TAMENESS AND COMPLEXITY OF FINITE GROUP SCHEMES

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Abstract. Using a representation-theoretic interpretation of support varieties due to Friedlander-Pevtsova [18], we show that the complexity of tame blocks of finite group schemes is bounded by 2. In this context, our result salvages a theorem by Rickard [23], whose proof is flawed.

1. Introduction and Preliminaries

In his paper [23] J. Rickard established an important result implying that the complexity $\mathrm{cx}_\Lambda(M)$ of a finite-dimensional module M over a self-injective tame algebra Λ is bounded by 2. His geometric arguments require a test for wildness [23, Lemma 1], which was recently falsified by K. Erdmann, A. Skowroński and G. Zwara.[1] As many classification results (cf. [13, 14, 15, 16, 20, 24]) concerning reduced enveloping algebras and infinitesimal group schemes of tame representation type ultimately rest on the boundedness of complexity, it seems desirable to salvage Rickard's Theorem in these contexts. The main result of this article, Theorem 3.1, accomplishes this for blocks of cocommutative Hopf algebras, or algebras of measures associated to finite group schemes. Our methods are based on J. Carlson's seminal papers [4, 5] concerning support varieties for finite groups as well as recent work by Friedlander and Pevtsova [18].

Throughout, we will be working over an algebraically closed field k of characteristic $p > 0$. If $f_1, \dots, f_n \in k[V]$ are elements of the coordinate ring of an affine variety V, then

$$Z(f_1, \dots, f_n) := \{v \in V \ ; \ f_i(v) = 0 \ , \ 1 \le i \le n\}$$

denotes the closed subvariety of their common zeros. Let $\mathcal{V}$ be a conical affine variety, that is, an affine variety, whose coordinate ring $k[\mathcal{V}]$ affords a gradation

$$k[\mathcal{V}] = \bigoplus_{i \ge 0} k[\mathcal{V}]_i$$

with $k[\mathcal{V}]_0 = k$. A closed subset $V \subset \mathcal{V}$ is called *conical* if there exist homogeneous elements $f_1, \dots, f_n \in k[\mathcal{V}]$ such that

$$V = Z(f_1, \dots, f_n).$$

In that case, the comorphism $\iota^* : k[\mathcal{V}] \longrightarrow k[V]$ is homogeneous of degree 0.

We begin with the following elementary Lemma.

Lemma 1.1. *Let $V \subset \mathcal{V}$ be a conical variety of dimension $n \ge 2$. Then there exist homogeneous elements $(f_\alpha)_{\alpha \in k} \in k[\mathcal{V}]$ of degree $d > 0$, such that*
(a) $\dim Z(f_\alpha) \cap V = n - 1$ *for $\alpha \in k$, and*
(b) $\dim Z(f_\alpha) \cap Z(f_\beta) \cap V = n - 2$ *for $\alpha \neq \beta \in k$.*

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[1] In an e-mail message dated November 25, 2004 A. Skowroński informed the author about K. Erdmann's observation that the three dimensional $k[X, Y]/(X^2, Y^2)$-modules having one indecomposable direct summand of dimension 2 provide a counter-example.

Proof. The Noether Normalization Lemma [9, (13.3)] provides homogeneous elements $f_1, \dots, f_n \in k[\mathcal{V}]$ of degree $d > 0$ such that the images $\bar{f}_i \in k[V]$ have the following properties:

(1) The set $\{\bar{f}_1, \dots, \bar{f}_n\}$ is algebraically independent, and
(2) $k[V]$ is a finitely generated $k[\bar{f}_1, \dots, \bar{f}_n]$-module.

As a result, the morphism

$$\varphi : V \longrightarrow \mathbb{A}^n \quad ; \quad v \mapsto (f_1(v), \dots, f_n(v))$$

is dominant and finite. In particular, the map φ is closed and surjective, and we have

$$(*) \qquad \dim \varphi^{-1}(W) = \dim W$$

for any closed subset $W \subset \mathbb{A}^n$ (cf. [9, Cor.9.3]).

