SUPPORT VARIETIES, p-POINTS AND JORDAN TYPE

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1. Lecture I: Survey

The purpose of these lectures is to provide an introduction to support varieties and their representation-theoretic interpretations. The origins of this work date back to Quillen's 1971 articles in which he analyzed the maximal ideal spectrum $\mathcal{V}_G(k)$ of the even cohomology ring

$$\mathrm{H}^\bullet(G,k) := \bigoplus_{n\geq 0} \mathrm{H}^{2n}(G,k)$$

of a finite group G. Here the base field k is assumed to have characteristic $p > 0$. If $G' \subseteq G$ is a subgroup of G, then the canonical restriction map $\mathrm{res} : \mathrm{H}^\bullet(G,k) \longrightarrow \mathrm{H}^\bullet(G',k)$ is finite and thus induces a closed map $\mathrm{res}^*_{G'} : \mathcal{V}_{G'}(k) \longrightarrow \mathcal{V}_G(k)$. Quillen showed that

$$\mathcal{V}_G(k) = \bigcup_{E\subseteq G} \mathrm{res}^*_E(\mathcal{V}_E(k)),$$

where E runs through the set of p-elementary abelian subgroups of G. This readily implies

$$\dim \mathcal{V}_G(k) = \max_{E\subseteq G} \dim \mathcal{V}_E(k).$$

Recall that an abelian group E is called *p-elementary abelian* if $px = 0$ for every $x \in E$. Consequently, such a group is an $\mathbb{F}_p$-vector space $(\cong (\mathbb{Z}/(p))^r$, whose dimension r is called the *rank of E*. The cohomology ring of $E = (\mathbb{Z}/(p))^r$ is well understood; in particular, $\mathrm{H}^\bullet(E,k)_{\mathrm{red}} = \mathrm{H}^\bullet(E,k)/\operatorname{Rad}(\mathrm{H}^\bullet(E,k))$ is isomorphic to the polynomial ring $k[X_1,\dots,X_r]$, where $\deg(X_i) = 2$. As a result, $\mathcal{V}_E(k) \cong \mathbb{A}^r$, and $\dim \mathcal{V}_E(k) = r$, so that $\dim \mathcal{V}_G(k)$ coincides with the maximal rank of all p-elementary abelian subgroups of G, the so-called *p-rank $r_p(G)$ of G*.

Theorem 1.1 (Quillen's Dimension Theorem,1971)**.**

$$\dim \mathcal{V}_G(k) = r_p(G).$$

Elementary abelian subgroup also arise in a different context, namely in connection with the detection of projectivity. *Henceforth all modules are assumed to be finite-dimensional.*

If M is a projective G-module and $G' \subseteq G$ is a subgroup, then the restriction $M|_{G'}$ is also projective. The question is, on which class of subgroups can projectivity effectively be detected. It is fairly easy to see that a module is projective, if its restriction $M|_P$ to a Sylow-p-subgroup enjoys the same property. The following refinement of this result is much harder to prove:

Theorem 1.2 (Chouinard,1976)**.** *A G-module M is projective if and only if its restriction $M|_E$ to every p-elementary abelian subgroup $E \subseteq G$ is projective.*

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Let $E \cong \mathbb{Z}/(p)^r$ be p-elementary abelian with generators $g_1, \dots, g_r$. Setting $x_i := g_i - 1$, the map $X_i \longrightarrow x_i$ induces an isomorphism

$$k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p) \xrightarrow{\sim} kE.$$

For $\lambda = (\lambda_1, \dots, \lambda_r) \in k^r$, we put $u_\lambda := \sum_{i=1}^r \lambda_i x_i$. The following result provides a criterion for the projectivity of an E-module:

Lemma 1.3 (Dade's Lemma,1978)**.** *Suppose that k is algebraically closed. The E-module M is projective if and only if $M|_{k[u_\lambda]}$ is projective for every $\lambda \in k^r$.*

Two important developments ensued in quick succession:

- In 1981 Alperin and Evens employed the notion of the *complexity* $\mathrm{cx}_G(M)$ of a G-module M to present a common generalization of (1.1) and (1.2). The verified the identity
$$\mathrm{cx}_G(M) = \max_{E \subseteq G} \mathrm{cx}_E(M).$$
- In his seminal 1981 paper, Jon Carlson introduced cohomological support varieties and rank varieties for p-elementary abelian groups. He found a new proof of Dade's Lemma and conjectured that rank varieties and support varieties coincide. Shortly thereafter, Avrunin and Scott verified Carlson's conjecture.

Group algebras of finite groups and their representations have always served as a paradigm for other classes of algebras, such as restricted enveloping algebras of restricted Lie algebras. It is clear that support varieties can also be defined in that context. In fact, if H is a finite-dimensional Hopf algebra and M is an H-module, one always has a map

$$\Phi_M : \mathrm{H}^\bullet(H, k) \longrightarrow \mathrm{Ext}^*_H(M, M) \ \ ; \ \ [f] \mapsto [f \otimes \mathrm{id}_M]$$

from the even cohomology ring $\mathrm{H}^\bullet(H, k) := \bigoplus_{n \geq 0} \mathrm{H}^{2n}(H, k)$ to the Yoneda algebra $\mathrm{Ext}^*_H(M, M)$.

Let me briefly recall the definition of this map: If $(P_n)_{n \geq 0}$ is a projective resolution of the trivial H-module k, then $(P_n \otimes_k M)_{n \geq 0}$ is a projective resolution of M, and a cocycle $f : P_n \longrightarrow k$ defines a cocycle $f \otimes \mathrm{id}_M : P_n \otimes_k M \longrightarrow M$. Hence the usual definition

$$\mathcal{V}_H(M) := Z(\ker \Phi_M)$$

of the *support variety* as the zero locus (i.e., the maximal ideals of $\mathrm{H}^\bullet(H, k)$ containing $\ker \Phi_M$) can be given in general, without referring to groups or Lie algebras.

There are two problems with such a general approach:

- If the commutative algebra $\mathrm{H}^\bullet(H, k)$ is not finitely generated, then $\mathcal{V}_H(M)$ is not a variety, and the powerful results from algebraic geometry cannot be brought to bear.
- If $\mathrm{Ext}^*_H(M, M)$ is not a finitely generated $\mathrm{H}^\bullet(H, k)$-module, then $V_H(M)$ does not coincide with the usual notion of a support. Moreover, $\mathcal{V}_H(M)$ doesn't tell us much about M. In particular, one does not have an interpretation of $\dim \mathcal{V}_H(M)$.

Accordingly, one always needs a fundamental result of the following nature:

Theorem 1.4. (1) *$\mathrm{H}^\bullet(H, k)$ is a finitely generated k-algebra.*

(2) *For every H-module M, the $\mathrm{H}^\bullet(H, k)$-module $\mathrm{H}^*(H, M)$ is finitely generated.*

Let me delineate the historical development:

- If $H = kG$ is the group algebra of a finite group, then (1.4(1)) was verified by Evens and Venkov in 1961, with Evens also proving (2).

- The case of restricted enveloping algebras $H = U_0(\mathfrak{g})$ was settled by Friedlander and Parshall in 1986.
- Eleven years later, Friedlander and Suslin showed the validity of (1.4) for finite group schemes, that is, for cocommutative Hopf algebras.

A cocommutative Hopf algebra is $H = k\mathcal{G}$ is the *algebra of measures* on a finite group scheme $\mathcal{G}$. We will speak of $\mathcal{G}$-modules and denote the cohomology groups by $\mathrm{H}^n(\mathcal{G}, M)$. Henceforth, $\mathrm{mod}\,\mathcal{G}$ denotes the category of finite-dimensional $\mathcal{G}$-modules. As before, we consider

$$\Phi_M : \mathrm{H}^\bullet(\mathcal{G}, k) \longrightarrow \mathrm{Ext}^*_{\mathcal{G}}(M, M) \ \ ; \ \ [f] \mapsto [f \otimes \mathrm{id}_M].$$

Owing to (1.4), the zero locus

$$\mathcal{V}_{\mathcal{G}}(M) := Z(\ker \Phi_M)$$

is a conical affine variety. By definition, $\mathcal{V}_{\mathcal{G}}(M) \subseteq \mathcal{V}_{\mathcal{G}}(k)$ for every $\mathcal{G}$-module M.

Let me begin with a general remark concerning the rôle of the tensor product. A priori, the cohomology ring $\mathrm{H}^\bullet(\mathcal{G}, k)$ only provides information on the principal block $\mathcal{B}_0(\mathcal{G})$ of the Hopf algebra $k\mathcal{G}$. In fact, the canonical restriction map induces an isomorphism

$$\mathrm{H}^\bullet(\mathcal{B}_0(\mathcal{G}), k) \cong \mathrm{H}^\bullet(\mathcal{G}, k).$$

The tensor product takes us from $\mathcal{B}_0(\mathcal{G})$ to any other block, and the above definition shows that factor algebras of $\mathrm{H}^\bullet(\mathcal{G}, k)$ actually yield information on other blocks.

Support varieties can be interpreted in the module category $\mathrm{mod}\,\mathcal{G}$ of the Frobenius algebra $k\mathcal{G}$:

- $\dim \mathcal{V}_{\mathcal{G}}(M) = \mathrm{cx}_{\mathcal{G}}(M)$ is the *complexity* of the $\mathcal{G}$-module M. In particular, M is projective if and only if $\mathcal{V}_{\mathcal{G}}(M) = \{0\}$.
- If $(0) \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow (0)$ is exact, then $\mathcal{V}_{\mathcal{G}}(M_2) \subseteq \mathcal{V}_{\mathcal{G}}(M_1) \cup \mathcal{V}_{\mathcal{G}}(M_3)$, with equality holding in case the sequence splits.
- Let $\Gamma_s(\mathcal{G})$ be the *stable Auslander-Reiten quiver* of $\mathcal{G}$. If $[M], [N] \in \Gamma_s(\mathcal{G})$ belong to the same component $\Theta \subseteq \Gamma_s(\mathcal{G})$, then $\mathcal{V}_{\mathcal{G}}(M) = \mathcal{V}_{\mathcal{G}}(N)$, so that one can speak of the variety $\mathcal{V}_{\mathcal{G}}(\Theta)$ of Θ.

