Directional Compression of Helmholtz Potentials

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Overview

Overview

Helmholtz equation

Goal: Solve the Helmholtz equation

$$\Delta u(x) + \kappa^2 u(x) = 0 \qquad \text{for all } x \in \Omega,$$
$$u(x) = f(x) \qquad \text{for all } x \in \partial\Omega.$$

Approach: Boundary integral formulation

$$u(x) = \int_{\partial\Omega} g(x,y)\frac{\partial u}{\partial n}(y)\,dy - \int_{\partial\Omega} \frac{\partial g}{\partial n(y)}(x,y)u(y)\,dy \quad \text{for all } x \in \Omega$$

with fundamental solution

$$g(x,y) = \frac{\exp(\iota\kappa\|x-y\|)}{4\pi\|x-y\|}.$$

Challenge: Standard discretization leads to dense matrix.

Fast summation

Standard approach: Approximate g by degenerate kernel,

$$g(x,y) \approx \sum_{\nu=1}^{k} v_\nu(x) \sum_{\mu=1}^{k} s_{\nu\mu} \overline{w_\mu(y)}$$

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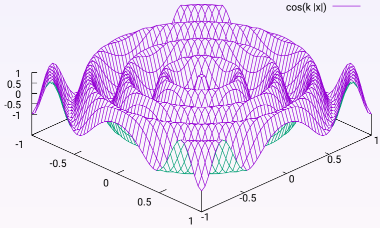
Challenge: Helmholtz kernel oscillates rapidly if κ large.



$$g(x,y) = \frac{\exp(\iota\kappa\|x-y\|)}{\|x-y\|}.$$

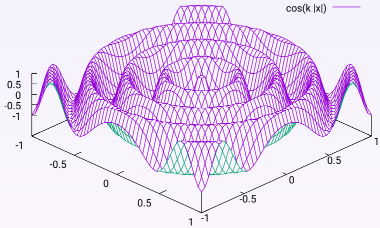
Consequence: Standard approximation schemes require high rank k.

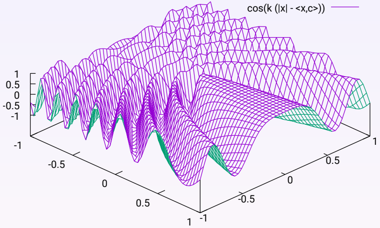
Overview

Directional approximation

Idea: Divide spherical wave by plane wave in direction c, $\|c\| = 1$:

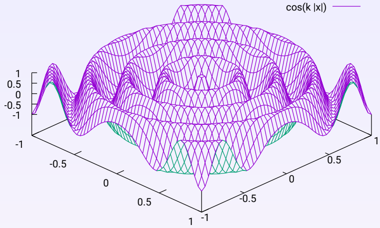
$$g(x,y) = \exp(\iota\kappa\langle x-y, c\rangle)\,\underbrace{\frac{\exp(\iota\kappa(\|x-y\| - \langle x-y, c\rangle))}{4\pi\|x-y\|}}_{=g_c(x,y)}.$$





✎ Brandt (1991), Engquist/Ying (2007), Messner/Schanz/Darve (2012), Bebendorf/Kuske/Venn (2015)

Directional smoothness



Oscillatory term: $\exp(\iota\kappa \; \|x - y\| \qquad\qquad\qquad)$.

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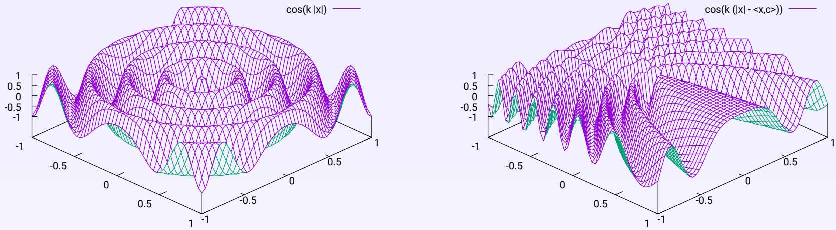

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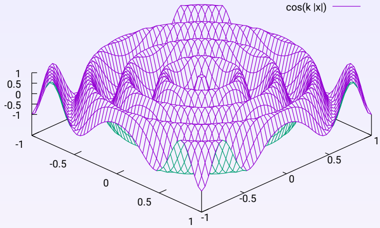

