Fixed Points with Finite Variance of a Smoothing Transformation

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Abstract
Let $T = (T_1, T_2, T_3, ...)$ be a sequence of real random variables. We investigate the following fixed point equation for distributions $\mu$: $W \overset{\sim}{=} \sum_{j=1}^{\infty} T_j W_j$, where $W, W_1, W_2, ...$ have distribution $\mu$ and $T, W_1, W_2, ...$ are independent. The corresponding functional equation is $\phi(t) = E \prod_{j=1}^{\infty} \phi(tT_j)$, where $\phi$ is a characteristic function. We consider solutions of the fixed point equation with finite variance. Results about existence and uniqueness are derived. In the situation of solutions with zero expectation we give a representation of the characteristic functions of solutions and treat the question of moments and $C^\infty$-Lebesgue densities. The article extends results on the case of non-negative $T$ and non-negative solutions.

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1 Introduction

Suppose the sequence $T = (T_1, T_2, T_3, \ldots)$ of real random variables is given. Denote the map

$$M : \mathcal{D} \to \mathcal{D}; \quad M(\mu) = \sum_{j=1}^{\infty} T_j W_j$$

with $\mathcal{D}$ a subspace of distributions and $W_1, W_2, \ldots$ random variables with distribution $\mu$ and $T, W_1, W_2, \ldots$ independent. We are interested in fixed points of $M$. Note that the expression $\sum_{j=1}^{\infty} T_j W_j$ is not necessarily well defined. In Section 2 we take care of this problem in our set-up by considering it as the $L_2$ limit of $\sum_{j=1}^{n} T_j W_j$ under certain conditions.

If we consider the following equation for distributions $\mu$

$$W \equiv \sum_{j=1}^{\infty} T_j W_j ,$$

where $W, W_1, W_2, \ldots$ have distribution $\mu$ and $T, W_1, W_2, \ldots$ are independent, then the fixed points of $M$ are exactly the solutions of Eq. (1). $M$ is called smoothing transformation. Eq. (1) is also referred to as fixed point equation and solutions of Eq. (1) as fixed points of Eq. (1). With slight abuse of notation, we speak also of a random variable as a fixed point if its distribution solves Eq. (1). If $W = 0$ a.s., $W$ is called a trivial fixed point or a trivial solution of Eq. (1).

The fixed point equation (1) appears in several fields. We mention here only infinite particle systems and turbulence models (Dawson, 1977; Holley and Liggett, 1981; Liggett, 1978; Liggett and Spitzer, 1981; Mandelbrot, 1974a,b),

In the context of infinite particle systems first only non-negative solutions and non-negative coefficients $T$ of the fixed point equation (1) were considered; In the essential article of Durrett and Liggett (1983) the problem of existence of solutions was solved for bounded $N := \sum_{j=1}^{\infty} 1_{T_j \neq 0}$. Moreover these solutions were characterized and their tail behaviour and other properties investigated. The results were extended and generalized for $N < \infty \ a.s.$ in several articles of Liu. The case of iid $T_j$ and solutions of finite mean were considered in Guivarc’h (1990) and Kahane and Peyrière (1976). In Holley and Liggett (1981) solutions were given for $T_j$ being a constant multiple of a fixed random variable. Biggins and Kyprianou (2002) proved uniqueness and properties of solutions in a special boundary case employing the corresponding branching random walk. In Lyons (1997) (based on Lyons et al., 1995) convergence for related martingales was treated.

Solutions and coefficients $T$ which are not necessarily non-negative are less intensely investigated. Some results for general solutions are given in Liu (2001). Roesler (1998) characterized general solutions in the case that the $T_j$ are fixed real constants. In the context of algorithms the contraction method is often applied to solve distributional equations (Hwang and Neininger, 2002;
Rachev and Rueschendorf, 1994; Roesler, 1992; Roesler and Rueschendorf, 2001). These equations usually have an additional additive random term as is common for divide-and-conquer algorithms.

Here, we investigate solutions with finite variance of the smoothing transformation, i.e. probability measures on \((-\infty, \infty)\) with finite variance. We do not require the coefficients \(T\) to be non-negative. The problem of general solutions and coefficients differs in a high degree from the case of non-negative ones since the latter can be treated by using the Laplace transform. This method was applied thoroughly in Durrett and Liggett (1983). For the general case new methods have to be developed. We employ here the classical theory of convergence of infinitesimal independent triangular schemes to infinitely divisible distributions as, e.g., developed in Gnedenko and Kolmogorov (1968). By this method we can determine exactly when solutions with finite variance exist. It is necessary to distinguish two cases: Solutions with non-zero and zero expectation (always with finite variance). We shall call the former solutions of class I and the latter solutions of class II. Obviously, if non-negative fixed points are considered, only solutions of class I can occur. The occurrence of solutions of class II is a new phenomenon due to the extended set of solutions. We show uniqueness of solutions of class I and class II (up to multiplicative constants). From the convergence of schemes arises an interesting correspondence between solutions of class II of Eq. (1) and non-negative solutions of the following fixed point equation for non-negative distributions \(\mu\) on \([0, \infty)\)

\[
X \equiv \sum_{j=1}^{\infty} T_j^2 X_j ,
\]

(2)
where $X, X_1, X_2, \ldots$ have distribution $\mu$ and $T, X_1, X_2, \ldots$ are independent. This is exploited to establish results about moments and $C^\infty$-Lebesgue densities of solutions of class II.

After some introductory notation and remarks in Section 2, we give a detailed formulation of the main results in Section 3. Section 4 gives some elementary statements about solutions with finite variance and related martingales. In Section 5 statements about the convergence of infinitesimal independent triangular schemes are summarized which lead to the convergence results in Section 6. Finally, Section 7 is devoted to the proofs of the theorems stated in Section 3.

2 Assumptions and Mathematical Notation

Let $N := \sum_{j=1}^\infty 1_{T_j \neq 0}$.

The following assumptions are of importance for this article:

(A0) $P(N \leq 1) < 1$.

(A1) $P(\sum_{j=1}^\infty T_j = 1) < 1$.

(A2a) $E \sum_{j=1}^\infty T_j^2 < \infty$ and $E \left| \sum_{j=1}^\infty T_j^2 \log T_j^2 \right| < \infty$.

(A2b) $E \sum_{j=1}^\infty T_j^2 < \infty$ and $\left( \sum_{j=1}^n T_j \right)_{n \in \mathbb{N}}$ converges in $L_2$.

If $Z$ is a random variable with finite variance, assumption (A2a) or (A2b), respectively, ensure that $M(Z)$ is well defined as the $L_2$ limit of $\sum_{j=1}^n T_j Z_j$: 
Lemma 1 Let $Z$ be a random variable with finite variance. If $EZ = 0$ and (A2a) is satisfied or if $EZ \neq 0$ and (A2b) is satisfied then $\sum_{j=1}^{n} T_j Z_j$, with $Z_1, Z_2, \ldots$ distributed as $Z$ and $T, Z_1, Z_2, \ldots$ independent, converges in $L_2$ for $n \to \infty$.

PROOF. We will show that under the given conditions $(\sum_{j=1}^{n} T_j Z_j)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L_2$. Let $n, m \in \mathbb{N}$, $n > m$. Then

$$
E \left( \sum_{j=1}^{n} T_j Z_j - \sum_{j=1}^{m} T_j Z_j \right)^2 = E \left( \sum_{j=m+1}^{n} T_j Z_j \right)^2 
$$

$$
= E \sum_{j=m+1}^{n} T_j Z_j T_k Z_k + E \sum_{j=m+1}^{n} T_j^2 Z_j^2 
$$

$$
= (EZ)^2 E \sum_{j=m+1}^{n} T_j T_k + E(Z^2)E \sum_{j=m+1}^{n} T_j^2 . \quad \square
$$

When we apply $M$ on distributions of random variables $Z$ with finite variance or consider fixed points $Z$ with finite variance we always suppose (A2a) (if $EZ = 0$) or (A2b) (if $EZ \neq 0$) to be satisfied and regard $\sum_{j=1}^{\infty} T_j Z_j$ as the $L_2$ limit.

If (A0) does not hold the set of solutions can easily be determined:

Lemma 2 Assume $P(N \leq 1) = 1$. Then the fixed point equation (1) has a non-trivial solution if and only if $P \left( |\sum_{j=1}^{\infty} T_j| = 1 \right) = 1$.

If $P \left( |\sum_{j=1}^{\infty} T_j| = 1 \right) = 1$ and $P(\sum_{j=1}^{\infty} T_j = 1) < 1$ exactly the symmetric distributions solve Eq. (1).

If $P(\sum_{j=1}^{\infty} T_j = 1) = 1$ every distribution solves Eq. (1).