One of the main problems resides in the computational intractability of support varieties. We are thus interested in finding varieties with a similar behaviour that avoid reference to the cohomology ring. These so-called *representation-theoretic support spaces* can be easily explained for restricted enveloping algebras. As before, we are working over a field of characteristic $p > 0$.

Recall that a restricted Lie algebra $(\mathfrak{g}, [p])$ is a pair, consisting of a finite-dimensional Lie algebra $\mathfrak{g}$ and an operator $x \mapsto x^{[p]}$ of $\mathfrak{g}$ that has the formal properties of an associative p-th power map. In fact, we may consider the special linear Lie algebra $\mathfrak{sl}(n)$ together with the ordinary p-th power of matrices as an example. Nevertheless, we shall denote this map by $[p]$ in order to distinguish the p-th power $x^{[p]} \in \mathfrak{g}$ from the p-th power x^p of x in the universal enveloping algebra $U(\mathfrak{g})$. Jacobson defined the *restricted enveloping algebra*

$$U_0(\mathfrak{g}) = U(\mathfrak{g})/(\{x^p - x^{[p]} \ ; x \in \mathfrak{g}\}).$$

Contrary to the ordinary enveloping algebra $U(\mathfrak{g})$, the restricted enveloping algebra $U_0(\mathfrak{g})$ is a finite-dimensional algebra. In fact, it is a cocommutative Hopf algebra and thus the algebra of measures of a group scheme. The algebra $U_0(\mathfrak{g})$ takes on the rôle of the group algebra, and we shall write $\mathrm{H}^n(\mathfrak{g}, M)$ for the n-th cohomology group of $U_0(\mathfrak{g})$ with coefficients in the $U_0(\mathfrak{g})$-module M.

Friedlander and Parshall proved in 1986 that, in case k is algebraically closed, $\mathfrak{g}$ is classical simple (such as $\mathfrak{g} = \mathfrak{sl}(n)$) and p is large enough, there exists an isomorphism

$$\mathrm{H}^\bullet(U_0(\mathfrak{g}), k) \cong k[\widehat{\mathcal{V}}_{\mathfrak{g}}],$$

where the right-hand side is the coordinate ring of the *nullcone* $\widehat{\mathcal{V}}_{\mathfrak{g}} := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}$. Note that $\widehat{\mathcal{V}}_{\mathfrak{g}} \subseteq \mathfrak{g}$ is a conical affine variety.

Given any restricted Lie algebra $(\mathfrak{g}, [p])$, there exists a natural algebra homomorphism

$$S(\mathfrak{g}^*) \longrightarrow \mathrm{H}^\bullet(U_0(\mathfrak{g}), k),$$

induced by a linear map $\mathfrak{g}^* \longrightarrow \mathrm{H}^2(U_0(\mathfrak{g}), k)$ coming from certain central extensions.

Theorem 1.5 (Friedlander-Parshall,1986). *Let M be a $U_0(\mathfrak{g})$-module. Then $\mathrm{H}^*(\mathfrak{g}, M)$ is a finitely generated $S(\mathfrak{g}^*)$-module.*

We shall see that this result leads to a concrete description of support varieties via subvarieties of the nullcone. Let me digress for a moment to illustrate how results like (1.5) can be exploited in representation theory.

The Theorem tells us that $\mathrm{H}^\bullet(\mathfrak{g}, k)$ is essentially generated in degree 2. As we shall learn later, this entails that periodic modules have period 1 or 2. Recall that an indecomposable $U_0(\mathfrak{g})$-module M is *periodic* if there exists $n > 0$ with $\Omega^n(M) \cong M$. Here Ω denotes the *Heller operator* of the Frobenius algebra $U_0(\mathfrak{g})$. Since Ω permutes the isoclasses of non-projective indecomposables, the non-projective indecomposable modules belonging to a representation-finite block $\mathcal{B} \subseteq U_0(\mathfrak{g})$ are periodic. If $\mathcal{B}$ is not simple, then we have $\Omega^2(S) \cong S$ for every simple $\mathcal{B}$-module S. This is a rather strong condition, which implies that $\mathcal{B}$ is a Nakayama algebra, whose simple modules constitute an orbit of the Nakayama functor $\mathcal{N}$. In our case $\mathcal{N}$ has order 1 or p, so the Gabriel quiver of $\mathcal{B}$ is $\tilde{A}_0$ or $\tilde{A}_{p-1}$.

In 1997 Friedlander and Suslin proved the finite generation of $\mathrm{H}^\bullet(\mathcal{G}, k)$. So let us take a closer look at the main part of that result:

Theorem 1.6 (Friedlander-Suslin,1997). *Let $\mathcal{G}$ be an infinitesimal group scheme of height $\leq r$. Then there exists a graded subalgebra $S \subseteq \mathrm{H}^\bullet(\mathcal{G}, k)$ such that*

(a) *S is generated by $S_{2p^{r-1}}$, and*

(b) *for every $M \in \operatorname{mod} \mathcal{G}$, the space $\mathrm{H}^*(\mathcal{G}, M)$ is a finitely generated S-module.*

The restricted Lie algebras correspond to infinitesimal groups of height ≤ 1, so we have a natural generalization of (1.5). However, this result is much harder to prove, involving a detailed analysis of the cohomology of polynomial functors.

The rank varieties of infinitesimal groups were defined by Suslin-Friedlander-Bendel via of homomorphisms $\mathbb{G}_{a(r)} \longrightarrow \mathcal{G}$, where $\mathbb{G}_{a(r)}$ is the r-th Frobenius kernel of the additive group $\mathbb{G}_a$. The reason for choosing the $\mathbb{G}_{a(r)}$ is their similarity with p-elementary abelian subgroups: The Hopf algebras $k\mathbb{G}_{a(r)}$ accociated to $\mathbb{G}_{a(r)}$ are isomorphic to truncated polynomial rings

$$k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p).$$

Suslin, Friedlander and Bendel characterized $\mathrm{H}^\bullet(\mathcal{G}, k)$ in terms of the coordinate ring of the rank variety of $\mathcal{G}$.

The common theme of all these approaches is to study modules M of some cocommutative Hopf algebra $k\mathcal{G}$ by considering their pullbacks $\alpha^*(M)$ along certain algebra homomorphisms $\alpha : k[X]/(X^p) \longrightarrow k\mathcal{G}$, where $p > 0$ denotes the characteristic of the base field k.

- For finite groups, the homomorphisms arise via cyclic shifted subgroups, as motivated by Dade's Lemma.

- For restricted Lie algebras $(\mathfrak{g}, [p])$, they correspond to $x \in \mathfrak{g} \setminus \{0\}$ with $x^{[p]} = 0$.
- In case $\mathcal{G}$ is an infinitesimal group, one considers infinitesimal one-parameter subgroups, defined by homomorphisms $\alpha : \mathbb{G}_{a(r)} \longrightarrow \mathcal{G}$, originating in the r-th Frobenius kernel of the additive group $\mathbb{G}_a$.

Friedlander-Pevtsova have subsumed the aforementioned approaches under their concept of a p-point, thereby generalizing everything to the setting of finite group schemes. The basic idea is to give precision to the common theme.

Let $\mathfrak{A}_p := k[X]/(X^p)$ be the truncated polynomial ring, and let x be its canonical generator. An algebra homomorphism $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ into the algebra of measures of a finite group scheme $\mathcal{G}$ is given by an element $u \in k\mathcal{G}$ with $u^p = 0$. Dade's Lemma already shows that only some of these elements are really useful, namely those for which $k\mathcal{G}$ is a projective $k[u]$-module. Accordingly, one considers homomorphisms α that are *left flat*. Let $\alpha^* : \operatorname{mod} \mathcal{G} \longrightarrow \operatorname{mod} \mathfrak{A}_p$ be the associated pull-back functor. Flatness means that α^* takes projectives to projectives. Given an arbitrary $\mathcal{G}$-module M, we want to study M by looking at all pull-backs $\alpha^*(M) \in \operatorname{mod} \mathfrak{A}_p$. This leads to two notions involving the space $P(\mathcal{G})$ of homomorphisms $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$:

- *Support spaces* $P(\mathcal{G})_M$ are defined by looking at those homomorphisms $\alpha \in P(\mathcal{G})$ for which $\alpha^*(M)$ is not projective. Friedlander and Pevtsova show that the $P(\mathcal{G})_M$ define a topology on $P(\mathcal{G})$, such that $P(\mathcal{G})$ and $\operatorname{Proj}(\mathcal{V}_{\mathcal{G}}(k))$ are homeomorphic, with $P(\mathcal{G})_M$ corresponding to $\operatorname{Proj}(\mathcal{V}_{\mathcal{G}}(M))$.
- In subsequent work, Friedlander, Pevtsova and Suslin have studied the behaviour of the isoclasses $\alpha^*(M)$. Note that such a class is given by a partition $\lambda_{\alpha^*(M)}$ of $\dim_k M$, called a *Jordan type of* M. We have the dominance order on partitions and can thus speak of a maximal Jordan type of M.

In related work, Carlson-Friedlander-Pevtsova have studied modules of constant Jordan type. The trivial module and the projective module, as well as the so-called endo-trivial modules belong to this class.

2. Lecture II: Support Varieties and Rank Varieties

In this lecture we outline the geometric approach to the representation theory of finite algebraic groups. We will be working over an algebraically closed field k of characteristic $p > 0$. Given a module M of a commutative ring R, we recall that the *support*

$$\operatorname{Supp}(M) := \{\mathfrak{M} \in \operatorname{Max}(R) \ ; \ M_{\mathfrak{M}} \neq (0)\}$$

of M is the set of those maximal ideals $\mathfrak{M} \lhd R$ for which the localization of M at $\mathfrak{M}$ is not trivial. If R is noetherian and M is finitely generated, then

$$\operatorname{Supp}(M) = Z(\operatorname{ann}_R(M)) := \{\mathfrak{M} \in \operatorname{Max}(R) \ ; \ \operatorname{ann}_R(M) \subseteq \mathfrak{M}\}$$

is the zero locus of the annihilator of M. Thus, if R is a finitely generated k-algebra, then $\operatorname{Supp}(M)$ is an affine variety. In our context, the even cohomology ring $\mathrm{H}^\bullet(\mathcal{G}, k)$ of a finite group scheme $\mathcal{G}$ assumes the rôle of R, with M being given by the cohomology space $\mathrm{H}^*(\mathcal{G}, \operatorname{Hom}_k(N, N))$ associated to a $\mathcal{G}$-module N.