Now let $\mathrm{pr} : \mathbb{A}^n \longrightarrow \mathbb{A}^2$ be the projection onto the first two coordinates, and consider $\Psi := \mathrm{pr} \circ \varphi$. Thus,

$$\Psi : V \longrightarrow \mathbb{A}^2 \quad ; \quad v \mapsto (f_1(v), f_2(v))$$

is a surjective morphism.

Given $\alpha \in k$, we put $\mathcal{L}_\alpha := \{(x, \alpha x) \ ; \ x \in k\}$. Then $Q_\alpha := \mathrm{pr}^{-1}(\mathcal{L}_\alpha) \subset \mathbb{A}^n$ is a subspace of dimension $n-1$, and $(*)$ yields

$$\dim Z(f_2 - \alpha f_1) \cap V = \dim \Psi^{-1}(\mathcal{L}_\alpha) = \dim \varphi^{-1}(Q_\alpha) = n - 1.$$

In particular, $f_\alpha := f_2 - \alpha f_1 \in k[\mathcal{V}]_d$ is a non-zero homogeneous element of degree d. For $\alpha \neq \beta \in k$ we have

$$Z(f_\alpha) \cap Z(f_\beta) \cap V = \Psi^{-1}(\mathcal{L}_\alpha) \cap \Psi^{-1}(\mathcal{L}_\beta) = \Psi^{-1}(\mathcal{L}_\alpha \cap \mathcal{L}_\beta) = \Psi^{-1}(0).$$

Since $\ker \mathrm{pr} \subset \mathbb{A}^n$ is a subspace of dimension $n-2$, another application of $(*)$ implies $\dim \Psi^{-1}(0) = n - 2$, as desired. □

2. Realizability of Conical Varieties

Let $\mathcal{G}$ be a finite k-group scheme. By general theory (cf. [22, (I.8.6)]), the module category of $\mathcal{G}$ is equivalent to that of the algebra $k\mathcal{G}$ of measures on $\mathcal{G}$. By definition, $k\mathcal{G}$ is the dual of the coordinate $k[\mathcal{G}]$ ring of $\mathcal{G}$. It coincides with the group algebra $k\mathcal{G}(k)$ of the group of k-rational points of $\mathcal{G}$ in the case $\mathcal{G}$ is reduced, and with the algebra $\mathrm{Dist}(\mathcal{G})$ of distributions in the case $\mathcal{G}$ is infinitesimal. Given a finite group G, we denote by G_k the reduced finite group scheme satisfying $G_k(k) = G$.

In this Section we define for every block $\mathcal{B} \subset k\mathcal{G}$ a conical variety $\mathcal{V}_\mathcal{B}$, such that the closed, conical subvarieties of $\mathcal{V}_\mathcal{B}$ occur as support varieties of $\mathcal{B}$-modules. Given a finite dimensional $\mathcal{G}$-module M, we let $\mathrm{Ext}^*_\mathcal{G}(M, M) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_\mathcal{G}(M, M)$ be its Yoneda algebra, and define

$$\mathrm{Ext}^\bullet_\mathcal{G}(M, M) := \begin{cases} \bigoplus_{n \geq 0} \mathrm{Ext}^{2n}_\mathcal{G}(M, M) & \text{for } p \neq 2 \\ \mathrm{Ext}^*_\mathcal{G}(M, M) & \text{for } p = 2 \end{cases}.$$

The Yoneda composition induces a canonical homomorphism

$$\Phi_M : H^\bullet(\mathcal{G}, k) \longrightarrow \mathrm{Ext}^\bullet_\mathcal{G}(M, M) \quad ; \quad [f] \mapsto [f \otimes \mathrm{id}_M]$$

