\iota\kappa\|x-y\|)}{\|x-y\|}.$$

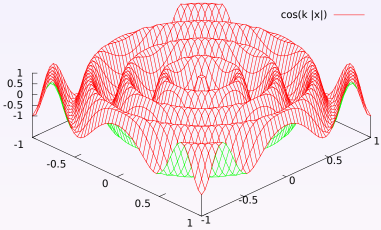
Consequence: Standard approximation schemes require $k \gg 1$.

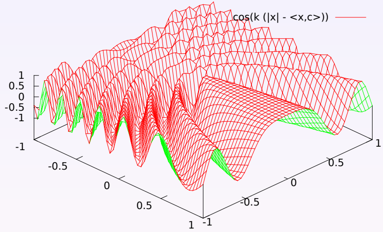
# Overview

# Directional approximation

Idea: Divide spherical wave by plane wave in direction $c$, $\|c\| = 1$:

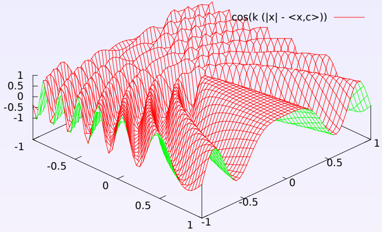
$$g(x,y) = \exp(\iota\kappa\langle x-y, c\rangle) \underbrace{\frac{\exp(\iota\kappa(\|x-y\| - \langle x-y, c\rangle))}{4\pi\|x-y\|}}_{=g_c(x,y)}.$$




✎ Brandt (1991), Engquist/Ying (2007), Messner/Schanz/Darve (2012), Bebendorf/Kuske/Venn (2015)

# Directional smoothness




Oscillatory term: $\exp(\iota\kappa \; \|x - y\| \qquad\qquad\qquad\qquad )$.

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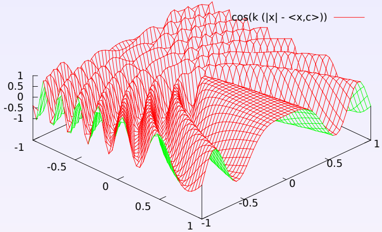


Oscillatory term: $\exp(\iota\kappa(\|x-y\| - \langle x-y, c\rangle))$.

$$\begin{aligned}\kappa(\|x-y\| - \langle x-y, c\rangle) &= \kappa\langle x-y, \frac{x-y}{\|x-y\|} - c\rangle \\ &\leq \kappa\|x-y\| \left\| \frac{x-y}{\|x-y\|} - c \right\|.\end{aligned}$$

# Admissibility

Given subsets $\tau, \sigma \subseteq \Omega$, can we approximate the directionally smoothed kernel function $g_c|_{\tau\times\sigma}$?

Directional condition:

$$\kappa\|x-y\|\left\|\frac{x-y}{\|x-y\|}-c\right\| \leq \eta_1$$

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Result: If both conditions are satisfied, $g_c$ should be smooth.

# Factorization

Approach: Apply interpolation to smoothed function $g_c$.

$$\tilde{g}_c(x,y) = \sum_{\nu,\mu} \mathcal{L}_{\tau,\nu}(x) g_c(\xi_{\tau,\nu}, \xi_{\sigma,\mu}) \mathcal{L}_{\sigma,\mu}(y).$$

Directional approximation given by

$$\begin{aligned} g(x,y) &= \exp(\iota\kappa\langle x-y, c\rangle) g_c(x,y) \\ &\approx \exp(\iota\kappa\langle x-y, c\rangle) \tilde{g}_c(x,y) \\ &= \sum_{\nu,\mu} \underbrace{\exp(\iota\kappa\langle x, c\rangle)\mathcal{L}_{\tau,\nu}(x)}_{=:\mathcal{L}_{\tau c,\nu}(x)} g_c(\xi_{\tau,\nu}, \xi_{\sigma\mu}) \underbrace{\exp(-\iota\kappa\langle y, c\rangle)\mathcal{L}_{\sigma\mu}(y)}_{=:\overline{\mathcal{L}_{\sigma c,\mu}(y)}} \\ &= \sum_{\nu,\mu} \mathcal{L}_{\tau c,\nu}(x) g_c(\xi_{\tau,\nu}, \xi_{\sigma,\mu}) \overline{\mathcal{L}_{\sigma c,\mu}(y)}. \end{aligned}$$

