Adaptive Compression of Helmholtz Potentials

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Overview

Overview

Helmholtz equation

Goal: Approximate boundary element matrices for the high-frequency Helmholtz equation.

$$g_{ij} = \int_\Gamma \varphi_i(x) \int_\Gamma \frac{\exp(i\kappa\|x-y\|)}{4\pi\|x-y\|} \psi_j(y)\, dy\, dx \qquad \text{for } i,j \in \mathcal{I}.$$

Challenges:

- Matrix dense.
- Standard techniques like Taylor expansion or interpolation unattractive due to oscillations.

Approach: Use $\mathcal{DH}^2$-matrices to get $\mathcal{O}(nk^2 \log n)$ complexity.

Time vs Storage

Experiment: Compare construction of $\mathcal{DH}^2$-matrices by

- directional interpolation and
- singular value decomposition applied to dense matrix.

	Interpolation			SVD		
n	Bld$/s$	Mem$/MB$	Err	Bld$/s$	Mem$/MB$	Err
4608	9.0	878.5	1.9_{-4}	12.0	265.7	1.8_{-6}
8192	20.6	2 965.4	4.2_{-4}	27.5	707.6	1.9_{-6}
18432	64.8	15 630.3	1.2_{-3}	140.4	2 631.8	2.2_{-6}
32768	163.3	41 806.0	1.4_{-3}	451.2	6 215.0	2.4_{-6}
73728	409.0	133 685.9	1.2_{-3}	2 581.4	15 767.0	2.4_{-6}
131072	610.4	295 524.6	1.6_{-3}	8 482.38	31 227.8	2.5_{-6}

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Goal: As fast as interpolation, as economical as SVD.

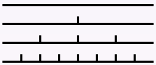
Overview

Directional trees

Goal: Organize subdomains $\tau, \sigma \subseteq \mathbb{R}^3$, and $\tau \times \sigma \subseteq \mathbb{R}^6$ in hierarchies to improve efficiency.

Directional cluster tree $\mathcal{T}_{\mathcal{I}}$ for an index set $\mathcal{I}$:

- Each node is a subdomain $\tau \subseteq \mathbb{R}^3$, associated with $\hat{\tau} \subseteq \mathcal{I}$.
- The root contains $\partial\Omega$ and is associated with $\mathcal{I}$.
- Father clusters are the union of their sons.
- Each node is associated with a set $\mathcal{D}_\tau \subseteq \mathbb{R}^3$ of directions c.



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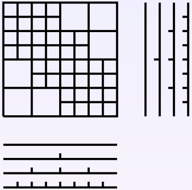


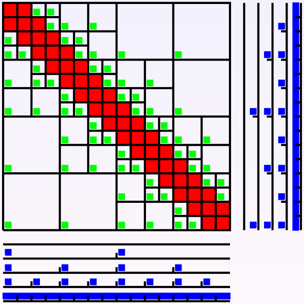
Directional block tree $\mathcal{T}_{\mathcal{I}\times\mathcal{I}}$ for the Cartesian product $\mathcal{I} \times \mathcal{I}$:

- Each node has the form $\tau \times \sigma$ with clusters $\tau, \sigma \in \mathcal{T}_{\mathcal{I}}$.
- The sons of $\tau \times \sigma$ are $\tau' \times \sigma'$ with $\tau' \in \mathrm{sons}(\tau)$ and $\sigma' \in \mathrm{sons}(\sigma)$.
- Each block is associated with a direction $c_{\tau\sigma} \in \mathcal{D}_\tau \cap \mathcal{D}_\sigma$.

$\mathcal{DH}^2$-matrix

Assume that $\mathcal{T}_{\mathcal{I}}$ and $\mathcal{T}_{\mathcal{I}\times\mathcal{I}}$ are given.

- Directional cluster bases represented by leaf matrices and transfer matrices, $V_{\tau c}|_{\widehat{\tau}'\times k} = V_{\tau' c'} E_{\tau' c}$.
- Admissible blocks represented by cluster bases and coupling matrices, $G|_{\widehat{\tau}\times\widehat{\sigma}} = V_{\tau c} S_{\tau\sigma} W_{\sigma c}^{*}$.