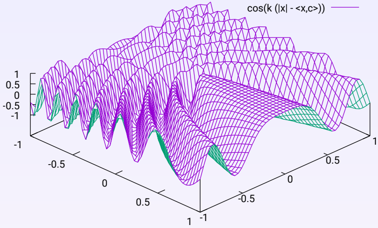

Oscillatory term: $\exp(\iota\kappa(\|x - y\| - \langle x - y, c\rangle))$.

$$\begin{aligned}\kappa(\|x - y\| - \langle x - y, c\rangle) &= \kappa\|x - y\|(1 - \cos\angle(x - y, c)) \\ &\approx \frac{\kappa}{2}\|x - y\|\sin^2\angle(x - y, c).\end{aligned}$$

Admissibility

Given subsets $\tau, \sigma \subseteq \Omega$, can we approximate the directionally smoothed kernel function $g_c|_{\tau \times \sigma}$?

Directional condition: Given midpoints $m_\tau \in \tau$, $m_\sigma \in \sigma$, we require

$$\kappa \operatorname{diam} \left\| \frac{m_\tau - m_\sigma}{\|m_\tau - m_\sigma\|} - c \right\| \leq \eta_1, \qquad \kappa \operatorname{diam}^2 \leq \eta_2 \operatorname{dist}(\tau, \sigma)$$

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Result: If both conditions are satisfied, g_c should be smooth.

Factorization

Approach: Apply interpolation to smoothed function g_c.

$$\tilde{g}_c(x,y) = \sum_{\nu,\mu} \mathcal{L}_{\tau,\nu}(x) g_c(\xi_{\tau,\nu}, \xi_{\sigma,\mu}) \mathcal{L}_{\sigma,\mu}(y).$$

Directional approximation given by

$$\begin{aligned} g(x,y) &= \exp(\iota\kappa\langle x-y, c\rangle) g_c(x,y) \\ &\approx \exp(\iota\kappa\langle x-y, c\rangle) \tilde{g}_c(x,y) \\ &= \sum_{\nu,\mu} \underbrace{\exp(\iota\kappa\langle x, c\rangle)\mathcal{L}_{\tau,\nu}(x)}_{=:\mathcal{L}_{\tau c,\nu}(x)} g_c(\xi_{\tau,\nu}, \xi_{\sigma\mu}) \underbrace{\exp(-\iota\kappa\langle y, c\rangle)\mathcal{L}_{\sigma\mu}(y)}_{=:\overline{\mathcal{L}_{\sigma c,\mu}(y)}} \\ &= \sum_{\nu,\mu} \mathcal{L}_{\tau c,\nu}(x) g_c(\xi_{\tau,\nu}, \xi_{\sigma,\mu}) \overline{\mathcal{L}_{\sigma c,\mu}(y)}. \end{aligned}$$

Reinterpolation

Goal: Establish nested hierarchy of directional bases to make our algorithms more efficient.

Approach: Choose $\tau' \subseteq \tau$, directions c' and c, and reinterpolate:

$$\begin{aligned}
\mathcal{L}_{\tau c,\nu}(x) &= \exp(\iota\kappa\langle x, c\rangle)\,\mathcal{L}_{\tau,\nu}(x) \\
&= \exp(\iota\kappa\langle x, c'\rangle)\exp(\iota\kappa\langle x, c - c'\rangle)\,\mathcal{L}_{\tau,\nu}(x) \\
&\approx \exp(\iota\kappa\langle x, c'\rangle)\sum_{\nu'} \underbrace{\exp(\iota\kappa\langle \xi_{\tau',\nu'}, c - c'\rangle)\mathcal{L}_{\tau,\nu}(\xi_{\tau',\nu'})}_{=:e_{\tau' c,\nu'\nu}}\,\mathcal{L}_{\tau',\nu'}(x) \\
&= \exp(\iota\kappa\langle x, c'\rangle)\sum_{\nu'} e_{\tau' c,\nu'\nu}\mathcal{L}_{\tau',\nu'}(x) = \sum_{\nu'} e_{\tau' c,\nu'\nu}\mathcal{L}_{\tau' c',\nu'}(x).
\end{aligned}$$

Result: We only have to handle $\mathcal{L}_{\tau c,\nu}$ for small domains τ and can use transfer matrices $E_{\tau' c}$ for larger ones.