2.1. **Support Varieties and the Friedlander-Suslin Theorem.** Let $\mathcal{G}$ be a finite algebraic k-group. If M is a $\mathcal{G}$-module, we denote by

$$\mathrm{H}^n(\mathcal{G}, M) := \mathrm{Ext}^n_{\mathcal{G}}(k, M) \quad (n \geq 0)$$

the *n-th cohomology group of* $\mathcal{G}$ *with coefficients in* M. Note that these groups are just the Hochschild cohomology groups of the augmented algebra $(k\mathcal{G}, \varepsilon)$.

Given three $\mathcal{G}$-modules X, Y, Z, we recall the *Yoneda product*

$$\mathrm{Ext}^m_{\mathcal{G}}(Y, Z) \times \mathrm{Ext}^n_{\mathcal{G}}(X, Y) \longrightarrow \mathrm{Ext}^{m+n}_{\mathcal{G}}(X, Z).$$

This product endows $\mathrm{Ext}^*_{\mathcal{G}}(X, X) := \bigoplus_{n\geq 0} \mathrm{Ext}^n_{\mathcal{G}}(X, X)$ with the structure of a $\mathbb{Z}$-graded k-algebra. Moreover, the spaces $\mathrm{Ext}^*_{\mathcal{G}}(Y, X)$ and $\mathrm{Ext}^*_{\mathcal{G}}(X, Y)$ are graded left and right $\mathrm{Ext}^*_{\mathcal{G}}(X, X)$-modules, respectively. In particular, $\mathrm{H}^*(\mathcal{G}, M)$ is a graded right module over the *cohomology ring* $\mathrm{H}^*(\mathcal{G}, k)$. This ring is known to be *graded commutative*, i.e., we have

$$yx = (-1)^{\deg(x)\deg(y)} xy$$

for any two homogeneous elements $x, y \in \mathrm{H}^*(\mathcal{G}, k)$. Consequently, the subring

$$\mathrm{H}^\bullet(\mathcal{G}, k) := \begin{cases} \bigoplus_{i\geq 0} \mathrm{H}^{2i}(\mathcal{G}, k) & \text{if } p > 2 \\ \bigoplus_{i\geq 0} \mathrm{H}^{i}(\mathcal{G}, k) & \text{if } p = 2 \end{cases}$$

is a commutative, $\mathbb{Z}$-graded k-algebra.

The following result by Friedlander and Suslin, which generalizes earlier work by Venkov and Evens for finite groups, and Friedlander-Parshall for infinitesimal groups of height ≤ 1, is fundamental for everything that follows.

Theorem 2.1 (Friedlander-Suslin)**.** *Let $\mathcal{G}$ be a finite algebraic k-group, M be a finite-dimensional $\mathcal{G}$-module. Then the following statements hold:*

(1) $\mathrm{H}^\bullet(\mathcal{G}, k)$ *is a finitely generated k-algebra.*

(2) $\mathrm{H}^*(\mathcal{G}, M)$ *is a finitely generated* $\mathrm{H}^\bullet(\mathcal{G}, k)$*-module.* □

Let M be a finite-dimensional $\mathcal{G}$-module. Recall the homomorphism

$$\Phi_M : \mathrm{H}^\bullet(\mathcal{G}, k) \longrightarrow \mathrm{Ext}^*_{\mathcal{G}}(M, M) \;\; ; \;\; [f] \mapsto [f \otimes \mathrm{id}_M]$$

of graded k-algebras. In view of the standard isomorphism $\mathrm{Ext}^*_{\mathcal{G}}(M, M) \cong \mathrm{H}^*(\mathcal{G}, \mathrm{Hom}_k(M, M))$, Φ_M endows the Yoneda algebra with the structure of a finitely generated $\mathrm{H}^\bullet(\mathcal{G}, k)$-module. We define the *cohomological support variety* of M via

$$\mathcal{V}_{\mathcal{G}}(M) := Z(\ker \Phi_M) \subseteq \mathrm{Max}(\mathrm{H}^\bullet(\mathcal{G}, k)).$$

Since $\ker \Phi_M$ is a homogeneous ideal, the affine variety $\mathcal{V}_{\mathcal{G}}(M)$ is conical.

Our first result concerns the interpretation of the dimension of a support variety. Let

$$A = \bigoplus_{n\geq 0} A_n$$

be a finitely generated, commutative graded k-algebra. We want to find the growth $\mathrm{gr}(A)$ of the sequence $(A_n)_{n\geq 0}$.

Let $(a_i)_{i\geq 0}$ be a sequence of natural numbers. We call

$$\mathrm{gr}((a_i)_{i\geq 0}) := \min\{s \in \mathbb{N} \cup \{\infty\} \; ; \; \exists \; \lambda > 0 \text{ such that } a_n \leq \lambda n^{s-1} \quad \forall \; n \geq 1\}$$

the *polynomial rate of growth* of the sequence $(a_i)_{i\geq 0}$. If $\mathcal{V} := (V_i)_{i\geq 0}$ is a sequence of finite dimensional k-vector spaces, then we write $\mathrm{gr}(\mathcal{V}) := \mathrm{gr}((\dim_k V_i)_{i\geq 0})$.

By the Noether Normalization Lemma there exists a graded subalgebra

$$R = \bigoplus_{n \geq 0} R_n$$

of A such that
(a) A is a finitely generated R-module, and
(b) $R \cong k[X_1, \dots, X_\ell]$, where $\deg(X_i) = d$ for some $d \geq 1$.
Owing to (a), we have $\mathrm{gr}(R) = \mathrm{gr}(A)$, while property (b) implies $\mathrm{gr}(R) = \ell$. The number ℓ is the *Krull dimension* of R, which, by the Cohen-Seidenberg Theorems, coincides with the Krull dimension $\dim A$ of A. We therefore obtain

$$\mathrm{gr}(A) = \dim A.$$

Let M be a $\mathcal{G}$-module, $P := (P_n)_{n\geq 0}$ be a minimal projective resolution of M, then

$$\mathrm{cx}_{\mathcal{G}}(M) := \mathrm{gr}(P)$$

is called the *complexity of M.*

Lemma 2.2. *Let $\mathcal{G}' \subseteq \mathcal{G}$ be a subgroup of the finite algebraic group $\mathcal{G}$, $M \in \mathrm{mod}\,\mathcal{G}$ be a $\mathcal{G}$-module.*
(1) $\dim \mathcal{V}_{\mathcal{G}}(M) = \mathrm{cx}_{\mathcal{G}}(M)$.
(2) $\dim \mathcal{V}_{\mathcal{G}'}(M) \leq \dim \mathcal{V}_{\mathcal{G}}(M)$.

Proof. (1) Let $\mathcal{S}$ be a complete set of representatives for the isoclasses of the simple $\mathcal{G}$-modules. Thanks to (2.1) the Yoneda algebra $\mathrm{Ext}^*_{\mathcal{G}}(M,M)$ is a finitely generated $\mathrm{H}^\bullet(\mathcal{G},k)$-module. Consequently, we have

$$\begin{aligned} \dim \mathcal{V}_{\mathcal{G}}(M) &= \dim \mathrm{H}^\bullet(\mathcal{G},k)/\ker \Phi_M = \mathrm{gr}(\mathrm{H}^\bullet(\mathcal{G},k)/\ker \Phi_M) \\ &= \mathrm{gr}(\mathrm{Ext}^*_{\mathcal{G}}(M,M)) \leq \max_{S\in\mathcal{S}} \mathrm{gr}(\mathrm{Ext}^*_{\mathcal{G}}(M,S)), \end{aligned}$$

where the last inequality follows from the long exact sequence in cohomology. The latter number is known to coincide with $\mathrm{cx}_{\mathcal{G}}(M)$.

The verification of the reverse inequality employs similar arguments.

(2) This follows directly from (1), and the fact that $k\mathcal{G}$ is a free $k\mathcal{G}'$-module. □

Example. Suppose that $p \geq 3$. Let $G = (\mathbb{Z}/(p))^r$ be a p-elementary abelian group of rank r. The Künneth formula furnishes an isomorphism

$$\mathrm{H}^*(G,k) \cong k[X_1,\dots,X_r] \otimes_k \Lambda(Y_1,\dots,Y_r),$$

where the X_i and Y_i have degrees 2 and 1, respectively. Consequently, $k[X_1,\dots,X_r]$ is a Noether normalization of $\mathrm{H}^\bullet(G,k)$, and $\dim \mathcal{V}_G(k) = r$.

In the above example, we have $\mathcal{V}_G(k) \cong \mathbb{A}^r$. In general, support varieties are hard to come by and we thus look for closely related varieties that avoid reference to the cohomology ring.

2.2. **Cyclic Shifted Subgroups.** For p-elementary elementary abelian groups, Carlson's rank varieties, whose definition is motivated by Dade's Lemma, can be described as follows: Given a p-elementary abelian group E, one considers a subspace $V \subseteq kE$ with $\dim_k V = \mathrm{rk}(E)$, whose nonzero elements $u \in V \setminus \{0\}$ have the following property:

- $u^p = 0$, and $k[u]$ is a local algebra of dimension p,
- the module $kE|_{k[u]}$ is free.

In fact, the second property is guaranteed by Dade's Lemma, if one takes the elements u_λ defined in my first lecture.

For such an element u, the element $1 + u$ generates a subgroup $C_u \subseteq kE^\times$ of order p, a *cyclic shifted subgroup* of E. Given an E-module M, one defines the *rank variety* of M via

$$\widehat{\mathcal{V}}_E(M) := \{u \in V \ ; \ M|_{k[u]} \text{ is not free}\} \cup \{0\}.$$

The name derives from the equivalence

$$u \in \widehat{\mathcal{V}}_E(M) \ \Leftrightarrow \mathrm{rk}(u_M) < \frac{(p-1)\dim_k M}{p},$$

which also shows that $\widehat{\mathcal{V}}_E(M)$ is a closed subset of V. Dade's Lemma may now be rephrased by saying that M is projective if and only if $\widehat{\mathcal{V}}_E(M) = \{0\}$. In view of (2.2), the same holds for the support variety $\mathcal{V}_E(M)$. In 1982 Avrunin and Scott showed that $\widehat{\mathcal{V}}_E(M) \cong \mathcal{V}_E(M)$, and they thus obtained a representation-theoretic characterization of the cohomological support variety.