of k-algebras. Thanks to the Friedlander-Suslin Theorem [19, Theorem 1.1] the algebra $H^\bullet(\mathcal{G}, k)$ is finitely generated, and $\mathrm{Ext}^*_\mathcal{G}(M, M)$ is a finite $H^\bullet(\mathcal{G}, k)$-module. The *cohomological support variety* $\mathcal{V}_\mathcal{G}(M)$ is the variety associated to the ideal $\sqrt{\ker \Phi_M}$. Since Φ_M is a graded homomorphism, the variety $\mathcal{V}_\mathcal{G}(M) \subset \mathcal{V}_\mathcal{G}(k)$ is conical. We put $\mathcal{V} := \mathcal{V}_\mathcal{G}(k)$, so that the affine k-algebra $k[\mathcal{V}] := H^\bullet(\mathcal{G}, k)/\sqrt{(0)}$ is graded via $k[\mathcal{V}] = \bigoplus_{i \geq 0} k[\mathcal{V}]_{2i}$ for $p \geq 3$ and $k[\mathcal{V}] = \bigoplus_{i \geq 0} k[\mathcal{V}]_i$ for $p = 2$. Given $\zeta_1, \dots, \zeta_n \in H^\bullet(\mathcal{G}, k)$, we denote by $Z(\zeta_1, \dots, \zeta_n)$ the set of common zeros of the corresponding images $\bar{\zeta}_1, \dots, \bar{\zeta}_n \in k[\mathcal{V}]$.

Let $\mathcal{B} \subset k\mathcal{G}$ be a block and $\{S_1, \ldots, S_n\}$ a complete set of representatives of the simple $\mathcal{B}$-modules. Then

$$\mathcal{V}_{\mathcal{B}} := \bigcup_{i=1}^{n} \mathcal{V}_{\mathcal{G}}(S_i)$$

is the *support variety* of the block $\mathcal{B}$. By general properties of support varieties [3, §5.7], we have

$$\mathcal{V}_{\mathcal{G}}(M) \subset \mathcal{V}_{\mathcal{B}}$$

for every finite-dimensional $\mathcal{B}$-module M.

In the sequel, we shall prove our results only in case $p \geq 3$, leaving the simple modifications for the case $p = 2$ to the reader. References to [2, 3] are supposed to pertain to results on finite group schemes, whose proofs are verbatim copies of those in [2, 3].

Let $\Omega_{\mathcal{G}}$ be the Heller operator of the category of finite dimensional $\mathcal{G}$-modules (see [2, p.29]). According to [2, (2.5.4)] an element $\zeta \in H^{2n}(\mathcal{G}, k)$ corresponds to a linear map $\hat{\zeta} : \Omega^{2n}_{\mathcal{G}}(k) \longrightarrow k$, whose kernel we denote by L_ζ. We require the following generalization of [5, (2.3)]:

Lemma 2.1. *We have $\mathcal{V}_{\mathcal{G}}(L_\zeta) = Z(\zeta)$ for any $\zeta \in H^{2n}(\mathcal{G}, k)$.*

Proof. Let $\alpha : k(\mathbb{Z}/(p)) \longrightarrow k\mathcal{G}$ be a p-point of $\mathcal{G}$ (cf. [18, §3]), and consider the induced map $\alpha^* : H^{\mathrm{ev}}(\mathcal{G}, k) \longrightarrow H^{\mathrm{ev}}(\mathbb{Z}/(p), k)$. As argued in the proof of [18, (2.3)], we have $\zeta \in \ker \alpha^*$ if and only if the pull-back $L_\zeta|_{k(\mathbb{Z}/(p))}$ of L_ζ along α is not projective. Thanks to [18, (5.1)], the latter condition is equivalent to $\ker \alpha^* \supset \ker \Phi_{L_\zeta}$, so that $(\zeta) \subset \ker \alpha^* \Leftrightarrow \ker \Phi_M \subset \ker \alpha^*$. In view of [18, (4.11)], this implies $\mathrm{Proj}(Z(\zeta)) = \mathrm{Proj}(\mathcal{V}_{\mathcal{G}}(L_\zeta))$, whence $Z(\zeta) = \mathcal{V}_{\mathcal{G}}(L_\zeta)$. □

The following results refine [25, (7.5)].

