

EXTENSIONS OF TAME ALGEBRAS AND FINITE GROUP SCHEMES OF DOMESTIC REPRESENTATION TYPE

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ABSTRACT. Let k be an algebraically closed field. Given an extension $A : B$ of finite-dimensional k -algebras, we establish criteria ensuring that the representation-theoretic notion of polynomial growth is preserved under ascent and descent. These results are then used to show that principal blocks of finite group schemes of odd characteristic are of polynomial growth if and only if they are Morita equivalent to trivial extensions of radical square zero tame hereditary algebras. In that case, the blocks are of domestic representation type and the underlying group schemes are closely related to binary polyhedral group schemes.

INTRODUCTION

Let k be an algebraically closed field. Given a finite-dimensional k -algebra A , we denote by mod_A the category of finitely generated left A -modules. We will write $[M]$ for the isoclass of $M \in \text{mod}_A$. If mod_A affords only finitely many isoclasses of indecomposable objects, then A is called *representation-finite*. A fundamental result of Drozd [11] asserts that A is either tame or wild in the sense of the following definition:

Definition. An algebra A is referred to as *tame*, if for every $d > 0$, there exist $(A, k[T])$ -bimodules $X_1, \dots, X_{n(d)}$ which are free $k[T]$ -modules of rank d , such that all but finitely many isoclasses of indecomposable A -modules of dimension d are of the form $[X_i \otimes_{k[T]} k_\lambda]$ for $i \in \{1, \dots, n(d)\}$ and $\lambda : k[T] \rightarrow k$.

We say that A is *wild*, if there exists an $(A, k\langle x, y \rangle)$ -bimodule X , which is a finitely generated free right $k\langle x, y \rangle$ -module, such that the functor

$$\text{mod } k\langle x, y \rangle \longrightarrow \text{mod } A \quad ; \quad M \mapsto X \otimes_{k\langle x, y \rangle} M$$

preserves indecomposables and reflects isomorphisms.

The notion of wildness derives from the fact that the module category of a wild algebra A is at least as complicated as that of any other algebra, rendering the classification of indecomposable A -modules a rather hopeless endeavor.

Remark. In the definition of tameness, it suffices to require the parametrizing $(A, k[T])$ -bimodules $(X_i)_{1 \leq i \leq n(d)}$ of the d -dimensional indecomposable modules to be finitely generated over $k[T]$. In that case, each torsion submodule $t(X_i)$ of the $k[T]$ -module X_i is an $(A, k[T])$ -submodule, and there exist non-zero polynomials $f_i \in k[T]$ such that $t(X_i)f_i = (0)$. Let $f := f_1 f_2 \cdots f_{n(d)}$. Then the principal open subset $D(f) := \{\lambda \in \text{Hom}_{\text{alg}}(k[T], k) ; \lambda(f) \neq 0\}$ is cofinite.

Let M be a d -dimensional indecomposable A -module. Assume that, with only finitely many exceptions, we have $[M] = [X_i \otimes_{k[T]} k_\lambda]$ for some $i \in \{1, \dots, n(d)\}$ and $\lambda : k[T] \rightarrow k$. If $\lambda \in D(f)$, then

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$t(X_i) \otimes_{k[T]} k_\lambda = (0)$, and the right exactness of the tensor product implies that $[M] = [Y_i \otimes_{k[T]} k_\lambda]$, where $Y_i := X_i/t(X_i)$ is an $(A, k[T])$ -bimodule that is a free $k[T]$ -module of rank d .

This implies in particular that tameness is preserved under Morita equivalence. In [20] the authors show that this even holds for stable equivalence.

Let A be a tame algebra. For each $d > 0$, we let $\mu_A(d)$ be the minimum of all possible numbers $n(d)$ occurring in the definition. We thus obtain a function $\mu_A : \mathbb{N} \rightarrow \mathbb{N}_0$, which we refer to as the *growth function* of the tame algebra A . The bimodules $X_1, \dots, X_{n(d)}$ define morphisms

$$f_{i,d} : \mathbb{A}^1 \rightarrow \text{mod}_A^d ; \quad \lambda \mapsto X_i \otimes_{k[T]} k_\lambda,$$

with values in the variety mod_A^d of d -dimensional A -modules. Accordingly, these bimodules provide continuous one-parameter families of indecomposable A -modules of dimension d . This fact notwithstanding, there exist prominent examples of tame algebras, where a classification of the indecomposables has remained elusive. This has led to the definition of various subclasses of tame algebras, whose module categories promise to be better behaved. In this paper, we are ultimately concerned with the following class, introduced by Ringel [28]:

Definition. An algebra A is called *domestic*, if there exist $(A, k[T])$ -bimodules X_1, \dots, X_m that are free of finite rank over $k[T]$ such that for every $d > 0$, all but finitely many isoclasses of d -dimensional indecomposable A -modules are of the form $[X_i \otimes_{k[T]} (k[T]/(T-\lambda)^j)]$ for some $i \in \{1, \dots, m\}, j \in \mathbb{N}$ and $\lambda \in k$.

