SUPPORT VARIETIES, AR-COMPONENTS AND GOOD FILTRATIONS

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The leitmotiv underlying these lectures is a reduction technique, which appears in different guises in Lie theory and the representation theory of quivers. The idea is to study questions concerning module categories via passage to categories that are better behaved. In abstract representation theory this is achieved by means of covering techniques; the most prominent examples in Lie theory are the BGG category $\mathcal{O}$ of a complex semi-simple Lie algebra and the categories of G_rT-modules associated to a reductive algebraic group of positive characteristic. The common basic concept is that of gradations of algebras and modules. In the context of Artin algebras, categories of graded modules were first systematically studied by Gordon and Green [21, 22, 23]. In the preface to their article [22], the authors write: "Our point of view is to study graded Λ-modules, which we believe are more tractable than ungraded ones, in order to obtain information on all Λ-modules."

1. Lecture I: Representations of Graded Algebras

In this lecture, we provide basic results on graded modules over algebras that are graded relative to a free abelian group of rank n. Our presentation is based on work by Gordon and Green [21, 22], who developed the theory within the more general framework of Artin algebras. In [28], one can find an account that focuses more on the special case of reduced enveloping algebras of reductive Lie algebras. We refer to [2] or [1] for general facts on finite-dimensional algebras and to [3, 25] for notions concerning categories.

The results to be presented here were also obtained by Gabriel [20], who used the language of coverings introduced in [33]. In view of [24], coverings of graphs and group-graded algebras are equivalent concepts.

1.1. **Preliminaries.** Throughout, Λ denotes a finite-dimensional algebra over a field k. We let $\operatorname{mod}\Lambda$ be the category of finite-dimensional left Λ-modules. We say that Λ is $\mathbb{Z}^n$*-graded*, provided

(a) $\Lambda = \bigoplus_{i\in\mathbb{Z}^n} \Lambda_i$, and

(b) $\Lambda_i\Lambda_j \subseteq \Lambda_{i+j}$ for $i, j \in \mathbb{Z}^n$.

Graded ideals and graded subalgebras are defined canonically. For instance, the center $Z(\Lambda)$ of Λ is a graded subalgebra of Λ.

A Λ-module M is $\mathbb{Z}^n$*-graded* if

(a) $M = \bigoplus_{j\in\mathbb{Z}^n} M_j$, and

(b) $\Lambda_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}^n$

We shall usually suppress explicit reference to the group $\mathbb{Z}^n$ and just talk about graded modules and algebras.

The category $\operatorname{mod}\operatorname{gr}\Lambda$ has objects the graded Λ-modules and morphisms $f : M \longrightarrow N$ the Λ-linear maps satisfying $f(M_j) \subseteq N_j$ for every $j \in \mathbb{Z}^n$. We denote the space of these morphisms by $\operatorname{Hom}_{\operatorname{gr}\Lambda}(M, N)$.

The category $\operatorname{mod}\operatorname{gr}\Lambda$ shares some important properties with $\operatorname{mod}\Lambda$:

- $\operatorname{mod}\operatorname{gr}\Lambda$ is an abelian category.
- $\operatorname{mod}\operatorname{gr}\Lambda$ is a Krull-Schmidt category (Fitting's Lemma).

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We introduce some important functors:

- The *forgetful functor* $\mathfrak{F} : \operatorname{mod}\operatorname{gr}\Lambda \longrightarrow \operatorname{mod}\Lambda$. We let $\mathfrak{F}(\operatorname{mod}\operatorname{gr}\Lambda)$ be the full subcategory of $\operatorname{mod}\Lambda$, whose objects are isomorphic to objects of the form $\mathfrak{F}(M)$. The objects of $\mathfrak{F}(\operatorname{mod}\operatorname{gr}\Lambda)$ are referred to as *gradable* Λ-modules.
- Given $i \in \mathbb{Z}^n$, the *i-th shift functor* $[i] : \operatorname{mod}\operatorname{gr}\Lambda \longrightarrow \operatorname{mod}\operatorname{gr}\Lambda$ associates to $M \in \operatorname{mod}\operatorname{gr}\Lambda$ the object $M[i]$, where $M[i]_j := M_{j-i}$, and leaves the morphisms unchanged. Note that we have $\mathfrak{F} \circ [i] = \mathfrak{F}$.

For $M, N \in \operatorname{mod}\operatorname{gr}\Lambda$ and $i \in \mathbb{Z}^n$, we put

$$\operatorname{Hom}_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N))_i := \{f \in \operatorname{Hom}_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N)) \ ; \ f(M_j) \subseteq N_{j+i} \ \ \forall\, j \in \mathbb{Z}^n\}.$$

Then

$$\operatorname{Hom}_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N)) = \bigoplus_{i \in \mathbb{Z}^n} \operatorname{Hom}_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N))_i$$

is a graded module over the graded algebra $\operatorname{End}_\Lambda(\mathfrak{F}(N))$. Directly from the definitions we obtain

$$\operatorname{Hom}_{\operatorname{gr}\Lambda}(M[i], N) = \operatorname{Hom}_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N))_i = \operatorname{Hom}_{\operatorname{gr}\Lambda}(M, N[-i]),$$

so that in particular $\operatorname{Hom}_{\operatorname{gr}\Lambda}(M, N)) = \operatorname{Hom}_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N))_0$.

Given $M \in \operatorname{mod}\operatorname{gr}\Lambda$, the dual space M^* is a left module over the graded algebra Λ^{op}. Setting

$$M^*_j := \{f \in M^* \ ; \ f(M_i) = (0) \ \ \forall\, i \neq -j\}$$

we endow M^* with the structure of a graded Λ^{op}-module. In this fashion, the ordinary duality $D : \operatorname{mod}\Lambda \longrightarrow \operatorname{mod}\Lambda^{\mathrm{op}}$ induces a duality $D : \operatorname{mod}\operatorname{gr}\Lambda \longrightarrow \operatorname{mod}\operatorname{gr}\Lambda^{\mathrm{op}}$ such that

$$D \circ \mathfrak{F} = \mathfrak{F} \circ D.$$

1.2. **Indecomposable Modules.** The main result of this section asserts that the forgetful functor $\mathfrak{F} : \operatorname{mod}\operatorname{gr}\Lambda \longrightarrow \operatorname{mod}\Lambda$ preserves indecomposables. We begin by recalling a basic result from Lie theory.

Let Λ be a finite-dimensional k-algebra with associated Lie bracket $[x, y] := xy - yx$. A subset $S \subseteq \Lambda$ is a *Lie subset* if $[s, t] \in S$ for all $s, t \in S$.

Theorem 1.1 (Engel-Jacobson,[36])**.** *Let Λ be a finite-dimensional k-algebra, $S \subseteq \Lambda$ be a Lie subset. If s is nilpotent for every $s \in S$, then the associative algebra $\operatorname{alg}(S)$ without identity generated by S is nilpotent.* □

Recall that a Λ-module affording a unique maximal submodule is referred to as *local*. We say that Λ is local if the regular module Λ local.

Theorem 1.2. *Suppose that Λ_0 is local. Then $\Lambda/\operatorname{Rad}(\Lambda) \cong \Lambda_0/\operatorname{Rad}(\Lambda_0)$, and Λ is local.*

Proof. For $i \in \mathbb{Z}^n$, we let

$$N_i := \{x \in \Lambda_i \ ; \ x \text{ is nilpotent}\}.$$

Since $\mathbb{Z}^n$ is torsion-free, we have $N_i = \Lambda_i$ for all $i \neq 0$. Given $x \in \Lambda_i$ and $y \in N_{-i}$, the element $xy \in \Lambda_0$ is not invertible. As Λ_0 is local, we conclude that $xy \in N_0$. It follows that the Lie subset $\bigcup_{i \in \mathbb{Z}^n} N_i \subseteq \Lambda$ consists of nilpotent elements. By the Engel-Jacobson Theorem, the ideal $N := \bigoplus_{i \in \mathbb{Z}^n} N_i$ is nilpotent, and

$$\Lambda/N \cong \Lambda_0/N_0$$

is a division algebra. As a result, Λ is local. □

Remark. A suitable modification of the above arguments yields an analogous result for algebras over algebraically closed fields of characteristic $p > 0$ that are graded by p-groups (cf. [13]).

Corollary 1.3. *The functor $\mathfrak{F} : \operatorname{mod} \operatorname{gr} \Lambda \longrightarrow \operatorname{mod} \Lambda$ preserves and reflects indecomposables.*

Proof. Since $\mathfrak{F}$ is an additive functor, it reflects indecomposables.

Suppose that $M \in \operatorname{mod} \operatorname{gr} \Lambda$ is indecomposable. Then

$$\operatorname{End}_\Lambda(\mathfrak{F}(M)) = \bigoplus_{i \in \mathbb{Z}^n} \operatorname{End}_\Lambda(\mathfrak{F}(M))_i$$

is a $\mathbb{Z}^n$-graded algebra with $\operatorname{End}_\Lambda(\mathfrak{F}(M))_0 \cong \operatorname{End}_{\operatorname{gr} \Lambda}(M)$. Since M is indecomposable, the latter algebra is local. Our assertion now follows from Theorem 1.2. □

Let $M \in \operatorname{mod} \operatorname{gr} \Lambda$ be graded. A submodule $N \subseteq \mathfrak{F}(M)$ is said to be *homogeneous* if

$$N = \bigoplus_{i \in \mathbb{Z}^n} (N \cap M_i).$$

We write $X \,|\, Y$ to indicate that X is isomorphic to a direct summand of Y.

Proposition 1.4. *Let Λ be a graded k-algebra.*

(1) *If $M \in \operatorname{mod} \Lambda$ is gradable and $N \,|\, M$, then N is gradable.*
(2) *If $P \in \operatorname{mod} \Lambda$ is projective (injective), then P is gradable.*
(3) *If $M \in \operatorname{mod} \operatorname{gr} \Lambda$, then $\operatorname{Rad}(\mathfrak{F}(M))$ is a homogeneous submodule of $\mathfrak{F}(M)$.*
(4) *The forgetful functor preserves and reflects simple objects.*
(5) *If $S \in \operatorname{mod} \Lambda$ is simple, then S is gradable.*

Proof. (1) By assumption, there exists $X \in \operatorname{mod} \operatorname{gr} \Lambda$ such that $\mathfrak{F}(X) \cong M$. If

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_m$$

is a decomposition of X into indecomposables, then Corollary 1.3 implies that

$$M \cong \mathfrak{F}(X_1) \oplus \mathfrak{F}(X_2) \oplus \cdots\cdots \oplus \mathfrak{F}(X_m)$$

is a similar decomposition of M. The theorem of Krull-Remak-Schmidt now yields $N \cong \mathfrak{F}(Y)$ for some direct summand $Y \,|\, X$.

(2) This is a direct consequence of (1) and our remarks on duality.

(3) Let $J := \operatorname{Rad}(\Lambda)$ be the Jacobson radical of Λ. Then $\varphi(J) = J$ for every automorphism φ of Λ.

