1. The space of $p$-points

Let me begin by fixing the notation that will be in force throughout this talk:

- $k$ is an algebraically closed field, $\text{char}(k) = p > 0$.
- $\mathcal{G}$ denotes a finite group scheme over $k$, with coordinate ring $k[\mathcal{G}]$ and group algebra $k\mathcal{G} := k[[\mathcal{G}]]$. In other words, $k\mathcal{G}$ is a finite-dimensional cocommutative Hopf algebra. There are two well-known special cases:
  - The group algebra $kG$ of a finite group $G$, and
  - the restricted enveloping algebra $U_0(\mathfrak{g})$ of a restricted Lie algebra $\mathfrak{g}$.
- $\text{mod} \mathcal{G}$ denotes the category of finite-dimensional $k\mathcal{G}$-modules.
- $H^{\bullet}(\mathcal{G}, k) := \bigoplus_{n \geq 0} H^{2n}(\mathcal{G}, k)$ is the even cohomology ring; this is a commutative $k$-algebra. By the Friedlander-Suslin Theorem, $H^{\bullet}(\mathcal{G}, k)$ is finitely generated.
- $\text{mod} \mathcal{G}$ denotes the category of finite-dimensional $k\mathcal{G}$-modules.
- $V_G(M) := \{ \alpha \in P_G \mid \alpha^*(M) \text{ is not projective} \}$ is called the $p$-support of $M$.

Support varieties are important invariants, which are usually hard to compute. This poses the problem of finding a non-cohomological characterization while retaining the homological features. At this stage, the fact that $k$ is algebraically closed enters.

For notational convenience we put $A_p := k[T]/(T^p)$. Given any homomorphism $\alpha : A_p \rightarrow k\mathcal{G}$ of $k$-algebras, we denote by $\alpha^* : \text{mod} \mathcal{G} \rightarrow \text{mod} A_p$ the functor given by pull-back along $\alpha$.

**Definition.** An algebra homomorphism $\alpha : A_p \rightarrow k\mathcal{G}$ is called a $p$-point if

- (P1) $\alpha$ is left flat, and
- (P2) there exists an abelian unipotent subgroup $U \subseteq \mathcal{G}$ with $\text{im} \alpha \subseteq kU$.

Two $p$-points $\alpha, \beta$ are equivalent ($\alpha \sim \beta$) if for every $M \in \text{mod} \mathcal{G}$ we have $\alpha^*(M)$ is projective $\iff \beta^*(M)$ is projective.

The set of $p$-points will be denoted $P_G$, and we write $P(\mathcal{G}) := P_G/\sim$.

Since this definition looks somewhat contrived, let us spend a few moments on the defining conditions. Property (P1) means that the pull-back functor $\alpha^* : \text{mod} k\mathcal{G} \rightarrow \text{mod} A_p$ sends projectives to projectives. This is motivated by the fact that projective modules have trivial support varieties.

Condition (P2) allows us to reduce many questions to the consideration of $p$-points of abelian unipotent group schemes. Such a group scheme $U$ is the Cartier dual of an infinitesimal group. Consequently, we have

$$kU \cong k[X_1, \ldots, X_n]/(X_1^{p^r_1}, \ldots, X_n^{p^r_n}).$$

Accordingly, $U$ is the analogue of an abelian $p$-subgroup of a finite group $G$.

Let $M \in \text{mod} \mathcal{G}$. Then

$$P_G(M) := \{ [\alpha] \in P_G : \alpha^*(M) \text{ is not projective} \}$$

is called the $p$-support of $M$. 

Theorem 2.1. Suppose that unipotent subgroups of complexity 1.

Example. Carlson module associated to $\zeta$ is the $G$-module $\Omega^0_G(k)$ such that

$$\text{dim } \Omega^0_G(k) = \text{proj}(\mathcal{V}_G(k)) \quad \forall M \in \text{mod } G.$$ 

In particular, we have $\dim P(G)_M = \text{cx}_G(M) - 1$, where $\text{cx}_G(M)$ denotes the complexity of the $G$-module $M$, that is, the polynomial rate of growth of a minimal projective resolution of $M$.

