Transformations of Galton-Watson processes
and linear fractional reproduction*

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June 30, 2007

*Research supported by the Australian Research Council
†Partially supported by the Bank of Sweden Tercentenary Foundation
Abstract

By establishing general relations between branching transformations (Harris-Sevastyanov, Lamperti-Ney, time reversals, Asmussen-Sigman) and Markov chains transforms (Doob $h$-transform, time reversal, cone dual) we discover a deeper connection of these transformations with harmonic functions and invariant measures for the process itself and its space-time process. We give classification of the duals into Doob’s $h$-transforms, pathwise time-reversal and cone reversal. Explicit results are obtained for the linear fractional offspring distribution. Remarkably for this case all reversals turn out to be a GW process with a dual reproduction law and eternal particle or some kind of immigration. In particular, we generalize a result of [7] in which only a geometric offspring distribution was considered. A new graphical representation in terms of an associated simple random walk on $\mathbb{N}^2$ allows for illuminating picture proofs of our main results concerning transformations of the linear fractional GWP.

AMS 1991 subject classifications. 60J80

Key words and phrases. Galton-Watson process, Markov chains transforms, time-reversal, Doob’s transform, Cone dual, linear fractional generating function, $Q$-process.
1 Introduction

A famous model for population growth is the Galton-Watson process (GWP). In genealogy and population genetics it is important to be able to look back in time, and this leads to consideration of time reversals of this process. Various time reversals of the classical Galton-Watson process have been given in the literature. To provide a synoptic survey of these and establish new is the purpose of the present paper. More precisely, the first part of this paper does so for general GWP, while the second part deals with the linear fractional case where results are more explicit. Linear fractional distributions are in fact modified geometric distributions, this view is more suitable for our purposes and we use it throughout this paper. Since the Galton-Watson process is a Markov chain, we review Markov chains (MC) transforms and establish new relationships, which become important for imbedding branching transformations into more general Markov chains transforms. We consider the following branching transformations: Harris-Sevastyanov, Lamperti-Ney, time reversals and show that they are in fact well-known Markov chains transforms: Doob’s $h$-transform, time reversals and the cone dual. Specifically, we establish that Doob’s $h$-transform is the Harris-Sevastyanov transformation, Doob’s $h$-transform with the space-time harmonic function is the Lamperti-Ney transformation; Doob’s $h$-transform for the space-time process is also a Lyons-Pemantle-Peres (LPP) size-biased GWP process; the Lamperti-Ney transformation is a composition of a Harris-Sevastyanov and the LPP transformations, and Asmussen-Sigman transformation is the cone dual. For the case of modified geometric offspring distributions the results are more explicit and show that branching transformations result in branching processes in the same class with possibly eternal particles; and the time reversal of a GWP with a modified geometric offspring distribution is a Lamperti-Ney process. This generalizes earlier results of authors in [7], where only a geometric offspring distribution was considered. Construction of the time-reversed process allows to answer some questions about the original branching population. For example, the question of the age of the population can be answered by considering the hitting time of 1 by the time-reversed process, see [7].

We close this section with the definitions of the Galton-Watson process, and its variants that allow immigration (GWPI) and processes with an eternal particle (GWPE).
In the Galton-Watson process the number \( Z_n \) of individuals in \( n \)-th generation is given recursively by \( Z_0 = 1 \) and for \( n > 0 \)

\[
Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i},
\]

where \( \xi_{n,i} \) denotes the offspring of the \( i \)-th individual in the \( n \)-th generation, see eg [5]. All offspring random variables \( \xi_{n,i} \) are independent and identically distributed. In the Galton-Watson process with immigration (GWPI) \( (Y_n)_n \)
the \( n + 1 \)-th generation consists of the offspring of the \( n \)-th generation plus an immigration variable \( C_n \),

\[
Y_{n+1} = \sum_{i=1}^{Y_n} X_{n,i} + C_n,
\]

where \( C_n \)'s are iid and \( P(C_n = 0) = 0 \). Alternatively, we may think of one eternal particle with offspring \( C_n \) at time \( n \) and all other particles have offspring like \( \xi \). Now we should count the eternal particle, \( \hat{Y}_n = Y_n + 1 \), and obtain representation

\[
\hat{Y}_{n+1} - 1 = \sum_{i=1}^{\hat{Y}_n-1} X_{n,i} + C_n,
\]

which we use in this paper and refer to as GWPE.

The paper is organized as follows. In the rest of the Introduction we review Markov chains transforms, branching transformations, time reversals and give the concept of cone dual. Section 2 contains results on the linear fractional or modified geometric distributions and GWP with such offspring distributions, and Section 3 contains main results on branching transformations as Markov chains transforms with explicit results for linear fractional processes.

1.1 Doob’s \( h \)- transform

Let \( X = (X_n)_{n \in \mathbb{N}_0} \) be a time homogeneous MC on a countable state space \( S \) and let all states communicate \( (P_x(\exists n X_n = y) > 0 \) for all \( x, y \in S \)). Let \( h : S \to R \) be a strict positive harmonic function \( (Ph = h) \) for the transition probability kernel \( P = (p(x,y))_{x,y \in S} \). The Doob \( h \)-transform [11] is a MC with transition probabilities

\[
q(x, y) := \frac{h(y)}{h(x)} p(x, y)
\]
and the semi group \( q^n(y, x) = \frac{h(y)}{h(x)} p^n(x, y) \).

The Doob \( h \)-transform is a measure transform. Let \( M = (M_n)_n \) be a positive martingale of expectation 1. Let \( P \) be the measure of the MC and \( P_n \) be the restriction of \( P \) to the \( \sigma \)-field \( A_n \) up to time \( n \). Then \( Q_n := M_n P_n \) defines a consistent family of probability measures with a projective limit \( Q \). If the martingale is regular (\( L^1 \) convergent) with limit \( M_\infty \) then \( Q = M_\infty P \).

In our case \( (h(X_n))_n \) is a positive martingale and the corresponding measure transform is the MC with transitions given in (3). If \( A \) is a tail event, then \( h(x) := P_x(A) \) is a bounded harmonic function and the Doob \( h \)-transformed process \( X^h \) is the original process conditioned to end in the terminal set \( A \).