✎ B./Melenk (2015), B. (2015)

Overview

Idea

Approach: Start with a preliminary $\mathcal{DH}^2$-matrix and apply algebraic techniques to reduce the rank.

Goal: For the row basis $(V_{\tau c})$, we are looking for an improved basis $(Q_{\tau c})$ and translation matrices $(R_{\tau c})$ such that

$$V_{\tau c} S_{\tau\sigma} W_{\sigma c}^* \approx Q_{\tau c} \underbrace{R_{\tau c} S_{\tau\sigma}}_{=:\widetilde{S}_{\tau\sigma}} W_{\sigma c}^*.$$

On-the-fly recompression: Preliminary coupling matrices $(S_{\tau\sigma})$, leaf matrices $(V_{\tau c})$ and transfer matrices $(E_{\tau' c})$ are not stored, only created when necessary and immediately discarded.

Weighted singular value decomposition

Naive approach: Compute SVD of $V_{\tau c}$, eliminate small singular values.

$\rightarrow$ disappointing compression rate.

Weighted singular value decomposition

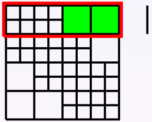
Weighted SVD: Compute singular value decomposition of

$$V_{\tau c}\widetilde{Z}_{\tau c}$$

with the weight matrix

$$\widetilde{Z}_{\tau c} = \begin{pmatrix} S_{\tau\sigma_1} W^*_{\sigma_1, c_{\tau\sigma_1}} & \cdots & S_{\tau\sigma_m} W^*_{\sigma_m, c_{\tau\sigma_m}} \end{pmatrix},$$

where $\tau \times \sigma_1, \ldots, \tau \times \sigma_m$ are the admissible blocks using $V_{\tau c}$.



Condensed weight: Compute thin Householder factorization $P_{\tau c} Z_{\tau c} = \widetilde{Z}^*_{\tau c}$ and replace $\widetilde{Z}_{\tau c}$ by $Z^*_{\tau c} \in \mathbb{R}^{k \times k}$.

Weight matrices

Goal: Compute weight matrices $Z_{\tau c}$ efficiently.

Cluster weights: Apply recursive thin Householder factorizations to find orthogonal $\widehat{P}_{\sigma c}$ and upper triangular $\widehat{W}_{\sigma c} \in \mathbb{R}^{k \times k}$ with $W_{\sigma c} = \widehat{P}_{\sigma c} \widehat{W}_{\sigma c}$.

Intermediate condensation: Construct

$$\widetilde{Z}_{\tau c} = \begin{pmatrix} & S_{\tau\sigma_1} \widehat{W}^*_{\sigma_1, c_{\tau\sigma_1}} & \cdots & S_{\tau\sigma_m} \widehat{W}^*_{\sigma_m, c_{\tau\sigma_m}} \end{pmatrix}$$

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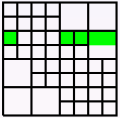
Intermediate condensation: Construct

$$\widetilde{Z}_{\tau c} = \begin{pmatrix} E_{\tau c_1} Z^*_{\tau_+ c_1} & \cdots & E_{\tau c_p} Z^*_{\tau_+ c_p} & S_{\tau \sigma_1} \widehat{W}^*_{\sigma_1, c_{\tau \sigma_1}} & \cdots & S_{\tau \sigma_m} \widehat{W}^*_{\sigma_m, c_{\tau \sigma_m}} \end{pmatrix}$$

taking contributions of the father cluster τ_+ and those directions $c_1, \ldots, c_p \in \mathcal{D}_{\tau_+}$ into account that correspond to c in τ.



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taking contributions of the father cluster τ_+ and those directions $c_1, \ldots, c_p \in \mathcal{D}_{\tau_+}$ into account that correspond to c in τ.

Total weights: Compute thin Householder factorization $P_{\tau c} Z_{\tau c} = \widetilde{Z}^*_{\tau c}$.