Overview

Tensor interpolation

Algorithm: Use *m*-th order tensor Chebyshev interpolation

$$\tilde{g}_c := \mathfrak{I}_{\tau\times\sigma}[g_c]$$

in the axis-parallel box $\tau \times \sigma$.

Tensor analysis: We only have to investigate interpolation of one-dimensional functions

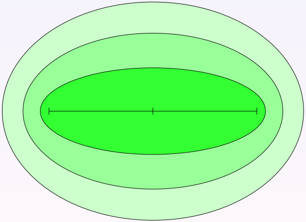
$$g_{cdp} \colon [-1, 1] \to \mathbb{C}, \qquad t \mapsto g_c(d + tp).$$

Interpolation of analytic functions

Best approximation result: If f is analytic in the Bernstein ellipse

$$\mathcal{E}_\varrho := \left\{ x + iy \; : \; \left(\frac{2x}{\varrho + 1/\varrho}\right)^2 + \left(\frac{2y}{\varrho - 1/\varrho}\right)^2 \leq 1 \right\}, \qquad \varrho > 1,$$

polynomial approximations in $[-1, 1]$ converge at a rate of $1/\varrho$.



Idea: Construct holomorphic extension of g_{cdp} in $\mathcal{E}_\varrho$.

Challenge: $\exp(\iota\kappa z)$ grows exponentially as $\Im(z) \to \infty$.

Convergence result

Directional interpolation: Assume $\tau \times \sigma$ admissible. We have

$$\|g - \tilde{g}\|_{\infty,\tau\times\sigma} \leq \frac{C_{\mathrm{in}}(\eta_1, \eta_2)}{4\pi \operatorname{dist}(\tau, \sigma)} \varrho^{-m} \qquad \text{for all } m \in \mathbb{N}$$

with the convergence rate

$$\varrho := \min\left\{2, \frac{3}{2\eta_2} + 1\right\}.$$

The estimate is independent of the wave number κ.

Remark: We can reach any rate $\varrho < \frac{2}{\eta_2} + 1$ if we are prepared to make the constant $C_{\mathrm{in}}(\eta_1, \eta_2)$ larger.

✎ B./Melenk (2015)

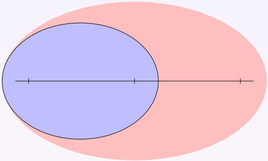
Reinterpolation

Problem: Instead of $\mathcal{L}_{\tau c}$, we use reinterpolated functions

$$\widetilde{\mathcal{L}}_{\tau c}(x) := \begin{cases} \mathcal{L}_{\tau c}(x) & \text{if } \tau \text{ small,} \\ \sum_{\nu'} e_{\tau' c, \nu' \nu} \widetilde{\mathcal{L}}_{\tau' c'}(x) & \text{otherwise, with } \tau' \ni x. \end{cases}$$

If we reinterpolate L times, each step contributes to the total error.

Idea: Assume $\operatorname{diam}(\tau') \leq q \operatorname{diam}(\tau)$ with $q < 1$.
We can find $\tilde{q} < 1$ such that
ellipses of radius $\varrho/\tilde{q}$ around τ' contained in
ellipses of radius ϱ around τ.
$\rightarrow m \sim \log(L)$ yields stability and convergence.



✎ B./Melenk (2015)

Overview

Discretization

Goal: Approximate Galerkin matrix $G \in \mathbb{C}^{\mathcal{I}\times\mathcal{I}}$ given by

$$g_{ij} = \int_{\partial\Omega} \varphi_i(x) \int_{\partial\Omega} g(x,y)\psi_j(y)\,dy\,dx.$$

Standard approach: Choose admissible $\tau \times \sigma$, fix subsets $\widehat{\tau}, \widehat{\sigma} \subseteq \mathcal{I}$ with

$$\operatorname{supp}\varphi_i \subseteq \tau, \qquad \operatorname{supp}\psi_j \subseteq \sigma \qquad \text{for all } i \in \widehat{\tau},\ j \in \widehat{\sigma}.$$