For a corresponding intuitive description in the general case we need Martin boundary theory. Let \( \nu \) be a Radon measure on the state space. Let \( \mathcal{P} \) be the set of positive potentials (measurable functions \( g \) on the state space satisfying \( \lim_{n \to \infty} P^n g = 0 \) pointwise) integrable with respect to \( \nu \) and let \( \overline{\mathcal{P}} \) be the closure with respect to pointwise convergence. Let \( \mathcal{H} \) be the set of \( \nu \)-integrable harmonic functions (\( Ph = h \geq 0 \)). Then \( \mathcal{P}, \mathcal{H} \) and \( \overline{\mathcal{P}} \) are convex, positive cones with a unique integral representation. There is a bijection of functions \( g \) in the convex cone \( C \) and measures \( \mu \) on the set \( E_C \subset C \) of extremals of the cone endowed with the cone topology (here Martin topology of pointwise convergence). The bijection \( g \leftrightarrow \mu_g \) can be given by

\[
g = \int_{E_C} e\mu_g(de).
\]

Further \( \overline{\mathcal{P}} = \mathcal{P} \oplus \mathcal{H} \) with a unique representation \( \overline{\mathcal{P}} \ni g = g + h \) and a unique decomposition \( \mu_{\overline{g}} = \mu_g + \mu_h \) with respect to the disjoint union \( E_{\overline{\mathcal{P}}} = E_{\mathcal{P}} \cup E_{\mathcal{H}} \).

The Martin boundary theory provides a tool in order to find the extremals \( E_\mathcal{H} \). The extremals for the potentials are the Greens kernel \( G(., y) = \sum_{n \geq 0} p^n(., y) \) for \( y \in S \). (For simplicity we assume finiteness.) Form the Martin kernel

\[
K(x, y) := \frac{G(x, y)}{G(\nu, y)}
\]

where \( G(\nu, y) = \int G(s, y) \nu(ds) \) and consider all limits of the Martin functions \( K(., y) \) with respect to pointwise convergence. The set of Martin limits contains the extremals \( \mathcal{H} \) of harmonic functions. In most cases Martin limits are extremals. Finally we note the standard normalization is by a point measure \( \nu \) on some state \( x_0 \) and the Martin kernel becomes \( K(x, y) := \frac{G(x, y)}{G(x_0, y)} \).
Coming back to the interpretation, the Doob’s $h$-transform is the MC $X^h$, such that $X^h_n$ converges weakly in Martin topology to the representing measure $\mu_h$. If $h$ is extremal then $X^h$ converges to a single point in the Martin boundary. The intuitive description is that the paths of the transformed process are Martin sequences for the extremal function.

We also use Doob’s $h$-transform for the space-time process defined as follows. Let $(P^{m,n})_{m \leq n \in \mathbb{N}_0}$ be a semi group of a time dependent MC on $S$. The space-time transition probability kernel $R$ is a stochastic matrix on $S \times \mathbb{Z}$ defined by
\[
r((x, s), (y, t)) = \mathbb{1}_{s+1=t} p^{s,t}(x, y).
\]
The corresponding semigroup to $R$ is given by $r^n((x, s), (y, t)) = \mathbb{1}_{s+n=t} p^{s,t}(x, y)$. The Greens kernel becomes $G((x, s), (y, t)) = \mathbb{1}_{s\leq t} p^{s,t}(x, y)$. A space-time harmonic function (parabolic) is a positive function $h$ on space-time $S \times \mathbb{Z}$ satisfying $Rh = h$. The Doob’s $h$-transform for the space-time setting becomes
\[
q((x, s), (y, t)) = \mathbb{1}_{s\leq t} \frac{h(y, t)}{h(x, s)} p^{t-s}(x, y).
\] (4)

Of special interest for a time homogenous MC are factorizing parabolic functions of the special form
\[h(x, s) = g(x)e^{\lambda s}.
\]
$h$ is space-time harmonic if and only if the function $g$ on $S$ is a right eigenfunction of $P$ with eigenvalue $e^{-\lambda}$, $Pg = e^{-\lambda}g$. The Doob’s $h$-transform becomes
\[
q((x, s), (y, s+1)) = \frac{g(y)e^{\lambda}}{g(x)} p(x, y).
\] (5)

This is a measure transform corresponding to the martingale $h(X_n, n)$. For the GWP, for example, the function $h(x, n) = x/m^n$ is space-time harmonic.

The Doob’s $h$-transform is a pathwise transform. This means that the probability of every path $(i_0, i_1, \ldots, i_n)$ is changed by a factor depending on $i_0, i_1, \ldots, i_n$ only. In the case of GWP it depends on $i_0, i_n$ only, more formally, the probability of a cylindrical set $\{\omega \mid X_j(\omega) = i_j, j = 0, 1, \ldots, n\}$ is changed by the factor $\frac{h(i_n)}{h(i_0)}$. 
1.2 Harris-Sevastyanov transformation

The Harris-Sevastyanov transformation $Z^*_n$ of a supercritical GWP $(Z_n)_{n \geq 0}$ with offspring generating function $f$ satisfying $f(0) > 0$ is the subcritical GWP with generating function $f^*(s) := f(sq)/q$, where $q < 1$ denotes the extinction probability of $(Z_n)_{n \geq 0}$ and thus the smallest fixed point of $f$ in $[0, 1]$ ([5], p. 47-53). The transformation is obtained by conditioning $(Z_n)_{n \geq 0}$ upon ultimate extinction.

Therefore it is a Doob’s $h$-transform with the harmonic function being the extinction probability starting with $x$ individuals, $h(x) = P_x(\lim_n Z_n = 0) = q^x$. We shall see that the Harris-Sevastyanov transformation of a supercritical GWP with a modified geometric $\text{Geom}(u_0, u)$ offspring distribution is a subcritical GWP again with a modified geometric $\text{Geom}(1-u, 1-u_0)$ distribution.

1.3 Lamperti-Ney transformation

The Lamperti-Ney transformation, also known as the $Q$ transform, introduced in [8], (see [5], p. 58) is a Markov chain on $\mathbb{N}$ with transition matrix $Q = (q(i, j))$ on $\mathbb{N}$ defined by

$$q(i, j) := \frac{jq^{i-1}p(i, j)}{i f'(q)},$$

where it is assumed that $q > 0$. It is easy to see that the function

$$h(x, s) = xq^{x-1}f'(q)^{-s}$$

is space-time harmonic. This follows from the formal calculation, which can be justified,

$$E_x(Z_1 s^{Z_1-1}) = E_x \left( \frac{d}{ds} s^{Z_1} \right) = \frac{d}{ds} E_x(s^{Z_1}) = \frac{d}{ds} f^x(s) = xf^{x-1}(s)f'(s).$$

Taking $s = q$ leads to $h$. Now it is clear from (5) that the $Q$-transform (6) is the Doob’s $h$-transform of the space-time process with the above $h$.