Remarks. (1) A group algebra kG of a finite group G is representation-infinite and domestic if and only if $\text{char}(k) = 2$, and the Sylow 2-subgroups of G are Klein four groups, cf. [5, (4.4.4)]. The restricted enveloping algebra $U_0(\mathfrak{sl}(2))$ of the restricted Lie algebra $\mathfrak{sl}(2)$ over a field of characteristic $p > 0$ is domestic.

(2) By the above remark, domesticity is preserved under Morita equivalence.

Finite groups and restricted Lie algebras are examples of finite group schemes. The purpose of this paper is determine those finite k -group schemes \mathcal{G} of characteristic $p \geq 3$, whose algebra $k\mathcal{G}$ of measures has domestic representation type. To that end, we study particular algebra extensions $A:B$, and show that the more general notion of polynomial growth is preserved under ascent and descent.

Given a function $f : \mathbb{N} \rightarrow \mathbb{N}_0$, we say that f has *polynomial growth*, if there exist $c > 0$ and $n \geq 0$ such that $f(d) \leq cd^{n-1}$ for all $d \geq 1$. The minimal number $n \in \mathbb{N}_0$ with this property is the *rate of growth* γ_f of the function f . The following class of algebras was introduced by Skowroński [29]:

Definition. A tame algebra A is said to be of *polynomial growth*, if μ_A has polynomial growth. In that case, we refer to $\gamma_A := \gamma_{\mu_A}$ as the *rate of growth* of the tame algebra A .

Remarks. Let A be tame.

- (a) In view of Brauer-Thrall II (cf. [3, (IV.5)]), the algebra A is representation-finite if and only if A is tame and $\gamma_A = 0$.
- (b) Thanks to [8, (5.7)], the algebra A is domestic if and only if A is tame and $\gamma_A \leq 1$.
- (c) Suppose that A is of polynomial growth. If B is a k -algebra that is Morita equivalent to A , then the above remark implies that B is also of polynomial growth.

- (d) If A is of polynomial growth and $I \trianglelefteq A$ is an ideal, then parametrizing $(A, k[T])$ -bimodules $X_1, \dots, X_{\mu_A(d)}$ induce parametrizing $(A/I, k[T])$ -bimodules $(X_j/I X_j)_{1 \leq j \leq \mu_A(d)}$, and the above remark implies that A/I is of polynomial growth of rate $\gamma_{A/I} \leq \gamma_A$.
- (e) As will be shown in Section 4, group schemes of polynomial growth are of domestic representation type.

Our main result provides a criterion guaranteeing that polynomial growth is preserved under passage between the constituents of certain extensions $A : B$. We say that an extension $A : B$ of k -algebras is *split*, if B is a direct summand of the (B, B) -bimodule A . We refer to $A : B$ as *separable*, if the multiplication of A induces a split surjective morphism $\mu : A \otimes_B A \rightarrow A$ of (A, A) -bimodules.

Theorem. *Suppose that $A : B$ is an extension of algebras.*

- (1) *If $A : B$ is split and A is of polynomial growth, then B is of polynomial growth, and $\gamma_B \leq \gamma_A + 1$.*
- (2) *If $A : B$ is separable and B is of polynomial growth, then A is of polynomial growth and $\gamma_A \leq \gamma_B + 1$.*

It is well-known that finite representation type is preserved when descending via a split extension or ascending via a separable extension, cf. [27].

Our approach involves a geometric interpretation of growth functions that elaborates on a result by de la Peña [25]. Section 1 provides the requisite tools which are then applied in Section 2, where the above result and some refinements are established. The concluding Section presents examples and applications concerning smash products and finite group schemes. In particular, we show that the principal block $\mathcal{B}_0(\mathcal{G})$ of a finite group scheme of characteristic $p \geq 3$ is representation-infinite and domestic if and only if $\mathcal{B}_0(\mathcal{G})$ is Morita equivalent to a trivial extension of a radical square zero tame hereditary algebra. Accordingly, one has a fairly good understanding of the module categories of these blocks (cf. [17, (V.3.2)]).