2.3. **Nullcones of Restricted Lie algebras.** In this section we consider restricted Lie algebras $(\mathfrak{g}, [p])$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ and recall that $U(\mathfrak{g})$ is a Hopf algebra, whose comultiplication $\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes_k U(\mathfrak{g})$ and co-unit $\varepsilon : U(\mathfrak{g}) \longrightarrow k$ are determined by

$$\Delta(x) = x \otimes 1 + 1 \otimes x \ \ ; \ \ \varepsilon(x) = 0 \quad \forall\, x \in \mathfrak{g},$$

respectively. The ideal $I := (\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\})$ satisfies

$$\Delta(I) \subseteq I \otimes_k U(\mathfrak{g}) + U(\mathfrak{g}) \otimes_k I \ \text{ and } \ \varepsilon(I) = (0).$$

This implies that the *restricted enveloping algebra*

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/I$$

of $(\mathfrak{g}, [p])$ inherits from $U(\mathfrak{g})$ the structure of a cocommutative Hopf algebra. By the Theorem of Poincaré-Birkhoff-Witt, we have $\dim_k U_0(\mathfrak{g}) = p^{\dim_k \mathfrak{g}}$. Thus, $U_0(\mathfrak{g})$ corresponds to the algebra of measures of a finite group scheme, an infinitesimal group of height ≤ 1.

In his 1954 paper, Hochschild showed that the cohomology groups $\mathrm{H}^2(U_0(\mathfrak{g}), k)$ correspond to central extensions

$$(0) \longrightarrow k \longrightarrow E \longrightarrow \mathfrak{g} \longrightarrow (0)$$

of restricted Lie algebras, where $k \subseteq C(E)$ is assumed to have a trivial p-map. If such a sequence splits as a sequence of ordinary Lie algebras, then

$$E = \mathfrak{g} \oplus k \quad ; \quad (e, \alpha)^{[p]} = (e^{[p]}, \psi(e))$$

for some p-semilinear map $\psi : \mathfrak{g} \longrightarrow k$. By twisting the k-action on $\mathfrak{g}$, we can think of $\mathfrak{g}$ as a linear map. This point is of importance if some group is acting on $\mathfrak{g}$, but we won't worry about such matters. In this fashion one obtains a linear map

$$\mathfrak{g}^* \longrightarrow \mathrm{H}^2(U_0(\mathfrak{g}), k),$$

the so-called *Hochschild map*. It induces an algebra homomorphism

$$S(\mathfrak{g}^*) \longrightarrow \mathrm{H}^\bullet(U_0(\mathfrak{g}), k).$$

Theorem 2.3 (Friedlander-Parshall,1986). *Let M be a $U_0(\mathfrak{g})$-module. Then $\mathrm{H}^*(U_0(\mathfrak{g}), M)$ is a finitely generated $S(\mathfrak{g}^*)$-module.*

Since $S(\mathfrak{g}^*) = k[\mathfrak{g}]$ is the coordinate ring of the affine space $\mathfrak{g}$, we thus obtain in particular a finite morphism

$$\mathcal{V}_{\mathfrak{g}}(k) \longrightarrow \mathfrak{g}.$$

In 1986 Jantzen showed that the image of this map is the nullcone $\widehat{\mathcal{V}}_{\mathfrak{g}} := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}$. Shortly thereafter, Friedlander and Parshall proved that the above morphism takes the support variety $\mathcal{V}_{\mathfrak{g}}(M)$ onto the rank variety

$$\widehat{\mathcal{V}}_{\mathfrak{g}}(M) := \{x \in \widehat{\mathcal{V}}_{\mathfrak{g}} \ ; \ M|_{k[x]} \text{ is not free}\} \cup \{0\}.$$

Given $x \in \widehat{\mathcal{V}}_{\mathfrak{g}}$, the Poincaré-Birkhoff-Witt Theorem shows that $U_0(\mathfrak{g})$ is a free $k[x]$-module. Thus, this description of $\mathcal{V}_{\mathfrak{g}}$ via the nullcone parallels that of Carlson's cyclic shifted subgroups.

Examples. (1) Consider the restricted Lie algebra $\mathfrak{g} = \mathfrak{sl}(n)$. If p is large enough, then $\widehat{\mathcal{V}}_{\mathfrak{g}}$ is the set of nilpotent $(n \times n)$-matrices. For $n = 2$, the matrix x belongs to $\widehat{\mathcal{V}}_{\mathfrak{g}}$ if and only if $\mathrm{tr}(x) = 0 = \det(x)$. Thus, the variety $\widehat{\mathcal{V}}_{\mathfrak{sl}(2)}$ is given by

$$\widehat{\mathcal{V}}_{\mathfrak{sl}(2)} = \{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{Mat}_2(k) \ ; \ a^2 + bc = 0\},$$

so that $\widehat{\mathcal{V}}_{\mathfrak{sl}(2)}$ is an irreducible variety of dimension $\dim \widehat{\mathcal{V}}_{\mathfrak{sl}(2)} = 2$.

Let M be an indecomposable $U_0(\mathfrak{sl}(2))$-module:

- If $\dim \widehat{\mathcal{V}}_{\mathfrak{sl}(2)}(M) = 2$, then M belongs to an AR-component of type $\mathbb{Z}[\tilde{A}_{12}]$.
- If $\dim \widehat{\mathcal{V}}_{\mathfrak{sl}(2)}(M) = 1$, then M is periodic and belongs to an AR-component of type $\mathbb{Z}[A_\infty]/(\tau)$.
- If $\dim \widehat{\mathcal{V}}_{\mathfrak{sl}(2)}(M) = 0$, then M is projective.

For instance, a baby Verma module $Z(\lambda) := U_0(\mathfrak{sl}(2)) \otimes_{U_0(\mathfrak{b})} k_\lambda$, induced by a one-dimensional representation of a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{sl}(2)$, has rank variety $\widehat{\mathcal{V}}_{\mathfrak{sl}(2)}(Z(\lambda)) \subseteq \widehat{\mathcal{V}}_{\mathfrak{b}}$, and is thus periodic or projective.

(2) Let $\mathfrak{g} = \mathfrak{sl}(2)_n := \mathfrak{sl}(2) \oplus k$ be a central extension of $\mathfrak{sl}(2)$, which splits as an extension of ordinary Lie algebras, and whose p-map is defined via the p-semilinear map

$$\psi_n : \mathfrak{sl}(2) \longrightarrow k \ \ ; \ \ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto a^p.$$

Then

$$\widehat{\mathcal{V}}_{\mathfrak{sl}(2)_n} = (\ker \psi_n \cap \widehat{\mathcal{V}}_{\mathfrak{sl}(2)}) \times k$$

is a two-dimensional, reducible variety.

2.4. **Infinitesimal One-Parameter Subgroups.** In 1997 Friedlander, Suslin and Bendel generalized the above to the setting of infinitesimal group schemes. Their "higher nullcones" are the schemes $\mathcal{HOM}(\mathbb{G}_a(r), \mathcal{G})$ of infinitesimal one-parameter subgroups. We shall only briefly consider the corresponding variety of k-rational points.

Definition. Let $\mathcal{G}$ be an algebraic group. For $r \geq 1$, we define

$$V_r(\mathcal{G}) := \{\alpha : \mathbb{G}_{a(r)} \longrightarrow \mathcal{G} \ ; \ \alpha \text{ homomorphism of group schemes}\}.$$

Examples. (1) $V_r(\mathbb{G}_a) \cong k^r$
(2) $V_r(\mathrm{GL}(n)) \cong \{(v_0, \ldots, v_{r-1}) \in \mathrm{Mat}_n(k)^r \ ; \ v_i^p = 0 \text{ and } [v_i, v_j] = 0 \text{ for } 0 \leq i, j \leq r-1\}$.

Our examples suggest a relationship with the commuting varieties of nullcones of $\mathrm{Lie}(\mathcal{G})$.

Proposition 2.4. *If there exists a nice embedding $\mathcal{G} \hookrightarrow \mathrm{GL}(n)$, then*

$$V_r(\mathcal{G}) \cong \{(v_0, \ldots, v_{r-1}) \in \widehat{\mathcal{V}}^r_{\mathrm{Lie}(\mathcal{G})} \ ; \ [v_i, v_j] = 0 \text{ for } 0 \leq i, j \leq r-1\}.$$

Theorem 2.5 (Suslin-Friedlander-Bendel,1997). *Let $\mathcal{G}$ be an infinitesimal group of height $\leq r$, that is, $\mathcal{G} = \mathcal{G}_r$.*
(1) *There exists a homomorphism $\Psi : \mathrm{H}^\bullet(\mathcal{G}, k) \longrightarrow k[V_r(\mathcal{G})]$ of k-algebras such that* $\ker \Psi$ *is nilpotent and $k[V_r(\mathcal{G})]^{p^r} \subseteq \mathrm{im}\, \Psi$.*
(2) *If $\mathcal{H} \subseteq \mathcal{G}$ is a subgroup, then the image of* $\mathrm{res} : \mathrm{H}^\bullet(\mathcal{G}, k) \longrightarrow \mathrm{H}^\bullet(\mathcal{H}, k)$ *contains all p^r-th powers.*

In particular, the map Ψ induces a homeomorphism

$$\Psi^* : V_r(\mathcal{G}) \longrightarrow \mathcal{V}_{\mathcal{G}}(k).$$

Example. We consider the infinitesimal group $\mathbb{G}_{a(r)}$. Then $\mathrm{Lie}(\mathbb{G}_{a(r)}) = kx$, where $x^{[p]} = 0$, so that the Proposition gives $V_r(\mathbb{G}_{a(r)}) = k^r$. We denote the coordinate ring by $k[T_0, \ldots, T_{r-1}]$. Since $k\mathbb{G}_{a(r)} \cong k(\mathbb{Z}/(p))^r$, we have

$$\mathrm{H}^*(\mathbb{G}_{a(r)}, k) \cong k[X_1, \ldots, X_r] \otimes_k \Lambda(Y_0, \ldots, Y_{r-1}).$$

Then $\Psi(X_i) = T_{r-i}^{p^i}$.