1.4 Time reversals

Given a MC and the knowledge of the starting measure $\mu_{-n}$ at time $-n$ we may reverse the process by conditioning on specific events today, like entering a specific state and looking back in history $(Y_m) = (X_{-m})$. Any distributional
limit as $-n \to -\infty$ of $Y_m$ is called a time reversal. We present the time reversals for the GWP, which lead to quasi-invariant measures [1]. The Esty-reversal is the limit for conditioning to just entering the state 0 at time 0 and starting in the state 1 when $-n \to -\infty$. The Esty-reversal is included in the standard approach.

1.4.1 Time reversal based on an invariant measure

Let $\mu$ be an invariant Radon measure ($\mu P = \mu$) for a time homogenous MC $^*$ (For simplicity take a finite measure and assume $\mu(x) > 0$ for all $x \in S$.) The time reversal of a MC with respect to $\mu$ is a MC with transition probability kernel

$$q(y, x) := \frac{\mu(x)p(x, y)}{\mu(y)}$$

and the semi group $q^n(y, x) = \frac{\mu(x)p^n(x, y)}{\mu(y)}$.

If $\mu$ is a stationary probability measure and $X = (X_n)_{n \in \mathbb{Z}}$ is a MC to $\mu$ on the whole time scale $\mathbb{Z}$ then reversing the time $Y_n := X_{-n}$ will do the job. If $\mu$ is only stationary, but not finite, the same interpretation holds although the construction of $X = (X_n)_{n \in \mathbb{Z}}$ is somewhat different, leaving probability spaces.

Notice the analogy to harmonic function interchanging first and second coordinate. This is more than a formal similarity. The function $x \mapsto \frac{1}{\mu(x)}$ is a harmonic function for the $Q$-kernel and vice versa. This provides a one-to-one correspondence between invariant measures of $P$ and harmonic functions of $Q$. The Martin boundary theory as above outlined is applicable.

For a GWP, since 0 is absorbing, the time reversal has to be done for the space-time process and consequently via the quasi-stationary measures [5],[1]. Notice there are no invariant Radon measures besides multiples of $\delta_0$.

Let $\tau$ be the extinction time for a subcritical GWP. The general approach to the time reverse $Y_n := Z_{\tau-n}$ for $n \in \mathbb{N}$ was considered in Alsmeyer-Roesler [1] via quasi-stationary measures. The set of quasi-invariant measures is a convex cone and has a unique integral representation over the extremals. Therefore a characterization of the extremals including the Martin topology, (here pointwise convergence), suffices for the knowledge of all quasi-invariant measures by integration. Alsmeyer and Roesler [1] give a complete description of all extremal quasi-invariant Radon measures including the Martin topology without

$^*$ $\mu$ is a Radon measure if $\mu$ is $\sigma$-finite and for any measurable set $A$ of finite measure and $\varepsilon > 0$ there exists a compact set $K \subset A$ such that $\mu(A \setminus K) < \varepsilon$. 

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any $L \log L$ condition. The Martin boundary stated in \[5\] and \[3\] is stated correctly only under the $L \log L$ condition, basically referring to an announcement of Spitzer \[12\]. Note the construction in \[1\] of the quasi-invariant measures as pointwise limits of the Martin (measures) functions $G(x,.) \over G(x,0)$ now for sequences in $x$.

### 1.4.2 Esty time reversal

The Esty-reversal is obtained by conditioning a GWP in negative time upon entering state 0 (extinction) at time 0 when starting at state 1 at time $-n$ and letting $n$ tend to infinity. (Esty has the additional technical condition $P(\xi = 1) > 0$, which we think can in his setting be replaced by the smallest additive group generated by the support of $\xi$ are the integers.) Provided the additional condition $P(\xi = 1) > 0$, the following limits exist (Monotone Ratio lemma \[5\], p.12)

$$a_j = \lim_{N \to \infty} \frac{P(Z_N = j, Z_0 = 1)}{P(Z_N = 1, Z_0 = 1)},$$

$j \geq 1$, and the Esty-reversal is a time homogeneous MC with transition probabilities

$$P(X_{n+1} = i \mid X_n = j) = \frac{a_j}{a_i b} p(i,j),$$

(7) $b = f'(q)$, $n \in \mathbb{N}_0$, $i, j \in \mathbb{N}$. The vector $(a_j)_{j \in \mathbb{N}}$ is a quasi stationary measure, ([5], p.12) for the GWP,

$$\sum_{i \in \mathbb{N}} a_i P(Z_{n+1} = j \mid Z_n = i) = ba_j, \ j \in \mathbb{N}. \quad \text{(8)}$$

A closer look shows the limit

$$\lim_{N \to \infty} \frac{P(Z_N = j \mid Z_0 = i)}{P(Z_N = 1 \mid Z_0 = 1)}$$

is independent of $i$ and is equal to $a_j$. This allows us to view the Esty-reversal as a time reversal based on an invariant measure for the space-time MC on $\mathbb{N}$. It is a remarkable result that in the linear fractional case (modified geometric offspring distribution) the Esty-reversal is the same as the Lamperti-Ney process, shown in Theorem 3.2. This generalizes earlier results of authors in [7], where only a geometric offspring distribution was considered.
1.5 Cone dual

Another transform via stochastic monotonicity was introduced by Asmussen and Sigman [4]. We call this the cone dual, since we will imbed this dual in some more general approach using cones with unique integral representations. (Uwe Roesler has learnt this from Hans Foellmer, private communication.) We are not aware of any reference to the following. Let $C$ be a cone of functions on some nice topological space $S$ with a unique integral representation, i.e. for every $c \in C$ there exists a unique Radon measure $\mu_c$ on the set $E$ of extremals, such that $c(\cdot) = \int_E e(\cdot) \mu_c(de)$. Let $P$ be a probability kernel on $S$. We will use $P(x, f)$ for the integral $\int f(y)P(x, dy)$ and use $P$ as a map of function on $S$ to itself. The important assumption for this duality to work is

$$PC \subset C.$$  

For every extremal $e \in E$ there exists a unique probability measure $\mu_e$ on $E$ such that

$$P(\cdot, e) = \int_E f(\cdot) \mu_e(df).$$

By the same argument exists a unique $\mu^n_e \in E$ such that

$$P^n(x, e) = \int_E f(x) \mu^n_e(df).$$

The family $(e, \cdot) \mapsto \mu^n_e(\cdot), n \in \mathbb{N}$ is a transition kernel semigroup, as the following calculation shows.