The proof is given in the appendix. (A0) is assumed throughout the text.
Note that if assumption (A1) is not met, i.e. \( P(\sum_{j=1}^{\infty} T_j = 1) = 1 \), we have a special case since then constants are solutions of Eq. (1). In order not to obscure our results and not to get lost in a jungle of different cases, we suppose (A1) to be satisfied everywhere in the text save the appendix, where the results for \( P(\sum_{j=1}^{\infty} T_j = 1) = 1 \) are stated (Theorem 18).

We now introduce some notation to obtain different forms of the fixed point equation (1), which are often useful.

Denote

\[
V := \bigcup_{n \in \mathbb{Z}^+} \mathbb{N}^n \quad \mathbb{N}^0 := \emptyset .
\]

Let \( T(v) : \Omega \to \mathbb{R}^n \), \( v \in V \), be independent copies of \( T \). Denote for \( v = (v_1, ..., v_n) \in V \) and \( j \in \mathbb{N}, j < n \), by \( v_{[j]} = (v_1, ..., v_j) \) the restriction of \( v \) to the first \( j \) components, \( v_{[0]} := \emptyset \). Define the random variables \( L(v) : \Omega \to \mathbb{R}, \quad v \in V \), for \( v = (v_1, v_2, ..., v_n) \) by

\[
L(v) = \prod_{j=1}^{n} T_{v_j} (v_{[j-1]}) .
\]

The fixed point equation (1) can be written for \( n \in \mathbb{N} \) by using recursion as

\[
W \cong \sum_{|v|=n} L(v) W(v) ,
\]

where \( W(v), v \in V \), are independent copies of \( W \) and \( W(v), |v| = n \), \( L(w), |w| = n \) are independent. Defining \( M^1 := M \) and \( M^n := M^{n-1} \circ M \) for \( n \in \mathbb{N} \), \( \sum_{|v|=n} L(v) W(v) \) is understood as \( M^n(W) \), i.e.

\[
\sum_{t_1=1}^{\infty} \left( \sum_{t_2=1}^{\infty} \left( \sum_{t_{n-1}=1}^{\infty} L(v_1...v_{n-1}) W(v_1...v_{n}) \right) \right) ... .
\]

However, one can show that \( \sum_{|v|=n} L(v) W(v) \) is well defined as a recursive \( L_2 \) limit independent
of the order of the summation (use the argument of Cauchy sequences).

It is natural to write the fixed point equation (1) in terms of characteristic functions or Laplace transforms. For each random variable $Y$ let $\phi_Y$ be the corresponding characteristic function and (if $Y$ is non-negative) $\psi_Y$ the Laplace transform. Eq. (1) can be written in terms of characteristic functions $\phi$ as ($t \in \mathbb{R}$)

$$\phi(t) = E \prod_{j=1}^{\infty} \phi(tT_j) .$$

(4)

The same equation applies for Laplace transforms with $\phi$ substituted by $\psi$.

In the following we use the convention

$$\frac{u}{0} = \begin{cases} \infty & : \ u > 0 \\ -\infty & : \ u \leq 0 \end{cases}.$$

For each random variable $Y$ denote by $F_Y$ its distribution function and by $P_Y$ its distribution.

For a random variable $Y$ with finite variance let $||Y||_2 = (EY^2)^{1/2}$ be the usual $L_2$ norm.

3 Main Results

Denote by $\mathcal{F}_1$, resp. $\mathcal{F}_2$, the set of non-trivial solutions of the fixed point equation (1) with finite variance and non-zero, resp. zero, expectation. Elements of $\mathcal{F}_1$ are called solutions of class I, elements of $\mathcal{F}_2$ solutions of class II. Let $\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2$ be the set of non-trivial solutions of Eq. (1) with finite variance.
Furthermore we denote by $\mathcal{P}$ the set of non-negative, non-trivial solutions of Eq. (2) with finite expectation.

Our first main result determines under which conditions on $T$ there exist non-trivial solutions of Eq. (1) with finite variance. These solutions turn out to be unique.

**Theorem 3 (Existence of solutions with finite variance)**

Assume (A0), (A1) and (A2a) or (A2b), respectively.

(i) $\mathcal{F}_1 \neq \emptyset \iff E \sum_{j=1}^{\infty} T_j^2 < 1 \text{ and } E \sum_{j=1}^{\infty} T_j = 1$.

(ii) $\mathcal{F}_2 \neq \emptyset \iff E \sum_{j=1}^{\infty} T_j^2 = 1, E \sum_{j=1}^{\infty} T_j^2 \log T_j^2 < 0$

and $E \left( (\sum_{j=1}^{\infty} T_j^2) \log^+ \left( \sum_{j=1}^{\infty} T_j^2 \right) \right) < \infty$

$\iff \mathcal{P} \neq \emptyset$.

(iii) If $\mathcal{F} \neq \emptyset$, the solution of Eq. (1) with finite variance is unique up to multiplicative constants.

The next theorem is our second main result and gives a description of solutions of $\mathcal{F}$.

**Theorem 4 (Representation of Solutions)**

Assume (A0), (A1) and (A2a) or (A2b), respectively.

(i) Let $\mathcal{F}_1 \neq 0$. Then $\sum_{\{v\} \sim n} L(v)$ is a martingale and converges a.s. and in $L_2$ to a random variable $L^\infty$ with $E L^\infty = 1$. For any $W \in \mathcal{F}_1$

\[ W \sim L^\infty EW \text{ and } \]

\[ \text{Var} W = \frac{(EW)^2 \text{Var} \left( \sum_{j=1}^{\infty} T_j \right)}{1 - E \sum_{j=1}^{\infty} T_j^2} \].

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(ii) Let $\mathcal{F}_2 \neq 0$. Then $\sum_{|v|=n} L^2(v)$ is a martingale and converges a.s. and in $L_1$ to a random variable with expectation 1. For any $W \in \mathcal{F}_2$
\[ \phi_W(t) = \mathbb{E}\exp \left( -\frac{1}{2} t^2 X \right) \quad (t \in \mathbb{R}) \] (5)
where $X = \text{Var}(W) \lim_{n \to \infty} \sum_{|v|=n} L^2(v)$ is the unique solution in $\mathcal{P}$ with $\text{Var}(W) = \mathbb{E}X$.

**Remark 5 (Special cases)**

(i) Non-negative coefficients, non-negative solutions
Assume $T_j \geq 0$ for all $j \in \mathbb{N}$ and consider the set $\mathcal{F}_{pos}$ of non-trivial solutions $\mu$ of Eq. (1) with finite variance and $\mu([0, \infty)) = 1$. Then obviously $\mathcal{F}_2 \cap \mathcal{F}_{pos} = \emptyset$. For $\mathcal{F}_1$ the existence, uniqueness and representation of non-negative solutions with finite variance were proved in Durrett and Liggett (1983), Theorems 1, 2, and 3, under the additional assumption that $N = \sum_{j=1}^{\infty} 1_{T_j \neq 0}$ is bounded. Connected results are given in Liu (1997) Theorems 1.1, 3.1, and 5.1 and in Liu (1998) Theorem 1.1 and Corollary 1.3 under the additional assumption that $N < \infty$ a.s.

(ii) Positive coefficients
Assume $T_j \geq 0$ for all $j \in \{1, ..., N\}$. Then $L^\infty \geq 0$ a.s. Since the solutions of class 1 are multiplicative constants of $L^\infty$ they have thus support $(-\infty, 0]$ or $[0, +\infty)$.

(iii) Positive solutions
Let $\mathcal{F}_{pos}$ be defined as in (i). Since as above $\mathcal{F}_2 \cap \mathcal{F}_{pos} = \emptyset$ we obtain
\[ \mathcal{F}_{pos} \neq \emptyset \iff \mathcal{F}_1 \neq \emptyset \quad \text{and} \quad L^\infty \geq 0 \quad \text{a.s.} \]
Up to now mainly non-negative solutions and coefficients have been considered in the literature. By the remark it becomes clear that solutions of class II are of a new, uninvestigated kind since their support is not contained in $\mathbb{R}^+$ or $\mathbb{R}^-$. They are, however, closely connected to non-negative solutions of the fixed point equation (2), see Eq. (5) for the characteristic function of solutions of class II.

The next theorem presents our main convergence result.

**Theorem 6 (Convergence)**

Assume $(A0)$, $(A1)$ and $(A2a)$ or $(A2b)$, respectively.

(i) Let $\mathcal{F}_1 \neq \emptyset$. Then for each distribution $\mu$ with finite variance and non-zero expectation, $M^n(\mu)$ converges for $n \to \infty$ in distribution to the (up to multiplicative constants) unique solution of Eq. (1).