In sum, we have produced varieties for finite groups, restricted Lie algebras (and infinitesimal groups) that are closely related to cohomological support varieties. In particular, their dimensions coincide with the complexities of the relevant modules.

3. Lecture III: The Space of p-Points

Throughout, k denotes an algebraically closed field of characteristic $p > 0$. For notational convenience we put $\mathfrak{A}_p := k[X]/(X^p)$.

Given any homomorphism $\alpha : A \longrightarrow B$ of algebras, we denote by $\alpha^* : \mathrm{mod}\, B \longrightarrow \mathrm{mod}\, A$ the functor given by pull-back along α.

Definition. Let $\mathcal{G}$ be a finite k-group scheme. An algebra homomorphism $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ is called a *p-point* if
(P1) α is left flat, and
(P2) there exists an abelian unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}$ with $\mathrm{im}\, \alpha \subseteq k\mathcal{U}$.
Two p-points α, β are *equivalent* if for every $M \in \mathrm{mod}\, k\mathcal{G}$ we have

$$\alpha^*(M) \text{ is projective} \Leftrightarrow \beta^*(M) \text{ is projective}.$$

The set of equivalence classes of p-points will be denoted $P(\mathcal{G})$.

Since this definition may look somewhat contrived, let us spend a few moments on the defining conditions. Property (P1) means that the pull-back functor $\alpha^* : \operatorname{mod} k\mathcal{G} \longrightarrow \operatorname{mod} \mathfrak{A}_p$ sends projectives to projectives. This is one implication of Dade's Lemma.

Condition (P2) is more subtle. It allows us to reduce many questions concerning p-points to the consideration of abelian unipotent groups, which are easier to understand. Recall that a finite group scheme $\mathcal{U}$ is *unipotent* if the algebra $k\mathcal{U}$ is local. In the theory of finite groups, these are just the p-groups. An abelian p-group U is of the form $U = \bigoplus_{i=1}^n \mathbb{Z}/(p^{r_i})$, so that

$$kU \cong k[X_1, \dots, X_n]/(X_1^{p^{r_1}}, \dots, X_n^{p^{r_n}}).$$

The latter isomorphism turns out to hold for any finite abelian unipotent group scheme. The following subsidiary result shows that the above equivalence relation is more tractable in this context.

Proposition 3.1. *Let $n \geq 1$ and consider the algebra $A := k[X, Y, Z]/(X^p, Y^p, Z^n)$, whose generators are denoted x, y, z. Let M be an A-module. Then M is projective as a $k[x]$-module if and only if M is projective as a $k[x + yz]$-module.*

Proof. Observe that the subalgebras $k[x]$ and $k[x+yz]$ are both isomorphic to $\mathfrak{A}_p$. By decomposing an $\mathfrak{A}_p$-module N into its cyclic direct summands, we see that N is projective if and only if the endomorphism t_N of a generator $t \in \mathfrak{A}_p$ satisfies the equivalent conditions $\ker t_N = \operatorname{im} t_N^{p-1}$ and $\ker t_N^{p-1} = \operatorname{im} t_N$.

Suppose that $M|_{k[x]}$ is projective and consider the A-module $N := \ker(x + yz)_M / \operatorname{im}(x + yz)_M^{p-1}$. Direct computation shows that the operator z_N is injective. As z is nilpotent, we obtain $N = (0)$, so that $M|_{k[x+yz]}$ is projective. □

Example. We illustrate the relevance of condition (P2) by considering the case of infinitesimal groups $\mathcal{G}$ of height 1. Recall that $k\mathcal{G} \cong U_0(\mathfrak{g})$ is the restricted enveloping algebra of the restricted Lie algebra $\mathfrak{g} := \operatorname{Lie}(\mathcal{G})$.

Let $\alpha : \mathfrak{A}_p \longrightarrow U_0(\mathfrak{g})$ be a p-point. Then α factors through the restricted enveloping algebra $U_0(\mathfrak{u})$ of a suitable abelian, p-unipotent p-subalgebra $\mathfrak{u} \subseteq \mathfrak{g}$. In this case, the generators $x_1, \dots, x_n$ of the algebra $U_0(\mathfrak{u})$ are elements of $\mathfrak{u}$ that satisfy $x_i^{p^{r_i}} = 0 = x^{[p]^{r_i}}$. Repeated application of (3.1) shows that α is equivalent to a p-point $\alpha_0 : \mathfrak{A}_p \longrightarrow U_0(\mathfrak{u})$ given by

$$\alpha_0(x) = \sum_{i=1}^n \zeta_i x_i^{p^{r_i - 1}},$$

where $(\zeta_1, \dots, \zeta_n) \in k^n \setminus \{0\}$. Thus $u := \alpha_0(x)$ belongs to $\mathfrak{u} \subseteq \mathfrak{g}$ and satisfies the identity $u^{[p]} = 0$. Consequently, the element u belongs to the nullcone $\widehat{\mathcal{V}}_{\mathfrak{g}}$ of $\mathfrak{g}$.

Given $v, w \in \widehat{\mathcal{V}}_{\mathfrak{g}} \setminus \{0\}$, the $U_0(\mathfrak{g})$-module $M_v := U_0(\mathfrak{g}) \otimes_{k[v]} k$ is $k[w]$-projective if and only if $w \notin kv$ (One direction follows from the PBW-Theorem. If $kw = kv$, then $k[v] = k[w]$. The augmentation $\varepsilon : U_0(\mathfrak{g}) \longrightarrow k$ induces a $U_0(\mathfrak{g})$-linear surjection $M_v \longrightarrow k$ for which $1 \mapsto 1 \otimes 1$ defines a $k[v]$-linear splitting. Hence $M_v|_{k[v]}$ is not projective.) Thus, the line $k\alpha_0(x)$ only depends on the equivalence class $[\alpha]$, and we obtain a bijection

$$P(\mathcal{G}) \xrightarrow{\sim} \operatorname{Proj}(\widehat{\mathcal{V}}_{\mathfrak{g}}) \ \ ; \ \ [\alpha] \mapsto k\alpha_0(x).$$

Since any homomorphism $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ is a homomorphism of augmented algebras, we have an induced homomorphism

$$\alpha^\bullet : \mathrm{H}^\bullet(\mathcal{G}, k) \longrightarrow \mathrm{H}^\bullet(\mathfrak{A}_p, k)$$

of graded k-algebras. Recall that $\mathrm{H}^\bullet(\mathfrak{A}_p, k) = k[X]$ is a polynomial ring, where X is given cohomological degree 2.

Given any finitely generated, commutative, graded k-algebra

$$A = \bigoplus_{n\geq 0} A_n \quad ; \quad A_0 = k$$

we recall that

$$\mathrm{Proj}(A) = \{\ker\varphi \ ; \ \varphi \in \mathrm{Hom}_{\mathrm{gr}}(A, k[X]), \ \mathrm{im}\,\varphi \neq k\}.$$

The following Proposition elicits the cohomological aspects of the equivalence relation on p-points. The key notion is that of the Carlson module associated to a cohomology class $[\zeta] \in \mathrm{H}^{2n}(\mathcal{G}, k)$. By general theory, we have an isomorphism $\mathrm{H}^{2n}(\mathcal{G}, k) \cong \mathrm{Hom}_{\mathcal{G}}(\Omega^{2n}_{\mathcal{G}}(k), k)$, allowing us to view the representing cocycle as a map $\zeta : \Omega^{2n}_{\mathcal{G}}(k) \longrightarrow k$. Then

$$L_\zeta := \ker\zeta$$

is called the *Carlson module* of ζ.

Proposition 3.2. *Let $\mathcal{G}$ be a finite group scheme and $\alpha, \beta : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ be flat maps such that $\ker\alpha^\bullet \not\subseteq \ker\beta^\bullet$. Then there exists $\zeta \in \mathrm{Hom}_{\mathcal{G}}(\Omega^{2n}_{\mathcal{G}}(k), k)$ such that $\alpha^*(L_\zeta)$ is not projective and $\beta^*(L_\zeta)$ is projective.*

Proof. As α is flat, the functor α^* sends a minimal projective resolution of $M \in \mathrm{mod}\,\mathcal{G}$ to a (not necessarily minimal) projective resolution of $\alpha^*(M)$, so that

$$\alpha^*(\Omega^m_{\mathcal{G}}(M)) \cong \Omega^m_{\mathfrak{A}_p}(\alpha^*(M)) \oplus (\mathrm{proj.}) \qquad \forall\, m \in \mathbb{N}.$$

Let $\zeta : \Omega^{2n}_{\mathcal{G}}(k) \longrightarrow k$ be an arbitrary non-zero element. Upon application of α^* to the exact sequence $(0) \longrightarrow L_\zeta \longrightarrow \Omega^{2n}_{\mathcal{G}}(k) \stackrel{\zeta}{\longrightarrow} k \longrightarrow (0)$ we obtain, observing $\Omega^{2n}_{\mathfrak{A}_p}(k) \cong k$, an exact sequence

$$(*) \qquad\qquad (0) \longrightarrow \alpha^*(L_\zeta) \longrightarrow k \oplus (\mathrm{proj.}) \stackrel{\alpha^*(\zeta)}{\longrightarrow} k \longrightarrow (0).$$

The definition of $\alpha^\bullet$ yields

$$\alpha^\bullet([\zeta]) = 0 \Leftrightarrow k \subseteq \ker\alpha^*(\zeta).$$

If $\alpha^*(L_\zeta)$ is projective, then it is injective and thus belongs to the projective part of the middle term of $(*)$. Consequently, $(*)$ implies $k \not\subseteq \ker\alpha^*(\zeta)$, so that $[\zeta] \notin \ker\alpha^\bullet$. Conversely, $[\zeta] \notin \ker\alpha^\bullet$ yields the splitting of the sequence $(*)$, whence

$$k \oplus (\mathrm{proj.}) \cong \alpha^*(L_\zeta) \oplus k.$$

By the Theorem of Krull-Remak-Schmidt, the $\mathfrak{A}_p$-module $\alpha^*(L_\zeta)$ is projective.

By assumption, there exists an element $[\zeta] \in \mathrm{H}^{2n}(\mathcal{G}, k)$ with $[\zeta] \in \ker\alpha^\bullet \setminus \ker\beta^\bullet$. The foregoing observations now show that the Carlson module L_ζ has the requisite properties. □

The following result, which employs (3.2) but does not require property (P2), indicates a connection between p-points and the variety $\mathrm{Proj}(\mathcal{V}_{\mathcal{G}}(k))$.