$$\int f(x) \mu^{s+t}_e(df) = P^{s+t}(x, e)$$

$$= \int \int e(z) P^t(y, dz) P^s(x, dy)$$

$$= \int P^t(y, e) P^s(x, dy)$$

$$= \int \int g(y) \mu^t_e(dy) P^s(x, dy)$$

$$= \int P^s(x, g) \mu^t_e(dg)$$

$$= \int \int h(x) \mu^s_g(dh) \mu^t_e(dg)$$

The uniqueness of the representation provides the semigroup property $\mu^{s+t} = \mu^s \ast \mu^t$. We call the MC with transition $Q(e, \cdot) = \mu_e(\cdot)$ the cone dual of a MC
to \( P \) relative to the cone \( C \). We specialize this setting to stochastic monotone matrices on \( \mathbb{N}_0 \). A MC is called stochastically monotone, if \( P(X_{n+1} \geq y \mid X_n = x) \) is monotone increasing in \( x \) for every \( y \). Consider the cone \( C \) of positive and increasing functions. This cone has a unique integral representation over the extremals \( e_y = \mathbb{1}_{[y, \infty)} \), \( y \in \mathbb{N}_0 \). The property \( PC \subset C \) follows by stochastic monotonicity. Thus we obtain a cone dual with respect to \( C \) via

\[
Q(e, \{e_z \mid z \in [0, x]\}) = P(x, e).
\]

The cone dual corresponds to a transition matrix on the set \( E \) of extremals. We identify now the extremal function \( e = e_y \) with the representing integer \( y = \varphi(e_y) \). Then the cone-dual \( \overline{Q}(y, A) := Q(e_y, \varphi^{-1}(A)) \) is a transition kernel on the integers,

\[
\overline{Q}(y, [0, x]) = P(x, [y, \infty)).
\]

This leads to the process defined by the transition probabilities

\[
\overline{q}(y, x) = P(x, [y, \infty)) - P(x - 1, [y, \infty)),
\]

as in Asmussen and Sigman [4].

## 2 Linear fractional or modified geometric reproduction

Let \( p = (p_0, p_1, \ldots) \) be a probability vector not concentrated at a point. Let \( \xi_{n,i}, n, i \in \mathbb{N} \) be independent rvs with distribution \( p = (P(\xi = j))_j \). The associated GWP \( (Z_n)_{n \geq 0} \)

\[
Z_{n+1} := \sum_{i=1}^{z_n} \xi_{n,i}
\]

is a time homogeneous MC with transitions

\[
p(i, j) = P(S_i = j),
\]

where \( S_i \) is the \( i \)-th partial sum of iid copies of \( \xi \).

Let \( f \) denote the probability generating function of the offspring distribution \( p \),

\[
f(s) = E(s^\xi) = \sum_{i=0}^{\infty} p_i s^i.
\]
For our purpose we take a positive \( s \) and allow infinite values of the generating function. The generating function of \( Z_n \) for given \( Z_0 = i \) is

\[
E(s^{Z_n} \mid Z_0 = i) = f^i_n(s),
\]

where \( f_n \) is the \( n \)-th iterate of \( f \). These generating functions are all strict increasing, whenever finite. These have at most two fixed points, one of them is always 1.

The extinction probability \( q \)

\[
q := \lim_{n \to \infty} P(Z_n = 0 \mid Z_0 = 1)
\]

is the smallest fixed point of the generating function \( f \). Depending on the offspring mean \( m = E(\xi) = f'(1) \) (left hand derivative) the extinction probability \( q \) is either one, if \( m \leq 1 \), or strict less than one, if \( m > 1 \). The derivative \( b := f'(q) \) is strictly larger than 1, if \( q < 1 \), and less or equal 1 for \( q = 1 \).

A geometric distribution \( \text{Geom}(u) \), \( u \in (0,1) \) is a distribution \( p \) on \( \mathbb{N}_0 \) given by

\[
p_n = vu^n, \quad n \in \mathbb{N}, \quad v = 1 - u. \]

A modified geometric distribution \( \text{Geom}(u_0, u) \), \( u_0, u \in (0,1) \) or zero-modified geometric distribution is a distribution on \( \mathbb{N}_0 \) given by

\[
p_0 = v_0, \quad p_n = u_0vu^{n-1}, \quad n \in \mathbb{N}, \quad v_0 = 1 - u_0, \quad \text{e.g.} \quad [10].
\]

It is easy to see the following

**Proposition 2.1** The \( \text{Geom}(u_0, u) \) distribution is obtained as a product of independent rv.’s \( B \) and \( \xi \), with \( B \) Bernoulli distributed with parameter \( u_0 \) and \( \xi \) geometric distributed with parameter \( u \). Then \( B(\xi + 1) \) has a modified geometric \( \text{Geom}(u_0, u) \) distribution.

It follows that the expectation of a \( \text{Geom}(u_0, u) \) distribution is \( m = \frac{u_0}{v} \). Notice \( \text{Geom}(u, u) = \text{Geom}(u) \) is a solution of the stochastic fixed point equation

\[
\xi \overset{D}{=} B(\xi + 1)
\]

where \( B, \xi \) are independent and \( B \) is distributed Bernoulli with parameter \( u \).
Often Geom\((u_0, u)\) is referred to as a linear fractional distribution due to the form of its generating function

\[
f(s) = v_0 + \frac{u_0 vs}{1 - us} = v_0 + \frac{(u_0 - u)s}{1 - us} = \frac{v_0 + (v - v_0)s}{1 - us}
\]

(12)

if \(s < \frac{1}{u}\) and \(f(s) = \infty\) for \(s \geq \frac{1}{u}\).