We consider the group $T := (k^\times)^n$. For $d \in \mathbb{Z}^n$, we define the homomorphism

$$\chi_d : T \longrightarrow k^\times \quad ; \quad (\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1^{d_1} \cdots \alpha_n^{d_n}.$$

The torus T acts on Λ via

$$t.x := \chi_i(t)x \quad \forall\, x \in \Lambda_i,\ i \in \mathbb{Z}^n.$$

A submodule $X \subseteq \Lambda$ is homogeneous if and only if X is T-invariant. Since $\chi_i(t)\chi_j(t) = \chi_{i+j}(t)$, the map $x \mapsto t.x$ is an automorphism of k-algebras. Thus, J is T-invariant and therefore homogeneous. It readily follows that $\operatorname{Rad}(\mathfrak{F}(M)) = JM$ is a homogeneous submodule of $\mathfrak{F}(M)$.

(4) Let $S \in \operatorname{mod} \operatorname{gr} \Lambda$ be simple. According to (1.3), the Λ-module $\mathfrak{F}(S)$ is indecomposable. By (3), $\operatorname{Rad}(\mathfrak{F}(S))$ is a homogeneous submodule of $\mathfrak{F}(S)$ and the simplicity of S forces $\operatorname{Rad}(\mathfrak{F}(S)) = (0)$. Thus, the indecomposable Λ-module $\mathfrak{F}(S)$ is semi-simple and therefore simple.

(5) Let $P(S)$ be the projective cover of S. Thanks to (2), there exists a graded module P with $\mathfrak{F}(P) \cong P(S)$. Now (3) ensures that $\operatorname{Rad}(P(S))$ is a homogeneous submodule of $P(S)$. It follows that $S \cong \mathfrak{F}(P/\operatorname{Rad}(P(S)))$ is gradable. □

Our next result is fundamental for much of what is going to follow:

Theorem 1.5. *Let M, $N \in \operatorname{mod} \operatorname{gr} \Lambda$ be indecomposable graded Λ-modules. If $\mathfrak{F}(M) \cong \mathfrak{F}(N)$, then there exists exactly one $i \in \mathbb{Z}^n$ such that $M \cong N[i]$.*

Proof. Recall that $\operatorname{Hom}_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N)) = \bigoplus_{i \in \mathbb{Z}^n} \operatorname{Hom}_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N))_i$ is a graded left module of the graded algebra $\operatorname{End}_\Lambda(\mathfrak{F}(N))$. In view of Corollary 1.3 and our current assumption, the space $\operatorname{Hom}_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N))$ is a local $\operatorname{End}_\Lambda(\mathfrak{F}(N))$-module, and Proposition 1.4(3) implies that its radical is homogeneous. Since the given isomorphism $f : \mathfrak{F}(M) \longrightarrow \mathfrak{F}(N)$ does not belong to $\operatorname{Rad}(\operatorname{Hom}_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N))$, there exists a homogeneous component $f_{-i} \in \operatorname{Hom}_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N))_{-i}$ which is not in the radical. This also applies to $f_{-i} \circ f^{-1} \in \operatorname{End}_\Lambda(\mathfrak{F}(N))$, so that this map is invertible. Consequently, f_{-i} is surjective and thus an isomorphism. As a result, $M \cong N[i]$.

If we also have $M \cong N[j]$, then $M[j-i] \cong N[j] \cong M$, so that there exists an invertible element of $\operatorname{End}_\Lambda(\mathfrak{F}(M))$ which belongs to $\operatorname{End}_\Lambda(\mathfrak{F}(M))_{j-i}$. This readily implies $j = i$, as desired. □

Given $M \in \operatorname{mod} \operatorname{gr} \Lambda$, we define $\operatorname{Rad}_{\operatorname{gr} \Lambda}(M)$ and $\operatorname{Soc}_{\operatorname{gr} \Lambda}(M)$ is the canonical fashion. Since the exact functor $\mathfrak{F}$ is additive and preserves simples, we have

$$\mathfrak{F}(\operatorname{Rad}_{\operatorname{gr} \Lambda}(M)) = \operatorname{Rad}_\Lambda(\mathfrak{F}(M)) \ \text{ and } \ \mathfrak{F}(\operatorname{Soc}_{\operatorname{gr} \Lambda}(M)) = \operatorname{Soc}_\Lambda(\mathfrak{F}(M)).$$

Proposition 1.6. *The following statements hold:*

(1) *The category* $\operatorname{mod} \operatorname{gr} \Lambda$ *has projective covers.*

(2) *If $P \xrightarrow{f} M$ is a projective cover in $\operatorname{mod} \operatorname{gr} \Lambda$, then $\mathfrak{F}(P) \xrightarrow{\mathfrak{F}(f)} \mathfrak{F}(M)$ is a projective cover.*

Proof. (1) Let $S \in \operatorname{mod} \operatorname{gr} \Lambda$ be a simple object, and consider a projective cover $P(\mathfrak{F}(S)) \longrightarrow \mathfrak{F}(S)$. In virtue of Proposition 1.4, there exists a projective object $P \in \operatorname{mod} \operatorname{gr} \Lambda$ such that $\mathfrak{F}(P) \cong P(\mathfrak{F}(S))$. Hence $\mathfrak{F}(P) \xrightarrow{\lambda} \mathfrak{F}(S)$ is a projective cover.

We write $\lambda = \sum_{i \in \mathbb{Z}^n} \lambda_i$, with $\lambda_i \in \operatorname{Hom}_\Lambda(\mathfrak{F}(P), \mathfrak{F}(S))_i$. If λ_j is non-zero, then it is surjective (cf. (1.4)) and hence defines a surjective morphism $\lambda_j : P[j] \longrightarrow S$. Since the object $P[j]$ is local, we obtain $\ker \lambda_j = \operatorname{Rad}_{\operatorname{gr} \Lambda}(P[j])$, so that $P[j]$ is a projective cover of S.

Thus, every semi-simple object has a projective cover and so does every object $M \in \operatorname{mod} \operatorname{gr} \Lambda$. □

We can now define the *transpose* $\operatorname{Tr}_{\operatorname{gr} \Lambda}(M)$ of an object $M \in \operatorname{mod} \operatorname{gr} \Lambda$: Given M, we consider a minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow (0).$$

Application of $\operatorname{Hom}_\Lambda(-, \Lambda) \circ \mathfrak{F}$ yields an exact sequence

$$\operatorname{Hom}_\Lambda(\mathfrak{F}(P_0), \Lambda) \xrightarrow{\mathfrak{F}(f)^*} \operatorname{Hom}_\Lambda(\mathfrak{F}(P_1), \Lambda) \longrightarrow \operatorname{coker} \mathfrak{F}(f)^* \longrightarrow (0)$$

of graded Λ^{op}-modules. We then define

$$\operatorname{Tr}_{\operatorname{gr} \Lambda}(M) := \operatorname{coker} \mathfrak{F}(f)^*$$

and observe that the isoclass of $\operatorname{Tr}_{\operatorname{gr} \Lambda}(M)$ does not depend on the choice of the minimal presentation.

Lemma 1.7. (1) *If $M \in \operatorname{mod} \operatorname{gr} \Lambda$, then $\mathfrak{F}(D \operatorname{Tr}_{\operatorname{gr} \Lambda}(M)) \cong D \operatorname{Tr}_\Lambda(\mathfrak{F}(M))$.*

(2) *If $M \in \operatorname{mod} \Lambda$ is gradable, so is $D \operatorname{Tr}_\Lambda(M)$.*

Proof. (1) If

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow (0)$$

is a minimal projective presentation of M, then, by Proposition 1.6,

$$\mathfrak{F}(P_1) \xrightarrow{\mathfrak{F}(f)} \mathfrak{F}(P_0) \longrightarrow \mathfrak{F}(M) \longrightarrow (0)$$

is a minimal projective presentation of $\mathfrak{F}(M)$. Consequently,

$$\mathrm{Tr}_\Lambda(\mathfrak{F}(M)) \cong \mathfrak{F}(\mathrm{Tr}_{\mathrm{gr}\,\Lambda}(M)).$$

Since $\mathfrak{F}$ commutes with taking duals, we obtain the desired result.

(2) This is a direct consequence of (1). $\square$

Let $M, N \in \mathrm{mod}\,\mathrm{gr}\,\Lambda$. Given $m \in \mathbb{N}$, the space $\mathrm{Ext}^m_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N))$ is graded and we have

$$\mathrm{Ext}^m_{\mathrm{gr}\,\Lambda}(M[i], N) \cong \mathrm{Ext}^m_\Lambda(\mathfrak{F}(M), \mathfrak{F}(N))_i \cong \mathrm{Ext}^m_{\mathrm{gr}\,\Lambda}(M, N[-i])$$

for every $i \in \mathbb{Z}^n$.

2. Lecture II: Almost Split Sequences and G_rT-modules

2.1. **Almost split sequences in modgrΛ.** As before, we consider a finite-dimensional $\mathbb{Z}^n$-graded algebra

$$\Lambda = \bigoplus_{i \in \mathbb{Z}^n} \Lambda_i$$

Given a simple module $S \in \mathrm{mod}\,\mathrm{gr}\,\Lambda$, the shifts $S[i]$ are also simple and pairwise non-isomorphic. Hence $\mathrm{mod}\,\mathrm{gr}\,\Lambda$ is not the module category of a finite-dimensional algebra, so the existence of almost split sequences within $\mathrm{mod}\,\mathrm{gr}\,\Lambda$ is not automatic.

On the other hand, almost split sequence are defined locally and thus do not involve all of $\mathrm{mod}\,\mathrm{gr}\,\Lambda$: Every $M \in \mathrm{mod}\,\mathrm{gr}\,\Lambda$ has a finite *support*

$$\mathrm{supp}(M) := \{i \in \mathbb{Z}^n \ ; \ M_i \neq (0)\}.$$

For a finite subset $I \subseteq \mathbb{Z}^n$, we let $\mathrm{mod}_I\,\mathrm{gr}\,\Lambda$ be the full subcategory of $\mathrm{mod}\,\mathrm{gr}\,\Lambda$, whose objects M satisfy $\mathrm{supp}(M) \subseteq I$.

Theorem 2.1. *Let $I \subseteq \mathbb{Z}^n$ be a finite subset. The category $\mathrm{mod}_I\,\mathrm{gr}\,\Lambda$ is equivalent to the module category of some finite-dimensional algebra Γ.*

Sketch of Proof. The category $\mathrm{mod}_I\,\mathrm{gr}\,\Lambda$ is a length category such that

(a) $\mathrm{mod}_I\,\mathrm{gr}\,\Lambda$ has finitely many simple objects, and

(b) $\dim_k \mathrm{Ext}^1_{\mathrm{mod}_I\,\mathrm{gr}\,\Lambda}(M, N) < \infty$ for all $M, N \in \mathrm{mod}_I\,\mathrm{gr}\,\Lambda$, and

(c) there is a common bound for the Loewy lengths of the objects of $\mathrm{mod}_I\,\mathrm{gr}\,\Lambda$.