Proposition 3.3. *Let $\mathcal{G}$ be a finite group scheme. Then*

$$\Psi_{\mathcal{G}} : \begin{cases} P(\mathcal{G}) & \longrightarrow \quad \mathrm{Proj}(\mathcal{V}_{\mathcal{G}}(k)) \\ [\alpha] & \mapsto \quad \ker\alpha^\bullet \end{cases}$$

is a well-defined map which is natural relative to flat morphisms $\mathcal{G} \longrightarrow \mathcal{H}$.

The hope is that $\Psi_{\mathcal{G}}$ is a bijection. This turns out to necessitate condition (P2).

Let M be a $\mathcal{G}$-module. Then we define

$$P(\mathcal{G})_M := \{[\alpha] \in P(\mathcal{G}) \ ; \ \alpha^*(M) \text{ is not projective}\}.$$

The following result shows that $P(\mathcal{G})$ is a topological space which serves as a good representation-theoretic model for the support varieties:

Theorem 3.4 (Friedlander-Pevtsova,2005). *Let $\mathcal{G}$ be a finite group scheme.*

(1) *The sets $P(\mathcal{G})_M$ form the closed sets of a noetherian topology on $P(\mathcal{G})$.*

(2) *The map $\Psi_{\mathcal{G}} : P(\mathcal{G}) \longrightarrow \mathrm{Proj}(\mathcal{V}_{\mathcal{G}}(k))$ is a homeomorphism such that*

$$\Psi_{\mathcal{G}}(P(\mathcal{G})_M) = \mathrm{Proj}(\mathcal{V}_{\mathcal{G}}(M)) \qquad \forall\ M \in \operatorname{mod} \mathcal{G}.$$

The proof of this result requires information on $P(\mathcal{G})_{M_1} \cap P(\mathcal{G})_{M_2}$. Work on rank varieties suggests that $P(\mathcal{G})_{M_1} \cap P(\mathcal{G})_{M_2} = P(\mathcal{G})_{M_1 \otimes_k M_2}$. Since α is an algebra homomorphism, α^* does not necessarily commute with tensor products (provided we have given $\mathfrak{A}_p$ the structure of a Hopf algebra) and it is not clear why such an equality should hold. In fact, it is false if one does not require (P2).

We summarize a few properties of $M \mapsto P(\mathcal{G})_M$:

- $\dim P(\mathcal{G})_M = \mathrm{cx}_{\mathcal{G}}(M) - 1$, where $\mathrm{cx}_{\mathcal{G}}(M)$ denotes the *complexity* of the $\mathcal{G}$-module M. In particular, M is projective if and only if $P(\mathcal{G})_M = \emptyset$.
- If $(0) \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow (0)$ is exact, then $P(\mathcal{G})_{M_2} \subseteq P(\mathcal{G})_{M_1} \cup P(\mathcal{G})_{M_3}$, with equality holding in case the sequence splits.
- $P(\mathcal{G})_{M_1 \otimes_k M_2} = P(\mathcal{G})_{M_1} \cap P(\mathcal{G})_{M_2}$.

We have already seen the utility of the Carlson modules L_ζ. Another important feature concerns their rôle concerning realizability of varieties and periodicity of modules:

Corollary 3.5. *The following statements hold:*

(1) *We have $\mathcal{V}_{\mathcal{G}}(L_\zeta) = Z([\zeta])$ for any $\zeta \in \mathrm{Hom}_{\mathcal{G}}(\Omega^{2n}_{\mathcal{G}}(k), k)$.*

(2) *If M is a non-projective indecomposable $\mathcal{G}$-module and $\zeta \in \mathrm{H}^{2n}(\mathcal{G}, k)$ satisfies $Z(\zeta) \cap \mathcal{V}_{\mathcal{G}}(M) = \{0\}$, then $\Omega^{2n}_{\mathcal{G}}(M) \cong M$.*

Proof. (1) Let $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ be a p-point of $\mathcal{G}$, and consider the induced map $\alpha^\bullet : \mathrm{H}^\bullet(\mathcal{G}, k) \longrightarrow \mathrm{H}^\bullet(\mathfrak{A}_p, k)$. As argued in the proof of (3.2), we have $[\zeta] \in \ker \alpha^\bullet$ if and only if the pull-back $\alpha^*(L_\zeta)$ is not projective. Thanks to (3.4), the latter condition is equivalent to $\ker \alpha^\bullet \supseteq \ker \Phi_{L_\zeta}$, so that

$$([\zeta]) \subseteq \ker \alpha^\bullet \Leftrightarrow \ker \Phi_{L_\zeta} \subseteq \ker \alpha^\bullet.$$

In view of (3.4), this implies $\mathrm{Proj}(Z([\zeta])) = \mathrm{Proj}(\mathcal{V}_{\mathcal{G}}(L_\zeta))$, whence $Z([\zeta]) = \mathcal{V}_{\mathcal{G}}(L_\zeta)$.

(2) Upon tensoring the exact sequence $(0) \longrightarrow L_\zeta \longrightarrow \Omega^{2n}_{\mathcal{G}}(k) \longrightarrow k \longrightarrow (0)$ with M we obtain an exact sequence

$$(0) \longrightarrow L_\zeta \otimes_k M \longrightarrow \Omega^{2n}_{\mathcal{G}}(M) \oplus (\text{proj.}) \longrightarrow M \longrightarrow (0).$$

Since $\mathcal{V}_{\mathcal{G}}(L_\zeta \otimes_k M) = \mathcal{V}_{\mathcal{G}}(L_\zeta) \cap \mathcal{V}_{\mathcal{G}}(M) = \{0\}$, the module $L_\zeta \otimes_k M$ is injective, so that the above sequence splits and

$$\Omega^{2n}_{\mathcal{G}}(M) \oplus (\text{proj.}) \cong M \oplus (L_\zeta \otimes_k M).$$

The result follows from the Theorem of Krull-Remak-Schmidt, which provides an isomorphism between the projective-free parts. □

Let M be an indecomposable $\mathcal{G}$-module of complexity 1, $S \subseteq \mathrm{H}^\bullet(\mathcal{G}, k)$ be a graded subalgebra with homogeneous generators $\zeta_1, \ldots, \zeta_n$ and such that $\mathrm{H}^\bullet(\mathcal{G}, k)$ is a finitely generated S-module. Then $\mathcal{V}_{\mathcal{G}}(M)$ is a line, and there exists $i \in \{1, \ldots, n\}$ with $Z(\zeta_i) \cap \mathcal{V}_{\mathcal{G}}(M) = \{0\}$.

The computation of $P(\mathcal{G})$ is reduced to the case of finite groups and infinitesimal groups via a version of Quillen stratification. We shall consider a somewhat weaker statement. Recall that a closed embedding $\iota : \mathcal{G}' \hookrightarrow \mathcal{G}$ of finite group schemes induces a continuous map

$$\iota_* : P(\mathcal{G}') \hookrightarrow P(\mathcal{G}) \quad ; \quad [\alpha] \mapsto [\iota \circ \alpha],$$

such that $\iota_*(P(\mathcal{G}')) \subseteq P(\mathcal{G})$ is closed. We apply this to special subgroups: For every p-elementary abelian subgroup $E \subseteq \mathcal{G}(k)$ we consider the group scheme $(\mathcal{G}^0)^E \times E \subseteq \mathcal{G} \cong \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}}$ and denote the corresponding embedding by $\iota_{*,E}$. (Strictly speaking, we take closed subgroups of $\mathcal{G}_{\mathrm{red}}$, whose groups of k-rational points are p-elementary abelian.) At the level of Hopf algebras this amounts to considering the group algebra $(k(\mathcal{G}^0)^E)E \subseteq (k\mathcal{G}^0)\mathcal{G}(k) = k\mathcal{G}$.

Theorem 3.6. *We have*

$$P(\mathcal{G}) = \bigcup_{E \subseteq \mathcal{G}(k)} \iota_{*,E}(P((\mathcal{G}^0)^E \times E)),$$

where the union is taken with respect to the p-elementary abelian subgroups of $\mathcal{G}(k)$.

Example. Let $p \geq 3$. Consider the finite group scheme $\mathcal{L} : M_k \longrightarrow \mathrm{Gr}$, given by

$$\mathcal{L}(R) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)(R) \ ; \ a^p = 1 = d^p \ , \ b^{p^2} = b^p \ , \ c^p = 0\}$$

for every $R \in M_k$. Then we have

$$\mathcal{L}(k) := \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2)(k) \ ; \ b \in \mathbb{F}_p\} \cong \mathbb{Z}/(p).$$

The connected component $\mathcal{L}^0$ of $\mathcal{L}$ is an infinitesimal group that coincides with some Frobenius kernel of $\mathcal{L}$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{L}_r$ for $r \geq 2$, then $0 = b^{p^r} = b^p$, so that $\mathcal{L}_r = \mathcal{L}_1 = \mathrm{SL}(2)_1$. Consequently,

$$k\mathcal{L} \cong U_0(\mathfrak{sl}(2))\mathbb{Z}/(p)$$

is a skew group algebra, with an element $x \in \mathbb{Z}/(p)$ acting on a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{sl}(2)$ via

$$x.\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}.$$

One computes $\mathfrak{sl}(2)^{\mathbb{Z}/(p)} = \mathfrak{u}$, the subalgebra of strictly upper triangular matrices. In view of Theorem 3.6 we thus have to find the p-points of the algebras $U_0(\mathfrak{u}) \otimes_k k\mathbb{Z}/(p) \cong U_0(\mathfrak{u} \times \mathfrak{u})$ and $U_0(\mathfrak{sl}(2))$, respectively. It follows that $\dim P(\mathcal{L}) = 1$, so that $\mathrm{cx}_{\mathcal{L}}(k) = 2$. (The principal block of $k\mathcal{L}$ turns out to be wild.)

4. Lecture IV: π-Points and Jordan Type

In this lecture, we address a number of generalizations concerning the representation-theoretic support spaces considered so far, namely

- group schemes and modules that are defined over arbitrary base fields,
- infinite dimensional modules,
- refinements of the projectivity condition.

The first two issues are addressed via the notion of a π-point, the last one involves the so-called Jordan types of a module.