(ii) Let $\mathcal{F}_2 \neq \emptyset$. Then for each non-trivial distribution $\mu$ with finite variance and zero expectation, $M^n(\mu)$ converges for $n \to \infty$ in distribution to the (up to multiplicative constants) unique solution of Eq. (1).

By exploiting the representation of the characteristic function of solutions of class II in Eq. (5) we derive results about densities and moments.

Denote the generating function of $N$ by

$$g : [0, 1] \to [0, 1]; \quad t \mapsto \sum_{j=0}^{\infty} t^j P(N = j).$$

(6)
Theorem 7 (Lebesgue density for class II solutions) Assume (A0), (A1) and (A2a). Let \( W \in \mathcal{F}_2 \) with representation as in Eq. (5) in Theorem 4. Then \( W \) has on \( \mathbb{R} \setminus \{0\} \) the Lebesgue density

\[
f : \mathbb{R} \setminus \{0\} \to \mathbb{R}_{\geq 0}; \ t \mapsto E \varphi_{\alpha, \Theta}(t) 1_{X>0}
\]

with \( \varphi_{m, \sigma^2} \) being the density of the normal distribution with expectation \( m \) and variance \( \sigma^2 \). Additionally \( g \) has a unique fixed point \( q \) in \( [0,1) \) and \( P(W = 0) = q \).

Let \( \tau := \min\{j \in \mathbb{N} : T_j \neq 0\} \). If \( N < \infty \) a.s., \( P(N = 0) = 0 \), \( E T_{\tau}^{-a} < \infty \) and \( E T_{\tau}^{-a} 1_{N=1} < 1 \) for all \( a > 0 \) then the above Lebesgue density is \( C^\infty \).

**Remark 8** In the situation of non-negative coefficients the existence of a Lebesgue density for solutions of Eq. (1) (not necessarily with finite variance) was proved under essentially the assumptions that \( N < \infty \) a.s., \( P(N = 0) = 0 \), \( E \sum_{j=1}^{\infty} T_j^a \leq 1 \) for \( \alpha \in (0,2] \) and \( E T_{\tau}^{-a} < \infty \) for some \( a > 0 \) (with \( \tau \) as in Theorem 7) in Theorem 6.4 of Liu (2001).

**Theorem 9 (Moments)** Assume (A0), (A1) and (A2a). Let \( W \in \mathcal{F}_2 \) and \( 2 < \beta < \infty \). Assume \( N < \infty \) a.s.. If \( \beta \in \mathbb{N} \) or \( N \) is bounded then

\[
E|W|^\beta < \infty \iff E \sum_{j=1}^{\infty} |T_j|^\beta < 1 \quad \text{and} \quad E \left( \sum_{j=1}^{\infty} T_j^2 \right)^{\beta/2} < \infty .
\]
4 Preliminary Results

In this section some easily obtainable results are stated. The first lemma characterizes $T$ in the case $\mathcal{F} \neq \emptyset$.

**Lemma 10** Let $W \in \mathcal{F}$. Then

(i) $EW = 0$ or $E \sum_{j=1}^{\infty} T_j = 1$.

(ii) If $EW = 0$ then $E \sum_{j=1}^{\infty} T_j^2 = 1$.

If $EW \neq 0$ then $E \sum_{j=1}^{\infty} T_j^2 < 1$ and $\text{Var}W = \frac{(EW)^2 \text{Var} \left( \sum_{j=1}^{\infty} T_j \right)}{1 - E \sum_{j=1}^{\infty} T_j^2}$.

**PROOF.** Let $W \in \mathcal{F}$.

(i) We obtain from Eq. (1)

$$EW = E \left( \sum_{j=1}^{\infty} T_j W_j \right) = EW \left( \sum_{j=1}^{\infty} T_j \right).$$

(ii): We obtain from Eq. (1) and (i)

$$\text{Var}W = EW^2 - (EW)^2$$

$$= E \left( \sum_{j=1}^{\infty} T_j W_j \right)^2 - (EW)^2$$

$$= \lim_{n \to \infty} E \left( \sum_{j=1}^{n} T_j W_j \right)^2 - (EW)^2$$

$$= (EW)^2 \lim_{n \to \infty} E \left( \sum_{j=1}^{n} \sum_{k=1}^{n} T_j T_k \right) + EW^2 \lim_{n \to \infty} E \left( \sum_{j=1}^{n} T_j^2 \right) - (EW)^2$$

$$= (EW)^2 \left( E \left( \sum_{j=1}^{\infty} T_j \right)^2 - 1 \right) + E \left( \sum_{j=1}^{\infty} T_j^2 \right) \left( EW^2 - (EW)^2 \right). \quad \square$$

The next two lemmata contain martingale results. Martingales are a useful tool for studying solutions of the fixed point equation (1), see, e.g., Biggins and Kyprianou (2002), Liu (1997) and Lyons (1997).
Define $\mathcal{F}_n := \sigma\{T(v) : v \in V, |v| < n\}$ for $n \in \mathbb{N}$.

**Lemma 11** Let $\alpha \in \{1, 2\}$ and assume for $\alpha = 1$ that $\left(\sum_{j=1}^{n} T_{j}\right)_{n \in \mathbb{N}}$ converges in $L_2$ and for $\alpha = 2$ that $\sum_{j=1}^{\infty} T_{j}^{2} < \infty$. Then $\sum_{|v|=n} L^\alpha(v)$ is a martingale with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$ if and only if $E\left(\sum_{j=1}^{\infty} T_{j}^{\alpha}\right) = 1$.

**PROOF.** Let $n \in \mathbb{N}$. $\sum_{|v|=n} L^\alpha(v)$ is well defined because of the corresponding assumptions.

$$
E\left(\sum_{|v|=n} L^\alpha(v) \Big| \mathcal{F}_{n-1}\right) = \sum_{|v|=n} E\left(T^\alpha_{vn}(v_{|n-1}) \Big| \mathcal{F}_{n-1}\right) \cdot T^\alpha_{vn-1}(v_{|n-2}) \cdots T^\alpha_{v1}(\emptyset)
$$

$$
= \sum_{j=1}^{\infty} E\left(T^\alpha_{j}\right) \cdot \sum_{|v|=n-1} T^\alpha_{vn-1}(v_{|n-2}) \cdots T^\alpha_{v1}(\emptyset)
$$

$$
= \left(E\left(\sum_{j=1}^{\infty} T_{j}^{\alpha}\right)\right) \sum_{|v|=n-1} L^\alpha(v).
$$

**Lemma 12** Let $E\sum_{j=1}^{\infty} T_{j}^{2} < 1$.

(i) $\sum_{|v|=n} L^2(v) \xrightarrow{n \to \infty} 0$ a.s..

(ii) Assume $(\sum_{j=1}^{n} T_{j})_{n \in \mathbb{N}}$ converges in $L_2$. If $E\sum_{j=1}^{\infty} T_{j} = 1$ then $\sum_{|v|=n} L(v)$ converges a.s. and in $L_2$.

**PROOF.** Let $C := E\sum_{j=1}^{\infty} T_{j}^{2}$. Then $E\sum_{|v|=n} L^2(v) = C^n$ for $n \in \mathbb{N}$.

(i): By the Chebyshev-Markov inequality we obtain

$$
P\left(\sum_{|v|=n} L^2(v) \geq \frac{1}{n}\right) \leq nC^n.
$$

Since $\sum_{n=1}^{\infty} nC^n < \infty$ the statement of the lemma follows by using the Borel-Cantelli lemma.
(ii): Assume \( \left( \sum_{j=1}^{n} T_j \right)_{n \in \mathbb{N}} \) converges in \( L_2 \) and \( \mathbb{E} \sum_{j=1}^{\infty} T_j = 1 \) and let \( Z_n := \sum_{|v|=n} L(v) \). Note that \( (Z_n)_{n \in \mathbb{N}} \) is a martingale due to the preceding lemma.