General theory [19] now provides an algebra Γ with the requisite property. $\square$

With this tool in hand we can prove the following:

Proposition 2.2 ([22]). *Let $(0) \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow (0)$ be an almost split sequence in $\mathrm{mod}\,\Lambda$. If M or N are gradable, then E is gradable.*

Sketch of Proof. Thanks to Lemma 1.7, both modules, M and N are gradable, and there exists an indecomposable graded module $X \in \operatorname{mod}\operatorname{gr}\Lambda$ with $\mathfrak{F}(X) \cong N$. Since $M \cong \operatorname{Tr}_\Lambda(D(N))$, Lemma 1.7 ensures that the graded module $Z := \operatorname{Tr}_{\operatorname{gr}\Lambda}(D(X))$ satisfies $\mathfrak{F}(Z) \cong M$.

We have

$$(0) \neq \operatorname{Ext}^1_\Lambda(\mathfrak{F}(Z), \mathfrak{F}(X)) \cong \bigoplus_{i \in \mathbb{Z}^n} \operatorname{Ext}^1_\Lambda(\mathfrak{F}(Z), \mathfrak{F}(X))_i \cong \bigoplus_{i \in \mathbb{Z}^n} \operatorname{Ext}^1_{\operatorname{mod}\operatorname{gr}\Lambda}(Z[i], X).$$

Accordingly, $\operatorname{Ext}^1_{\operatorname{mod}\operatorname{gr}\Lambda}(Z[i_0], X) \neq (0)$ for some $i_0 \in \mathbb{Z}^n$, and for $I := \operatorname{supp}(X) \cup \operatorname{supp}(Z[i_0])$, the object $X \in \operatorname{mod}_I \operatorname{gr}\Lambda$ is not injective. Let $J := I + (\operatorname{supp}(\operatorname{End}_\Lambda(\mathfrak{F}(Z))) - i_0)$. Theorem 2.1 now yields the existence of an almost split sequence

$$(*) \qquad (0) \longrightarrow X \longrightarrow Y \longrightarrow X' \longrightarrow (0)$$

in $\operatorname{mod}_J \operatorname{gr}\Lambda$. As before, general theory implies that $X' \cong \operatorname{Tr}_{\operatorname{gr}\Lambda}(D(X)) = Z$. Since the sequence

$$(0) \longrightarrow \mathfrak{F}(X) \longrightarrow \mathfrak{F}(Y) \longrightarrow \mathfrak{F}(Z) \longrightarrow (0)$$

does not split, there results a commutative diagram

$$\begin{tikzcd}
(0) \arrow[r] & N \arrow[r, "f"] \arrow[d, equal] & E \arrow[r] \arrow[d, "\lambda"'] & M \arrow[r] \arrow[d, "\mu"'] & (0) \\
(0) \arrow[r] & N \arrow[r] & \mathfrak{F}(Y) \arrow[r] & M \arrow[r] & (0).
\end{tikzcd}$$

Let $\mu = \sum_{i \in \mathbb{Z}^n} \mu_i$. If $\mu_j \neq 0$, then $j \in \operatorname{supp}(\operatorname{End}_\Lambda(\mathfrak{F}(Z)))$ and

$$\operatorname{supp}(Z[j]) = j + \operatorname{supp}(Z) = j - i_0 + \operatorname{supp}(Z[i_0]) \subseteq J.$$

Consequently, $\mu_j : Z[j] \longrightarrow Z$ is a morphism in $\operatorname{mod}_J \operatorname{gr}\Lambda$. If none of these μ_j is a split epimorphism, then $(*)$ being almost split implies that f is a split monomorphism, a contradiction. Hence there is j with μ_j being bijective. Thus, $j = 0$ and Theorem 1.2 shows that μ is bijective. Hence λ is also an isomorphism and E is gradable. □

Theorem 2.3 ([22])**.** *The following statements hold:*

(1) *The category* $\operatorname{mod}\operatorname{gr}\Lambda$ *has almost split sequences.*

(2) *The exact functor* $\mathfrak{F} : \operatorname{mod}\operatorname{gr}\Lambda \longrightarrow \operatorname{mod}\Lambda$ *sends almost split sequences to almost split sequences.*

Proof. Let M be a non-projective, indecomposable, graded Λ-module. By the proof of Proposition 2.2, there exists an exact sequence $(0) \longrightarrow N \longrightarrow E \xrightarrow{\pi} M \longrightarrow (0)$ in $\operatorname{mod}\operatorname{gr}\Lambda$ such that the sequence $(0) \longrightarrow \mathfrak{F}(N) \longrightarrow \mathfrak{F}(E) \xrightarrow{\mathfrak{F}(\pi)} \mathfrak{F}(M) \longrightarrow (0)$ is almost split.

Suppose that $\varphi \in \operatorname{Hom}_{\operatorname{gr}\Lambda}(X, M)$ is not a split epimorphism. Then $\mathfrak{F}(\varphi)$ is not a split epimorphism, and there exists $g \in \operatorname{Hom}_\Lambda(\mathfrak{F}(X), \mathfrak{F}(E))$ such that

$$\mathfrak{F}(\pi) \circ g = \mathfrak{F}(\varphi).$$

By decomposing g into its constituents we see that

$$\mathfrak{F}(\pi) \circ g_0 = \mathfrak{F}(\varphi),$$

whence $\mathfrak{F}(\varphi) = \mathfrak{F}(\pi \circ \gamma)$ for some $\gamma \in \operatorname{Hom}_{\operatorname{gr}\Lambda}(X, E)$. Since $\mathfrak{F}$ is faithful, we have $\varphi = \pi \circ \gamma$, implying that the above sequence is almost split. □

Corollary 2.4 ([22])**.** *Let* $\Theta \subseteq \Gamma(\Lambda)$ *be a component of the AR-quiver of* Λ *containing a gradable module. Then all modules belonging to* Θ *are gradable.* □

2.2. **$\mathbf{G_rT}$-modules.** Let us begin by taking another look at graded algebras. For $\Lambda = \bigoplus_{i\in\mathbb{Z}^n} \Lambda_i$, we consider for every $t = (t_1, \ldots, t_n) \in (k \setminus \{0\})^n$, the graded automorphism $\varphi_t : \Lambda \longrightarrow \Lambda$, given by

$$\varphi_t(x) = t^i x \quad \forall\ x \in \Lambda_i,\ i \in \mathbb{Z}^n,$$

where $t^i := t_1^{i_1} \cdots t_n^{i_n}$. In this fashion we obtain an action of the n-dimensional torus $T := (k \setminus \{0\})^n$ on Λ via automorphisms. Conversely, if T acts on Λ via automorphisms, then Λ is a semi-simple T-module, which decomposes into weight spaces

$$\Lambda = \bigoplus_{\lambda\in X(T)} \Lambda_\lambda.$$

Here $X(T) \cong \mathbb{Z}^n$ denotes the character group of T. As an upshot of our discussion, we see that a graded algebra is an algebra together with a torus acting via automorphisms.

Similarly, a graded module M is a Λ-module, which also has the structure of a T-module such that

$$t.(a.m) = (t.a).(t.m) \qquad \forall\ t \in T, m \in M, a \in \Lambda.$$

Example. Suppose that $\mathrm{char}(k) = p > 0$, and consider the Lie algebra $\mathfrak{sl}(2)$ of trace zero (2×2)-matrices with its standard basis $\{e, h, f\}$. Recall that the p-th power map of matrices endows $\mathfrak{sl}(2)$ with the structure of a restricted Lie algebra, whose restricted enveloping algebra may be defined via

$$U_0(\mathfrak{sl}(2)) := U(\mathfrak{sl}(2))/\langle\{e, h^p - h, f\}\rangle.$$

The Lie algebra $\mathfrak{sl}(2)$ is $\mathbb{Z}$-graded with $\deg(e) = 2$, $\deg(h) = 0$, $\deg(f) = -2$. Being a factor algebra of the tensor algebra by a homogeneous ideal, the ordinary enveloping algebra $U(\mathfrak{sl}(2))$ is $\mathbb{Z}$-graded. By the same token, the algebra $U_0(\mathfrak{sl}(2))$ inherits this grading. Our grading comes from the adjoint action of the standard maximal torus

$$T := \{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\ ;\ \alpha \in k \setminus \{0\}\} \subseteq \mathrm{SL}(2).$$

Denoting the elements of T by t_α, we have $t_\alpha e t_\alpha^{-1} = \alpha^2 e$, etc.. By the universal property of $U_0(\mathfrak{sl}(2))$, this action naturally extends to an action of T on $U_0(\mathfrak{sl}(2))$.

Note that there is a compatibility between the action of T and that of $\mathrm{Lie}(T) = kh$: The differential of the T-action on $\mathfrak{sl}(2)$ is the adjoint action of the Lie algebra.

If we interpret $U_0(\mathfrak{sl}(2))$ as the Hopf algebra $k\,\mathrm{SL}(2)_1$ of the first Frobenius kernel $\mathrm{SL}(2)_1$, then a graded $k\,\mathrm{SL}(2)_1$-module is just a module for the group $\mathrm{SL}(2)_1 \rtimes T$, while compatibility of the actions induced by T and kh means that the subgroup $\mathrm{SL}(2)_1 \cap T \subseteq \mathrm{SL}(2)_1 \rtimes T$ acts trivially. We thus obtain an operation of the group $\mathrm{SL}(2)_1 T := (\mathrm{SL}(2)_1 \rtimes T)/(\mathrm{SL}(2)_1 \cap T)$. Since $\mathrm{SL}(2)_1 T \subseteq \mathrm{SL}(2)$, the category $\mathrm{mod}\,\mathrm{SL}(2)_1 T$ contains the restrictions of the rational $\mathrm{SL}(2)$-modules.

From now on we assume our base field k to be algebraically closed, with $\mathrm{char}(k) = p > 0$. We fix an algebraic group G as well as a maximal torus $T \subseteq G$. The coordinate ring $k[G]$ is a finitely generated reduced k-algebra. Moreover, $k[G]$ is a Hopf algebra with augmentation ideal $k[G]^\dagger \subseteq k[G]$. Given $r \in \mathbb{N}$, the ideal

$$k[G]_r^\dagger := \langle\{x^{p^r}\ ;\ x \in k[G]^\dagger\}\rangle$$

is a Hopf ideal, so that

$$k[G_r] := k[G]/k[G]_r^\dagger$$

is a finite-dimensional commutative Hopf algebra. We are interested in the dual Hopf algebra

$$kG_r := k[G_r]^*,$$

which is the *algebra of measures* of the associated infinitesimal group scheme G_r. This group scheme is called the r-th *Frobenius kernel* of G.