4.1. **π-Points.** Throughout, k denotes an arbitrary field of positive characteristic $p > 0$. If $\mathcal{G} := \mathrm{Spec}_k(k[\mathcal{G}])$ is an affine group scheme over k and K is an extension field of k, then we denote by $\mathcal{G}_K := \mathrm{Spec}_K(k[\mathcal{G}] \otimes_k K)$ the affine K-group scheme defined by $k[\mathcal{G}] \otimes_k K$. If $\mathcal{G}$ is a finite k-group, then $\mathcal{G}_K$ is a finite K-group with algebra of measures

$$K\mathcal{G} := k\mathcal{G} \otimes_k K.$$

We shall write $\mathfrak{A}_{p,K} := K[X]/(X^p)$.

Definition. Let $\mathcal{G}$ be a finite group scheme over k. Given a field extension $K\!:\!k$, a *π-point*

$$\alpha_K : \mathfrak{A}_{p,K} \longrightarrow K\mathcal{G}$$

is a left flat homomorphism of K-algebras, such that there exists an abelian, unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}_K$ with $\mathrm{im}\,\alpha_K \subseteq K\mathcal{U}$.

We may now investigate $k\mathcal{G}$-modules M by considering their extensions $M_K := M \otimes_k K$ as $K\mathcal{G}$-modules as well as the structure of their pullbacks $\alpha_K^*(M_K) \in \mathrm{mod}\,\mathfrak{A}_{p,K}$. Friedlander and Pevtsova define an equivalence relation for π-points $\alpha_K : \mathfrak{A}_{p,K} \longrightarrow K\mathcal{G}$ and $\beta_L : \mathfrak{A}_{p,L} \longrightarrow L\mathcal{G}$ via

$$\alpha_K \sim \beta_L \;:\Leftrightarrow\; \alpha_K^*(M_K) \text{ is projective } \Leftrightarrow\; \beta_L^*(M_L) \text{ is projective } \quad \forall\, M \in \mathrm{mod}\,k\mathcal{G}.$$

We let $\Pi(\mathcal{G})$ be the set of equivalence classes of π-points.

Let me comment on the technical aspect of the definition involving the existence of an abelian, unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}_K$. By not insisting on $\mathcal{U}$ to be defined over k, we introduce a degree of freedom that enables us to reduce many questions to the consideration of p-points over algebraically closed fields. Here is the relevant Lemma:

Lemma 4.1. *Let $\mathcal{G}$ be a finite k-group scheme, $L \supseteq K$ be extension fields of k.*

1. *A π-point $\alpha_L : \mathfrak{A}_{p,L} \longrightarrow L\mathcal{G}_K$ of $\mathcal{G}_K$ can be considered as a π-point $\widehat{\alpha}_L : \mathfrak{A}_{p,L} \longrightarrow L\mathcal{G}$ of $\mathcal{G}$.*
2. *The map $\Pi(\mathcal{G}_K) \longrightarrow \Pi(\mathcal{G})$; $[\alpha_L] \mapsto [\widehat{\alpha}_L]$ is well-defined.*

By the Universal Coefficient Theorem, we may consider $\mathrm{H}^\bullet(\mathcal{G}, k)$ canonically as a k-subalgebra of the K-algebra $\mathrm{H}^\bullet(\mathcal{G}_K, K)$. As before, every π-point $\alpha_K : \mathfrak{A}_{p,K} \longrightarrow K\mathcal{G}$ defines a non-trivial homomorphism

$$\alpha_K^\bullet : \mathrm{H}^\bullet(\mathcal{G}_K, K) \longrightarrow \mathrm{H}^\bullet(\mathfrak{A}_{p,K}, K).$$

In particular, $\ker \alpha_K^\bullet \cap \mathrm{H}^\bullet(\mathcal{G}, k)$ is a homogeneous prime ideal of $\mathrm{H}^\bullet(\mathcal{G}, k)$ which does not coincide with the augmentation ideal $\bigoplus_{n\geq 1} \mathrm{H}^{2n}(\mathcal{G}, k)$ of $\mathrm{H}^\bullet(\mathcal{G}, k)$. We let $\mathrm{Proj}(\mathrm{H}^\bullet(\mathcal{G}, k))$ be the set of these ideals.

Theorem 4.2 (Friedlander-Pevtsova,2006)**.** *Let $\mathcal{G}$ be a finite group scheme over k. Then*

$$\Psi_{\mathcal{G}} : \Pi(\mathcal{G}) \longrightarrow \mathrm{Proj}(\mathrm{H}^\bullet(\mathcal{G},k)) \ \ ; \ \ [\alpha_K] \mapsto \ker \alpha_K^\bullet \cap \mathrm{H}^\bullet(\mathcal{G},k)$$

is a bijective map.

Proof. By way of illustration, let me outline the argument for the surjectivity of the map $\Psi_{\mathcal{G}}$. Let $\mathfrak{p} \in \mathrm{Proj}(\mathrm{H}^\bullet(\mathcal{G},k))$ be a prime ideal. We define K to be the algebraic closure of the residue field of $\mathfrak{p}$. Then there exists a maximal element $\mathfrak{P} \in \mathrm{Proj}(\mathrm{H}^\bullet(\mathcal{G}_K,K))$ with $\mathfrak{p} = \mathfrak{P} \cap \mathrm{H}^\bullet(\mathcal{G},k)$.

The field extension $K\!:\!k$ gives rise to a commutative diagram

$$\begin{array}{ccc} \Pi(\mathcal{G}_K) & \xrightarrow{\Psi_{\mathcal{G}_K}} & \mathrm{Proj}(\mathrm{H}^\bullet(\mathcal{G}_K,K)) \\ \Big\downarrow{\scriptstyle \mathrm{res}_1} & & \Big\downarrow{\scriptstyle \mathrm{res}_2} \\ \Pi(\mathcal{G}) & \xrightarrow{\Psi_{\mathcal{G}}} & \mathrm{Proj}(\mathrm{H}^\bullet(\mathcal{G},k)), \end{array}$$

where res_2 is induced by base change and res_1 is defined in Lemma 4.1. Thanks to (3.4), we can find $x \in \Pi(\mathcal{G}_K)$ with $\Psi_{\mathcal{G}_K}(x) = \mathfrak{P}$. Consequently, $\mathrm{res}_1(x) \in \Pi(\mathcal{G})$ is the desired pre-image of $\mathfrak{p}$ under $\Psi_{\mathcal{G}}$. □

Let M be an arbitrary $k\mathcal{G}$-module (not necessarily finite-dimensional). In analogy with Section 2 one defines the Π-*support of M* via

$$\Pi(\mathcal{G})_M := \{[\alpha_K] \in \Pi(\mathcal{G}) \ ; \ \alpha_K^*(M_K) \text{ is not projective}\}.$$

The sets $\Pi(\mathcal{G})_M$ enjoy the usual properties with respect to tensor products and short exact sequences. We summarize the new features in the following:

Theorem 4.3. *Let $\mathcal{G}$ be a finite group scheme over a field k.*

(1) *The set $\{\Pi(\mathcal{G})_M \ ; \ \dim_k M < \infty\}$ defines the closed sets of a noetherian topology on $\Pi(\mathcal{G})$.*
(2) *$\Psi_{\mathcal{G}}$ is a homeomorphism.*
(3) *A $\mathcal{G}$-module M is projective if and only if $\Pi(\mathcal{G})_M = \emptyset$.*
(4) *For every subset $\mathcal{X} \subseteq \Pi(\mathcal{G})$ there exists a $k\mathcal{G}$-module M such that $\mathcal{X} = \Pi(\mathcal{G})_M$.*
(5) *If k is algebraically closed, then $P(\mathcal{G})$ is the set of closed points of $\Pi(\mathcal{G})$.*

Property (4) is related to Rickard's idempotent modules: If α_K is a π-point, then $\overline{[\alpha_K]}$ defines a thick tensor ideal that can be described via two infinite dimensional modules E_α and F_α. Their tensor product has Π-support $\{[\alpha_K]\}$.

We shall allude to the feature of "genericity" by briefly discussing two examples involving restricted Lie algebras. Recall that the equivalence classes of p-points of a restricted Lie algebra $(\mathfrak{g},[p])$ correspond to elements of the projective space $\mathrm{Proj}(\widehat{\mathcal{V}}_{\mathfrak{g}})$ associated to the nullcone $\widehat{\mathcal{V}}_{\mathfrak{g}}$ of $\mathfrak{g}$. This description does not require the ground field to be algebraically closed, so that we may assume that a π-point $\alpha_K : \mathfrak{A}_{p,K} \longrightarrow U_0(\mathfrak{g} \otimes_k K)$ is given by $\alpha_K(x) \in \widehat{\mathcal{V}}_{\mathfrak{g}\otimes_k K}$.

Examples. We fix a field k and assume that $p \geq 3$.

(1) Let $\mathfrak{g} := ky_1 \oplus ky_2 \oplus kz$ be the three-dimensional *Heisenberg algebra*, given by $[y_1,y_2] = z$ and $[z,\mathfrak{g}] = (0)$, as well as $y_1^{[p]} = 0 = y_2^{[p]}$ and $z^{[p]} = z$. Since $p \geq 3$, the resulting p-map is p-semilinear, so that $\widehat{\mathcal{V}}_{\mathfrak{g}} = ky_1 \oplus ky_2 \cong \mathbb{A}^2$. Note that this holds for every field extension $K\!:\!k$. Let $K := k(X_1,X_2)$ be the field of rational functions in two variables, and consider the π-point

$$\alpha_K : \mathfrak{A}_{p,K} \longrightarrow U_0(\mathfrak{g} \otimes_k K) \ \ ; \ \ x \mapsto X_1y_1 + X_2y_2.$$

Up to equivalence, every π-point β_L of $\mathfrak{g}$ arises by substituting X_1 and X_2 by elements of L. Direct computation shows that, given a $U_0(\mathfrak{g})$-module M, we have $\beta_L^*(M_L)$ free $\Rightarrow \alpha_K^*(M_K)$ free. If two π-points possess this property, then β_L is referred to as a *specialization* of α_K.