The key feature of the modified geometric distribution Geom\((u_0, u)\) is an explicit formula \[5\], p. 7 for the iterations

\[
f_n(s) = 1 - m^n v_n + m^n v_n^2 s,
\]

(13)

\[
v_n = \begin{cases} \frac{v - v_0}{um^n - v_0} & \text{if } m \neq 1 \\ \frac{v}{v + nu} & \text{if } m = 1 \end{cases}, \quad u_n = 1 - v_n, \quad n \geq 1.
\]

(14)

In particular \(f_1 = f\) and \(u_1 = u\). Thus given \(Z_0 = 1\), the size of the \(n\)-th generation \(Z_n\) has the zero-modified geometric distribution Geom\((m^n v_n, u_n)\) with the following counterpart of (11)

\[
P(Z_n = 0|Z_0 = 1) = 1 - m^n v_n, \quad P(Z_n = i|Z_0 = 1) = m^n v_n^2 u_n^{i-1}.
\]

(15)

More generally, in the case of \(i \geq 1\) initial particles we have

\[
P(Z_n = 0|Z_0 = i) = (1 - m^n v_n)^i
\]

\[
P(Z_n = j|Z_0 = i) = \sum_{l=1}^{j} \binom{j - 1}{l - 1} \binom{i}{l} u_n^l v_n^{j-l} (1 - m^n v_n)^{i-l} m^n
\]

\(i, j \in \mathbb{N}\). These formulae follow from

**Proposition 2.2** If \(S_i\) denotes the sum of \(i\) independent random variables with a common Geom\((u_0, u)\) distribution, then

\[
P(S_i = 0) = u_0^i, \quad P(S_i = j) = \sum_{l=1}^{j} \binom{j - 1}{l - 1} \binom{i}{l} u_0^l v_n^{j-l} u_0^{i-l}, \quad i, j \in \mathbb{N}.
\]

(16)

Here and elsewhere we put \(\binom{i}{l} = 0\) for \(j < l\).

**Proof** To verify (16) we use the representation \(S_i = \sum_{l=1}^{i} B_l(\xi_l + 1)\), where \(B_l, \xi_l\) are independent rvs, the \(B\)'s have a Bernoulli Ber\((u_0)\) distribution and the \(\xi\)'s a geometric distribution Geom\((u)\) distribution by Proposition 2.1. The number of strict positive summands in \(S_i\) is a binomially distributed random
variable \( L \sim \text{Bin}(i, u_0) \) with \( P(L = l) = \binom{i}{l} u_0^{l} v_0^{i-l}, \) \( 0 \leq l \leq i. \) Given \( L = l, \) the distribution of \((S_i - l) \sim \text{NB}(l, p)\) is Negative Binomial with \( P(S_i - L = k|L = l) = \binom{k+l-1}{k} v^l u^{k}, \) \( k \in \mathbb{N}_0. \) Applying the law of total probabilities we arrive at (16).

\[
\square
\]

The extinction probability of the GWP with Geom\((u_0, u)\) reproduction equals \( q = \min\left(\frac{v_0}{u}, 1\right) \) in accordance with the first part of (15). Notice that \( q \) is the smallest of the two fixed points \( \frac{v_0}{u} \) and 1 of the generating function \( f. \) The parameter \( b = f'(q) \) computes to

\[
b = \begin{cases} 
  v/u_0 & \text{if } m > 1 \\
  u_0/v & \text{if } m \leq 1 
\end{cases}.
\]

Let us point out the mathematical clue in order to establish the formulas ([5]). Consider the set of generating functions of the form \( f(s) = \frac{as + b}{cs + d} \) where \( a, b, c, d > 0, a + b = c + d. \) Identify this generating function with a matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \)

Observe, that the composition of these generating functions corresponds to the multiplication of the associated matrices. In our setting of a modified geometric distribution Geom\((u_0, u)\) the associated matrix is \( A = \begin{pmatrix} v - v_0 & v_0 \\ -u & 1 \end{pmatrix}. \)

We consider first the case \( u \neq v_0. \) The matrix has eigenvalues \( v, u_0 \) with the eigenvectors \((1, 1)\) and \((1, u/v_0)\). A diagonal form is given by \( O^{-1}AO = D \)

where

\[
O = \begin{pmatrix} 1 & 1 \\ 1 & u/v_0 \end{pmatrix}, \quad D = \begin{pmatrix} v & 0 \\ 0 & u_0 \end{pmatrix}.
\]

The inverse of \( O \) is

\[
O^{-1} = \frac{1}{u - v_0} \begin{pmatrix} u & -v_0 \\ -v_0 & v_0 \end{pmatrix}.
\]
The powers $A^n$

$$A^n = \frac{1}{u-v_0} \begin{pmatrix} v^n u - u_0 v_0 & u_0^n v_0 - v_0 v^n \\ v^n u - uu_0^n & uu_0^n - v_0 v^n \end{pmatrix}.$$ 

correspond to the iterates $f_n$.

The argument in case $u = v_0$ is similar. The matrix has the eigenvalue $v$ with multiplicity 2. The eigenvector is $(1,1)$ (up to multiplication) and $y = (1,1 + \frac{1}{v})$ satisfies $Ay = vy + (1,1)$. The powers $A^n$ are

$$A^n = v^{n-1} \begin{pmatrix} v - nu & nu \\ -nu & v + nu \end{pmatrix}.$$ 

3 Branching Transformations as MC transforms

3.1 Harris-Sevastyanov and Doob’s $h$-transform

Here we give a more general version of the Harris-Sevastyanov transformation, which also applies to subcritical processes and show that in fact it is a Doob’s $h$-transform with an appropriate function $h$.

**Definition 3.1** Let $r > 0$ be a fixed point of the offspring distribution generating function $f$. The Harris-Sevastyanov transformation ([5], p. 47-53) is the GWP with the transformed offspring generating function

$$f^*(s) = \frac{f(rs)}{r}.$$ 

The iterates of $f^*$ are

$$f^*_n(s) = \frac{f_n(rs)}{r}.$$ 

If $r = 1$ the Harris-Sevastyanov transform is the process itself. The Harris-Sevastyanov transform is symmetric in the sense $f = f^{**}$, where $f^{**}(s) = \frac{f(f^*(r^*))}{r}$ is the Harris-Sevastyanov transform of $f^*$ with the fixed point $r^* := \frac{1}{r}$.

If the generating function $f$ has exactly two fixed points $0 < q < r$, then the Harris-Sevastyanov transform $f^*$ has exactly the two fixed points $q^* = 1 < \frac{1}{q} = r^*$ in case $q < r = 1$ and exactly the two fixed points $q^* = \frac{1}{r} < 1 = r^*$ in case $q = 1 < r$. If the GWP is supercritical (subcritical) then the Harris-Sevastyanov transform is subcritical (supercritical) and vice versa. (The generating function $f$ defined on the positive reals and allowing the value $\infty$
is strictly convex and strictly increasing whenever finite. This function has at most one fixed point \( r \in [1, \infty) \) besides the extinction probability \( q \in [0, 1] \). In the case \( m > 1 \) there are two fixed points \( 0 \leq q < r = 1 \), in the case \( m = 1 \) there is only the fixed point 1, so we put \( r = 1 \), and in the case \( m < 1 \) the smallest fixed point is \( q = 1 \) and there may be another one \( r > 1 \) depending on the finiteness range of \( f \).