2. The set of Jordan Types

New features arise when one considers the isomorphism type of $\alpha^*(M)$, rather than just asking whether this module is projective. If $M \in \text{mod } G$ and $\alpha \in \text{Pt}(G)$, then

$$\alpha^*(M) \cong \bigoplus_{i=1}^{\text{dim } P(G)_M} \alpha_i(M)[i],$$

where $[i]$ denotes the $i$-dimensional indecomposable $A_p$-module. We call the isoclass of $\alpha^*(M)$ the Jordan type $\text{Jt}(M, \alpha)$ of $M$ relative to $\alpha$. Then

$$\text{Jt}(M) := \{\text{Jt}(M, \alpha) ; \alpha \in \text{Pt}(G)\}$$

is the finite set of Jordan types of $M$.

The theory of Jordan types was initiated by Friedlander-Pevtsova-Suslin and its ramifications are only emerging. At this juncture, the following three aspects are being investigated:

- Jordan types give rise to finer invariants than support varieties do,
- they have led to the introduction of new classes of modules,
- they may be used to show that certain modules are indecomposable.

In the geometric approach via support varieties, the following modules play a prominent rôle. Let $\zeta \in H^n(G, k) \setminus \{0\}$ be a non-zero element. By general theory, $\zeta$ corresponds to a non-zero homomorphism $\hat{\zeta} : \Omega^0_G(k) \to k$, and

$$L_{\zeta} := \ker \hat{\zeta}$$

is the Carlson module associated to $\zeta$. If $n$ is even, then $\mathcal{V}_G(L_{\zeta}) = Z(\zeta)$.

Example. Suppose that $p \geq 3$, and let $\zeta \in H^n(G, k) \setminus \{0\}$ be such that $L_{\zeta} \neq (0)$.

1. If $n$ is even and $\zeta$ is not nilpotent, then $\text{Jt}(L_{\zeta}) = \{m_{\zeta}[p], [1] \oplus [p-1] \oplus n_{\zeta}[p]\}$.
2. If $n$ is even and $\zeta$ is nilpotent, then $\text{Jt}(L_{\zeta}) = \{[1] \oplus [p-1] \oplus n_{\zeta}[p]\}$. $G$-modules $M$ with $|\text{Jt}(M)| = 1$ are said to have constant Jordan type.
3. If $n$ is odd, then $\text{Jt}(L_{\zeta}, \alpha) = \begin{cases} 2[p-1] \oplus m_{\zeta}[p] & \text{if } \alpha^*(\zeta) = 0 \\ [p-2] \oplus n_{\zeta}[p] & \text{if } \alpha^*(\zeta) \neq 0. \end{cases}$

Theorem 2.1. Suppose that $p \geq 3$. Let $\zeta \in H^n(G, k) \setminus \{0\}$ be nilpotent and such that $L_{\zeta} \neq (0)$.

1. If $n$ is even, then $L_{\zeta}$ is indecomposable.
2. If $n$ is odd, and $P(G) = \bigcup_{\text{cx}_G(l) \geq 2} P(l)$, then $L_{\zeta}$ is indecomposable.

If $G$ is a finite group, then the technical condition given in (2) is superfluous. The example of the restricted Lie algebra $\mathfrak{sl}(2)$ shows that infinitesimal groups of complexity 2 may only have abelian unipotent subgroups of complexity 1.
3. Invariants of Auslander-Reiten Components

We let $\Gamma_s(\mathcal{S})$ be the stable Auslander-Reiten quiver of $\mathcal{S}$. By definition, $\Gamma_s(\mathcal{S})$ is a directed graph, given by

- **vertices**: the isomorphism classes of non-projective indecomposable modules, and
- **arrows** $M \to N$: the irreducible morphisms $M \to N$, and
- **the Auslander-Reiten translation**: $\tau_\mathcal{S}: \Gamma_s(\mathcal{S}) \to \Gamma_s(\mathcal{S})$; $M \mapsto \nu_\mathcal{S} \circ \Omega^2_\mathcal{S}(M)$. Here $\nu_\mathcal{S}$ is the Nakayama functor of $\mathcal{S}$ (which is the identity for finite groups).

A theorem by Riedtmann concerning stable representation quivers ensures that a connected component $\Theta \subseteq \Gamma_s(\mathcal{S})$ can be described by a directed tree $T_\Theta$, whose underlying graph $T_\Theta$, the so-called **tree class**, is uniquely determined.