Nested basis

Goal: If $\operatorname{sons}(\tau) \neq \emptyset$, we have to represent $Q_{\tau c}$ by transfer matrices.

Idea: Assume that we first compute the basis and the translation matrices for $\operatorname{sons}(\tau) = \{\tau_1, \tau_2\}$.

$$V_{\tau c} Z_{\tau c}^* = \begin{pmatrix} V_{\tau_1 c_1} E_{\tau_1 c} \\ V_{\tau_2 c_2} E_{\tau_2 c} \end{pmatrix} Z_{\tau c}^* \approx \begin{pmatrix} Q_{\tau_1 c_1} R_{\tau_1 c_1} E_{\tau_1 c} \\ Q_{\tau_2 c_2} R_{\tau_2 c_2} E_{\tau_2 c} \end{pmatrix} Z_{\tau c}^*$$
$$= \begin{pmatrix} Q_{\tau_1 c_1} & \\ & Q_{\tau_2 c_2} \end{pmatrix} \widehat{V}_{\tau c} Z_{\tau c}^*, \qquad \widehat{V}_{\tau c} := \begin{pmatrix} R_{\tau_1 c_1} E_{\tau_1 c} \\ R_{\tau_2 c_2} E_{\tau_2 c} \end{pmatrix}.$$

Transfer matrices: Compute the SVD of $\widehat{V}_{\tau c} Z_{\tau c}^*$ and construct $\widehat{Q}_{\tau c}$ and $R_{\tau c}$ with $\widehat{Q}_{\tau c} R_{\tau c} Z_{\tau c}^* \approx \widehat{V}_{\tau c} Z_{\tau c}^*$, then choose

$$\begin{pmatrix} F_{\tau_1 c_1} \\ F_{\tau_2 c_2} \end{pmatrix} := \widehat{Q}_{\tau c}, \qquad Q_{\tau c} := \begin{pmatrix} Q_{\tau_1 c_1} F_{\tau_1 c_1} \\ Q_{\tau_2 c_2} F_{\tau_2 c_2} \end{pmatrix}.$$

Complexity

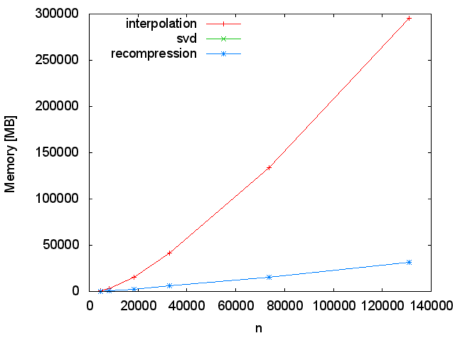
Recompression proceeds in five phases:

- compute cluster weights $\widehat{W}_{\sigma c}$,
- compute total weights $Z_{\tau c}$,
- compute improved row basis,
- compute improved column basis,
- transform coupling matrices.

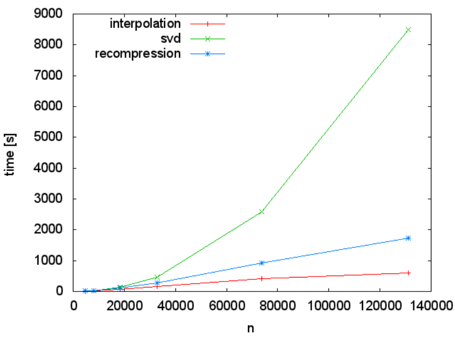
Under standard assumptions, each phase requires $\mathcal{O}(nk^2 \log n)$ operations.

Experiments

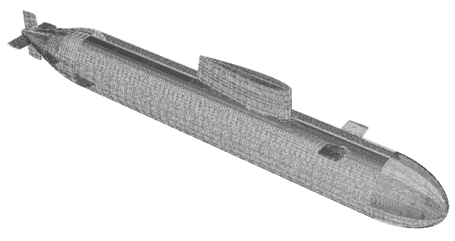
Storage



Time



Experiments (submarine)



n	Bld[s]	Mem[MB]
1 784 510	15 725.4	82 401.6