Proposition 2.2. *Let $\mathcal{B} \subset k\mathcal{G}$ be a block, M a $\mathcal{B}$-module. Given $\zeta \in H^{2n}(\mathcal{G}, k)$, there exists a $\mathcal{B}$-module N of dimension $\dim_k N \leq (\dim_k M)(\dim_k \Omega^{2n}_{\mathcal{G}}(k))$ such that $\mathcal{V}_{\mathcal{G}}(N) = Z(\zeta) \cap \mathcal{V}_{\mathcal{G}}(M)$.*

Proof. Upon tensoring the exact sequence

$$(0) \longrightarrow L_\zeta \longrightarrow \Omega^{2n}_{\mathcal{G}}(k) \xrightarrow{\hat{\zeta}} k \longrightarrow (0)$$

with the $\mathcal{B}$-module M, we obtain an exact sequence

$$(*) \qquad (0) \longrightarrow L_\zeta \otimes_k M \longrightarrow \Omega^{2n}_{\mathcal{G}}(k) \otimes_k M \longrightarrow M \longrightarrow (0).$$

By general theory [2, (3.1.6)], there exists a projective $\mathcal{G}$-module P such that $\Omega^{2n}_{\mathcal{G}}(M) \oplus P \cong \Omega^{2n}_{\mathcal{G}}(k) \otimes_k M$. Let $e \in k\mathcal{G}$ be the central primitive idempotent associated to $\mathcal{B}$. Since $e \cdot M = M$ and $e \cdot \Omega^{2n}_{\mathcal{G}}(M) = \Omega^{2n}_{\mathcal{G}}(M)$, multiplication of $(*)$ by e and $1 - e$ yields short exact sequences

$$(0) \longrightarrow e \cdot (L_\zeta \otimes_k M) \longrightarrow \Omega^{2n}_{\mathcal{G}}(M) \oplus e \cdot P \longrightarrow M \longrightarrow (0),$$

and

$$(0) \longrightarrow (1-e) \cdot (L_\zeta \otimes_k M) \longrightarrow (1-e) \cdot P \longrightarrow (0) \longrightarrow (0).$$

As $(1-e)\cdot P$ is a projective $\mathcal{G}$-module, we conclude that $(1-e)\cdot(L_\zeta \otimes_k M)$ is projective. Consequently, $N := e \cdot (L_\zeta \otimes_k M)$ is a $\mathcal{B}$-module of dimension $\dim_k N \leq (\dim_k M)(\dim_k \Omega^{2n}_{\mathcal{G}}(k))$ such that $L_\zeta \otimes_k M = N \oplus P'$ for some projective $\mathcal{G}$-module P'. Thanks to Lemma 2.1, we have $\mathcal{V}_{\mathcal{G}}(L_\zeta) = Z(\zeta)$. We may now apply [18, (5.6(3))] and [18, (4.11)] to obtain

$$Z(\zeta) \cap \mathcal{V}_{\mathcal{G}}(M) = \mathcal{V}_{\mathcal{G}}(L_\zeta \otimes_k M) = \mathcal{V}_{\mathcal{G}}(N),$$

as desired. □

Corollary 2.3. *Let $\mathcal{B} \subset k\mathcal{G}$ be a block and $V \subset \mathcal{V}_\mathcal{B}$ a closed conical subvariety of $\mathcal{V}$. Then there exists a $\mathcal{B}$-module M such that*

$$V = \mathcal{V}_\mathcal{G}(M).$$

Proof. By assumption there exist homogeneous elements $\zeta_1, \dots, \zeta_\ell \in H^{\mathrm{ev}}(\mathcal{G}, k)$ such that $V = \bigcap_{j=1}^{\ell} Z(\zeta_j) \cap \mathcal{V}_\mathcal{B}$. General properties of support varieties [3, §5.7] yield $\mathcal{V}_\mathcal{B} = \mathcal{V}_\mathcal{G}(M_0)$, where $M_0 := \bigoplus_{i=1}^{n} S_i$ is a $\mathcal{B}$-module. Proposition 2.2 now provides a finite sequence $(M_j)_{1 \le j \le \ell}$ of $\mathcal{B}$-modules with $\mathcal{V}_\mathcal{G}(M_j) = Z(\zeta_j) \cap \mathcal{V}_\mathcal{G}(M_{j-1})$. In particular, we obtain $\mathcal{V}_\mathcal{G}(M_\ell) = \bigcap_{j=1}^{\ell} Z(\zeta_j) \cap \mathcal{V}_\mathcal{B} = V$. □

