SUPPORT SPACES, JORDAN TYPES AND INDECOMPOSABLE MODULES

Isaac Newton Institute, Cambridge, March 17, 2009

1. The space of p-points

Let me begin by fixing the notation that will be in force throughout this talk:

- k is an algebraically closed field, $\mathrm{char}(k) = p > 0$.
- $\mathcal{G}$ denotes a finite group scheme over k, with coordinate ring $k[\mathcal{G}]$ and group algebra $k\mathcal{G} := k[\mathcal{G}]^*$. In other words, $k\mathcal{G}$ is a finite-dimensional cocommutative Hopf algebra. There are two well- known special cases:
  - The group algebra kG of a finite group G, and
  - the *restricted enveloping algebra* $U_0(\mathfrak{g})$ of a restricted Lie algebra $\mathfrak{g}$.
- $\mathrm{mod}\,\mathcal{G}$ denotes the category of finite-dimensional $k\mathcal{G}$-modules.
- $\mathrm{H}^\bullet(\mathcal{G}, k) := \bigoplus_{n\geq 0} \mathrm{H}^{2n}(\mathcal{G}, k)$ is the even cohomology ring; this is a commutative k-algebra. By the Friedlander-Suslin Theorem, $\mathrm{H}^\bullet(\mathcal{G}, k)$ is finitely generated.
- Given $M \in \mathrm{mod}\,\mathcal{G}$, $\mathcal{V}_\mathcal{G}(M) := Z(\mathrm{ann}_{\mathrm{H}^\bullet(\mathcal{G},k)}(\mathrm{Ext}^*_\mathcal{G}(M, M))) \subseteq \mathrm{Maxspec}(\mathrm{H}^\bullet(\mathcal{G}, k))$ is the *cohomological support variety* of M. By definition, $\mathcal{V}_\mathcal{G}(M) \subseteq \mathcal{V}_\mathcal{G}(k)$ is a conical variety.

Support varieties are important invariants, which are usually hard to compute. This poses the problem of finding a non-cohomological characterization while retaining the homological features. At this stage, the fact that k is algebraically closed enters.

For notational convenience we put $\mathfrak{A}_p := k[T]/(T^p)$. Given any homomorphism $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ of k-algebras, we denote by $\alpha^* : \mathrm{mod}\,\mathcal{G} \longrightarrow \mathrm{mod}\,\mathfrak{A}_p$ the functor given by pull-back along α.

Definition. An algebra homomorphism $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ is called a *p-point* if
(P1) α is left flat, and
(P2) there exists an abelian unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}$ with $\mathrm{im}\,\alpha \subseteq k\mathcal{U}$.
Two p-points α, β are *equivalent* ($\alpha \sim \beta$) if for every $M \in \mathrm{mod}\,k\mathcal{G}$ we have

$$\alpha^*(M) \text{ is projective} \Leftrightarrow \beta^*(M) \text{ is projective.}$$

The set of p-points will be denoted $\mathrm{Pt}(\mathcal{G})$, and we write $P(\mathcal{G}) := \mathrm{Pt}(\mathcal{G})/\sim$.

Since this definition looks somewhat contrived, let us spend a few moments on the defining conditions. Property (P1) means that the pull-back functor $\alpha^* : \mathrm{mod}\,k\mathcal{G} \longrightarrow \mathrm{mod}\,\mathfrak{A}_p$ sends projectives to projectives. This is motivated by the fact that projective modules have trivial support varieties.

Condition (P2) allows us to reduce many questions to the consideration of p-points of abelian unipotent group schemes. Such a group scheme $\mathcal{U}$ is the Cartier dual of an infinitesimal group. Consequently, we have

$$k\mathcal{U} \cong k[X_1, \ldots, X_n]/(X_1^{p^{r_1}}, \ldots, X_n^{p^{r_n}}).$$

Accordingly, $\mathcal{U}$ is the analogue of an abelian p-subgroup of a finite group G.

Let $M \in \mathrm{mod}\,\mathcal{G}$. Then

$$P(\mathcal{G})_M := \{[\alpha] \in P(\mathcal{G}) \ ; \ \alpha^*(M) \text{ is not projective}\}$$

is called the *p-support of M*.