(2) We consider $\mathfrak{g} = \mathfrak{sl}(2)$ with its standard basis $\{e, h, f\}$ and recall that

$$\widehat{\mathcal{V}}_{\mathfrak{sl}(2)} = \{\alpha e + \beta h + \gamma f \ ; \ \beta^2 + \alpha\gamma = 0\}.$$

As before, we let K be the field of fractions of the coordinate ring $k[\widehat{\mathcal{V}}_{\mathfrak{sl}(2)}] \cong k[X_1, X_2, X_3]/(X_2^2 + X_1X_3)$, and denote its canonical generators by x_1, x_2, x_3. Then

$$\alpha_K : \mathfrak{A}_{p,K} \longrightarrow U_0(\mathfrak{sl}(2) \otimes_k K) \ \ ; \ \ x \mapsto x_1e + x_2h + x_3f$$

is a "generic" π-point in the above sense.

4.2. **Jordan Type.** Throughout this section, we let k be a field of characteristic $p > 0$. Instead of asking whether the pull-back $\alpha_K^*(M_K)$ of a finite-dimensional $\mathcal{G}_K$-module is projective, one can equally well investigate the isomorphism type of the $\mathfrak{A}_{p,K}$-module $\alpha_K^*(M_K)$. This point of view is adopted in recent articles by Friedlander-Pevtsova-Suslin and Carlson-Friedlander-Pevtsova.

Let N be a finite-dimensional $\mathfrak{A}_{p,K}$-module. The isoclass $[N]$ of N is given by a partition of $n := \dim_K N$, called the *Jordan type* of N. Note that the Jordan type does not change under base field extension.

Recall that the *dominance order* on the set of partitions of n is given by

$$\lambda \trianglelefteq \mu \ :\Leftrightarrow \ \sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i \qquad \text{for } 1 \leq j \leq n.$$

Let $\mathcal{G}$ be a finite group scheme over k, M be a finite-dimensional $\mathcal{G}$-module. If $\alpha_K : \mathfrak{A}_{p,K} \longrightarrow K\mathcal{G}$ is a π-point, then we can consider the Jordan type $\mathrm{Jt}(M, \alpha_K) := [\alpha_K^*(M_K)]$ of M relative to α_K. In general, the Jordan type depends on the π-point α_K and not just on its equivalence class. We let $\mathrm{Jt}(M)$ be the set of all Jordan types of M. A maximal element of $\mathrm{Jt}(M)$ is called a *maximal Jordan type for M*.

In their recent paper, Friedlander-Pevtsova-Suslin have proved the following remarkable result:

Theorem 4.4 (Friedlander-Pevtsova-Suslin,2006)**.** *Let $\mathcal{G}$ be a finite group scheme, M be a finite dimensional $\mathcal{G}$-module.*

(1) *If $[\alpha_K] \in \Pi(\mathcal{G})$ is a generic point, then $\mathrm{Jt}(M, \beta_L) = \mathrm{Jt}(M, \alpha_K)$ for all $\beta_L \in [\alpha_K]$.*
(2) *The set $\widetilde{\Pi}(\mathcal{G})_M := \{x \in \Pi(\mathcal{G}) \ ; \ \mathrm{Jt}(M, \alpha_K) \text{ is not maximal for some } \alpha_K \in x\}$ is closed.*

Example. Let G be a smooth (=reduced) reductice group with Lie algebra $\mathfrak{g}$. If p exceeds the Coxeter number of G, then every nilpotent element $x \in \mathfrak{g}$ lies in the nullcone $\widehat{\mathcal{V}}_{\mathfrak{g}}$ and the so-called regular nilpotent elements lie dense in $\widehat{\mathcal{V}}_{\mathfrak{g}}$. If M is a $U_0(\mathfrak{g})$-module, then not all regular nilpotent elements can give rise to elements of $\widetilde{\Pi}(\mathcal{G})_M$, so that a maximal Jordan type is attained at one of these elements.

The investigations of Friedlander-Pevtsova-Suslin motivate the following notion, which was subsequently studied by Carlson-Friedlander-Pevtsova:

Definition. A finite-dimensional $\mathcal{G}$-module M is said to have *constant Jordan type* if $\mathrm{Jt}(M)$ is a singleton.

Examples. (1) If M is projective, then M has constant Jordan type. The trivial $\mathcal{G}$-module has constant Jordan type.

(2) If M is non-projective of constant Jordan type, then M is a module of full support, that is, $\Pi(\mathcal{G})_M = \Pi(\mathcal{G})$.

(3) Every *endo-trivial* $\mathcal{G}$-module M (i.e., $\mathrm{End}_k(M) \cong k \oplus (\mathrm{proj.})$), has constant Jordan type. This is a Theorem by Carlson-Friedlander-Pevtsova.

(4) If M has constant Jordan type, so does $\Omega^n_{\mathcal{G}}(M)$.

We have the following characterization of modules of constant Jordan type:

Theorem 4.5 (Carlson-Friedlander-Pevtsova,2006)**.** *A $\mathcal{G}$-module M has constant Jordan type if and only if $\widetilde{\Pi}(\mathcal{G})_M = \emptyset$.*

Let $\Gamma_s(\mathcal{G})$ be the stable Auslander-Reiten quiver of the self-injective algebra $k\mathcal{G}$. Given a component $\Theta \subseteq \Gamma_s(\mathcal{G})$, it is well-known that

$$\Pi(\mathcal{G})_M = \Pi(\mathcal{G})_N \quad \text{for all } [M], [N] \in \Theta.$$

The following result has a similar flavor:

Theorem 4.6 (CFP,2006)**.** *Suppose that k is perfect and let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component. If $[M] \in \Theta$ has constant Jordan type, then any module N belonging to Θ has constant Jordan type.*

The reason for the condition on k is that Auslander-Reiten sequences do not interact well with purely inseparable base field extensions.

Example. Suppose that $\mathcal{B} \subset k\mathcal{G}$ is a block of finite representation type. If $\mathcal{B}$ possesses a non-projective module M of constant Jordan type, all $\mathcal{B}$-modules have constant Jordan type: By general theory, the stable Auslander-Reiten quiver $\Gamma_s(\mathcal{B})$ is connected (as a stable translation quiver), so that (4.6) applies.

Thus, if the principal block $\mathcal{B}_0(\mathcal{G}) \subseteq k\mathcal{G}$ is representation-finite, then all $\mathcal{B}_0(\mathcal{G})$-modules have constant Jordan type.

By assumption, M is a module of full support and complexity 1, so that $\dim \mathcal{V}_{\mathcal{G}}(k) = 1$. Consequently, all non-projective indecomposable $\mathcal{G}$-modules are periodic and if $\mathcal{G}$ is an infinitesimal group, the algebra $k\mathcal{G}$ has finite representation type.

Let us look at the example $U_0(\mathfrak{sl}(2))$, defined over an algebraically closed field of characteristic $p \geq 3$. Recall that the algebra $U_0(\mathfrak{sl}(2))$ is tame, with one simple block, and with all other blocks being Morita equivalent to the trivial extension of the Kronecker algebra.

Corollary 4.7. *If M is an indecomposable $U_0(\mathfrak{sl}(2))$-module of full support, then M has constant Jordan type.*

Proof. By assumption, M belongs to a tame block $\mathcal{B} \subseteq U_0(\mathfrak{sl}(2))$. As M has full support, it has complexity $\mathrm{cx}_{U_0(\mathfrak{sl}(2))}(M) = 2$. Hence M is not periodic and thus belongs to a component of type $\mathbb{Z}[\tilde{A}_{12}]$. By general theory, such a component contains a simple $U_0(\mathfrak{sl}(2))$-module S.

Recall that $\mathrm{SL}(2)(k)$ acts on $\mathfrak{sl}(2)$ and hence on $U_0(\mathfrak{sl}(2))$ via automorphisms. This action leaves the nullcone $\widehat{\mathcal{V}}_{\mathfrak{sl}(2)}$ invariant with $\widehat{\mathcal{V}}_{\mathfrak{sl}(2)} \setminus \{0\}$ being one orbit. The same applies then to the canonical action of $\mathrm{SL}(2)(k)$ on the space $P(\mathfrak{sl}(2))$ of p-points of $\mathfrak{sl}(2)$.

The group $\mathrm{SL}(2)(k)$ also acts on the isoclasses of modules. For $g \in \mathrm{SL}(2)(k)$, we consider $g.S := (g^{-1})^*(S)$. Since non-isomorphic simple $U_0(\mathfrak{sl}(2))$-modules occur in different dimensions, we obtain

$$S \cong g.S \qquad \forall\, g \in \mathrm{SL}(2)(k).$$

Given a p-point $\alpha : \mathfrak{A}_p \longrightarrow U_0(\mathfrak{sl}(2))$, we thus have

$$(g.\alpha)^*(S) \cong \alpha^*(g^{-1}.S) \cong \alpha^*(S).$$

Hence we may assume that α factors through the subalgebra $U_0(kf) \cong \mathfrak{A}_p$ (where $\{e, h, f\}$ is the standard basis of $\mathfrak{sl}(2)$). Recall that the simple $U_0(\mathfrak{sl}(2))$-modules are cyclic f-spaces. Hence S has constant Jordan type $(\dim_k S)$. Theorem 4.6 then shows that M also has constant Jordan type. □

The just emerging theory of Jordan types offers many open problems that are currently not well understood. For instance, it is not clear which Jordan types can be realized by modules of constant Jordan type. Here is a first result in this direction:

Theorem 4.8 (CFP,2006). *Let $\mathcal{G}$ be a finite group scheme over an algebraically closed field such that* $\dim \Pi(\mathcal{G}) \geq 2$. *Then there exists for any $n \in \mathbb{N}$ an indecomposable module M_n of constant Jordan type* $(p, p, \ldots, p, 1, \ldots, 1)$, *with* 1 *occurring n times.*

Proof. Since $\dim \Pi(\mathcal{G}) \geq 2$, a theorem of mine ensures that the connected component $\Theta_0 \subseteq \Gamma_s(\mathcal{G})$ containing the trivial module k is isomorphic to $\mathbb{Z}[A_\infty]$. As k has constant Jordan type (1), every module belonging to Θ_0 also has constant Jordan type. The result now follows from the fact that sending a module to the projective-free part of its Jordan type behaves like an additive function. □

Let me conclude by mentioning a nice result on tensor products:

Theorem 4.9 (CFP,2006). *Let $\mathcal{G}$ be a finite group scheme. If the $\mathcal{G}$-modules M and N have constant Jordan type, then $M \otimes_k N$ also has constant Jordan type.*

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