**Theorem 3.1** Let \( r > 0 \) be a fixed point of \( f \) and \( h \) be the function \( h(n) := r^n \). Then \( h \) is a harmonic function for \( P \) and Doob’s \( h \)-transform is again a GWP with generating function

\[
f^*(s) = \frac{f(rs)}{r}.
\]

**Proof** The function \( h \) is harmonic for \( P \)

\[
\sum_{j \in \mathbb{N}_0} p(i, j)h(j) = E(r^{Z_1} | Z_0 = i) = f^i(r) = r^i = h(i)
\]

for \( i \in \mathbb{N} \) and similar for \( i = 0 \).

The generating function for the probability vector \( q^n(i, j), i, j \in \mathbb{N}_0 \) for Doob’s \( h \)-transform, see Introduction, is

\[
q^n(i, j) = \frac{1}{r^i} p^n(i, j)r^j,
\]

and

\[
\sum_j q^n(i, j)s^j = \left( \frac{f_n(rs)}{r} \right)^i.
\]

By the uniqueness of generating functions the Doob’s \( h \)-transform is the Harris-Sevastyanov process to \( r \).

Next we specialize the offspring distribution to a modified geometric \( \text{Geom}(u_0, u) \) distribution. The generating function has the two fixed points 1 and \( \frac{m}{u} \). The generating function \( f^* \) of the Harris-Sevastyanov transform with \( r = \frac{m}{u} \) is

\[
f^*(s) = \frac{f\left(\frac{vs}{u}\right)}{v_0/q} = \ldots = \frac{u + (u_0 - u)s}{1 - v_0s} = u + \frac{u_0vs}{1 - v_0s}.
\]

Thus we obtain

**Proposition 3.1** The Harris-Sevastyanov transform of a GWP with a modified geometric \( \text{Geom}(u_0, u) \) distribution is a GWP with the modified geometric \( \text{Geom}(v, v_0) \) distribution.
3.2 Immigration and space-time Doob’s $h$-transform

Let $Y_0, C_n, \xi_{n,i}$, $n \in \mathbb{N}_0$, $i \in \mathbb{N}$, be independent rvs with values in the positive integers. The $\xi$ rvs are identically distributed and the $C$ rvs are identically distributed. Let $g$ be the generating function of the immigration $C$. Taking the view off an eternal particle producing the immigration and counting this particle, we arrive at the generating function of GWPE $\hat{Y}_n$, see Introduction (2)

\[ E(s^{\hat{Y}_{n+1}} | \hat{Y}_n = i) = sg(s)f^{i-1}(s). \]

Let $Z_n$ be a GWP with offspring distributed as $\xi$. Since $\frac{Z_n}{m}$ is a martingale, we consider the Doob’s $h$-transform $(X_n)_{n \in \mathbb{N}_0}$ on statespace $\mathbb{N}$ with transition matrix $Q = (q(i, j))_{i,j \in \mathbb{N}}$ and $h(i) = i$

\[ q(i, j) = p(i, j) \frac{j}{mi} \quad \text{for } i, j \in \mathbb{N}. \]

Notice that 0 is not in the state space. Since $E(Z_1 | Z_0 = i) = mi$, it is clear that $Q$ is a transition matrix. Apparently this transform appears in Lyons, Pemantle and Peres [9] as a size-biased GWP, call it (LPP) transform.

The corresponding generating function for the transformed process $(X_n)_n$ is given by

\[ E(s^{X_{n+1}} | X_n = i) = \sum_{j \in \mathbb{N}} q(i, j)s^j = \frac{1}{im} \sum_{j=1}^{\infty} p(i, j)js^j \]

\[ = \frac{s}{im} \sum_{j=1}^{\infty} p(i, j)js^{j-1} \]

\[ = \frac{s}{im} D_s(f^i)(s) = \frac{s}{m} f'(s)f^{i-1}(s). \]

Thus $(X_n)_n$ is a GWPE with the same offspring distribution and the eternal particle offspring generating function $g(s) = \frac{s}{m} f'(s)$.

**Proposition 3.2** Lamperti-Ney [8] transform is a composition of a Harris-Sevastyanov transform with the fixed point $r > 0$ and the LPP transform. It is a MC on $\mathbb{N}$ with transition matrix $Q = (q(i, j))_{i,j \in \mathbb{N}}$

\[ q(i, j) = p(i, j) \frac{j}{cjr^i}, \]

for $i, j \in \mathbb{N}$, where $c = f'(r)$. It is a GWPE with the offspring generating function $\frac{s}{f'(sr)} f'(sr) \left( \frac{f(sr)}{r} \right)^{i-1}$ and the eternal particle offspring generating function $g(s) = \frac{s}{m} f'(sq)$.  

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The special case \( r = 1 \) is the LPP transform.

Using the fact that the function \( h : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}_\geq \)

\[
h(i, n) := im^{-n}
\]  

(19)
is space-time harmonic, and (5) we can see that the LPP transform is a Doob’s 
\( h \)-transform for the space-time process.

Notice \( h(i, .) > 0 \) for \( i > 0 \) and \( h(0, .) = 0 \). Therefore we take the space-time 
process on the state space \( \mathbb{N} \times \mathbb{N}_0 \). We use the notation from the Introduction.
The Doob’s \( h \)-transform has the space-time (time homogeneous) transition 
probabilities

\[
q(i, s, j, t) = \mathbb{I}_{s+1=t}p(i, j)\frac{j}{mi}.
\]  

(20)
The Doob’s \( h \)-transform is the space-time process of the Lyons-Pemantle-Peres 
process as MC with transitions on \( \mathbb{N} \) given in (20). (Notice, the space-time 
harmonic function (19) factors.) A pathwise interpretation as a measure trans-
form is obvious, the transform depending on space and time.

Consecutive Doob’s \( h \)-transforms is a Doob’s \( h \)-transform for the product 
of harmonic functions. We summarize our findings in the next Proposition for 
a modified geometric distribution.