Theorem 1.1 (Friedlander-Pevtsova). (1) *The sets $P(\mathcal{G})_M$ form the closed sets of a noetherian topology on $P(\mathcal{G})$.*

(2) *There is a homeomorphism $\Psi_{\mathcal{G}} : P(\mathcal{G}) \longrightarrow \mathrm{Proj}(\mathcal{V}_{\mathcal{G}}(k))$ such that*

$$\Psi_{\mathcal{G}}(P(\mathcal{G})_M) = \mathrm{Proj}(\mathcal{V}_{\mathcal{G}}(M)) \qquad \forall\ M \in \mathrm{mod}\,\mathcal{G}.$$

In particular, we have

$$\dim P(\mathcal{G})_M = \mathrm{cx}_{\mathcal{G}}(M) - 1,$$

where $\mathrm{cx}_{\mathcal{G}}(M)$ denotes the *complexity* of the $\mathcal{G}$-module M, that is, the polynomial rate of growth of a minimal projective resolution of M.

2. The set of Jordan Types

New features arise when one considers the isomorphism type of $\alpha^*(M)$, rather than just asking whether this module is projective. If $M \in \mathrm{mod}\,\mathcal{G}$ and $\alpha \in \mathrm{Pt}(\mathcal{G})$, then

$$\alpha^*(M) \cong \bigoplus_{i=1}^{p} \alpha_i(M)[i],$$

where $[i]$ denotes the i-dimensional indecomposable $\mathfrak{A}_p$-module. We call the isoclass of $\alpha^*(M)$ the *Jordan type* $\mathrm{Jt}(M,\alpha)$ *of M relative to α*. Then

$$\mathrm{Jt}(M) := \{\mathrm{Jt}(M,\alpha)\ ;\ \alpha \in \mathrm{Pt}(\mathcal{G})\}$$

is the finite set of Jordan types of M.

The theory of Jordan types was initiated by Friedlander-Pevtsova-Suslin and its ramifications are only emerging. At this juncture, the following three aspects are being investigated:

- Jordan types give rise to finer invariants than support varieties do,
- they have led to the introduction of new classes of modules,
- they may be used to show that certain modules are indecomposable.

In the geometric approach via support varieties, the following modules play a prominent rôle. Let $\zeta \in \mathrm{H}^n(\mathcal{G},k) \setminus \{0\}$ be a non-zero element. By general theory, ζ corresponds to a non-zero homomorphism $\hat{\zeta} : \Omega^n_{\mathcal{G}}(k) \longrightarrow k$, and

$$L_\zeta := \ker \hat{\zeta}$$

is the *Carlson module associated to* ζ. If n is even, then $\mathcal{V}_{\mathcal{G}}(L_\zeta) = Z(\zeta)$.

Example. Suppose that $p \geq 3$, and let $\zeta \in \mathrm{H}^n(\mathcal{G},k) \setminus \{0\}$ be such that $L_\zeta \neq (0)$.

(1) If n is even and ζ is not nilpotent, then $\mathrm{Jt}(L_\zeta) = \{m_\zeta[p], [1] \oplus [p-1] \oplus n_\zeta[p]\}$.

(2) If n is even and ζ is nilpotent, then $\mathrm{Jt}(L_\zeta) = \{[1] \oplus [p-1] \oplus n_\zeta[p]\}$. $\mathcal{G}$-modules M with $|\,\mathrm{Jt}(M)| = 1$ are said to have *constant Jordan type.*

(3) If n is odd, then $\mathrm{Jt}(L_\zeta, \alpha) = \begin{cases} 2[p-1] \oplus m_\zeta[p] & \text{if } \alpha^*(\zeta) = 0 \\ [p-2] \oplus n_\zeta[p] & \text{if } \alpha^*(\zeta) \neq 0. \end{cases}$

Theorem 2.1. *Suppose that $p \geq 3$. Let $\zeta \in \mathrm{H}^n(\mathcal{G},k) \setminus \{0\}$ be nilpotent and such that $L_\zeta \neq (0)$.*

(1) *If n is even, then L_ζ is indecomposable.*

(2) *If n is odd, and $P(\mathcal{G}) = \bigcup_{\mathcal{U};\mathrm{cx}_{\mathcal{U}}(k)\geq 2} P(\mathcal{U})$, then L_ζ is indecomposable.*

If G is a finite group, then the technical condition given in (2) is superfluous. The example of the restricted Lie algebra $\mathfrak{sl}(2)$ shows that infinitesimal groups of complexity 2 may only have abelian unipotent subgroups of complexity 1.