# $\mathcal{DH}^2$-matrix

Approach:

- Split $\partial\Omega \times \partial\Omega$ into subsets $\tau \times \sigma$ that satisfy the admissibility conditions or are small.
- Choose a direction $c$ for each subset.
- Evaluate $g_c$ in interpolation points.
- Prepare $\mathcal{L}_{\tau c,\nu}$ and $\mathcal{L}_{\sigma c,\mu}$.

Results:

- Storage $\mathcal{O}(kn + k^2\kappa^2 \log n)$ for $k$ interpolation points.
- Matrix-vector multiplication in $\mathcal{O}(kn + k^2\kappa^2 \log n)$ operations.
- High-frequency case: $\kappa^2 \sim n$, complexity $\mathcal{O}(nk^2 \log n)$.

✎ B. (2015)

# Overview

# Tensor interpolation

Algorithm: Use m-th order tensor Chebyshev interpolation

$$\tilde{g}_c := \mathfrak{I}_{\tau\times\sigma}[g_c]$$

in $\tau \times \sigma = [a_1, b_1] \times \ldots \times [a_6, b_6]$.

Tensor analysis: We only have to investigate interpolation of one-dimensional functions

$$t \mapsto g_c(d + tp), \qquad d = \begin{pmatrix} (b_1 + a_1)/2 \\ d_2 \\ \ldots \\ d_6 \end{pmatrix}, \qquad p = \begin{pmatrix} (b_1 - a_1)/2 \\ 0 \\ \ldots \\ 0 \end{pmatrix}$$

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Tensor analysis: We only have to investigate interpolation of one-dimensional functions

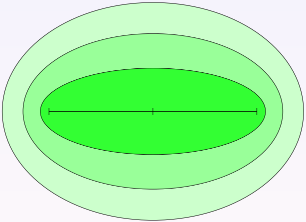
$$t \mapsto g_c(d + tp), \qquad d = \begin{pmatrix} d_1 \\ (b_2 + a_2)/2 \\ \ldots \\ d_6 \end{pmatrix}, \qquad p = \begin{pmatrix} 0 \\ (b_2 - a_2)/2 \\ \ldots \\ 0 \end{pmatrix}$$

# Interpolation of analytic functions

Best approximation result: If $f$ is analytic in the Bernstein ellipse

$$\mathcal{E}_\varrho := \left\{ x + iy \ : \ \left( \frac{2x}{\varrho + 1/\varrho} \right)^2 + \left( \frac{2y}{\varrho - 1/\varrho} \right)^2 \leq 1 \right\}, \qquad \varrho > 1,$$

polynomial approximations in $[-1, 1]$ converge at a rate of $1/\varrho$.



Idea: Construct holomorphic extension of $t \mapsto g_c(d + tp)$ in $\mathcal{E}_\varrho$.

Challenge: $\exp(\iota\kappa z)$ grows exponentially as $\Im(z) \to \infty$.

# Convergence result

Directional interpolation: Assume $\tau \times \sigma$ admissible, $r < 1/\eta_2$, $\varrho = \sqrt{r^2+1} + r$, interpolation order $m$.

$$\|g - \tilde{g}\|_{\infty,\tau\times\sigma} \leq C(r) \frac{\varrho^{-m}}{\operatorname{dist}(\tau,\sigma)},$$

$$C(r) \lesssim \frac{\exp(r(\eta_1+\eta_2) + r^2 \frac{\eta_2}{2(1-r\eta_2)})}{r}$$

Open question: What is the best choice for $r$?

✎ B./Melenk (2015)

# Reinterpolation

Goal: Establish nested hierarchy of directional bases to make our algorithms more efficient.