Given a component $\Theta \subseteq \Gamma_s(\mathcal{S})$, we have

$$P(\mathcal{S})_M = P(\mathcal{S})_N \quad \forall \, M, N \in \Theta.$$  

Hence we can speak of the support space $P(\mathcal{S})_\Theta$ of the component. This invariant tells us more about the tree classes of components:

- $T_\Theta$ is either a finite Dynkin diagram, an infinite Dynkin diagram, or a Euclidean diagram.
- If $\dim P(\mathcal{S})_\Theta \geq 2$, then $\Theta \cong \mathbb{Z}[A_\infty]$.

The following example concerns a family of groups that naturally arise in the classification of infinitesimal groups of tame representation type. We consider $\text{SL}(2)$ along with its standard maximal torus $T \subseteq \text{SL}(2)$ of diagonal matrices. Given $r \geq 1$, the group $\text{SL}(2)_1 T_r$ is the product of the first Frobenius kernel of $\text{SL}(2)_1$ with the $r$-th Frobenius kernel of $T$. Roughly speaking, one can think of its module category as the category of $\mathbb{Z}/(p^r)$-graded modules of $U_0(\mathfrak{sl}(2))$.

**Example.** Let $\Theta \subseteq \Gamma_s(\text{SL}(2)_1 T_r)$ be a component. Then the following statements hold:

1. If $\dim P(\text{SL}(2)_1 T_r)_\Theta = 1$, then $T_\Theta \cong A_\infty, \tilde{A}_{1,2}$, and there exists $s_\Theta \in \{1, \ldots, p-1\}$ such that

   $$\text{Jt}(M) = \{[s_\Theta] \oplus n_M[p]\}$$

   for every $M \in \Theta$.

2. If $\dim P(\text{SL}(2)_1 T_r)_\Theta = 0$, then $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau \rangle, \mathbb{Z}[A_\infty]/\langle \tau^{p-1} \rangle$, and there exists $i_\Theta \in \{1, \ldots, \frac{p-1}{2}\}$ such that

   $$\text{Jt}(M) = \{m_M[p], [i_\Theta] \oplus [p - i_\Theta] \oplus n_M[p]\}$$

   for every $M \in \Theta$.

Throughout the remainder of this talk, I will be concerned with results that explain some of these phenomena.

Let $M$ be a non-projective, indecomposable $\mathcal{S}$-module. Then

$$\mathcal{E}_M : (0) \longrightarrow \tau_\mathcal{S}(M) \longrightarrow E_M \longrightarrow M \longrightarrow (0)$$

denotes the **almost split sequence** terminating in $M$. These sequences are closely linked to the structure of the AR-quiver: The non-projective indecomposable summands of $E_M$ are precisely the predecessors of $M$ and the successors of $\tau_\mathcal{S}(M)$ in the stable AR-quiver $\Gamma_s(\mathcal{S})$.

Given $\alpha \in \text{Pt}(\mathcal{S})$, we say that a component $\Theta \subseteq \Gamma_s(\mathcal{S})$ is $\alpha$-**split** if $\alpha^*(\mathcal{E}_M)$ splits for every $M \in \Theta$. The component $\Theta$ is **locally split** if it is $\alpha$-split for every $\alpha \in \text{Pt}(\mathcal{S})$. 

Proposition 3.1. Let \( \Theta \subseteq \Gamma_s(\mathcal{G}) \) be a component, \( \alpha \in \text{Pt}(\mathcal{G}) \).

1. If \( \Theta \) is not \( \alpha \)-split, then \( P(\mathcal{G})_\Theta = \{ [\alpha] \} \), and \( \Theta \) is either finite or isomorphic to \( \mathbb{Z}[A_\infty]/(\tau^n) \) for some \( n \in \mathbb{N} \).
2. If \( \dim P(\mathcal{G})_\Theta \geq 1 \), then \( \Theta \) is locally split.
3. If \( \Theta \) is \( \alpha \)-split, then \( \alpha_1 : \Theta \rightarrow \mathbb{N}_0 \) is a \( \tau_s \)-invariant additive function for \( i \in \{1, \ldots, p - 1\} \).