1. GEOMETRIC CHARACTERIZATION OF μ_A

Let k be an algebraically closed field. Throughout, A is assumed to be a finite-dimensional k -algebra. Given $n \in \mathbb{N}$, we denote by mod_A^n the affine variety of n -dimensional A -modules. An element of mod_A^n is a homomorphism $\varrho : A \rightarrow \text{End}_k(k^n)$ of k -algebras. The general linear group $\text{GL}_n(k)$ acts on mod_A^n via

$$(g \cdot \varrho)(a) := g \circ \varrho(a) \circ g^{-1} \quad \forall g \in \text{GL}_n(k), \varrho \in \text{mod}_A^n, a \in A.$$

Note that the orbits under this action are just the isomorphism classes of the underlying A -modules. Consequently, the subset $\text{ind}_A^n \subseteq \text{mod}_A^n$ of n -dimensional indecomposable A -modules is $\text{GL}_n(k)$ -stable.

A subset C of a topological space X is *locally closed*, if it is the intersection of an open subset and a closed subset of X . We say that C is *constructible*, if it is a finite union of locally closed sets. By virtue of [7, (AG.1.3)], every constructible subset C of a noetherian topological space X contains a dense, open subset of its closure. In particular, constructible subsets of varieties have this important property.

The following geometric criterion (cf. [25, (1.2)]), is based on Drozd's Tame-Wild dichotomy [11]:

Proposition 1.1. *The following statements are equivalent:*

- (1) *The algebra A is tame.*
- (2) *For every natural number $d \in \mathbb{N}$ there exists a closed subset $C_d \subseteq \text{mod}_A^d$ such that $\dim C_d \leq 1$ and $\text{ind}_A^d \subseteq \text{GL}_d(k) \cdot C_d$. \square*

Suppose that A is tame with parametrizing bimodules $X_1, \dots, X_{\mu_A(d)}$ for some $d > 0$. Let $f_i : \mathbb{A}^1 \rightarrow \text{mod}_A^d$ be the morphism determined by X_i , and put $V_i := \overline{\text{im } f_i}$. By definition, there exists a finite subset $F_d \subseteq \text{mod}_A^d$ such that

$$\text{ind}_A^d \subseteq \bigcup_{i=1}^{\mu_A(d)} \text{GL}_d(k) \cdot V_i \cup \text{GL}_d(k) \cdot F_d.$$

We record the following elementary observation:

Lemma 1.2. *Let V be a one-dimensional, irreducible variety. If $C \subseteq V$ is constructible, then C is either finite or cofinite.*

Proof. Since C is constructible, it contains a dense open subset O of its closure \bar{C} .

If $\dim \bar{C} = 0$, then C is finite. Otherwise, $\dim \bar{C} = 1$, so that O is dense and open in $\bar{C} = V$. Thus, $V \setminus O$ is a proper, closed subset of V , whence $\dim V \setminus O = 0$. As a result, O and C are cofinite subsets of V . \square

Lemma 1.3. *Let $d > 0$. Suppose there exist one-dimensional closed, irreducible subsets $V_1, \dots, V_{\ell(d)}$ of mod_A^d , and a finite subset $F_d \subseteq \text{mod}_A^d$ such that*

$$\text{ind}_A^d \subseteq \bigcup_{i=1}^{\ell(d)} \text{GL}_d(k) \cdot V_i \cup \text{GL}_d(k) \cdot F_d,$$

with $\ell(d)$ being minimal subject to the above property. Then the following statements hold:

- (1) $|V_i \cap \text{GL}_d(k) \cdot x| < \infty \quad \forall x \in \text{mod}_A^d, 1 \leq i \leq \ell(d)$.
- (2) $V_i \cap \text{ind}_A^d$ is cofinite in V_i for every $i \in \{1, \dots, \ell(d)\}$.
- (3) $U_i := V_i \cap [\text{ind}_A^d \setminus (\bigcup_{\ell \neq i} \text{GL}_d(k) \cdot V_\ell)]$ is a dense, open subset of V_i for $1 \leq i \leq \ell(d)$.

Proof. (1) Let x be an element of mod_A^d . By Chevalley's theorem [16, (10.20)], $V_i \cap \text{GL}_d(k) \cdot x$ is a constructible subset of V_i , and therefore, by Lemma 1.2, finite or cofinite in V_i . If there exists an element $x_0 \in \text{mod}_A^d$ with $V_i \cap \text{GL}_d(k) \cdot x_0$ cofinite in V_i , then there is a finite set $H_d \subseteq \text{mod}_A^d$ such that $V_i \subseteq \text{GL}_d(k) \cdot x_0 \cup H_d$, so that $\text{GL}_d(k) \cdot V_i \subseteq \text{GL}_d(k) \cdot (H_d \cup \{x_0\})$. This contradicts the choice of $\ell(d)$.

(2) Note that ind_A^d is constructible (cf. [25, (1.1)]). If the constructible subset $V_i \cap \text{ind}_A^d \subseteq V_i$ is finite, then the $\text{GL}_d(k)$ -invariance of ind_A^d yields $\text{ind}_A^d \subseteq \bigcup_{\ell \neq i} \text{GL}_d(k) \cdot V_\ell \cup \text{GL}_d(k) \cdot (F_d \cup (V_i \cap \text{ind}_A^d))$, which contradicts the minimality of $\ell(d)$. Consequently, $V_i \cap \text{ind}_A^d \subseteq V_i$ is a cofinite subset of V_i .