Approach: Let $\tau' \subseteq \tau$, choose directions $c'$ and $c$.

$$
\begin{aligned}
\mathcal{L}_{\tau c,\nu}(x) &= \exp(\iota\kappa\langle x, c\rangle)\mathcal{L}_{\tau,\nu}(x) \\
&= \exp(\iota\kappa\langle x, c'\rangle)\exp(\iota\kappa\langle x, c-c'\rangle)\mathcal{L}_{\tau,\nu}(x) \\
&\approx \exp(\iota\kappa\langle x, c'\rangle)\sum_{\nu'} \underbrace{\exp(\iota\kappa\langle \xi_{\tau',\nu'}, c-c'\rangle)\mathcal{L}_{\tau,\nu}(\xi_{\tau',\nu'})}_{=:e_{\nu'\nu}} \mathcal{L}_{\tau',\nu'}(x) \\
&= \exp(\iota\kappa\langle x, c'\rangle)\sum_{\nu'} e_{\nu'\nu}\mathcal{L}_{\tau',\nu'}(x) = \sum_{\nu'} e_{\nu'\nu}\mathcal{L}_{\tau' c',\nu'}(x).
\end{aligned}
$$

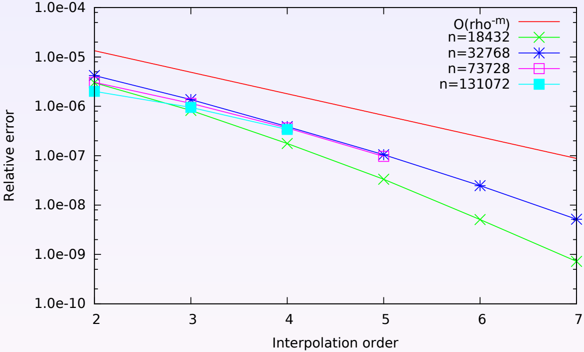
Challenge: Prove stability of resulting reinterpolation scheme.

✎ B./Melenk (2015)

# Overview

# Convergence

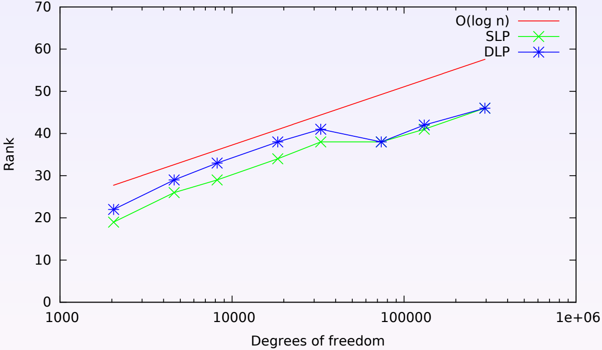
Experiment: Approximate Galerkin discretization of Helmholtz boundary element operator on the unit sphere.




Result: Exponential convergence, rate close to theoretical prediction.

# Algebraic compression

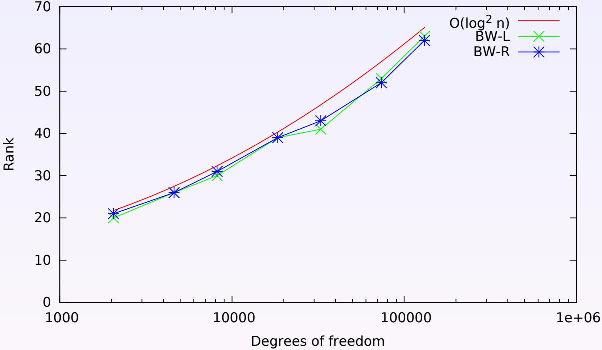
Experiment: Apply SVD-based algebraic compression algorithm to single and double layer operators.




Result: Maximal rank grows like $\mathcal{O}(\log n)$.

# Preconditioner

Experiment: Apply SVD-based algebraic compression algorithm to LR factorization of Brakhage-Werner operator.




Result: Maximal rank grows like $\mathcal{O}(\log^2 n)$.