The conjugation action of T on G is an action via morphisms of algebraic groups. There results an induced action of T on $k[G]$ via automorphisms of Hopf algebras. The ideal $k[G]_r^\dagger$ is T-invariant, so that we obtain actions of T on $k[G_r]$ and kG_r, respectively. In particular, kG_r is an $X(T)$-graded algebra, where $X(T) := \mathrm{Hom}(T, k \setminus \{0\}) \cong \mathbb{Z}^{\dim T}$ is the *character group* of the torus T. In fact, kG_r is a graded Hopf algebra. Note that $kT_r \subseteq kG_r$ for every $r \geq 1$.

We let $\mathrm{mod}\, G_r$ and $\mathrm{mod}(G_r \rtimes T)$ be the categories of kG_r-modules and graded kG_r-modules respectively. These categories have tensor products as well as dualities, defined by taking k-duals.

Given $\lambda \in X(T)$, we let k_λ be the one-dimensional $(G_r \rtimes T)$-module with trivial kG_r-action on which T acts via λ. Then k_λ is a graded module, concentrated in degree λ, and we have

$$M[\lambda] \cong M \otimes_k k_\lambda$$

for every $M \in \mathrm{mod}(G_r \rtimes T)$.

In the sequel, we shall consider $T_r = G_r \cap T$ a subgroup of $G_r \rtimes T$ via $t \mapsto (t^{-1}, t)$.

Definition. The category $\mathrm{mod}\, G_rT$ is the full subcategory of $\mathrm{mod}(G_r \rtimes T)$, whose objects are annihilated by the augmentation ideal $kT_r^\dagger$ of kT_r.

This category was first introduced by Jantzen [27] in order to establish a BGG reciprocity principle, see Theorem 2.8 below.

Lemma 2.5. (1) *The category* $\mathrm{mod}\, G_rT$ *is a sum of blocks of* $\mathrm{mod}(G_r \rtimes T)$.

(2) $\mathrm{mod}\, G_rT$ *is a Frobenius category.*

(3) *The category* $\mathrm{mod}\, G_rT$ *has almost split sequences.*

(4) *The canonical restriction functor* $\mathfrak{F} : \mathrm{mod}\, G_rT \longrightarrow \mathrm{mod}\, G_r$ *preserves indecomposables and almost split sequences.*

Sketch of Proof. (1) The group $T_r = G_r \cap T$ is a diagonalizable, normal subgroup of $G_1 \rtimes T$. The group scheme $G_r \rtimes T$ acts trivially on T_r. As a result, T_r acts via a single character $\gamma \in X(T_r)$ on every indecomposable object $M \in \mathrm{mod}(G_r \rtimes T)$. It follows that the full subcategory $\mathrm{mod}(G_r \rtimes T)_\gamma$ defined by γ, is a sum of blocks of $\mathrm{mod}(G_r \rtimes T)$.

(2) Let $P \in \mathrm{mod}(G_r \rtimes T)$ be projective. Thanks to Proposition 1.6, $\mathfrak{F}(P) \in \mathrm{mod}\, G_r$ is projective. By a theorem of Larson and Sweedler [30], the finite-dimensional Hopf algebra kG_r is self-injective. Hence $\mathfrak{F}(P)$ is also injective, and so is P. Consequently, $\mathrm{mod}(G_r \rtimes T)$ is a Frobenius category, and (1) implies that $\mathrm{mod}\, G_rT$ also enjoys this property.

(3) Thanks to Theorem 2.3, the category $\mathrm{mod}(G_r \rtimes T)$ has almost split sequences, hence so does $\mathrm{mod}\, G_rT$.

(4) This follows directly from Corollary 1.3 and Theorem 2.3. □

The shifts $M \mapsto M[\lambda]$ sending $\mathrm{mod}\, G_rT$ onto itself are given precisely by those $\lambda \in X(T)$ that vanish on the subgroup $T_r \subseteq T$. There is an exact sequence

$$(0) \longrightarrow p^r X(T) \xrightarrow{\text{can.}} X(T) \xrightarrow{\text{res}} X(T_r) \longrightarrow (0),$$

so that the relevant shifts are those belonging to the subgroup $p^r X(T) \subseteq X(T)$.

We have seen in Lecture I that the simple Λ-modules are gradable and that the simple objects of $\mathrm{mod\,gr}\,\Lambda$ correspond to their shifts. Likewise, the simple G_rT-modules are the simple G_r-modules and their shifts by elements of $p^r X(T)$. In the context of Frobenius kernels of reductive groups, simple modules are constructed by means of the so-called *baby Verma modules* $Z_r(\lambda)$.

Let G be reductive with maximal torus $T \subseteq G$. We pick a *Borel subgroup* $T \subseteq B \subseteq G$ along with its *opposite group* B^-. In case $G = \mathrm{GL}(n)$, we can take T, B and B^- to be the diagonal matrices, the upper triangular matrices, and the lower triangular matrices, respectively.

Thanks to the Lie-Kolchin theorem [39, (10.2)], the group B_r is *trigonalizable*, that is, its simple modules have the form k_λ, where $\lambda \in X(T_r) \cong X(T)/p^r X(T)$. Hence we may define a simple B_r-module for every $\lambda \in X(T)$. For such λ, we consider the kG_r-modules

$$Z_r(\lambda) := kG_r \otimes_{kB_r} k_\lambda \quad \text{and} \quad Z'_r(\lambda) := \mathrm{Hom}_{kB_r^-}(kG_r, k_\lambda).$$

The T-action on kG_r induced by the adjoint representation stabilizes $kB_r \subseteq kG_r$ and hence endows these modules with the structures of G_rT-modules. The corresponding G_rT-modules are customarily denoted $\widehat{Z}_r(\lambda)$ and $\widehat{Z}'_r(\lambda)$, respectively.

Proposition 2.6. *Let $\lambda \in X(T)$. The modules $\widehat{Z}_r(\lambda)$ and $\widehat{Z}'_r(\lambda)$ have simple tops and socles. More precisely, we have $\mathrm{Top}_{G_rT}(\widehat{Z}_r(\lambda)) \cong \mathrm{Soc}_{G_rT}(\widehat{Z}'_r(\lambda))$.*

Sketch of Proof. Let U^- be the unipotent radical of B^-. From the decomposition $kG_r = kU_r^- kB_r$, we obtain an isomorphism $Z_r(\lambda) \cong kU_r^- \otimes_k k_\lambda$ of T-modules. It follows that the λ-weight space $\widehat{Z}_r(\lambda)_\lambda$ is one-dimensional. If M is a submodule of $\mathrm{Top}_{G_rT}(\widehat{Z}_r(\lambda))$, then Frobenius reciprocity implies

$$(0) \neq \mathrm{Hom}_{G_r}(Z_r(\lambda), M)_0 \cong \mathrm{Hom}_{B_r}(k_\lambda, M)_0 \hookrightarrow M_\lambda.$$

Hence $\mathrm{Top}_{G_rT}(\widehat{Z}_r(\lambda))$ has only one simple constituent.

Being an extension of Hopf algebras, $kG_r : kB_r$ is a Frobenius extension (cf. [17]), so that induction and coinduction are equivalent functors (up to some twist). Hence both modules above have simple tops and simple socles. □

Definition. Given $\lambda \in X(T)$, we write

$$L_r(\lambda) := \mathrm{Top}_{G_r}(Z_r(\lambda)) \text{ and } \widehat{L}_r(\lambda) := \mathrm{Top}_{G_rT}(\widehat{Z}_r(\lambda)).$$

Proposition 2.7. *Suppose that G is reductive. Then $\{\widehat{L}_r(\lambda) \ ; \ \lambda \in X(T)\}$ is a complete set of representatives of the simple G_rT-modules.* □

The foregoing result tells us that the simple objects of our category $\mathrm{mod}\, G_rT$ are indexed by the partially ordered set $X(T)$. The ordering comes from the root space decomposition of $\mathfrak{g} := \mathrm{Lie}(G)$ relative to T:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $R \subseteq X(T) \setminus \{0\}$, $\mathfrak{g}_0 = \mathrm{Lie}(T)$ and $\dim_k \mathfrak{g}_\alpha = 1$ for all $\alpha \in R$. Once we have chosen a Borel subgroup $B \subseteq G$ containing T, we call the roots belonging to $\mathrm{Lie}(B) = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ *positive.* Let $\mathbb{N}_0 R^+ \subseteq X(T)$ be the semigroup generated by R^+. Given $\lambda, \gamma \in X(T)$, we write

$$\lambda \leq \gamma \ :\Leftrightarrow \ \gamma - \lambda \in \mathbb{N}_0 R^+.$$

We denote by $\mathcal{F}(\Delta)$ the category of Δ-*good* G_rT-modules. By definition, $\mathcal{F}(\Delta)$ is the full subcategory of $\mathrm{mod}\, G_rT$, whose objects afford filtrations by baby Verma modules. Given $M \in \mathcal{F}(\Delta)$ and $\lambda \in X(T)$, the filtration multiplicity

$$[M : \widehat{Z}_r(\lambda)]$$

is well-defined, cf. [27] or [29, (II.11.2)].

In view of Proposition 1.6, every simple G_rT-module $\widehat{L}_r(\lambda)$ has a projective cover $\widehat{P}_r(\lambda)$. For $M \in \operatorname{mod} G_rT$, we let $[M : \widehat{L}_r(\lambda)]$ denote the Jordan-Hölder multiplicity of $\widehat{L}_r(\lambda)$ in M. The following result is the BGG reciprocity principle for $\operatorname{mod} G_rT$.

Theorem 2.8 ([27])**.** *Let $\lambda \in X(T)$. Then $\widehat{P}_r(\lambda) \in \mathcal{F}(\Delta)$, and we have*

$$[\widehat{P}_r(\lambda) : \widehat{Z}_r(\gamma)] = [\widehat{Z}_r(\gamma) : \widehat{L}_r(\lambda)]$$

for all $\lambda, \gamma \in X(T)$. □

Reciprocity principles have been established in the wider context of a highest weight category, see [7]. The following fundamental result states that $\operatorname{mod} G_rT$ shares a lot of features with the module category of a quasi-hereditary algebra.

Theorem 2.9 ([7])**.** *Let G be reductive with maximal torus T and Borel subgroup $T \subseteq B \subseteq G$. Then $\operatorname{mod} G_rT$ is a highest weight category with standard objects $\widehat{Z}_r(\lambda)$ and co-standard objects $\widehat{Z}'_r(\lambda)$.* □

Remark. Although being closely related to $\operatorname{mod} G_rT$, the category $\operatorname{mod} G_r$ is usually not a highest weight category. If this is the case, then kG_r is quasi-hereditary (cf. [7, (3.6)]) and thus has finite global dimension (cf. [8]). On the other hand, the algebra kG_r is self-injectve, so that kG_r is actually semi-simple. By Nagata's Theorem [9, (IV,§3,3.6)], this can only happen if the infinitesimal group G_r is diagonalizable, whence $G_r = T_r$.

3. Lecture III: Support Spaces and AR-Components

Throughout, we shall be concerned with an algebraic group G, defined over an algebraically closed field k of characteristic $p > 0$. We fix a maximal torus $T \subseteq B$.