The additivity property refers to the additivity of \( \alpha_i \) on the almost split sequences of \( \Theta \).

Given a component \( \Theta \subseteq \Gamma_s(\mathcal{G}) \), we put

\[ \text{Pt}(\mathcal{G}, \Theta) := \{ \alpha \in \text{Pt}(\mathcal{G}) ; \Theta \text{ is } \alpha \text{-split} \} \]

Theorem 3.2. Let \( \Theta \subseteq \Gamma_s(\mathcal{G}) \) be an infinite component. Then there exist a \( \tau_s \)-invariant additive function \( f_\Theta : \Theta \rightarrow \mathbb{N} \) and a function \( d^\Theta : \text{Pt}(\mathcal{G}, \Theta) \rightarrow \mathbb{N}^{p-1}_0 \) with

\[ \alpha_i(M) = d^\Theta(f_\Theta(M)) \quad 1 \leq i \leq p - 1 \]

for every \( M \in \Theta \) and every \( \alpha \in \text{Pt}(\mathcal{G}, \Theta) \).

Examples. (1) If \( T_\Theta = A_\infty \), then \( f_\Theta(M) = q\ell(M) \) for every \( M \in \Theta \).

(2) If \( T_\Theta = A_\infty^{\infty}, A_{1,2} \), then \( f_\Theta \equiv 1 \).

Let us consider a component \( \Theta \subseteq \Gamma_s(\text{SL}(2)_1\Gamma) \) with \( \dim P(\text{SL}(2)_1\Gamma)_\Theta = 1 \). Then \( T_\Theta = A_\infty^{\infty}, A_{1,2} \).

Each of these components contains a simple module \( S \), whose Jordan type is completely determined by its restriction \( S|_{\text{SL}(2)_1} \).

We also obtain the following new invariant of AR-components.

Corollary 3.3. Let \( \Theta \subseteq \Gamma_s(\mathcal{G}) \) be a locally split component. Then

\[ |\text{Jt}(M)| = |\text{im}d^\Theta| \]

for every \( M \in \Theta \).

Corollary 3.4 (Carlson-Friedlander-Pevtsova). If a component \( \Theta \subseteq \Gamma_s(\mathcal{G}) \) contains a module of constant Jordan type, then all modules belonging to \( \Theta \) have constant Jordan type.

We finally turn to components \( \Theta \) that are not locally split. If such a \( \Theta \) is infinite, then \( \Theta \cong \mathbb{Z}[A_\infty]/(\tau^n) \), so we can speak of the quasi-length of a module. Let \( (a_{ij})_{1 \leq i, j \leq p - 1} \) be the Cartan matrix of the Dynkin diagram \( A_{p-1} \).

Theorem 3.5. Let \( \Theta \subseteq \Gamma_s(\mathcal{G}) \) be an infinite component, \( \alpha \in \text{Pt}(\mathcal{G}) \) be a \( p \)-point such that

(a) \( \Theta \) is not \( \alpha \)-split, and

(b) if \( N \in \Theta \) is such that \( \alpha^*(\mathcal{E}_N) \) does not split, then \( q\ell(N) = 1 \).

Then there exist a vector \( (n_1, \ldots, n_{p-1}) \in \mathbb{N}^{p-1}_0 \setminus \{0\} \) and \( M \in \Theta \) with \( q\ell(M) = 1 \) such that

\[ \alpha_i(X) = (\alpha_i(M) - \sum_{j=1}^{p-1} a_{ij} n_j) q\ell(X) + \sum_{j=1}^{p-1} a_{ij} n_j \quad 1 \leq i \leq p - 1 \]

for every \( X \in \Theta \).
The technical condition (b) of this result is known to hold in the following settings:

(1) $\Theta$ contains a module whose top or socle has a one-dimensional constituent. In this case, we have $n_i \in \{0, 1\}$.

(2) If $\mathcal{G}$ is a Frobenius kernel of a solvable algebraic group; then we have $n_i = \delta_{i,j}$ for some $j$ depending on $\Theta$.

(3) $G$ is a finite group and $\Theta$ contains a module with a cyclic vertex.

(4) $\mathcal{G} = \text{SL}(2)_T$ and $\dim P(\mathcal{G})_{\Theta} = 0$. In this case, translation functors may be used in conjunction with (1).