Then

\[
\text{Var} Z_n = \mathbb{E} (Z_n - 1)^2 = \mathbb{E} (Z_n - Z_{n-1} + Z_{n-1} - 1)^2 \\
= \mathbb{E} (Z_n - Z_{n-1})^2 + 2 \mathbb{E} (Z_n - Z_{n-1})(Z_{n-1} - 1) + \mathbb{E} (Z_{n-1} - 1)^2 .
\] (7)

The second term is equal to 0:

\[
\mathbb{E} (Z_n - Z_{n-1})(Z_{n-1} - 1) = \mathbb{E} \mathbb{E}((Z_n - Z_{n-1})(Z_{n-1} - 1) | \mathcal{F}_{n-1}) \\
\mathbb{E} [(Z_{n-1} - 1)(\mathbb{E} (Z_n | \mathcal{F}_{n-1}) - Z_{n-1})] = 0 .
\]

Furthermore with \( K := \text{Var} \sum_{j=1}^{\infty} T_j \)

\[
\mathbb{E} (Z_n - Z_{n-1})^2 = \mathbb{E} \mathbb{E} \left( (Z_n - Z_{n-1})^2 \bigg| \mathcal{F}_{n-1} \right) \\
= \mathbb{E} \mathbb{E} \left( \left( \sum_{|v|=n-1} L(v) \left( \sum_{j=1}^{\infty} T_j(v) - 1 \right) \right)^2 \bigg| \mathcal{F}_{n-1} \right) \\
= \mathbb{E} \mathbb{E} \left( \sum_{|v|=n-1} \sum_{|w|=n-1} L(v)L(w) \left( \sum_{j=1}^{\infty} T_j(v) - 1 \right) \left( \sum_{j=1}^{\infty} T_j(w) - 1 \right) \bigg| \mathcal{F}_{n-1} \right) \\
= \mathbb{E} \left( \sum_{|v|=n-1} \sum_{|w|=n-1} L(v)L(w) \mathbb{E} \left[ \left( \sum_{j=1}^{\infty} T_j(v) - 1 \right) \left( \sum_{j=1}^{\infty} T_j(w) - 1 \right) \right] \right) \\
= \mathbb{E} \left( \sum_{|v|=n-1} L^2(v) \right) \mathbb{E} \left[ \sum_{j=1}^{\infty} T_j - 1 \right]^2 = C^{n-1} K .
\]

Thus Eq. (7) becomes by recursion

\[
\text{Var} Z_n = \mathbb{E} (Z_n - 1)^2 = C^{n-1} K + \text{Var} Z_{n-1} = K \sum_{j=0}^{n-1} C^j < K \sum_{j=0}^{\infty} C^j < \infty .
\]

Therefore \( (Z_n)_{n \in \mathbb{N}} \) is an \( L_2 \)-bounded martingale. Thus it follows from general martingale theory that \( (Z_n)_{n \in \mathbb{N}} \) converges a.s. and in \( L_2 \). \( \square \)
5 Convergence of Independent Triangular Schemes

We present here results concerning the convergence of independent triangular schemes to infinitely divisible distributions and related statements connected with the set-up of fixed point equation (1).

A probability measure on $\mathbb{R}$ with characteristic function $\phi$ is called \textit{infinitely divisible} if for every $n \in \mathbb{N}$ there is a characteristic function $\phi_n$ with $\phi = (\phi_n)^n$.

The function $\phi : \mathbb{R} \to \mathbb{C}$ is a characteristic function of an infinitely divisible distribution of a random variable $X$ with finite variance if and only if there exist a constant $\gamma \in \mathbb{R}$ and a non-decreasing function $K$ on $\mathbb{R}$, which is continuous to the left, with $K(-\infty) = 0$, $K(\infty) < \infty$, such that

$$
\log \phi(t) = i\gamma t + \int \left( \frac{(e^{iut} - 1 - iut)}{u^2} \right) \frac{1}{u} \text{d}K(u). \quad (t \in \mathbb{R})
$$

(8)

This representation is unique. Furthermore $\gamma = \text{EX}$, $K(\infty) = \text{Var}X$ (Gnedenko and Kolmogorov, 1968, § 18, Formula (10)).

A \textit{triangular scheme} is an array of random variables $Z_{i,n}$, $i, n \in \mathbb{N}$, $1 \leq i \leq i_n$, for some $i_n \in \mathbb{N}$. A triangular scheme is called \textit{independent}, if for all $n \in \mathbb{N}$ the random variables $Z_{i,n}$, $1 \leq i \leq i_n$, are independent.
Theorem 13 (Gnedenko and Kolmogorov, 1968, § 21, Theorem 2)

Let \( (Z_{i,n})_{n \in \mathbb{N}, 1 \leq i \leq n} \) be an independent triangular scheme with finite expectations subject to the condition that for every \( \epsilon > 0 \)

\[
\sup_{1 \leq i \leq n} P(|Z_{i,n} - EZ_{i,n}| \geq \epsilon) \longrightarrow 0 .
\] (9)

Denote by \( F'_{i,n} \) the distribution function of \( Z_{i,n} - EZ_{i,n} \). Define

\[
Z_n := \sum_{i=1}^{n} Z_{i,n} , \quad K_n(x) := \sum_{i=1}^{n} \int_{-\infty}^{x} u^2 \, dF'_{i,n}(u) \quad (x \in \mathbb{R}) .
\]

For \( n \in \mathbb{N} \) let \( Z_n \) have finite variance. \((Z_n)_{n \in \mathbb{N}}\) converges in distribution and the variances of \( Z_n \) converge to the finite variance of the limit law if and only if there exist a constant \( \gamma \in \mathbb{R} \) and a non-decreasing function \( K \) on \( \mathbb{R} \), which is continuous to the left, with \( K(-\infty) = 0 \), \( K(\infty) < \infty \) such that for all continuity points \( x \) of \( K \)

\[
\lim_{n \to \infty} K_n(x) = K(x) , \quad \lim_{n \to \infty} K_n(\infty) = K(\infty) \quad \text{and}
\]

\[
\lim_{n \to \infty} \sum_{i=1}^{n} EZ_{i,n} = \gamma .
\]

Then the limit distribution of \( Z_n \) is infinitely divisible and its characteristic function can be written as in Eq. (8) with \( K \) and \( \gamma \) corresponding.

To apply the theory of convergence of independent triangular schemes to the context of fixed point equation (1), we first need the following preparatory

**Lemma 14** Assume \( E \sum_{j=1}^{\infty} T_j^2 = 1 \) and \( E \left| \sum_{j=1}^{\infty} T_j^2 \log T_j^2 \right| < \infty \). Then

\[
\sup_{|v| = n} |L(v)| \longrightarrow 0 \quad \text{a.s.}
\]
**Proof.** Assume first $E \sum_{j=1}^{\infty} T_j^2 \log T_j^2 \neq 0$. Define the following branching random walk on the real line: If $x$ is the position of a parent, let $\{x - \log |T_j| : j \in \mathbb{N}, T_j \neq 0\}$ be the positions of the offspring. As usually, let the offspring production process be independent for each parent and independent of former generations and start with one particle in 0. Then the first part of Theorem 2 of Biggins (1977b) can be applied to yield the desired result.

Note that associated branching random walks have frequently been regarded in the literature on similar distributional fixed point equations, e.g., by Durrett and Liggett (1983), Section 4, or Liu (1998), Section 2. Results similar to the lemma have been derived in Lemma 4.1 of Durrett and Liggett (1983) (for bounded $N$) and in Lemma 7.2 of Liu (1998) (for $N < \infty$ a.s.).

Now suppose $E \sum_{j=1}^{\infty} T_j^2 \log T_j^2 = 0$. Then $\sum_{|v|=n} L^2(v) \to 0$ by Lyons (1997).

Since

$$\sup_{|v|=n} L^2(v) \leq \sum_{|v|=n} L^2(v)$$

we obtain the stated result. \qed

Now we can determine the special independent triangular scheme in the set-up of Eq. (1).
Lemma 15 Assume $E \sum_{j=1}^{\infty} T_j^2 = 1$. Let $Z$ be a random variable with $E|Z| < \infty$ and for $n \in \mathbb{N}$ let $(Z(v))_{v \in V, |v| = n}$ be independent copies of $Z$. For $l \in \mathbb{R}^V$, $n \in \mathbb{N}$ choose a finite, non-empty set $V_n(l) \subset \{v \in V : |v| = n\}$. Let $L := (L(v))_{v \in V}$. Then $P^L$-a.s. with respect to $l \in \mathbb{R}^V$ the triangular scheme $(l(v)Z(v))_{n \in \mathbb{N}, v \in V_n(l)}$ is independent and asymptotically negligible (in the sense of condition (9) of Theorem 13).

PROOF. The independence of the scheme is obvious. Because of the preceding lemma, for every $\epsilon > 0$ we achieve $P^L$-a.s. with respect to $l \in \mathbb{R}^V$

\[
\sup_{v \in V_n(l)} P(|l(v)Z(v) - E(l)Z(v)| \geq \epsilon) = \sup_{v \in V_n(l)} P\left(\frac{|Z(v) - EZ|}{|l(v)|} \geq \frac{\epsilon}{|l(v)|}\right) = \sup_{v \in V_n(l)} P\left(\frac{|Z - EZ|}{|l(v)|} \geq \frac{\epsilon}{\sup_{v \in V_n(l)} |l(v)|}\right) \rightarrow 0.
\]

\[\square\]

6 Convergence Results

In this section we derive the convergence of $M^n(Z)$ for $Z$ appropriate. We heavily rely on the statements of the preceding section.