In Lecture II, we introduced Jantzen's category $\operatorname{mod} G_rT$ of G_rT-modules. Recall that $\operatorname{mod} G_rT$ is a sum of blocks of the category $\operatorname{mod} \operatorname{gr} kG_r$ of $X(T)$-graded modules of the r-th Frobenius kernel G_r of G.

3.1. **Support Spaces.** Consider the polynomial ring $k[X]$. This algebra obtains the structure of a commutative Hopf algebra by defining

$$\Delta(X) := X \otimes 1 + 1 \otimes X \quad ; \quad \eta(X) = -X \quad ; \quad \varepsilon(X) = 0.$$

Given $r \geq 1$, the ideal (X^{p^r}) is a Hopf-ideal, so that $k[\mathbb{G}_{a(r)}] := k[X]/(X^{p^r})$ is a Hopf algebra, with canonical basis $\mathcal{B} := \{x^i \ ; \ 0 \leq i \leq p^r - 1\}$. We denote its dual Hopf algebra by

$$k\mathbb{G}_{a(r)} := k[\mathbb{G}_{a(r)}]^*.$$

If $\{\delta_0, \dots, \delta_{p^r-1}\}$ denotes the basis dual to $\mathcal{B}$, then, setting $u_i := \delta_{p^i}$ for $0 \leq i \leq r-1$, we obtain an isomorphism

$$k[X_0, \dots, X_{r-1}]/(X_0^p, \dots, X_{r-1}^p) \xrightarrow{\sim} k\mathbb{G}_{a(r)} \quad ; \quad X_i \mapsto u_i$$

of associative k-algebras. Accordingly, the group schemes $\mathbb{G}_{a(r)}$ may be thought of as analogues of p-elementary abelian subgroups.

Given any homomorphism $\varphi : k\mathbb{G}_{a(r)} \longrightarrow kG_r$, we let $\varphi^* : \operatorname{mod} G_r \longrightarrow \operatorname{mod} \mathbb{G}_{a(r)}$ be the associated pull-back functor.

Definition. Let $M \in \operatorname{mod} G_r$. Then

$$V_r(G)_M := \{\varphi \in \mathrm{Hom}_{\mathrm{Hopf}}(k\mathbb{G}_{a(r)}, kG_r) \ ; \ \varphi^*(M)|_{k[u_{r-1}]} \text{ is not projective}\}$$

is called the *rank variety* of M.

This definition, which is due to Suslin-Friedlander-Bendel [37], specializes for $r = 1$ to something more tractable. Let $\mathfrak{g} := \mathrm{Lie}(G)$ be the Lie algebra of G. Then

$$\mathfrak{g} = \{x \in kG_r \ ; \ \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

is the Lie algebra of *primitive elements* of kG_r. We consider the associated *restricted enveloping algebra*

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/\langle\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\}\rangle,$$

which is a finite-dimensional factor algebra of the ordinary enveloping algebra $U(\mathfrak{g})$. General theory provides a canonical map

$$U_0(\mathfrak{g}) \hookrightarrow kG_r,$$

which is an isomorphism for $r = 1$. It follows that

$$V_1(G)_M \cong \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0 \text{ and } M|_{k[x]} \text{ is not projective}\} \cup \{0\}.$$

Since $G_r \trianglelefteq G$ is a normal subgroup of G, the group G acts on G_r via conjugation. There results an action $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(kG_r)$ of G on kG_r by automorphisms of the Hopf algebra kG_r, the so-called *adjoint representation.* This action induces an operation of G on

$$V_r(G) := V_r(G)_k = \mathrm{Hom}_{\mathrm{Hopf}}(k\mathbb{G}_{a(r)}, kG_r).$$

By way of illustration, let us consider the case $G = \mathrm{GL}(n)$ and $r = 1$, which corresponds to $\mathfrak{g} := \mathrm{Lie}(G) = \mathfrak{gl}(n)$. Then G acts on $\mathfrak{g}$ via

$$g.x := gxg^{-1} \qquad \forall \ g \in G, \ x \in \mathfrak{g}.$$

Note that $x \mapsto g.x$ is a homomorphism of restricted Lie algebras, i.e.,

$$g.x^{[p]} = g.x^p = (g.x)^p = (g.x)^{[p]}$$

for $g \in G$ and $x \in \mathfrak{g}$. As a result, the rank variety $V_1(\mathrm{GL}(n))$ is G-invariant.

For a G_r-module M and $g \in G$, we write

$$M^{(g)} := \mathrm{Ad}(g^{-1})^*(M).$$

Then we have

$$V_r(G)_{M^{(g)}} = g.(V_r(G)_M).$$

If G is connected, then $S^{(g)} \cong S$ for every simple G_r-module S, so that the varieties of simple modules are G-invariant. The above action also provides information of G_rT-modules:

Lemma 3.1. *Let M be a G_rT-module. Then $V_r(G)_M$ is T-invariant.* □

The importance of rank varieties derives from their connection with *cohomological support varieties.* Recall that $\mathrm{H}^\bullet(G_r, k) := \bigoplus_{n\geq 0} \mathrm{H}^{2n}(G_r, k)$ is a commutative k-algebra. Given a G_r-module M, there is an algebra homomorphism $\Phi_M : \mathrm{H}^\bullet(G_r, k) \longrightarrow \mathrm{Ext}^*_{G_r}(M, M)$ which takes values in the Yoneda algebra $\mathrm{Ext}^*_{G_r}(M, M)$ of M. We can thus consider the set

$$\mathcal{V}_{G_r}(M) := \{\mathfrak{M} \in \mathrm{Maxspec}(\mathrm{H}^\bullet(G_r, k)) \ ; \ \mathfrak{M} \supseteq \ker \Phi_M\}.$$

The following fundamental result shows that $\mathcal{V}_{G_r}(M)$ is indeed a variety.

Theorem 3.2. *The following statements hold:*

(1) (Friedlander-Suslin,[18]) *The algebra* $\mathrm{H}^\bullet(G_r, k)$ *is finitely generated, and* Φ_M *is a finite morphism for every* $M \in \operatorname{mod} G_r$.

(2) (Carlson,[4]) *Let* $M \in \operatorname{mod} G_r$. *We have* $\dim \mathcal{V}_{G_r}(M) = \mathrm{cx}_{G_r}(M)$, *the complexity of* M.

(3) (Carlson,[5]) *If* M *is indecomposable, then* $\operatorname{Proj}(\mathcal{V}_{G_r}(M))$ *is connected.*

(4) (Suslin-Friedlander-Bendel,[38]) *There is a homeomorphism* $V_r(G) \longrightarrow \mathcal{V}_{G_r}(k)$ *sending* $V_r(G)_M$ *onto* $\mathcal{V}_{G_r}(M)$ *for every* $M \in \operatorname{mod} G_r$. □

3.2. **Modules of complexity 1.** Since the algebra kG_r is self-injective, modules of complexity 0 are projective. Moreover, one can show that an indecomposable G_r-module M of complexity 1 is *periodic*, that is,

$$\Omega^n_{G_r}(M) \cong M \quad \text{for some } n \neq 0.$$

Here Ω_{G_r} denotes the *Heller operator* of the Frobenius category $\operatorname{mod} G_r$. Nakayama functors of finite-dimensional Hopf algebras are known to have finite order (cf. [17]), so that such modules are also τ_{G_r}-periodic. Giving rise to τ_{G_r}-invariant subadditive functions, periodic modules play an important rôle in determining the components of the Auslander-Reiten quiver. Let us see what happens for G_rT-modules.

If M is an indecomposable G_rT-module and $(P_n)_{n\geq 0}$ is a minimal projective resolution of M, then Proposition 1.6 shows that $(\mathfrak{F}(P_n))_{n\geq 0}$ is a minimal projective resolution of M, whence

$$\mathrm{cx}_{G_rT}(M) = \mathrm{cx}_{G_r}(\mathfrak{F}(M)).$$

Thus, if M has complexity 1, then there exists $n > 0$ with

$$\mathfrak{F}(\Omega^n_{G_rT}(M)) \cong \Omega^n_{G_r}(\mathfrak{F}(M)) \cong \mathfrak{F}(M),$$

and Theorem 1.5 tells us that

$$\Omega^n_{G_rT}(M) \cong M[\gamma].$$

for some $\gamma \in X(T)$. Recall that the shifts in $\operatorname{mod} G_rT$ are obtained by tensoring with $k_{p^r\lambda}$ for some $\lambda \in X(T)$.

For each root $\alpha \in R$, we consider the root subgroup $U_\alpha \subseteq G$, whose Lie algebra is given by $\operatorname{Lie}(U_\alpha) = \mathfrak{g}_\alpha$. Note that $k(U_\alpha)_s \subseteq k(U_\alpha)_r \subseteq kG_r$ for all $s \leq r$. If M is a G_r-module such that $M|_{k(U_\alpha)_r}$ is not projective, then we call

$$\mathrm{ph}_\alpha(M) := \min\{1 \leq s \leq r \ ; \ M|_{k(U_\alpha)_s} \text{ is not projective}\}$$

the *projective height of* M *relative to* α.

Theorem 3.3. *Suppose that* G *is reductive. Let* M *be an indecomposable* G_rT*-module such that* $\mathrm{cx}_{G_rT}(M) = 1$. *Then there exists a root* $\alpha \in R$ *such that*

$$\Omega^{2p^{r-\mathrm{ph}_\alpha(M)}}_{G_rT}(M) \cong M \otimes_k k_{p^r\alpha}.$$

Sketch of Proof. The Main Theorem of [6] provides a root $\alpha \in R$ such that $M|_{k(U_\alpha)_r}$ is not projective. Let $t := \mathrm{ph}_\alpha(M)$. A detailed analysis of the T-action on $\mathrm{H}^\bullet(G_r, k)$ furnishes an element $\zeta \in \mathrm{H}^{2p^{r-t}}(G_r, k)_{-p^r\alpha} \setminus \{0\}$ such that $Z(\zeta) \cap \mathcal{V}_{G_r}(M) \subsetneq \mathcal{V}_{G_r}(M)$. Thanks to Lemma 2.5, the module $M|_{G_r}$ is indecomposable, and Theorem 3.2(2),(3) implies that $\mathcal{V}_{G_r}(M)$ is a line. Consequently, $Z(\zeta) \cap \mathcal{V}_{G_r}(M) = \{0\}$.