**Proposition 3.3** Let \( (Z_n)_n \) be a GWP with a modified \( \text{Geom}(u_0, u) \) offspring 
distribution. Then the LPP process is a GWPE with the same offspring dis-
tribution and eternal particle offspring generating function

\[
\frac{s}{m}f'(s) = s\frac{v}{1-us}\frac{v}{1-us},
\]

which is the convolution of a point measure at 1 with two geometric \( \text{Geom}(u) \)
distributions.

The Lamperti-Ney process \( (X_n)_n \), equation (18), is the Doob’s \( h \)-transform 
with the space-time harmonic function \( h(i, n) = \frac{ir^{i-1}}{c^n} \), where \( c = f'(r) \). The 
process \( (X_n)_n \) is a GWPE process with offspring generating function \( \frac{f(sr)}{r} \) and 
eternal particle offspring distribution \( \delta_1 \ast \text{Geom}(ur) \ast \text{Geom}(ur) \).

### 3.3 Esty-reversal as a time reversal

The next theorem provides the Esty-reversal for a modified geometric offspring 
distribution.
Theorem 3.2 The Esty-reversal of a subcritical or critical GWP with a \( \text{Geom}(u_0, u) \) offspring distribution, \( u_0 \leq 1 - u \), is a GWPE with the same offspring distribution. The eternal offspring distribution is a convolution of the point measure at 1 and two geometric distributions with parameter \( u \). In the supercritical case the Esty-reversal of a GWP with a \( \text{Geom}(u_0, u) \) offspring distribution is a GWPE with the dual offspring distribution \( \text{Geom}(v, v_0) \). The eternal offspring distribution is a convolution of the point measure at 1 and two geometric distributions with parameter \( v \).

Using Proposition 3.2 we obtain

Corollary 3.1 The Esty-reversal of a GWP with a \( \text{Geom}(u_0, u) \), \( u_0 < 1 - u \) is a Lamperti-Ney process with \( \text{Geom}(v_0, v) \) reproduction.

The proof relies on a result of independent interest.

Proposition 3.4 Let \( S_n \) be the \( n \)-th partial sum of iid rvs with a \( \text{Geom}(u_0, u) \) distribution. Then

\[
\sum_{i=0}^{\infty} r^i P(S_i = j) = ru_0v \frac{1}{1 - rv_0} \frac{1}{1 - rv_0} \hat{f}^{j-1}(r)
\]

for \( j \in \mathbb{N} \), where \( \hat{f} \) is the generating function of a \( \text{Geom}(v, v_0) \) distribution.

Proof Let \( f \) be the generating function of a \( \text{Geom}(u_0, u) \) distribution and \( \hat{f} \) from a \( \text{Geom}(v, v_0) \) distribution. After some formula manipulation of the
power series
\[
\sum_{i,j \in \mathbb{N}_0} r^i P(S_i = j) s^j = \sum_i r^i f^i(s) = \frac{1}{1 - rf(s)} = \frac{1}{1 - r^{u_0 + s(v - v_0)} 1 - su} = \frac{1 - ru - s(v_0 + t(v - v_0))}{1 - su} = \frac{1}{1 - r v_0 1 - s u + r(u_0 - u)} 1 - ru
\]
\[
= \frac{1}{1 - r v_0} (1 - su) \sum_{j \in \mathbb{N}_0} s^j \hat{f}^j(r)
\]
\[
= \frac{1}{1 - r v_0} (1 + \sum_{j \in \mathbb{N}} s^j \hat{f}^j(r)(\hat{f}(r) - u)
\]
a coefficient comparison provides
\[
\sum_{i \in \mathbb{N}_0} r^i P(S_i = j) = \frac{1}{1 - r v_0} (\hat{f}(r) - u) \hat{f}^{j-1}(r)
\]
and therefore the partial claim.

\[\square\]

**Proof** of Theorem 3.2. Using (15) and (14) we obtain
\[
a_j = \lim_n u_j^{n-1} = \left( \frac{u}{v_0} \right)^{j-1}
\]
for \(j \in \mathbb{N}\). We shall show
\[
\sum_{i \in \mathbb{N}_0} q(j, i)r^i = r^\frac{v}{1 - ru} \frac{v}{1 - ru} f^{j-1}(r)
\]
for \(j \in \mathbb{N}\), where \(f\) is the generating function to a Geom\((u_0, u)\). Using the previous lemma we obtain
\[
\sum_{i \in \mathbb{N}_0} q(j, i)r^i = \frac{1}{b} \sum_i \left( \frac{r u}{v_0} i \right) p(i, j) = \frac{1}{b} \frac{r u}{v_0} u_0 v \frac{1}{1 - ru} \frac{1}{1 - ru} \hat{f}^{j-1}(\frac{r u}{v_0}) = r \frac{v}{1 - ru} \frac{v}{1 - ru} f^{j-1}(r)
\]
Notice the theorem works also in the reverse, since there is one-to-one correspondence of invariant distributions of a MC and the harmonic functions of the transformed process.

Since we have a pathwise transform, an interpretation like looking back where we came from, is obvious for a stationary probability measure. The same intuition works for stationary measures and stationary space-time measures.

### 3.4 Uniform prior time reversal of GWP

Let \((Z_n)_n\) be a GWP with a modified Geom\((u_0, u)\) distribution. Putting \(t = 1\) into (21) yields \(\sum_{i \geq 1} P(S_i = i) = v/u_0\). This provides a quasi-invariant measure, the uniform distribution on the integers. Thus we found another space-time invariant measure \(\mu\)

\[
\mu(i, n) = \left(\frac{v}{u_0}\right)^n.
\]

The space-time time reversal with respect to \(\mu\) is a space-time process of the time homogeneous MC with transitions

\[
q(j, i) = \frac{u_0}{v} p(i, j)
\]

\(i, j \in \mathbb{N}\). This corresponds, cf. Proposition 3.4, to a GWPE with a modified geometric Geom\((v, v_0)\) offspring distribution. The eternal offspring distribution is a convolution of the point measure at 1 and two geometric distributions with the parameter \(v_0\).

Taking this view, the time reversal process could be called the reversed chain with the uniform prior as the quasi stationary distribution.

**Proposition 3.5** The uniform prior time reversal of the GWP with Geom\((u_0, u)\) reproduction is a GWPE \((Y_n)_n\) with the Geom\((v, v_0)\) reproduction and a \(\delta_1 \ast \text{Geom}(v_0) \ast \text{Geom}(v_0)\) eternal offspring distribution.