First we treat the case $E \sum_{j=1}^{\infty} T_j^2 = 1$. 

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Lemma 16 Assume \( E\sum_{j=1}^{\infty} T_j^2 = 1 \). Then \( \sum_{|v|=n} L^2(v) \) is a martingale and converges a.s. Let \( Z \) be a random variable with finite variance and zero expectation. Then \( M^n(Z) \) converges weakly to a (possibly trivial) random variable \( W \) with

\[
\phi_W(t) = E\exp\left(-\frac{1}{2}t^2 X\right) \quad (t \in \mathbb{R})
\]

where \( X := \text{Var}Z \lim_{n \to \infty} \sum_{|v|=n} L^2(v) \).

**Proof.** That \( \sum_{|v|=n} L^2(v) \) is a martingale and converges therefore a.s. follows from Lemma 11.

The proof will be broken up in several steps:

(i) Notation

Let \( L := (L(v))_{v \in V} \) and \( Z(v), v \in V \), be copies of \( Z \) with \( L, Z(v), v \in V \), independent. We consider the set

\[
R_0 := \{ l \in \mathbb{R}^V : \sum_{|v|=n} l^2(v) < \infty, \sum_{|v|=n} l^2(v) \text{ converges for } n \to \infty, \sup_{|v|=n} |l(v)| \to 0 \}.
\]

For \( l \in R_0 \) choose a sequence of finite sets \( V_1(l), V_2(l), \ldots \) with \( V_n(l) \subset \{ v \in V : |v| = n \} \) and \( \lim_{n \to \infty} \sum_{v \in V_n(l)} l^2(v) = \lim_{n \to \infty} \sum_{|v|=n} l^2(v) \) and define for \( N \in \mathbb{N} \)

\[
Z_N^n(l) := \sum_{v \in V_n(l)} l(v)Z(v),
\]

\[
W_N^n(l) := \sum_{v \in V_n(l)} l(v)Z(v),
\]

\[
R := \{ l \in R_0 : (l(v)Z(v))_{v \in V_n(l)} \text{ is an independent triangular scheme} \}
\]

satisfying condition (9) of Theorem 13.
(ii) $P^L(R) = 1.$

**Proof.** From the assumption $E\sum_{j=1}^\infty T_j^2 = 1$ we obtain on the one hand $E\sum_{|v|=n} L^2(v) = 1$ and therefore $\sum_{|v|=n} L^2(v) < \infty$ a.s. and on the other hand that $\sum_{|v|=n} L^2(v)$ is a martingale by Lemma 11 and converges therefore a.s. Combined with the Lemmata 14 and 15 this renders the assertion.

(iii) Let $n \in \mathbb{N}$, $l \in R$. Then $(Z^n_N(l))_{N \in \mathbb{N}}$ converges in $L_2$. Denote the limit by $Z^n(l)$.

**Proof.** It can easily be shown that $(Z^n_N(l))_{N \in \mathbb{N}}$ is a Cauchy sequence.

(iv) Let $n \in \mathbb{N}$. Then $Z^n_N(L) \xrightarrow{d} M^n(Z)$ for $N \to \infty$.

**Proof.** Let $n \in \mathbb{N}$. Denote by $E^L$ the expectation with respect to the measure $P^L$ and for $N \in \mathbb{N}, t \in \mathbb{R}$ define $g^n_N(t) : R \to \mathbb{C}; l \to \phi_{Z^n_N(l)}(t)$. Considering the characteristic function of $Z^n_N(L)$ we derive by (ii)

$$\phi_{Z^n_N(L)}(t) = E^L E (\exp(it Z^n_N(L)) | L = l) = E^L g^n_N(t).$$

Taking the limit for $N \to \infty$ and employing (iii) gives the desired result.

(v) Let $l \in R$. Then $\forall \epsilon > 0 \exists n_0 \in \mathbb{N} \forall n > n_0 \exists N_0 = N_0(n) \in \mathbb{N} : N > N_0$ :

$$||Z^n_N(l) - W^n(l)|| < \epsilon.$$
\textbf{PROOF.} The squared $L_2$ distance between $Z_N^n(l)$ and $W^n(l)$ is

$$
||Z_N^n(l) - W^n(l)||_2^2 = E \left( \sum_{v \in V_n} l(v)Z(v) - \sum_{v \in V_n} l(v)Z(v) \right)^2
$$

$$
= E \left( \sum_{v \in V_n} l(v)Z(v) - \sum_{v \in V_n} l(v)Z(v) \right)^2
$$

$$
= E \sum_{v \in V_n} \sum_{v \in V_n} l(v)^2Z^2(v) + E \sum_{v \in V_n} l(v)^2Z^2(v)
$$

$$
+ 2E \sum_{v \in V_n} \sum_{v \in V_n} l(v)^2Z^2(v) + E \sum_{v \in V_n} l(v)^2Z^2(v)
$$

$$
= \text{Var} Z \left( \sum_{v \in V_n} l^2(v) + \sum_{v \in V_n} l^2(v) \right).
$$

Let now $\epsilon > 0$. Choose a $n_0 \in \mathbb{N}$ with $\sum_{v \in V_n} l^2(v) < \frac{\epsilon^2}{2\text{Var} Z}$ for all $n > n_0$. Let $n > n_0$. Choose $N_0 \in \mathbb{N}$ with $\sum_{v \in V_n} l^2(v) < \frac{\epsilon^2}{2\text{Var} Z}$ for all $N > N_0$. Then for all $N > N_0$

$$
||Z_N^n(l) - W^n(l)||_2^2 \leq \text{Var} Z \left( \sum_{v \in V_n} l^2(v) + \sum_{v \in V_n} l^2(v) \right) \leq \epsilon^2.
$$

(vi) Let $l \in R$. Then $||Z^n(l) - W^n(l)||_2 \rightarrow 0$ for $n \rightarrow \infty$.

\textbf{PROOF.} Let $\epsilon_0 > 0$ and $\epsilon := \frac{\epsilon_0}{2}$. Choose $n_0$ as in (v). Let $n > n_0$ and choose $N_0$ as in (v). By (iii) there exists an $N_1 \in \mathbb{N}$ with $||Z^n(l) - Z_{N_1}^n(l)||_2 < \epsilon$ for all $N > N_1$. $N_2$ be the maximum of $N_0$ and $N_1$. Then

$$
||Z^n(l) - W^n(l)||_2 \leq ||Z^n(l) - Z_{N_1}^n(l)||_2 + ||Z_{N_1}^n(l) - W^n(l)||_2 \leq \epsilon_0.
$$
(vii) Let \( l \in R \). Then \( W^n(l) \) converges weakly to a probability measure \( \mu(l) \). If 
\[
\lim_{n \to \infty} \sum_{|v|=n} l^2(v) > 0 \quad \text{then} \quad \mu(l) \quad \text{is a normal distribution with variance} \\
\text{Var}Z \lim_{n \to \infty} \sum_{|v|=n} l^2(v) \quad \text{and zero expectation. If} \quad \lim_{n \to \infty} \sum_{|v|=n} l^2(v) = 0 \\
\text{then} \quad \mu(l) = \delta_0.
\]

**PROOF.** This is the crucial step in the proof of the lemma. By \( l \in R \) we know 
that the triangular scheme \((l(v)Z(v))_{v \in V_n(l) \in \mathbb{N}} \) is independent and that condition 
(9) of Theorem 13 is satisfied. Therefore Theorem 13 can be applied. In the 
notation of the theorem \( Z_{t_{k,n}} \) is now replaced by \( l(v)Z(v) \). For the existence of 
a probability measure \( \mu(l) \) with finite variance and 
\[
W^n(l) \xrightarrow{w} \mu(l) \quad \text{and} \quad \text{Var}W^n(l) \xrightarrow{} \text{Var}\mu(l)
\]
it is thus necessary and sufficient that 
\[
K_n(l)(u) := \sum_{v \in V_n(l)} \int l^2(v)Z^2(v)1_{l(v)Z(v) < u} dP
\]
converges to an in \( u \) non-decreasing function \( K(l)(u) \) on some dense set of \( \mathbb{R} \) 
with \( K(l)(-\infty) = 0, K(l)(\infty) < \infty \),
\[
\lim_{n \to \infty} K_n(l)(\infty) = K(l)(\infty) \quad \text{and} \quad \lim_{n \to \infty} \sum_{v \in V_n(l)} E l(v)Z(v) = \gamma(l).
\]