Replacing g by directional approximation yields

$$g_{ij} \approx \sum_{\nu,\mu} \underbrace{\int_{\partial\Omega} \varphi_i(x)\mathcal{L}_{\tau c,\nu}(x)\,dx}_{=:v_{\tau c,i\nu}} \underbrace{g_c(\xi_{\tau,\nu},\xi_{\sigma,\mu})}_{=:s_{\tau\sigma,\nu\mu}} \underbrace{\int_{\partial\Omega} \psi_j(y)\overline{\mathcal{L}_{\sigma c,\mu}(y)}\,dy}_{=:w_{\sigma c,j\mu}}$$

$$= (V_{\tau c} S_{\tau\sigma} W_{\sigma c}^*)_{ij} \qquad \text{for all } i \in \widehat{\tau},\ j \in \widehat{\sigma}.$$

Directional cluster tree

Goal: Organize subdomains τ, σ, and $\tau \times \sigma$ in hierarchies to improve efficiency.

cluster tree $\mathcal{T}_{\mathcal{I}}$ for an index set $\mathcal{I}$:

- Each node is a subdomain $\tau \subseteq \mathbb{R}^3$, associated with $\hat{\tau} \subseteq \mathcal{I}$.
- Father clusters are the union of their sons.
- The root contains $\partial\Omega$ and is associated with $\mathcal{I}$.

block tree $\mathcal{T}_{\mathcal{I}\times\mathcal{I}}$ for the Cartesian product $\mathcal{I} \times \mathcal{I}$:

- Each node has the form $\tau \times \sigma$ with clusters $\tau, \sigma \in \mathcal{T}_{\mathcal{I}}$.
- The sons of $\tau \times \sigma$ are $\tau' \times \sigma'$ with $\tau' \in \text{sons}(\tau)$ and $\sigma' \in \text{sons}(\sigma)$.

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- Each node is associated with a set $\mathcal{D}_\tau \subseteq \mathbb{R}^3$ of directions c.

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- The sons of $\tau \times \sigma$ are $\tau' \times \sigma'$ with $\tau' \in \mathrm{sons}(\tau)$ and $\sigma' \in \mathrm{sons}(\sigma)$.
- Each block is associated with a direction $c_{\tau\sigma} \in \mathcal{D}_\tau \cap \mathcal{D}_\sigma$.

Directional cluster basis

Reinterpolation for $\tau' \in \mathrm{sons}(\tau)$ yields

$$\mathcal{L}_{\tau c,\nu}(x) \approx \sum_{\nu'} e_{\tau' c,\nu'\nu} \mathcal{L}_{\tau' c',\nu'}(x).$$

Discretization leads to

$$\begin{aligned} v_{\tau c,i\nu} &= \int_{\partial\Omega} \varphi_i(x) \mathcal{L}_{\tau c,\nu}(x)\,dx \approx \sum_{\nu'} e_{\tau' c,\nu'\nu} \int_{\partial\Omega} \varphi_i(x) \mathcal{L}_{\tau' c',\nu'}(x)\,dx \\ &= \sum_{\nu'} v_{\tau' c',i\nu'} e_{\tau' c,\nu'\nu} = (V_{\tau' c'} E_{\tau' c})_{i\nu}. \end{aligned}$$

Directional cluster basis $(V_{\tau c})$ described by

- leaf matrices $V_{\tau c}$ for leaf clusters τ,
- transfer matrices $E_{\tau' c}$ non-leaf clusters, $\tau' \in \mathrm{sons}(\tau)$.

$\mathcal{DH}^2$-matrix

Assume that $\mathcal{T}_{\mathcal{I}}$ and $\mathcal{T}_{\mathcal{I}\times\mathcal{I}}$ are given.

- Directional cluster bases represented by leaf matrices and transfer matrices, $V_{\tau c}|_{\hat{\tau}'\times k} = V_{\tau' c'} E_{\tau' c}$.
- Admissible blocks represented by cluster bases and coupling matrices, $G|_{\hat{\tau}\times\hat{\sigma}} = V_{\tau c} S_{\tau\sigma} W_{\sigma c}^*$.

Results:

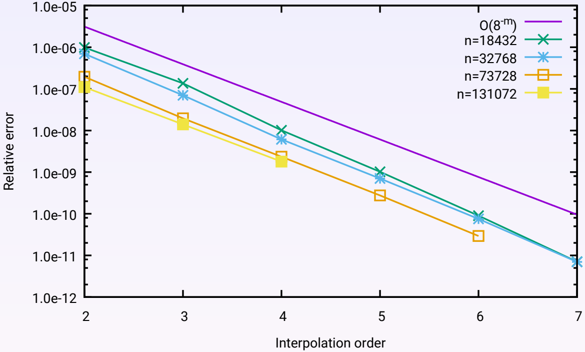
- Storage $\mathcal{O}(kn + k^2\kappa^2 \log n)$ for k interpolation points.
- Matrix-vector multiplication in $\mathcal{O}(kn + k^2\kappa^2 \log n)$ operations.
- High-frequency case: $\kappa^2 \sim n$, complexity $\mathcal{O}(nk^2 \log n)$.