Our cohomology class ζ corresponds to a nonzero element $\hat{\zeta} \in \operatorname{Hom}_{G_r}(\Omega^{2p^{r-t}}_{G_rT}(k), k)_{-p^r\alpha}$. Thus, defining $\widehat{L}_\zeta := \ker \hat{\zeta}$, we obtain an exact sequence

$$(0) \longrightarrow \widehat{L}_\zeta \longrightarrow \Omega^{2p^{r-t}}_{G_rT}(k) \otimes_k k_{-p^r\alpha} \xrightarrow{\hat{\zeta}} k \longrightarrow (0).$$

By the Tensor Product Theorem $\mathcal{V}_{G_r}(\widehat{L}_\zeta \otimes_k M) = Z(\zeta) \cap \mathcal{V}_{G_r}(M) = \{0\}$, so that $\widehat{L}_\zeta \otimes_k M$ is projective. Hence the above sequence splits upon tensoring with M. This implies

$$M \oplus (\text{proj.}) \cong (\Omega^{2p^{r-t}}_{G_rT}(M) \otimes_k k_{-p^r\alpha}) \oplus (\text{proj.}),$$

and our result follows by comparing the projective-free parts. □

Corollary 3.4. *Suppose that G is reductive. Let M be an indecomposable G_rT-module. Then M is not periodic.*

Proof. Suppose there is $n > 0$ such that $\Omega^n_{G_rT}(M) \cong M$. Then we have $\mathrm{cx}_{G_rT}(M) = 1$, and Theorem 3.3 provides a root $\alpha \in R$ and $\ell \geq 0$ such that

$$\Omega^{2p^\ell}_{G_rT}(M) \cong M \otimes_k k_{p^r\alpha}.$$

As a result,

$$M \cong \Omega^{2np^\ell}_{G_rT}(M) \cong M \otimes_k k_{np^r\alpha} \cong M[np^r\alpha],$$

so that Theorem 1.5 yields $np^r\alpha = 0$. As $X(T)$ is torsion-free, we have $n = 0$, a contradiction. □

3.3. **The stable AR-quiver $\mathbf{\Gamma_s(G_rT)}$.** According to Lemma 2.5, the category $\mathrm{mod}\, G_rT$ is a Frobenius category that possesses almost split sequences. We thus consider the stable Auslander-Reiten quiver $\Gamma_s(G_rT)$ as well as its ungraded version $\Gamma_s(G_r)$. Recall that the restriction functor $\mathfrak{F} : \mathrm{mod}\, G_rT \longrightarrow \mathrm{mod}\, G_r$ preserves and reflects indecomposables and projectives. Moreover, we have

$$\mathfrak{F} \circ \tau_{G_rT} = \tau_{G_r} \circ \mathfrak{F}.$$

Let M^+ and M^- be the sets of successors and predecessors of the vertex M, respectively.

Lemma 3.5. *The following statements hold:*

(1) *The canonical restriction functor $\mathfrak{F} : \mathrm{mod}\, G_rT \longrightarrow \mathrm{mod}\, G_r$ induces a morphism $\mathfrak{F} : \Gamma_s(G_rT) \longrightarrow \Gamma_s(G_r)$ of stable translation quivers such that $\mathfrak{F}(M^\pm) = \mathfrak{F}(M)^\pm$ for all $M \in \Gamma_s(G_rT)$.*

(2) *If $\Theta \subseteq \Gamma_s(G_rT)$ is a component, then $\mathfrak{F}(\Theta)$ is a component of $\Gamma_s(G_r)$.* □

Remark. The morphism $\mathfrak{F} : \Theta \longrightarrow \mathfrak{F}(\Theta)$ is usually not a covering, that is, $\mathfrak{F} : M^\pm \longrightarrow \mathfrak{F}(M)^\pm$ is not necessarily bijective.

Let $\Theta \subseteq \Gamma_s(G_rT)$ be a component. Since $\Gamma_s(G_rT)$ is a stable translation quiver, Riedtmann's theorem [33] provides a directed tree T_Θ and an admissible group $\Pi \subseteq \mathbb{Z}[T_\Theta]$ such that $\Theta \cong \mathbb{Z}[T_\Theta]/\Pi$. The underlying undirected tree $\bar{T}_\Theta$, the so-called *tree class* of Θ is uniquely determined by Θ. In view of

$$V_r(G)_{\mathfrak{F}(M)} = V_r(G)_{\mathfrak{F}(N)} \qquad \text{for all } M, N \in \Theta,$$

we can attach a variety $V_r(G)_\Theta$ to the component Θ.

Proposition 3.6. *Let $\Theta \subseteq \Gamma_s(G_rT)$ be a component. Then the tree class $\bar{T}_\Theta$ is a simply laced finite or infinite Dynkin diagram, a simply laced Euclidean diagram, or $\tilde{A}_{12}$.*

Sketch of Proof. Since Θ contains no projective modules, we have $V_r(G)_\Theta \neq \{0\}$. Given $\varphi \in V_r(G)_\Theta \setminus \{0\}$, the module $(kG_r)|_{k[u_{r-1}]}$ is projective. We consider the induced G_r-module $M_\varphi := kG_r \otimes_{k[u_{r-1}]} k$. Basic properties of the Heller operator and the Nakayama functor of G_r yield

$$(*) \qquad \tau_{G_r}(M_\varphi) \oplus (\text{proj.}) \cong M_\varphi.$$

Consider the function

$$d_\varphi : \Theta \longrightarrow \mathbb{N}_0 \ \ ; \ \ M \mapsto \dim_k \operatorname{Ext}^1_{G_r}(M_\varphi, \mathfrak{F}(M)).$$

Let $M \in \Theta$. Since $\mathfrak{F}(M)|_{k[u_{r-1}]}$ is not projective, it is not injective and Frobenius reciprocity yields

$$d_\varphi(M) = \dim_k \operatorname{Ext}^1_{k[u_{r-1}]}(k, \mathfrak{F}(M)|_{k[u_{r-1}]}) \neq 0.$$

Property $(*)$ readily implies that the subadditive function d_φ is τ_{G_rT}-invariant, that is, $d_\varphi \circ \tau_{G_rT} = d_\varphi$. Our result now follows from work by Happel-Preiser-Ringel [26] on trees affording subadditive functions. □

Theorem 3.7. *Suppose that G is reductive. If $\Theta \subseteq \Gamma_s(G_rT)$ is a stable AR-component, then the following statements hold:*
1. $\Theta \cong \mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty^\infty]$, *or* $\mathbb{Z}[D_\infty]$.
2. *If* $\dim V_r(G)_\Theta \neq 2$, *then* $\Theta \cong \mathbb{Z}[A_\infty]$.

Remark. It is not known, whether components of tree class D_∞ actually occur.

Sketch of Proof. (1) Since G is reductive, one can show that τ_{G_rT} coincides with $\Omega^2_{G_rT}$. Moreover, every representation-finite block of kG_r is simple, see [11].

Suppose that $\bar{T}_\Theta$ is a finite Dynkin diagram. Then $\bar{T}_{\mathfrak{F}(\Theta)}$ is also a finite Dynkin diagram, and $\mathfrak{F}(\Theta)$ is finite. By Auslander's theorem [2, (VI.1.4)], $\mathfrak{F}(\Theta)$ belongs to a representation-finite block of kG_r, a contradiction.

If $\bar{T}_\Theta = A_\infty$, then $\Theta \cong \mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty]/\langle\tau^m\rangle$. Thanks to Corollary 3.4, Θ contains no τ_{G_rT}-periodic modules, so that only the former case occurs.

If Θ is a component of Euclidean tree class or if $\Theta \cong \mathbb{Z}[\tilde{A}_{pq}]$, then $\dim V_r(G)_\Theta = 2$, and the structure theory of reductive groups reduces us to the case, where $G = \mathrm{SL}(2)$. The representation theory of $\mathrm{SL}(2)_r$ is sufficiently well understood to rule out this alternative. □

Example. Let $G = \mathrm{SL}(2)$ and $r = 1$. If $T \subseteq \mathrm{SL}(2)$ denotes the standard maximal torus of diagonal matrices, then $\operatorname{mod} G_1T$ is a sum of blocks of the category of $U_0(\mathfrak{sl}(2))$-modules that are graded by $X(T) \cong \mathbb{Z}$. Let $\mathfrak{b} := \operatorname{Lie}(B) \subseteq \mathfrak{sl}(2)$ be the Borel subalgebra of upper triangular matrices, and consider

$$\widehat{Z}_1(\lambda) := U_0(\mathfrak{sl}(2)) \otimes_{U_0(\mathfrak{b})} k_\lambda.$$

Let α be the positive root of $\mathrm{SL}(2)$. If $\lambda \notin \frac{p-1}{2}\alpha + pX(T)$, then $V_1(\mathrm{SL}(2))_{\widehat{Z}_1(\lambda)}$ is one-dimensional, and we have $\operatorname{cx}_{G_1T}(\widehat{Z}_1(\lambda)) = 1$. Moreover, Theorem 3.3 implies

$$\Omega^2_{\mathrm{SL}(2)_1T}(\widehat{Z}_1(\lambda)) \cong \widehat{Z}_1(\lambda) \otimes_k k_{p\alpha} \cong \widehat{Z}_1(\lambda + p\alpha),$$

In view of Theorem 3.7(2), the component Θ containing $\widehat{Z}_1(\lambda)$ is isomorphic to $\mathbb{Z}[A_\infty]$. We shall see later, that the baby Verma module $\widehat{Z}_1(\lambda)$ is actually quasi-simple.

Now suppose that Θ contains a simple module S. Then $\mathfrak{F}(\Theta) \cong \mathbb{Z}[\tilde{A}_{12}]$, so that every vertex in $\mathfrak{F}(\Theta)$ has two identical predecessors. According to Lemma 3.5, every M in Θ has at least two predecessors, and Proposition 3.6 implies $\bar{T}_\Theta = A_\infty^\infty, \tilde{A}_{12}$. An analysis of the principal indecomposable $\mathrm{SL}(2)_1$-modules shows that $S, S \otimes_k k_{2p\alpha} \in M^-$ for some $M \in \Theta$. Hence $\bar{T}_\Theta \cong A_\infty^\infty$.

4. Lecture IV: AR-Components containing Δ-good Modules

Throughout this lecture, G is assumed to be a reductive algebraic group. We fix a maximal torus $T \subseteq G$ as well as a Borel subgroup $T \subseteq B \subseteq G$. Recall that the category $\operatorname{mod} G_rT$ is a highest weight category with simple objects $\widehat{L}_r(\lambda)$, standard objects $\widehat{Z}_r(\lambda) := kG_r \otimes_{kB_r} k_\lambda$, and co-standard objects $\widehat{Z}'_r(\lambda) := \operatorname{Hom}_{kB_r^-}(kG_r, k_\lambda)$. Here B^- denotes the Borel subgroup of G opposite to B. If U and U^- are the unipotent radicals of these groups, then $B = UT$ and $B^- = U^-T$.

The canonical restriction functor $\mathfrak{F} : \operatorname{mod} G_rT \longrightarrow \operatorname{mod} G_r$ corresponds to the forgetful functor introduced earlier. Application of $\mathfrak{F}$ to the objects defined above yields G_r-modules $L_r(\lambda)$, $Z_r(\lambda)$ and $Z'_r(\lambda)$, respectively.

4.1. **Δ-good Modules.** Let us begin with an observation concerning the supports of these modules, cf. [31].