We obtain for \( u > 0 \)
\[
K_n(l)(u) = \sum_{\substack{v \in V_n(l) \in \mathbb{N} \\ l(v) > 0 \ \ l(v) \in \mathbb{N}}} l^2(v) \int_{-\infty}^{u/l(v)} x^2 dF_Z(x) + \sum_{\substack{v \in V_n(l) \in \mathbb{N} \\ l(v) < 0 \ \ l(v) \in \mathbb{N}}} l^2(v) \int_{(u/l(v), \infty)} x^2 dF_Z(x)
\]
\[
\leq \text{Var}Z \sum_{v \in V_n(l)} l^2(v) \xrightarrow{n \to \infty} \text{Var}Z \lim_{n \to \infty} \sum_{v \in V_n(l)} l^2(v) = \text{Var}Z \lim_{n \to \infty} \sum_{|v|=\infty} l^2(v)
\]

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and,
\[
K_n(l)(u) \geq \sum_{v \in V_n(l), l(v) > 0} l^2(v) \int_{-\infty}^{u/s} x^2 dF_Z(x) + \sum_{v \in V_n(l), l(v) < 0} l^2(v) \int_{u/\sup l(v) \leq l(v)} x^2 dF_Z(x)
\]
\[
\xrightarrow{n \to \infty} \text{Var}Z \lim_{n \to \infty} \sum_{|v|=n} l^2(v) .
\]

Hence
\[
K(l)(u) := \lim_{n \to \infty} K_n(l)(u) = \text{Var}Z \lim_{n \to \infty} \sum_{|v|=n} l^2(v) = K(l)(\infty) .
\]

It follows with the same arguments that
\[
K(l)(u) := \lim_{n \to \infty} K_n(l)(u) = 0 \quad (u < 0) \quad \text{and}
\]
\[
K(l)(\infty) = \lim_{n \to \infty} K_n(l)(\infty) , K(l)(-\infty) = 0 .
\]

Furthermore since E\text{Z} = 0
\[
\lim_{n \to \infty} \sum_{v \in V_n(l)} E l(v) Z(v) = 0 = \gamma(l) . \tag{10}
\]

This proves that $W^n(l)$ converges in distribution to a normal distribution with mean 0 and variance $\lim_{n \to \infty} \sum_{|v|=n} l^2(v)$ in the case that $\lim_{n \to \infty} \sum_{|v|=n} l^2(v) \neq 0$ and to 0 if $\lim_{n \to \infty} \sum_{|v|=n} l^2(v) = 0$.

(viii) Let $l \in R$. Then $Z^n(l)$ converges weakly to $\mu(l)$.

\textbf{PROOF.} This is a direct consequence of (vi) and (vii).
(ix) Let $S$ be a standard normal distributed random variable with $S, L$ independent. Define $Y : \Omega \times \mathbb{R} \rightarrow \mathbb{R}, (\omega, l) \mapsto Y(l)(\omega) := S(\omega) \left( \lim_{n \to \infty} \sum_{|v|=n} l^2(v) \text{Var}Z \right)^{1/2}$.

Then $M^n(Z)$ converges weakly to $W := Y(L)$.

**Proof.** Let $n \in \mathbb{N}$. Investigating the characteristic function $\phi_{M^n(Z)}$ and using (ii), (iii) and (iv) leads to ($t \in \mathbb{R}$)

$$\phi_{M^n(Z)}(t) = \lim_{N \to \infty} \phi_{Z^N,l}(t) = \lim_{N \to \infty} \mathbb{E} \exp \left( it \sum_{v \leq N,l \in \{1, \ldots, n\}} L(v) Z(v) \right) = \lim_{N \to \infty} \int \mathbb{E} \left( \exp \left( it \sum_{v \leq N,l \in \{1, \ldots, n\}} l(v) Z(v) \right) | L = l \right) P^L(\,dl)$$

$$= \lim_{N \to \infty} \int \mathbb{E} \left( \exp \left( it \sum_{v \leq N,l \in \{1, \ldots, n\}} l(v) Z(v) \right) \right) P^L(\,dl)$$

$$= \lim_{N \to \infty} \int \phi_{Z^N,l}(t) P^L(\,dl) = \int \phi_{Z^N,l}(t) P^L(\,dl).$$

Note that by (vii) $Y(l)$ has distribution $\mu(l)$. Passing to the limit and employing (ii) and (viii) gives

$$\lim_{n \to \infty} \phi_{M^n(Z)}(t) = \int \phi_{Y(l)}(t) P^L(\,dl) = \phi_W(t).$$

Thus $M^n(Z)$ converges weakly to $W$.

(x) $W$ has the characteristic function

$$\phi_W(t) = \mathbb{E} \exp \left( -\frac{1}{2} t^2 X \right) \quad (t \in \mathbb{R})$$

where $X := \text{Var}Z \lim_{n \to \infty} \sum_{|v|=n} L^2(v)$. 

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PROOF. By employing (ix) the characteristic function of \(W\) can be calculated as

\[
\phi_W(t) = \mathbb{E}\exp(itW) = \int \mathbb{E}\left(\exp(itW) | L = l\right) P^L(dl) = \int \mathbb{E}(\exp(itY(l))) P^L(dl)
\]

\[
= \int \exp \left(-\frac{1}{2} t^2 \text{Var}Z \lim_{n \to \infty} \sum_{|v| = n} l^2(v)\right) P^L(dl)
\]

\[
= \mathbb{E}\exp \left(-\frac{1}{2} t^2 \text{Var}Z \lim_{n \to \infty} \sum_{|v| = n} L^2(v)\right).
\]

This completes the proof of the lemma. \(\Box\)

Now we achieve a similar convergence result for the situation \(\mathbb{E} \sum_{j=1}^{\infty} T_j^2 < 1\) and \(\mathbb{E} \sum_{j=1}^{\infty} T_j = 1\).

**Lemma 17** Assume \(\mathbb{E} \sum_{j=1}^{\infty} T_j^2 < 1\) and \(\mathbb{E} \sum_{j=1}^{\infty} T_j = 1\). Then \(\sum_{|v| = n} L(v)\) is a martingale and converges a.s. and in \(L_2\) to a random variable \(L^\infty\) with \(\mathbb{E}L^\infty = 1\). Let \(Z\) be a random variable with finite variance and non-zero expectation. Then \(M^n(Z)\) converges weakly to a non-trivial fixed point \(W \in \mathcal{F}_1\) with

\[
W \doteq L^\infty EZ.
\]

**PROOF.**

In this case the contraction method can be applied. By the remark to Theorem 4 of Roesler (1992) we derive the convergence in distribution of \(M^n(Z)\) to a solution \(W\) of (1) with finite variance and expectation \(EZ\). Furthermore \(W\) is the unique fixed point of (1) with finite variance and expectation \(EZ\). That \(\sum_{|v| = n} L(v)\) is a martingale and converges a.s. and in \(L_2\) to a random variable \(L^\infty\) with \(\mathbb{E}L^\infty = 1\) follows from Lemma 12 (ii). The same contraction argument
as above yields that $M^n(1) \cong \sum_{|v|=n} L(v)$ converges in distribution to a fixed point $W_0 \cong L^\infty$ of (1) which is unique in the class of solutions with finite variance and expectation 1. Thus $EZL^\infty$ is the unique fixed point of (1) with finite variance and expectation $EZ$. □

7 Proofs of Main Results

In this section the statements of Section 3 are proved.

Proof of Theorem 3. (i): Assume first $\mathcal{F}_1 \neq \emptyset$. Since there exists a solution of Eq. (1) with finite variance and non-zero expectation, the desired statements follow from Lemma 10.

Now let $E\sum_{j=1}^\infty T_j < 1$ and $E\sum_{j=1}^\infty T_j = 1$. Due to Lemma 17 in this situation we already know that $L^\infty$ is a non-trivial solution of Eq. (1).

(ii): Equivalence $\mathcal{F}_2 \neq \emptyset \iff \mathcal{P} \neq \emptyset$:

Assume $\mathcal{F}_2 \neq \emptyset$. Let $W \in \mathcal{F}_2$. From Lemma 10 we derive $E\sum_{j=1}^\infty T_j = 1$. Since $W$ has finite variance and zero expectation, we can apply Lemma 16: Because $M^n(W) \cong W$ for all $n \in \mathbb{N}$ we obtain

$$\phi_W(t) = E \exp \left( -\frac{1}{2} t^2 X \right)$$

where $X = \text{Var}W \lim_{n \to \infty} \sum_{|v|=n} L^2(v)$. That $X$ is a solution of the fixed point equation (2) with $\text{Var}W = EX$ can be shown by substituting $\phi_W(t)$ in Eq. (4) and using the uniqueness of the Laplace transform. Since $W$ is non-trivial, $X$ is non-trivial, either. Therefore $X \in \mathcal{P}$. 