3. Tame Blocks of Finite Group Schemes

In this Section we give a proof of Rickard's Theorem [23, Theorem 2] for blocks of finite group schemes. Given a finite dimensional k-algebra Λ and a natural number $d > 0$, we denote by $\mathrm{mod}^d \Lambda$ the affine variety of d-dimensional Λ-modules. The group $\mathrm{GL}_d(k)$ acts on mod_Λ^d such that the orbits are the ismorphisms classes of d-dimensional Λ-modules. The notion of *representation type* is related to parametrizations of the constructible subset $\mathrm{ind}_\Lambda^d \subset \mathrm{mod}_\Lambda^d$ of indecomposable Λ-modules of dimension d: For every $d > 0$, let $C_d \subset \mathrm{mod}_\Lambda^d$ be a closed subset of minimal dimension subject to $\mathrm{ind}_\Lambda^d \subset \mathrm{GL}_d(k).C_d$. The algebra Λ is *tame*, if $\dim C_d \le 1$ for all $d > 0$. Otherwise Λ is referred to as *wild*. In view of Drozd's work [6, 8] the module category of a wild algebra is at least as complicated as that of any other algebra. Thus, there is no hope of classifying the indecomposables of such algebras.

In the representation theory of infinitesimal group schemes, support varieties and rank varieties play an impotant rôle in the determination of the components of the *stable Auslander-Reiten quiver* of $\mathrm{Dist}(\mathcal{G})$ (cf. [12]). Using p-points, the results of [12, §1-§3] generalize to finite group schemes. We shall only require one fact concerning a certain type of AR-components, the so-called *homogeneous tubes*, and refer the interested reader to [1, 10] for more details.

Given a finite-dimensional $\mathcal{G}$-module M, we denote its *complexity* by $\mathrm{cx}_\mathcal{G}(M)$ (cf. [3, p.157f]). By general theory, we have $\mathrm{cx}_\mathcal{G}(M) = \dim \mathcal{V}_\mathcal{G}(M)$ for every $\mathcal{G}$-module M (cf. for instance [3, (5.7.2)]). All modules belonging to an AR-component Θ have the same complexity. If Θ is a homogeneous tube, then the Auslander-Reiten translate coincides on Θ with the identity, and every module M belonging to Θ satisfies

$$\Omega_\mathcal{G}^2(M) \cong M \otimes_k k_\zeta.$$

Here, $\zeta : k\mathcal{G} \longrightarrow k$ is the *modular function* of the Hopf algebra $k\mathcal{G}$ (see [22, (I.8.8)]). Consequently, the non-projective module M has a minimal projective resolution, whose constituents have uniformely bounded dimensions. Accordingly, we have

$$\mathrm{cx}_\mathcal{G}(M) = 1 \quad \forall\ M \in \Theta.$$

We are now in a position to establish the following criterion for wildness:

Theorem 3.1. *Let $\mathcal{B} \subset k\mathcal{G}$ be a block. If $\dim \mathcal{V}_\mathcal{B} \ge 3$, then $\mathcal{B}$ is wild.*

Proof. As before, we let $\{S_1, \dots, S_n\}$ be a complete set of representatives for the isoclasses of simple $\mathcal{B}$-modules. Setting $M_0 := \bigoplus_{i=1}^{n} S_i$, we recall that $\mathcal{V}_\mathcal{B} = \mathcal{V}_\mathcal{G}(M_0)$. Since $\mathcal{V}_\mathcal{B}$ is a conical subvariety of $\mathcal{V}$, Lemma 1.1 provides homogeneous elements $(\zeta_\alpha)_{\alpha \in k}$ of $H^{\mathrm{ev}}(\mathcal{G}, k)$ of degree $2d > 0$ such that

(a) $\dim Z(\zeta_\alpha) \cap \mathcal{V}_\mathcal{B} = \dim \mathcal{V}_\mathcal{B} - 1 \ge 2$ for $\alpha \in k$, and
(b) $\dim Z(\zeta_\alpha) \cap Z(\zeta_\beta) \cap \mathcal{V}_\mathcal{B} = \dim \mathcal{V}_\mathcal{B} - 2$ for $\alpha \ne \beta \in k$.