3. Invariants of Auslander-Reiten Components

We let $\Gamma_s(\mathcal{G})$ be the *stable Auslander-Reiten quiver* of $\mathcal{G}$. By definition, $\Gamma_s(\mathcal{G})$ is a directed graph, given by

- *vertices*: the isomorphism classes of non-projective indecomposable modules, and
- *arrows* $M \to N$: the irreducible morphisms $M \longrightarrow N$, and
- the *Auslander-Reiten translation*: $\tau_{\mathcal{G}} : \Gamma_s(\mathcal{G}) \longrightarrow \Gamma_s(\mathcal{G})$; $M \mapsto \nu_{\mathcal{G}} \circ \Omega^2_{\mathcal{G}}(M)$. Here $\nu_{\mathcal{G}}$ is the Nakayama functor of $\mathcal{G}$ (which is the identity for finite groups).

A theorem by Riedtmann concerning stable representation quivers ensures that a connected component $\Theta \subseteq \Gamma_s(\mathcal{G})$ can be described by a directed tree T_Θ, whose underlying graph $\bar{T}_\Theta$, the so-called *tree class*, is uniquely determined.

Given a component $\Theta \subseteq \Gamma_s(\mathcal{G})$, we have

$$P(\mathcal{G})_M = P(\mathcal{G})_N \qquad \forall\ M, N \in \Theta.$$

Hence we can speak of the support space $P(\mathcal{G})_\Theta$ of the component. This invariant tells us more about the tree classes of components:

- $\bar{T}_\Theta$ is either a finite Dynkin diagram, an infinite Dynkin diagram, or a Euclidean diagram.
- If $\dim P(\mathcal{G})_\Theta \geq 2$, then $\Theta \cong \mathbb{Z}[A_\infty]$.

The following example concerns a family of groups that naturally arise in the classification of infinitesimal groups of tame representation type. We consider SL(2) along with is standard maximal torus $T \subseteq \mathrm{SL}(2)$ of diagonal matrices. Given $r \geq 1$, the group $\mathrm{SL}(2)_1T_r$ is the product of the first Frobenius kernel of $\mathrm{SL}(2)_1$ with the r-th Frobenius kernel of T. Roughly speaking, one can think of its module category as the category of $\mathbb{Z}/(p^r)$-graded modules of $U_0(\mathfrak{sl}(2))$.

Example. Let $\Theta \subseteq \Gamma_s(\mathrm{SL}(2)_1T_r)$ be a component. Then the following statements hold:

(1) If $\dim P(\mathrm{SL}(2)_1T_r)_\Theta = 1$, then $\bar{T}_\Theta \cong A_\infty^\infty, \tilde{A}_{1,2}$, and there exists $s_\Theta \in \{1, \dots, p-1\}$ such that

$$\mathrm{Jt}(M) = \{[s_\Theta] \oplus n_M[p]\}$$

for every $M \in \Theta$.

(2) If $\dim P(\mathrm{SL}(2)_1T_r)_\Theta = 0$, then $\Theta \cong \mathbb{Z}[A_\infty]/\langle\tau\rangle$, $\mathbb{Z}[A_\infty]/\langle\tau^{p^{r-1}}\rangle$, and there exists $i_\Theta \in \{1, \dots, \frac{p-1}{2}\}$ such that

$$\mathrm{Jt}(M) = \{m_M[p], [i_\Theta] \oplus [p - i_\Theta] \oplus n_M[p]\}$$

for every $M \in \Theta$.

Throughout the remainder of this talk, I will be concerned with results that explain some of these phenomena.

Let M be a non-projective, indecomposable $\mathcal{G}$-module. Then

$$\mathfrak{E}_M \ : \quad (0) \longrightarrow \tau_{\mathcal{G}}(M) \longrightarrow E_M \longrightarrow M \longrightarrow (0)$$

denotes the *almost split sequence* terminating in M. These sequences are closely linked to the structure of the AR-quiver: The non-projective indecomposable summands of E_M are precisely the predecessors of M and the successors of $\tau_{\mathcal{G}}(M)$ in the stable AR-quiver $\Gamma_s(\mathcal{G})$.

Given $\alpha \in \mathrm{Pt}(\mathcal{G})$, we say that a component $\Theta \subseteq \Gamma_s(\mathcal{G})$ is *α-split* if $\alpha^*(\mathfrak{E}_M)$ splits for every $M \in \Theta$. The component Θ is *locally split* if it is α-split for every $\alpha \in \mathrm{Pt}(\mathcal{G})$.