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On the other hand let \( X \in \mathcal{P} \). If \( Y \) is a standard normal random variable with \( X \) and \( Y \) independent it can easily be verified by applying Eq. (4) that \( W := Y \sqrt{X} \) is a solution of the fixed point equation (1) with characteristic function 
\[
\phi_W(t) = E \exp(-\frac{1}{2}t^2 X).
\]
(Compare similar arguments in Durrett and Liggett, 1983 (proof of Theorem 3.1) and in Guivarc’h, 1990 (Section II, Prop.1)).

Equivalence \( \mathcal{P} \neq \emptyset \iff 
E \sum_{j=1}^{\infty} T_j^2 = 1, E \sum_{j=1}^{\infty} T_j^2 \log T_j^2 < 0 \) and 
\[
E \left( \left( \sum_{j=1}^{\infty} T_j^2 \right) \log^+ \left( \sum_{j=1}^{\infty} T_j^2 \right) \right) < \infty:
\]

Let \( E \sum_{j=1}^{\infty} T_j^2 = 1, E \sum_{j=1}^{\infty} T_j^2 \log T_j^2 < 0 \) and 
\[
E \left( \left( \sum_{j=1}^{\infty} T_j^2 \right) \log^+ \left( \sum_{j=1}^{\infty} T_j^2 \right) \right) < \infty.
\]
Then it follows from Lyons (1997) that \( \mathcal{P} \neq \emptyset \).

Suppose now that \( \mathcal{P} \neq \emptyset \) and \( X \in \mathcal{P} \). As in the proof of the first equivalence we derive that \( W := Y \sqrt{X} \) with \( Y \) as above is a solution of (1) with
\[
\phi_W(t) = E \exp \left( -\frac{1}{2}t^2 X \right) \quad (t \in \mathbb{R}).
\]

Furthermore as above we can employ Lemma 16 to obtain the result 
\( X \overset{d}{=} \text{Var} W \lim_{n \to \infty} \sum_{|v|=n} L^2(v) \). Now an application of Lyons (1997) yields
\[
E \sum_{j=1}^{\infty} T_j^2 = 1, E \sum_{j=1}^{\infty} T_j^2 \log T_j^2 < 0 \text{ and } E \left( \left( \sum_{j=1}^{\infty} T_j^2 \right) \log^+ \left( \sum_{j=1}^{\infty} T_j^2 \right) \right) < \infty.
\]

(iii): Let \( W \) be a solution of Eq. (1) with finite variance. Then \( M^n(W) \overset{d}{=} W \). Applying the Lemmata 16 and 17 for the cases of zero resp. non-zero expectation results in the uniqueness of \( W \) up to multiplicative constants. \( \square \)

**Proof of Theorem 4.** (i): That \( \sum_{|v|=n} L(v) \) is a martingale and converges a.s. and in \( L_2 \) to a random variable \( L^\infty \) with \( EL^\infty = 1 \) is derived by the Lemmata 11 and 12 (ii) in combination with Theorem 3. Since for any \( W \in \mathcal{F}_1 \) we have 
\( M^n(W) \overset{d}{=} W \), Lemma 17 shows that \( W \overset{d}{=} L^\infty EW \). The expression for the
variance is given in Lemma 10 (ii).

(ii): Let $W \in \mathcal{F}_2$. That $\sum_{|v|=n} L^2(v)$ is an $L_1$-convergent martingale follows from Theorem 3 (ii) and Lyons (1997). From Lemma 10 (ii) we obtain $\mathbb{E}\sum_{j=1}^{\infty} T_j^2 = 1$. We once again use $M^n(W) \overset{a.s.}{=} W$ such that Lemma 16 can be applied to give the desired representation. For the uniqueness of $X$ observe that if $X_1, X_2 \in \mathcal{P}$, then define $W_1 := Y_1 \sqrt{X_1}, W_2 := Y_2 \sqrt{X_2}$ with $Y_1, Y_2$ standard normal random variables and $X_1, Y_1$ independent and $X_2, Y_2$ independent. $W_1$ and $W_2$ are solutions of Eq. (1) with characteristic functions $\phi_{W_1}(t) = \mathbb{E}\exp(-\frac{1}{2}t^2 X_1)$ and $\phi_{W_2}(t) = \mathbb{E}\exp(-\frac{1}{2}t^2 X_2)$ (see proof of Theorem 3). Thus by uniqueness (Theorem 3 (iii)) there exists a $c \in \mathbb{R}$ with $W_1 \overset{d}{=} c W_2$.

Uniqueness of the Laplace transform renders $X_1 \overset{d}{=} c^2 X_2$. □

Proof of Theorem 6. (i): Follows from Theorem 3 and Lemma 17.

(ii): Let $\mathcal{F}_2 \neq \emptyset$ and let $Z$ be a non-trivial random variable with finite variance and zero expectation. Due to the Lemmata 10 (ii) and 16, $M^n(Z)$ converges weakly to a random variable $W$ with

$$\phi_W(t) = \mathbb{E}\exp\left(-\frac{1}{2}t^2 X\right)$$

where $X := \text{Var}Z \lim_{n \to \infty} \sum_{|v|=n} L^2(v)$. Because of Theorem 4 (ii), $W$ is the unique solution of the fixed point equation (1) with $\text{Var}W = E X$. □

Proof of Theorem 7. Let $W \in \mathcal{F}_2$ with representation as in Eq. (5) of Theorem 4. We conclude

$$\phi_W(t) = P(X = 0) + \mathbb{E}\exp\left(-\frac{1}{2}t^2 X\right) 1_{X > 0} = P(X = 0) + \int \mathbb{E}(\varphi_{0,X}(u)) 1_{X > 0} \exp(iut) \, du .$$

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The right hand side is the characteristic function of a distribution with point mass \( P(X = 0) \) at 0 and Lebesgue density \( f \) on \( \mathbb{R} \setminus \{0\} \) as stated in the theorem. We now prove that \( P(X = 0) \) is the unique fixed point of \( g \) following the line of proof in Lemma 3.1 of Liu (1998). First we observe that by using Eq. (2)

\[
q := P(X = 0) = P \left( \sum_{j=1}^{\infty} T_j X_j = 0 \right) \\
= P(\forall j \in \mathbb{N} \text{ with } T_j \neq 0 : \quad X_j = 0) = \sum_{j=1}^{\infty} q^k P(N = k) = g(q).
\]

Thus \( q \in [0, 1] \) is a fixed point of \( g \). Because \( g(1) = P(N < \infty) \leq 1 \) and the convexity of \( g, q \) is either a unique fixed point of \( g \) in \([0, 1]\) or \( g = \text{id} \) on \([0, 1]\). The latter, however, implies \( N = 1 \) a.s., which is excluded by (A0). (For a similar result see Theorem 3.2 of Durrett and Liggett, 1983.) Applying the uniqueness of the characteristic function completes the proof of the first part.

If \( \varphi_{m,\sigma^2} \) denotes the density of the normal distribution with expectation \( m \) and variance \( \sigma^2 \) and \( \varphi^{(n)}_{m,\sigma^2} \) its \( n \)-th derivative, it can be shown by induction that for every \( n \in \mathbb{N} \) the following result holds:

\[
\varphi^{(n)}_{0,\sigma^2}(x) = p_n(x, \sigma^2) \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{11}
\]

where for each \( \sigma^2 > 0 \) the term \( p_n(\cdot, \sigma^2) \) is a polynomial of degree \( n \). Moreover,

\[
p_n(x, \sigma^2) = \sigma^{-n} \sum_{k=0}^{n/2} a_k x^{2k} \sigma^{-2k}, \quad n \text{ even },
\]

\[
p_n(x, \sigma^2) = x \sigma^{-(n+1)/2} \sum_{k=0}^{(n-1)/2} b_k x^{2k} \sigma^{-2k}, \quad n \text{ odd },
\]

where the coefficients \( a_k \) and \( b_k \) are real numbers and do not depend on \( \sigma \).
Now fix \( n \in \mathbb{N} \). By Eq. (11) it follows that for each \( \sigma^2 > 0 \) the function \( |\varphi_{0,\sigma^2}^{(n)}| \) possesses a global maximum \( x_m \in \mathbb{R} \) with \( \varphi_{0,\sigma^2}^{(n+1)}(x_m) = 0 \). Thus \( x_m \) is a root of the polynomial \( p_{n+1}(x,\sigma^2) \) and therefore can be written as \( x_m = y_m \sigma \), where \( y_m \) is a function of the coefficients \( a_k \) or \( b_k \), but does not depend on \( \sigma \). We obtain for each \( x \in \mathbb{R} \) in the case that \( n \) is even

\[
|\varphi_{0,\sigma^2}^{(n)}(x)| \leq |p_n(x_m,\sigma^2)\exp\left(-\frac{1}{2\sigma^2}x_m^2\right)| \leq |p_n(x_m,\sigma^2)|
\]

\[
= |\sigma^{-n} \sum_{k=0}^{n/2} a_k x_m^{2k} \sigma^{-2k}| = |\sigma^{-n} \sum_{k=0}^{n/2} a_k b_m^{2k}| = k_1 \sigma^{-n},
\]

where \( k_1 \) is a real constant independent of \( \sigma \). Similarly for \( n \) odd

\[
|\varphi_{0,\sigma^2}^{(n)}(x)| \leq k_2 \sigma^{-n},
\]

where \( k_2 \) is a real constant independent of \( \sigma \).