### 3.5 Asmussen-Sigman and the cone-dual

Asmussen and Sigman introduced a dual GW \((V_n)_n \in \mathbb{N}_0\) in [4], by the following formula for the transition probabilities

\[
P(V_{n+1} = i|V_n = j) = P(S_{i+1} > j \geq S_i)
\]

\(i, j \in \mathbb{N}_0\).
Proposition 3.6 The cone dual to the GWP with a Geom($u_0, u$) reproduction is a GWI process with a Geom($v, v_0$) reproduction and Geom($v_0$) immigration.

Proof Since (22) and
\[
\sum_{i=0}^{\infty} r^i P(S_i > j) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} r^i P(S_i = k)
\]
we obtain
\[
\sum_{i=0}^{\infty} r^i P(V_{n+1} = i|V_n = j) = \frac{1-r}{s} \sum_{i=0}^{\infty} r^i P(S_i > j) = \frac{v_0}{1-u_0 r} \left( v + \frac{uv_0 r}{1-u_0 r} \right)^j
\]

4 A picture proof of Theorem 3.2

In this section we present a graphical representation of the one step transitions in the linear fractional GWP using simple random walks on a $\mathbb{N}^2$ grid. This representation leads to illuminating picture proofs of several statements made in this paper concerning time-reversing and duality, including Theorem 3.2 which is somewhat counter intuitive in view of its corollary.

We visualize this as follows. Let $S_n$ be a sum of iid rvs with a Geom($u_0, u$) distribution. Define recursively $\tau_1 = 1$ and $\tau_{n+1} = S_{\tau_n}$ for $n \in \mathbb{N}$. Then the sequence $(Z_n)_n$ has the same distribution as $(\tau_n - \tau_{n-1})_n$, putting $\tau_0 = 0$. Draw the points $(n, S_n)$ on the $x, y$ plane. Connect two points $(n, y)$ and $(n, y+1)$ by drawing first a line from $(n, y)$ to $(n, z)$ and then a line from $(n, z)$ to $(n+1, z)$. Assign $v_0$ to the line $[(n, z), (n+1, z)]$ and $u_0$ to the line $[(n, y), (n, y+1)]$ and $u$ to every remaining line $[(n, .), (n, .+1)]$. This picture represents the path $S_n$. The probability of the path is the product of the letters written on the side. For the reverse process we have to take the same path, but replace the letter $u_0$ by $v$ and $u$ by $v_0$.

Panel 1 on Figure 1 gives an example of a one step transition for a linear fractional GWP with the Geom($u_0, u$) reproduction law. Here the generation size changes from seven to eight after a certain random walk on the $\mathbb{N}^2$ grid.
(which will be called a Geom\((u_0, u)\)-walk) makes seven horizontal jumps. As the picture indicates, there are two types of horizontal jumps in a Geom\((u_0, u)\)-walk. A \(u_0\)-jump says that a particle in the generation \(n\) has zero offspring and a \(u\)-jump correspond to a particle with at least one offspring.

Panel 1

\[ Z_{n+1} = 8 \]

\[ Z_n = 7 \]

\[ u_0, u_0, u_0 \]

Panel 2

\[ Z_n = 7 \]

\[ Z_{n+1} = 8 \]

\[ u_0, u_0, u_0, v, v \]

Panel 3

\[ Y_{n-1} = 8 \]

\[ Y_{n-1} = 7 \]

\[ v_0, v, v, v, v, v \]

Figure 1: Picture proof of Theorem 3.2 in the supercritical case.

Therefore each horizontal \(u\)-jump is preceded by several vertical jumps depicting the offspring of the successful particle in question. Since the offspring number conditional on being positive is one plus a Geom\((u)\) random variable, every streak of vertical arrows on Panel 1 starts with a \(v_0\) arrow and followed by a Geom\((u)\) number of \(v\) arrows. The number of horizontal \(u\) arrows equals to the number of vertical \(v_0\) arrows and corresponds to the number \(l\) of successful particles in the generation \(n\). Notice that every streak of horizontal \(u_0\) arrows has a Geom\((v_0)\) distribution.
The labels attached to the jumps give the probabilities so that the product of the labels along a trajectory with \( l = 3 \) stretches of vertical arrows gives the term \( u^3v^5u_0^4v_0^3 \) in the formula (16). The binomial coefficients in (16) reflect the fact that \( \binom{7}{2} \binom{7}{3} = 735 \) trajectories with the given number \( l = 3 \) may lead to the same transition from \( i = 7 \) to \( j = 8 \).

Panel 2 is the inverse random walk picture representing a naive attempt to get a reverse transition from eight to seven particles. There is no straightforward way to meaningfully interpret this panel even though the new trajectory look like a dual Geom\((v, v_0)\)-walk. The problem with this panel is that it is impossible to interpret the beginning and the end of the trajectory in a way consistent with the previous panel interpretation.

Panel 3 shows how the arrows in Panel 2 can be rearranged leading to a meaningful illustration of Theorem 3.2 in the supercritical case. Recall that according (7) Esty’s time-reversal is based on a transformation of the original GWP transitional probability by the factor \( \frac{a_j}{a_i} \), which in the supercritical case equals \( \frac{v_0}{v} \) for any pair \((i, j)\). It follows that we have to add a \( v_0\)-arrow and remove a \( u\)-arrow to arrive to a dual random walk trajectory describing the transition in the time-reversed process.

The change of measure shown in Panel 3 corresponds to the following rearrangements of the arrows in Panel 2:

1. remove the last \( u\)-arrow,
2. place an additional vertical \( v_0\)-arrow to the beginning of the path,
3. move the last \( u_0\) arrow(s) to the beginning of the path below the just added \( v_0\)-arrow.

In this way we get the correct transition probability for the reversed process and obtain its size-biased process interpretation which comes next.

The first horizontal \( v_0\)-arrow describes the eternal particle among eight particles in the current generation of the reverse process. Its offspring is depicted by the initial stretch of vertical arrows with all of them except one are \( u_0\)-arrows. The only vertical \( v_0\)-arrow describes the eternal particle among seven particles in the next generation of the reverse process. It separates two streaks of \( u_0\)-arrows each corresponding to a Geom\((v_0)\) random number of particles stemming from the eternal particle. The rest of the arrows have the same interpretation as in Panel 1 with the dual parameters: a \( v\)-arrow stands for
a childless particle and and a horizontal $v_0$-arrow is preceded by one plus a Geom($u$) number of vertical arrows representing the offspring of a successful particle.

References