Lemma 4.1. *Let $\lambda \in X(T)$. Then the following statements hold:*
(1) $V_r(G)_{Z_r(\lambda)} \subseteq V_r(U)$.
(2) $V_r(G)_{Z'_r(\lambda)} \subseteq V_r(U^-)$. $\square$

Recall that the category $\mathcal{F}(\Delta)$ of Δ-good modules is the full subcategory of $\operatorname{mod} G_rT$, whose objects afford a $\widehat{Z}_r$-filtration. Basic properties of rank varieties now yield

$$V_r(G)_{\mathfrak{F}(M)} \subseteq V_r(U) \qquad \forall\ M \in \mathcal{F}(\Delta).$$

By the same token, we have

$$V_r(G)_{\mathfrak{F}(M)} \subseteq V_r(U^-) \qquad \forall\ M \in \mathcal{F}(\nabla),$$

where $\mathcal{F}(\nabla)$ is the category of ∇-good modules, that is, modules with a $\widehat{Z}'_r$-filtration. Hence, if $X \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ is a *tilting object*, then $V_r(G)_{\mathfrak{F}(X)} \subseteq V_r(U) \cap V_r(U^-) = \{0\}$, so that Theorem 3.2(2) ensures that $\mathfrak{F}(X)$ is projective. As a result, X is also projective.

It turns out that the necessary conditions stated above are also sufficient:

Proposition 4.2. *Let $M \in \operatorname{mod} G_rT$. Then $M \in \mathcal{F}(\Delta)$ if and only if $V_r(G)_{\mathfrak{F}(M)} \subseteq V_r(U)$.* $\square$

We shall exploit this criterion in the proof of our next result:

Theorem 4.3. *Let M be a non-projective indecomposable G_rT-module, $\Theta \subseteq \Gamma_s(G_rT)$ and $\Psi \subseteq \Gamma_s(G_r)$ be the stable AR-components containing M and $\mathfrak{F}(M)$, respectively.*
(1) *Every vertex of Ψ has a G_rT-structure.*
(2) *If $M \in \mathcal{F}(\Delta)$, so is every indecomposable G_rT-module belonging to Θ.*
(3) *If $\mathfrak{F}(M)$ affords a Z_r-filtration, so does every indecomposable G_r-module belonging to Ψ.*

Proof. (1) According to Lemma 3.5, we have $\mathfrak{F}(\Theta) = \Psi$.

(2) Let $N \in \Theta$. Since $V_r(G)_{\mathfrak{F}(N)} = V_r(G)_{\mathfrak{F}(M)}$, the assertion follows from a twofold application of Proposition 4.2.

(3) In view of (2), we have $\Theta \subseteq \mathcal{F}(\Delta)$. Hence every $X \in \Psi = \mathfrak{F}(\Theta)$ has a Z_r-filtration. $\square$

Remark. (1) Part (3) of the foregoing result illustrates the reduction technique mentioned in the preface. The idea is to establish results for $\operatorname{mod} G_r$ by first proving analogous facts within $\operatorname{mod} G_rT$.

(2) Let Λ be a quasi-hereditary algebra. In [34], Ringel proved that the category $\mathcal{F}(\Delta)$ of Δ-good modules has relative almost split sequences. Part (2) of Theorem 4.3 tells us that in our case these sequences are the ordinary almost split sequences.

4.2. **AR-Sequences of Verma Modules.** Let $\lambda \in X(T)$ be a weight such that $Z_r(\lambda)$ is not projective. We are interested in the position of $Z_r(\lambda)$ within the stable Auslander-Reiten quiver $\Gamma_s(G_r)$. Following our reduction philosophy, we first address the corresponding question for G_rT-modules.

Given $M \in \mathcal{F}(\Delta)$, we define the Δ-*support* of M via

$$\mathrm{supp}_\Delta(M) := \{\lambda \in X(T) \ ; \ [M : \widehat{Z}_r(\lambda)] \neq 0\}.$$

For $\lambda \in X(T)$, we write $X(T)_{\geq\lambda} := \{\mu \in X(T) \ ; \ \mu \geq \lambda\}$ and $X(T)_{>\lambda} := X(T)_{\geq\lambda} \setminus \{\lambda\}$. We record the following basic properties, the first one being a direct consequence of BGG reciprocity (cf. Theorem 2.8):

Lemma 4.4. *The following statements hold:*

(1) $\mathrm{supp}_\Delta(\widehat{P}_r(\lambda)) \subseteq X(T)_{\geq\lambda}$ *and* $[\widehat{P}_r(\lambda) : \widehat{Z}_r(\lambda)] = 1$.

(2) *Let* $(0) \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow (0)$ *be an exact sequence of* G_rT*-modules. If two modules of the sequence belong to* $\mathcal{F}(\Delta)$*, so does the third. In that case, we have*

$$[M : \widehat{Z}_r(\lambda)] = [M' : \widehat{Z}_r(\lambda)] + [M'' : \widehat{Z}_r(\lambda)]$$

for every $\lambda \in X(T)$*.*

(3) *If* $M \in \mathcal{F}(\Delta)$*, then* $\mathrm{Top}_{G_rT}(M)$ *is a direct summand of* $\bigoplus_{\lambda \in \mathrm{supp}_\Delta(M)} [M : \widehat{Z}_r(\lambda)] \, \widehat{L}_r(\lambda)$. □

Recall that G being reductive implies $\tau_{G_rT} = \Omega^2_{G_rT}$.

Theorem 4.5. *Suppose that* $\widehat{Z}_r(\lambda)$ *is not projective and let*

$$(0) \longrightarrow \Omega^2_{G_rT}(\widehat{Z}_r(\lambda)) \longrightarrow E \longrightarrow \widehat{Z}_r(\lambda) \longrightarrow (0)$$

be the almost split sequence terminating in $\widehat{Z}_r(\lambda)$*. Then the following statements hold:*

(1) $\mathrm{supp}_\Delta(E) \subseteq X(T)_{\geq\lambda}$ *and* $[E : \widehat{Z}_r(\lambda)] = 1$.

(2) *The* G_rT*-module* E *is indecomposable.*

Proof. We consider a minimal projective presentation

$$(0) \longrightarrow \Omega^2_{G_rT}(\widehat{Z}_r(\lambda)) \longrightarrow P_1 \longrightarrow \widehat{P}_r(\lambda) \longrightarrow \widehat{Z}_r(\lambda) \longrightarrow (0)$$

of the G_rT-module $\widehat{Z}_r(\lambda)$. Since $[\widehat{Z}_r(\lambda) : \widehat{L}_r(\lambda)] = 1$, parts (1) and (2) of Lemma 4.4 imply the inclusion $\mathrm{supp}_\Delta(\Omega_{G_rT}(\widehat{Z}_r(\lambda))) \subseteq X(T)_{>\lambda}$. Owing to (3) of Lemma 4.4, the G_rT-module $\mathrm{Top}_{G_rT}(\Omega_{G_rT}(\widehat{Z}_r(\lambda)))$ is a direct summand of

$$\bigoplus_{\mu \in X(T)_{>\lambda}} [\Omega_{G_rT}(\widehat{Z}_r(\lambda)) : \widehat{Z}_r(\mu)] \, \widehat{L}_r(\mu).$$

Another application of (1) of Lemma 4.4 therefore yields $\mathrm{supp}_\Delta(P_1) \subseteq X(T)_{>\lambda}$. In particular, the module $\widehat{Z}_r(\lambda)$ is not a filtration factor of $\Omega^2_{G_rT}(\widehat{Z}_r(\lambda))$, and Lemma 4.4(2) implies

$$[E : \widehat{Z}_r(\lambda)] = 1.$$

In virtue of Proposition 4.2, every indecomposable constituent of E is Δ-good, so that each irreducible morphism terminating in $\widehat{Z}_r(\lambda)$ is surjective (see [2, (V.5.1)]). Thus, if E_i is an indecomposable constituent of E, then, by [2, (V.5.3)], our almost split sequence induces an exact sequence

$$(0) \longrightarrow X \longrightarrow E_i \longrightarrow \widehat{Z}_r(\lambda) \longrightarrow (0),$$

and Lemma 4.4(2) gives rise to

$$[E_i : \widehat{Z}_r(\lambda)] = [X : \widehat{Z}_r(\lambda)] + [\widehat{Z}_r(\lambda) : \widehat{Z}_r(\lambda)] = [X : \widehat{Z}_r(\lambda)] + 1.$$

Consequently, E can only have one indecomposable constituent. □

In view of Lemma 2.5, this result also implies that the middle term of the almost split sequence terminating in $Z_r(\lambda)$ is indecomposable.

Thanks to Theorem 3.7 and Theorem 4.5, the component $\Theta_r(\lambda)$ of $\Gamma_s(G_rT)$ containing $\widehat{Z}_r(\lambda)$ is isomorphic to $\mathbb{Z}[A_\infty]$ or $\mathbb{Z}[D_\infty]$, with the former occurring "most often". However, there are baby Verma modules with two-dimensional supports.

Theorem 4.6. *Let $\lambda \in X(T)$ be such that $\widehat{Z}_r(\lambda)$ is not projective. Then $\Theta_r(\lambda) \cong \mathbb{Z}[A_\infty]$.*

Proof. Assuming $\Theta_r(\lambda) \cong \mathbb{Z}[D_\infty]$, we consider the almost split sequence

$$(*) \qquad (0) \longrightarrow \Omega^2_{G_rT}(\widehat{Z}_r(\lambda)) \longrightarrow E \longrightarrow \widehat{Z}_r(\lambda) \longrightarrow (0)$$

terminating in $\widehat{Z}_r(\lambda)$. By assumption, there exists an indecomposable G_rT-module $Y_r(\lambda) \not\cong \widehat{Z}_r(\lambda)$ such that E is the only predecessor of $Y_r(\lambda) \in \Theta_r(\lambda)$. Let

$$(**) \qquad (0) \longrightarrow \Omega^2_{G_rT}(Y_r(\lambda)) \longrightarrow E \longrightarrow Y_r(\lambda) \longrightarrow (0)$$

be the almost split sequence terminating in $Y_r(\lambda)$. We now proceed in several steps:

(i) *We have $[Y_r(\lambda) : \widehat{Z}_r(\lambda)] = 1$.*

Consider the almost split sequence $(**)$. By virtue of Lemma 4.4(2) and Theorem 4.5(1), the assumption $[Y_r(\lambda) : \widehat{Z}_r(\lambda)] = 0$ implies

$$\mathrm{supp}_\Delta(Y_r(\lambda)) \subseteq X(T)_{>\lambda}.$$

Thus, by Lemma 4.4(3), the highest weights μ of the constituents $\widehat{L}_r(\mu)$ of $\mathrm{Top}_{G_rT}(Y_r(\lambda))$ belong to $X(T)_{>\lambda}$. Owing to Lemma 4.4(1), the projective cover $P(Y_r(\lambda))$ of $Y_r(\lambda)$ satisfies $\mathrm{supp}_\Delta(P(Y_r(\lambda))) \subseteq X(T)_{>\lambda}$ and, thanks to Lemma 4.4(2), the submodule $\Omega_{G_rT}(Y_r(\lambda))$ inherits this property. A repetition of this argument implies $\mathrm{supp}_\Delta(\Omega^2_{G_rT}(Y_r(\lambda))) \subseteq X(T)_{>\lambda}$. In virtue of Lemma 4.4(2), our almost split sequence $(**)$ then yields

$$[E : \widehat{Z}_r(\lambda)] = [\Omega^2_{G_rT}(Y_r(\lambda)) : \widehat{Z}_r(\lambda)] + [Y_r(\lambda) : \widehat{Z}_r(\lambda)] = 0,$$

which contradicts Theorem 4.5(1). Consequently, $[Y_r(\lambda) : \widehat{Z}_r(\lambda)] \geq 1$, and the reverse inequality follows from Lemma 4.4(2) and Theorem 4.5(1). ◇

(ii) *There exists a surjection $g : Y_r(\lambda) \longrightarrow \widehat{Z}_r(\lambda)$.*

Owing to Theorem 4.5 and (i), we have $\mathrm{supp}_\Delta(Y_r(\lambda)) \subseteq X(T)_{\geq\lambda}$ as well as $[Y_r(\lambda) : \widehat{Z}_r(\lambda)] = 1$. Consequently, there exist submodules $N \subseteq M \subseteq Y_r(\lambda)$ such that

(a) $M/N \cong \widehat{Z}_r(\lambda)$, and
(b) $Y_r(\lambda)/M$ belongs to $\mathcal{F}(\Delta)$ with $\mathrm{supp}_\Delta(Y_r(\lambda)/M) \subseteq X(T)_{>\lambda}$.

There results an exact sequence

$$(0) \longrightarrow \widehat{Z}_r(\lambda) \longrightarrow \widehat{Y}_r(\lambda)/N \longrightarrow \widehat{Y}_r(\lambda)/M \longrightarrow (0).$$

Lemma 4.4(1) implies $\mathrm{Ext}^1_{G_rT}(\widehat{Z}_r(\mu), \widehat{Z}_r(\lambda)) = (0)$ for all $\mu \in X(T)_{>\lambda}$. Hence our sequence splits, and

$$Y_r(\lambda)/N \cong (Y_r(\lambda)/M) \oplus \widehat{Z}_r(\lambda).$$

We may now define $g := \mathrm{pr} \circ \pi$ to be the composite of the projection $\mathrm{pr} : Y_r(\lambda)/N \longrightarrow \widehat{Z}_r(\lambda)$ with the canonical map $\pi : Y_r(\lambda) \longrightarrow Y_r(\lambda)/N$. ◇

(iii) *We have* $\dim_k \operatorname{Hom}_{G_rT}(E, \widehat{Z}_r(\lambda)) = 1$.

We write $X := \Omega^2_{G_rT}(\widehat{Z}_r(\lambda))$ for notational convenience. Application of the left exact functor $\operatorname{Hom}_{G_rT}(-, \widehat{Z}_r(\lambda))$ to the almost split sequence $(*)$ terminating in $\widehat{Z}_r(\lambda)$ gives an exact sequence

$$(0) \longrightarrow \operatorname{Hom}_{G_rT}(\widehat{Z}_r(\lambda), \widehat{Z}_r(\lambda)) \longrightarrow \operatorname{Hom}_{G_rT}(E, \widehat{Z}_r(\lambda)) \longrightarrow \operatorname{Hom}_{G_rT}(X, \widehat{Z}_r(\lambda)).$$

Since $\operatorname{Top}_{G_rT}(X)$ is a direct summand of $\bigoplus_{\mu > \lambda}[X : \widehat{Z}_r(\mu)]\widehat{L}_r(\mu)$, and all weights of $\widehat{Z}_r(\lambda)$ belong to $X(T)_{\leq \lambda}$, it follows that any homomorphism $X \longrightarrow \widehat{Z}_r(\lambda)$ sends the generators of X to zero. Consequently, $\operatorname{Hom}_{G_rT}(X, \widehat{Z}_r(\lambda)) = (0)$, implying

$$\operatorname{Hom}_{G_rT}(E, \widehat{Z}_r(\lambda)) \cong \operatorname{Hom}_{G_rT}(\widehat{Z}_r(\lambda), \widehat{Z}_r(\lambda)) \cong k,$$

as desired. $\diamond$

Let $f : E \longrightarrow \widehat{Z}_r(\lambda)$ and $h : E \longrightarrow Y_r(\lambda)$ be the irreducible morphisms given by the almost split sequences $(*)$ and $(**)$, respectively. By virtue of (ii) and (iii), there exists $\alpha \in k \setminus \{0\}$ such that

$$g \circ h = \alpha\, f.$$

Since αf is irreducible and h is not split injective, the map g is split surjective. Consequently, $\widehat{Z}_r(\lambda) \cong Y_r(\lambda)$, a contradiction. $\square$

Suppose that $\lambda, \mu \in X(T)$ are given such that $\Theta_r(\lambda) = \Theta_r(\mu)$. Theorems 4.5 and 4.6 tell us that $\widehat{Z}_r(\lambda)$ and $\widehat{Z}_r(\mu)$ belong to the same τ_{G_rT}-orbit.

Theorem 4.7. *The following statements hold:*

(1) *Let* $\lambda, \mu \in X(T)$ *be characters such that there exists* $m > 0$ *with*

$$\Omega^{2m}_{G_rT}(\widehat{Z}_r(\lambda)) \cong \widehat{Z}_r(\mu).$$

Then there exists a simple root α *such that* $\mu = \lambda + mp^r\alpha$.

(2) *Let* $\Theta \subseteq \Gamma_s(G_r)$ *be a connected component. Then* Θ *contains at most one baby Verma module.* $\square$

4.3. **Indecomposable $\mathbf{SL(2)_1T}$-Modules.** By way of illustration, let us consider the example $G = \mathrm{SL}(2)$ for $p \geq 3$. The representation theory of $\mathrm{SL}(2)_1$ is well understood: The algebra $k\,\mathrm{SL}(2)_1 \cong U_0(\mathfrak{sl}(2))$ affords $\frac{p-1}{2}$ non-simple blocks $\mathcal{B}_0, \ldots, \mathcal{B}_{\frac{p-3}{2}}$, with each block $\mathcal{B}_i$ possessing exactly two simple modules $L_1(i)$ and $L_1(p-i)$ of dimensions $i+1$ and $p-i-1$, respectively.

Given $d \geq 0$, we let $V(d)$ be the *Weyl module* of highest weight d. This module is obtained from the d-th symmetric power of the standard module $L(1)$ by twisting its dual by the Cartan involution $(g \mapsto g^{-\mathrm{tr}})$. Thus, if $\{e, h, f\} \subseteq \mathfrak{sl}(2)$ is the standard basis of $\mathfrak{sl}(2)$, then $V(d) = \bigoplus_{i=0}^{d} kv_i$ and

$$e.v_i = (i+1)v_{i+1} \quad ; \quad f.v_i = (d-i+1)v_{i-1} \quad ; \quad h.v_i = (2i-d)v_i.$$

Hence we have $\dim_k V(d) = d+1$ as well as $V(d) = L_1(d)$ for $d \in \{0, \ldots, p-1\}$. By construction, each $V(d)$ is a rational module for the group scheme $\mathrm{SL}(2)$. For $d = sp + a$, where $s \geq 1$ and $a \in \{0, \ldots, p-2\}$, Premet [32] defines the maximal $U_0(\mathfrak{sl}(2))$-submodule

$$W(d) := \bigoplus_{i=a+1}^{d} kv_i \subseteq V(d).$$

By definition, $W(d)$ is an sp-dimensional module that is stable under the standard Borel subgroup $B \subseteq \mathrm{SL}(2)$ of upper triangular matrices.

Given $g \in \mathrm{SL}(2)$, the subspace $g.W(d) \subseteq V(d)$ is $U_0(\mathfrak{sl}(2))$-stable and isomorphic to the twist $W(d)^{(g)}$ of $W(d)$ by the adjoint representation. In [32] Premet shows that

$$\mathcal{V}_{\mathfrak{sl}(2)}(g.W(d)) = k\mathrm{Ad}(g)(e) \qquad \forall\, g \in \mathrm{SL}(2).$$

The following result shows in particular that the modules $g.W(d)$ give rise to a complete list of the indecomposable $U_0(\mathfrak{sl}(2))$-modules with one-dimensional supports:

Theorem 4.8 ([32]). *Let $C \subseteq \mathrm{SL}(2)$ be a complete set of coset representatives of $\mathrm{SL}(2)/B$. Then any non-projective indecomposable $U_0(\mathfrak{sl}(2))$-module is isomorphic to exactly one of the modules of the following list:*

- *$V(d), V(d)^*$ for $d \geq p, d \not\equiv -1 \mod(p)$,*
- *$V(r)$ for $0 \leq r \leq p-1$,*
- *$g.W(d)$ for $g \in C$ and $d = sp + a$ with $s \geq 1$ and $a \in \{0, \dots, p-2\}$.* □

By work of Drozd [10], Fischer [16], and Rudakov [35], each of the $\frac{p-1}{2}$ non-simple blocks of $U_0(\mathfrak{sl}(2))$ is Morita equivalent to the trivial extension $\mathrm{Kr}^* \rtimes \mathrm{Kr}$ of the path algebra Kr associated to the Kronecker quiver $\bullet \rightrightarrows \bullet$. Consequently, the stable Auslander-Reiten quiver $\Gamma_s(\mathfrak{sl}(2))$ of $U_0(\mathfrak{sl}(2))$ is the disjoint union of $p-1$ components of type $\mathbb{Z}[\tilde{A}_{12}]$, and infinitely many homogeneous tubes.

In the following, we let $w_0 \in \mathrm{SL}(2)$ be a representative of the non-trivial element of the Weyl group of $\mathrm{SL}(2)$.

Corollary 4.9 ([13]). *Let $M \in \mathrm{mod}\,\mathrm{SL}(2)_1T$ be non-projective and indecomposable. Then the following statements hold*

(1) *The module M is isomorphic to exactly one of the following modules:*

(a) $V(d)\otimes_k k_\lambda$, $V(d)^*\otimes_k k_\lambda$, $V(r)\otimes_k k_\lambda$ *with* $\lambda \in pX(T)$ *and* $d \geq p$, $d \not\equiv -1 \mod(p), 0 \leq r \leq p-2$,

(b) $W(sp+i)\otimes_k k_\lambda$, $w_0.W(sp+i)\otimes_k k_\lambda$ *with* $\lambda \in pX(T)$, $i \in \{0, \dots, p-2\}$ *and* $s \geq 1$.

(2) *The modules of types (a) and (b) belong to components of types $\mathbb{Z}[A_\infty^\infty]$ and $\mathbb{Z}[A_\infty]$, respectively.* □

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