Proposition 3.1. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component, $\alpha \in \mathrm{Pt}(\mathcal{G})$.*

(1) *If Θ is not α-split, then $P(\mathcal{G})_\Theta = \{[\alpha]\}$, and Θ is either finite or isomorphic to $\mathbb{Z}[A_\infty]/\langle\tau^n\rangle$ for some $n \in \mathbb{N}$.*

(2) *If $\dim P(\mathcal{G})_\Theta \geq 1$, then Θ is locally split.*

(3) *If Θ is α-split, then $\alpha_i : \Theta \longrightarrow \mathbb{N}_0$ is a $\tau_\mathcal{G}$-invariant additive function for $i \in \{1, \dots, p-1\}$.*

The additivity property refers to the additivity of α_i on the almost split sequences of Θ.

Given a component $\Theta \subseteq \Gamma_s(\mathcal{G})$, we put

$$\mathrm{Pt}(\mathcal{G}, \Theta) := \{\alpha \in \mathrm{Pt}(\mathcal{G}) \ ; \ \Theta \text{ is } \alpha\text{-split}\}.$$

Theorem 3.2. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be an infinite component. Then there exist a $\tau_\mathcal{G}$-invariant additive function $f_\Theta : \Theta \longrightarrow \mathbb{N}$ and a function $d^\Theta : \mathrm{Pt}(\mathcal{G}, \Theta) \longrightarrow \mathbb{N}_0^{p-1}$ with*

$$\alpha_i(M) = d_i^\Theta(\alpha) f_\Theta(M) \qquad 1 \leq i \leq p-1$$

for every $M \in \Theta$ and every $\alpha \in \mathrm{Pt}(\mathcal{G}, \Theta)$.

Examples. (1) If $\bar{T}_\Theta = A_\infty$, then $f_\Theta(M) = \mathrm{q}\ell(M)$ for every $M \in \Theta$.

(2) If $\bar{T}_\Theta = A_\infty^\infty, \tilde{A}_{1,2}$, then $f_\Theta \equiv 1$.

Let us consider a component $\Theta \subseteq \Gamma_s(\mathrm{SL}(2)_1 T_r$ with $\dim P(\mathrm{SL}(2)_1 T_r)_\Theta = 1$. Then $\bar{T}_\Theta = A_\infty^\infty, \tilde{A}_{1,2}$. Each of these components contains a simple module S, whose Jordan type is completely determined by its restriction $S|_{\mathrm{SL}(2)_1}$.

We also obtain the following new invariant of AR-components.

Corollary 3.3. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a locally split component. Then*

$$|\,\mathrm{Jt}(M)| = |\,\mathrm{im}\, d^\Theta|$$

for every $M \in \Theta$.

Corollary 3.4 (Carlson-Friedlander-Pevtsova)**.** *If a component $\Theta \subseteq \Gamma_s(\mathcal{G})$ contains a module of constant Jordan type, then all modules belonging to Θ have constant Jordan type.*

We finally turn to components Θ that are not locally split. If such a Θ is infinite, then $\Theta \cong \mathbb{Z}[A_\infty]/\langle\tau^n\rangle$, so we can speak of the quasi-length of a module. Let $(a_{ij})_{1\leq i,j\leq p-1}$ be the Cartan matrix of the Dynkin diagram A_{p-1}.

Theorem 3.5. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be an infinite component, $\alpha \in \mathrm{Pt}(\mathcal{G})$ be a p-point such that*

(a) *Θ is not α-split, and*

(b) *if $N \in \Theta$ is such that $\alpha^*(\mathfrak{E}_N)$ does not split, then $\mathrm{q}\ell(N) = 1$.*

Then there exist a vector $(n_1, \dots, n_{p-1}) \in \mathbb{N}_0^{p-1} \setminus \{0\}$ and $M \in \Theta$ with $\mathrm{q}\ell(M) = 1$ such that

$$\alpha_i(X) = (\alpha_i(M) - \sum_{j=1}^{p-1} a_{ij} n_j)\, \mathrm{q}\ell(X) + \sum_{j=1}^{p-1} a_{ij} n_j \qquad 1 \leq i \leq p-1$$

for every $X \in \Theta$.

The technical condition (b) of this result is known to hold in the following settings:

(1) Θ contains a module whose top or socle has a one-dimensional constituent. In this case, we have $n_i \in \{0, 1\}$.
(2) If $\mathcal{G}$ is a Frobenius kernel of a solvable algebraic group; then we have $n_i = \delta_{ij}$ for some j depending on Θ.
(3) G is a finite group and Θ contains a module with a cyclic vertex.
(4) $\mathcal{G} = \mathrm{SL}(2)_1 T_r$ and $\dim P(\mathcal{G})_\Theta = 0$. In this case, translation functors may be used in conjunction with (1).