Proposition 2.2 furnishes $\mathcal{B}$-modules N_α of dimension $\dim_k N_\alpha \le (\dim_k \Omega_\mathcal{G}^{2d}(k))(\dim_k M_0) =: q_0$ such that

$$\mathcal{V}_\mathcal{G}(N_\alpha) = Z(\zeta_\alpha) \cap \mathcal{V}_\mathcal{B} \quad \forall\ \alpha \in k.$$

If $N_\alpha = \bigoplus_{j=1}^{m_\alpha} N_{\alpha,j}$ is a decomposition into indecomposable $\mathcal{B}$-modules, then we have $\mathcal{V}_\mathcal{G}(N_\alpha) = \bigcup_{j=1}^{m_\alpha} \mathcal{V}_\mathcal{G}(N_{\alpha,j})$, and there thus exists an indecomposable summand $X_\alpha := N_{\alpha,j(\alpha)}$ with

$$\mathcal{V}_\mathcal{G}(X_\alpha) \subset Z(\zeta_\alpha) \cap \mathcal{V}_\mathcal{B} \quad \text{and} \quad \dim \mathcal{V}_\mathcal{G}(X_\alpha) = \dim \mathcal{V}_\mathcal{B} - 1.$$

If $\alpha \neq \beta$, then the assumption $\mathcal{V}_\mathcal{G}(X_\alpha) = \mathcal{V}_\mathcal{G}(X_\beta)$ implies

$$\mathcal{V}_\mathcal{G}(X_\alpha) \subset Z(\zeta_\alpha) \cap Z(\zeta_\beta) \cap \mathcal{V}_\mathcal{B},$$

which contradicts property (b). As a result, the varieties $\mathcal{V}_\mathcal{G}(X_\alpha)$ are pairwise distinct, so that the indecomposable $\mathcal{B}$-modules X_α are pairwise non-isomorphic. Since $\dim_k X_\alpha \leq q_0$ for every $\alpha \in k$, infinitely many of the X_α occur in one dimension. By virtue of $\mathrm{cx}_\mathcal{G}(M) = \dim \mathcal{V}_\mathcal{G}(X_\alpha) \geq 2$, our observations above guarantee that these modules do not belong to homogeneous tubes of the stable Auslander-Reiten quiver of $\mathcal{B}$. Consequently, [6, Theorem D] implies the wildness of the block $\mathcal{B}$. □

We record the following consequence.

Corollary 3.2. *Let $\mathcal{B} \subset k\mathcal{G}$ be a tame block. Then we have*

$$\mathrm{cx}_\mathcal{G}(M) \leq 2$$

for every $\mathcal{B}$-module M.

Proof. Suppose there exists a $\mathcal{B}$-module M of complexity ≥ 3. Then we have

$$3 \leq \mathrm{cx}_\mathcal{G}(M) = \dim \mathcal{V}_\mathcal{G}(M) \leq \dim \mathcal{V}_\mathcal{B},$$

so that (3.1) implies the wildness of $\mathcal{B}$. □

The corresponding result for representation-finite blocks of $k\mathcal{G}$ is considerably easier to establish and actually holds for arbitrary self-injective algebras. In view of [21], the non-projective indecomposable modules of a representation-finite self-injective algebra satisfy $\Omega^n(M) \cong M$, so that $\mathrm{cx}(M) \leq 1$.

4. Tame Blocks of Reduced Enveloping Algebras

Given a finite dimensional restricted Lie algebra $(\mathfrak{g}, [p])$ and a linear form $\chi \in \mathfrak{g}^*$, we let I_χ be the two-sided ideal of the universal enveloping algebra $U(\mathfrak{g})$ generated by $\{x^p - x^{[p]} - \chi(x)^p 1 \ ; \ x \in \mathfrak{g}\}$. The finite dimensional factor algebra

$$U_\chi(\mathfrak{g}) := U(\mathfrak{g})/I_\chi$$

is called the *χ-reduced enveloping algebra* of $\mathfrak{g}$. The algebraic family $(U_\chi(\mathfrak{g}))_{\chi \in \mathfrak{g}^*}$ of Frobenius algebras of dimension $p^{\dim_k \mathfrak{g}}$ plays a prominent rôle in the representation theory of $\mathfrak{g}$. The algebra $U_0(\mathfrak{g})$ is a Hopf algebra, which coincides with the algebra $k\mathcal{G} = \mathrm{Dist}(\mathcal{G})$ of the infinitesimal group $\mathcal{G} = \mathrm{Spec}(U_0(\mathfrak{g})^*)$ (cf. [7, (II,§7,n° 4)]).