With \( X \) as in Eq. (5) of Theorem 4 we derived for all \( x \in \mathbb{R} \) that

\[
|\varphi_{0,X}^{(n)}(x)1_{X>0}| \leq k_3 X^{-n/2}1_{X>0},
\]

where \( k_3 := \max(k_1, k_2) \) is a real constant (no random variable). If now, with \( \tau \) defined as in the theorem, \( N < \infty \) a.s., \( P(N = 0) = 0 \), \( ET^{-a}_\tau < \infty \) and \( ET^{-a}_\tau 1_{N=1} < 1 \) for all \( a > 0 \), it can be concluded by Theorem 2.4 of Liu (2001) that \( EX^{-a}1_{X>0} < \infty \) for all \( a > 0 \). (Note that the assumption \( P(\forall j \in \mathbb{N} : T_j^2 \in \{0,1\}) < 1 \) of Liu, 2001, is satisfied because otherwise \( \mathcal{P} = \emptyset \) due to (A1) and Lemma 1.1 b) of Liu, 1998, in contradiction to \( \mathcal{F}_2 \neq \emptyset \) and Theorem 3.) Therefore integration and differentiation can be exchanged to give the desired statement. \( \square \)
Proof of Theorem 9. Let $2 < \beta < \infty$, $N < \infty$ a.s. and $W \in \mathcal{F}_2$ with the density of Theorem 7. Then

$$E|W|^\beta = \int |t|^\beta E \varphi_{\varrho, X}(t) 1_{X>0} \, dt = 2 \frac{1}{\sqrt{2\pi}} \left( \frac{1}{2} \right)^{(\beta+3)/2} \Gamma \left( \frac{\beta+1}{2} \right) E X^{\beta/2}.$$ 

We know that $X$ is a solution of the fixed point equation (2). If $\beta \in \mathbb{N}$ or $N$ is bounded we derive by Theorem 5.1 of Liu (1997) that $E X^{\beta/2} < \infty$ for $1 < \beta/2 < \infty$ if and only if $E \sum_{j=1}^{\infty} (T_j^2)^{\beta/2} < 1$ and $E \left( \sum_{j=1}^{\infty} T_j^2 \right)^{\beta/2} < \infty$. □

Appendix. Proof of Lemma 2 and the Case $P(\sum_{j=1}^{\infty} T_j = 1) = 1$

Now we give the remaining proof of Lemma 2.

Proof of Lemma 2. Assume $P(N \leq 1) = 1$. Denote $S := \sum_{j=1}^{\infty} T_j$. Then the fixed point equation (1) can be written as

$$W \overset{\sim}{=} SW, \quad (12)$$

where $W, \bar{W}$ have the same distribution and $\bar{W}, S$ are independent. Let $W$ be a non-trivial solution of Eq. (12). Then $|W| \overset{\sim}{=} |S||\bar{W}|$. Thus $|W|$ is a non-negative solution of the fixed point equation

$$X \overset{\sim}{=} |S|\bar{X}, \quad (13)$$

where $\bar{X}$ is a copy of $X$ and $\bar{X}, |S|$ are independent. Hence we are in the case of non-negative coefficients and solutions. For the fixed point equation (13) we have $\bar{N} := 1_{|S|>0} \leq 1$. By Lemma 1.1 a) of Liu (1998) it follows that $|S| = 1$ a.s.
Suppose now on the other hand that \( |S| = 1 \) a.s. Then for every random variable \( W \) for every \( t \in \mathbb{R} \)

\[
\mathbb{E}\phi_W(St) = \phi_W(t)P(S = 1) + \phi_W(-t)P(S = -1).
\]

Therefore \( W \) solves Eq. (4) if and only if for every \( t \in \mathbb{R} \)

\[
\phi_W(t) = \phi_W(t)P(S = 1) + \phi_W(-t)(1 - P(S = 1)),
\]

which is equivalent to

\[
\phi_W(t)(1 - P(S = 1)) = \phi_W(-t)(1 - P(S = 1)). \quad \square
\]

Up to now only the case \( P(\sum_{j=1}^{\infty} T_j = 1) < 1 \) has been considered. Results for the case \( P(\sum_{j=1}^{\infty} T_j = 1) = 1 \) are given in the next theorem.

**Theorem 18 (Case \( \sum_{j=1}^{\infty} T_j \equiv 1 \) a.s.)**

Assume \((A0), (A2a)\) and \( P(\sum_{j=1}^{\infty} T_j = 1) = 1.\)

Then the constants of \( \mathbb{R} \) are solutions of Eq. (1). Eq. (1) has non-constant solutions with finite variance if and only if

\[
\mathbb{E}\sum_{j=1}^{\infty} T_j^2 = 1, \mathbb{E}\sum_{j=1}^{\infty} T_j^2 \log T_j^2 < 0 \text{ and } \mathbb{E}\left(\left(\sum_{j=1}^{\infty} T_j^2\right)^\log^+\left(\sum_{j=1}^{\infty} T_j^2\right)\right) < \infty.
\]

This is equivalent to \( \mathcal{P} \neq \emptyset. \)

If \( W \) is a non-constant solution with finite variance then

\[
\phi_W(t) = e^{\gamma t} \mathbb{E}\exp\left(-\frac{1}{2}t^2 X\right), \quad (14)
\]

where \( X := \text{Var}W \lim_{n \to \infty} \sum_{|k|=n} L^2(v) \) is the unique solution of the fixed point equation (2) and \( \text{Var}W = \mathbb{E}X \) and \( \gamma = \mathbb{E}W, \) and the set of solutions of Eq. (1) is \( \{\lambda W + \kappa : \lambda, \kappa \in \mathbb{R}\}. \)
PROOF. Only a sketch of the proof is given since it can be done along the same lines as the proofs of Theorems 3 and 4 with minor modifications.

That the constants of $\mathbb{R}$ are solutions of Eq. (1) can be derived immediately. In the case $\mathbb{E} \sum_{j=1}^{\infty} T_j^2 < 1$ the contraction method via the remark to Theorem 4 of Roesler (1992) yields that the only solutions of Eq. (1) are the constants of $\mathbb{R}$. (This follows also directly from the proof of Lemma 11 (ii) adapted for the case $P(\sum_{j=1}^{\infty} T_j = 1) = 1$.) Thus it is sufficient to consider the case $\mathbb{E} \sum_{j=1}^{\infty} T_j^2 = 1$.

Suppose $W$ to be a solution of the fixed point equation (1). Due to the special situation $P(\sum_{j=1}^{\infty} T_j = 1) = 1$ the centered random variable $W' := W - \mathbb{E}W$ is also a solution of (1). Thus we can restrict ourselves to solutions with expectation zero.

In the case $P(\sum_{j=1}^{\infty} T_j = 1) < 1$ the main tool for solutions with zero expectation in the situation $\mathbb{E} \sum_{j=1}^{\infty} T_j^2 = 1$ was Lemma 16. For the case $P(\sum_{j=1}^{\infty} T_j = 1) = 1$ this lemma must be slightly changed to

Lemma 19 Assume $P(\sum_{j=1}^{\infty} T_j = 1) = 1$ and $\mathbb{E} \sum_{j=1}^{\infty} T_j^2 = 1$. Then $\sum_{|v|=n} L^2(v)$ is a martingale and converges a.s.. Let $Z$ be a random variable with finite variance and zero expectation. Then $M^n(Z)$ converges weakly to a (possibly trivial) random variable $W$ with

$$\phi_W(t) = \mathbb{E}\exp\left(-\frac{1}{2}t^2X\right) \quad (t \in \mathbb{R})$$

where $X := \text{Var}Z \lim_{n \to \infty} \sum_{|v|=n} L^2(v)$. 

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The lemma is proved in exactly the same way as Lemma 16. Then the proofs of the Theorems 3 and 4 carry over. The only difference is that we get the factor $e^{i\eta t}$ in the representation of the characteristic function because of the centering mentioned above. $\square$
References


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