✎ B. (2015)

Overview

Convergence: Sphere

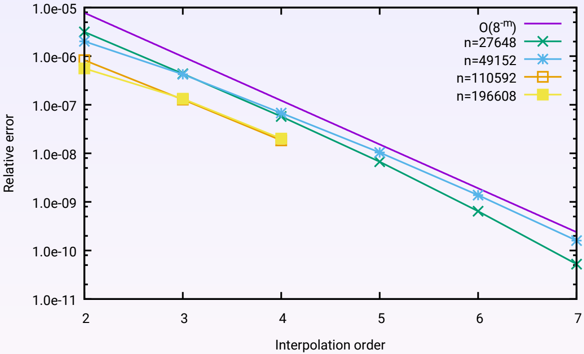
Experiment: Approximate Galerkin discretization of Helmholtz boundary element operator on the unit sphere.



Result: Exponential convergence, rate comparable to Laplace BEM.

Convergence: Cube

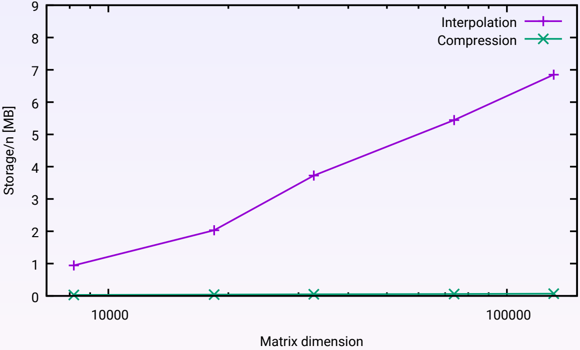
Experiment: Approximate Galerkin discretization of Helmholtz boundary element operator on the surface of the reference cube.



Result: Exponential convergence, rate similar to the sphere.

Algebraic compression

Experiment: Compare directional interpolation to algebraic compression based on SVD and orthgonal projections.



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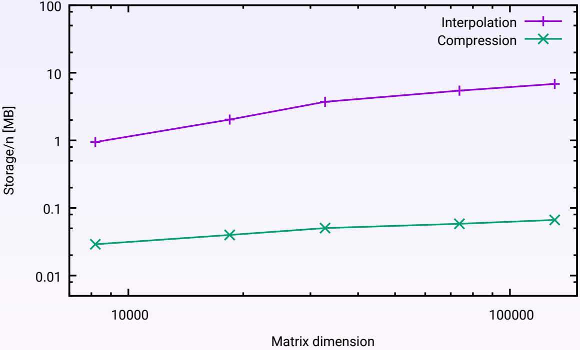

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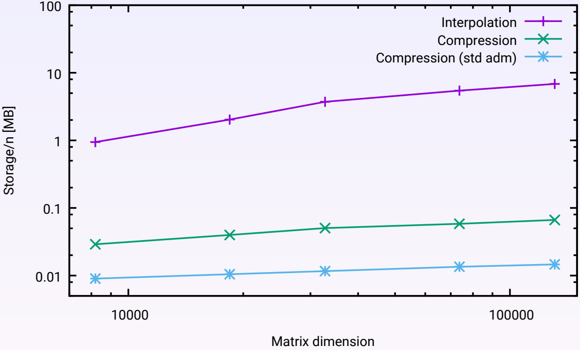

Result: Algebraic compression is necessary to reach useful efficiency.

Summary

Directional interpolation yields reliable and robust $\mathcal{DH}^2$-matrix approximation, complexity $\mathcal{O}(nk + \kappa^2 k^2 \log n)$.

Convergence analysis by extension to Bernstein ellipses in the complex plane yields κ-independent convergence rates.

Algebraic compression crucial to reduce storage requirements and allow us to treat large high-frequency problems.

Software available at `http://www.h2lib.org`
(open source, annual winter schools)