In [17, §§6-7] the authors developed the theory of support varieties for $U_\chi(\mathfrak{g})$-modules and showed than the fundamental results for supports of $U_0(\mathfrak{g})$-modules also obtain in this context.

Given a $U_\chi(\mathfrak{g})$-module M, the canonical action of $U(\mathfrak{g})$ on $\mathrm{End}_k(M)$ factors through $U_0(\mathfrak{g})$, endowing $\mathrm{End}_k(M)$ with the structure of a $U_0(\mathfrak{g})$-module algebra. The algebra homomorphism

$$k \longrightarrow \mathrm{End}_k(M) \quad ; \quad \alpha \mapsto \alpha\, \mathrm{id}_M$$

thus induces an algebra homomorphism

$$\Phi_M : H^\bullet(U_0(\mathfrak{g}), k) \longrightarrow H^\bullet(U_0(\mathfrak{g}), \mathrm{End}_k(M))$$

of degree zero. One then defines the *cohomological support variety* $\mathcal{V}_{\mathfrak{g}}(M)$ to be the variety associated to $\sqrt{\ker \Phi_M}$. In the case where $\chi = 0$, this definition coincides with the earlier one. In analogy with §2, we define for a block $\mathcal{B} \subset U_\chi(\mathfrak{g})$ and a complete set $\{S_1, \ldots, S_n\}$ of representatives for the isoclasses of simple $\mathcal{B}$-modules the variety of $\mathcal{B}$ via

$$\mathcal{V}_{\mathcal{B}} := \bigcup_{i=1}^{n} \mathcal{V}_{\mathfrak{g}}(S_i).$$

The methods of sections 2 and 3 now yield:

Theorem 4.1. *Let $\mathcal{B} \subset U_\chi(\mathfrak{g})$ be a block. If $\dim \mathcal{V}_{\mathcal{B}} \geq 3$, then $\mathcal{B}$ is wild.*

Proof. Thanks to [17, (6.3)], the complexity $\mathrm{cx}_{U_\chi(\mathfrak{g})}(M)$ of a $U_\chi(\mathfrak{g})$-module M coincides with the dimension of $\mathcal{V}_{\mathfrak{g}}(M)$. Moreover, our comments concerning homogeneous tubes of the stable Auslander-Reiten quiver continue to hold in our present context (cf. [11, §5]). Thus, every $U_\chi(\mathfrak{g})$-module M belonging to a homomgeneous tube has complexity 1. Consequently, up to the validity of (2.2), the proof of (3.1) can be transferred verbatim to our context.

Turning to the analogue of (2.2), we let M be a $\mathcal{B}$-module and consider $\zeta \in H^{2n}(U_0(\mathfrak{g}), k)$. Since $\Omega^{2n}_{U_0(\mathfrak{g})}(k)$ and L_ζ are $U_0(\mathfrak{g})$-modules, the tensor products $L_\zeta \otimes_k M$ and $\Omega^{2n}_{U_0(\mathfrak{g})}(k) \otimes_k M$ are $U_\chi(\mathfrak{g})$-modules, and the proof of [2, (3.1.6)] provides a projective $U_\chi(\mathfrak{g})$-module P such that

$$\Omega^{2n}_{U_\chi(\mathfrak{g})}(M) \oplus P \cong \Omega^{2n}_{U_0(\mathfrak{g})}(k) \otimes_k M.$$

We may thus proceed as in the proof of (2.2), observing that the identity

$$Z(\zeta) \cap \mathcal{V}_{\mathfrak{g}}(M) = \mathcal{V}_{\mathfrak{g}}(L_\zeta \otimes_k M) = \mathcal{V}_{\mathfrak{g}}(N)$$

is a consequence of [17, (7.1)]. □

The arguments of (3.2) now imply:

Corollary 4.2. *Let $\mathcal{B} \subset U_\chi(\mathfrak{g})$ be a tame block. Then we have*

$$\mathrm{cx}_{U_\chi(\mathfrak{g})}(M) \leq 2$$

for every $\mathcal{B}$-module M. □

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