(3) Let $i \in \{1, \dots, \ell(d)\}$. By Chevalley's Theorem [16, (10.20)], the image of $\text{GL}_d(k) \cdot V_\ell$ of V_ℓ under the multiplication $\text{GL}_d(k) \times \text{mod}_A^d \rightarrow \text{mod}_A^d$ is constructible. As ind_A^d is constructible, it follows that U_i is constructible. In view of Lemma 1.2, the set U_i is finite or cofinite in V_i .

By definition, we have

$$\text{ind}_A^d \subseteq \text{GL}_d(k) \cdot U_i \cup \bigcup_{\ell \neq i} \text{GL}_d(k) \cdot V_\ell \cup \text{GL}_d(k) \cdot F_d.$$

By choice of $\ell(d)$, the set U_i is not finite. Hence U_i is cofinite in V_i and thus a non-empty, open subset of the one-dimensional irreducible variety V_i . This implies that U_i lies dense in V_i . \square

We require the following refinement of Proposition 1.1:

Proposition 1.4. *The following statements are equivalent:*

- (1) *There exists a function $\ell : \mathbb{N} \rightarrow \mathbb{N}_0$ such that:*
- (a) *For every $d > 0$ there exist one-dimensional irreducible closed subsets $V_1, \dots, V_{\ell(d)} \subseteq \text{mod}_A^d$, and a finite subset $F_d \subseteq \text{ind}_A^d$ such that $\text{ind}_A^d \subseteq \bigcup_{i=1}^{\ell(d)} \text{GL}_d(k) \cdot V_i \cup \text{GL}_d(k) \cdot F_d$, and*
 - (b) *$\ell(d)$ is minimal subject to the properties listed in (a).*
- (2) *The algebra A is tame with growth function $\mu_A = \ell$.*

Proof. (1) \Rightarrow (2) Thanks to Proposition 1.1, we know that A is tame. Let $X_1, \dots, X_{\mu_A(d)}$ be $(A, k[T])$ -bimodules giving rise to morphisms

$$f_i : \mathbb{A}^1 \rightarrow \text{mod}_A^d$$

such that $\text{ind}_A^d \subseteq \text{GL}_d(k) \cdot (\bigcup_{i=1}^{\mu_A(d)} \text{im } f_i) \cup \text{GL}_d(k) \cdot G_d$ for some finite set $G_d \subseteq \text{ind}_A^d$. Condition (b) readily yields $\ell(d) \leq \mu_A(d)$ for all $d > 0$.

Let $d > 0$. By assumption, there exists a finite subset $F_d \subseteq \text{ind}_A^d$ such that

$$\text{ind}_A^d \subseteq \bigcup_{i=1}^{\ell(d)} \text{GL}_d(k) \cdot V_i \cup \text{GL}_d(k) \cdot F_d,$$

with $\ell(d)$ being minimal subject to this property. For each $j \in \{1, \dots, \ell(d)\}$ we have

$$V_j \cap \text{ind}_A^d \subseteq \bigcup_{i=1}^{\mu_A(d)} (V_j \cap \text{GL}_d(k) \cdot \text{im } f_i) \cup (V_j \cap \text{GL}_d(k) \cdot G_d),$$

with each constituent of the union being constructible. By Lemma 1.3(1), the set $V_j \cap \text{GL}_d(k) \cdot G_d$ is finite, while Lemma 1.3(2) shows that $V_j = \overline{V_j \cap \text{ind}_A^d}$. Hence there is an element $\zeta(j) \in \{1, \dots, \mu_A(d)\}$ such that

$$V_j = \overline{V_j \cap \text{GL}_d(k) \cdot \text{im } f_{\zeta(j)}}.$$

In particular, the constructible set $V_j \cap \text{GL}_d(k) \cdot \text{im } f_{\zeta(j)}$ is not finite. Thus, Lemma 1.2 provides a finite set R_j with $V_j = (V_j \cap \text{GL}_d(k) \cdot \text{im } f_{\zeta(j)}) \cup R_j \subseteq \text{GL}_d(k) \cdot \text{im } f_{\zeta(j)} \cup R_j$, whence

$$\text{GL}_d(k) \cdot V_j \subseteq \text{GL}_d(k) \cdot \text{im } f_{\zeta(j)} \cup \text{GL}_d(k) \cdot R_j.$$

As a result, there is a function $\zeta : \{1, \dots, \ell(d)\} \rightarrow \{1, \dots, \mu_A(d)\}$ such that

$$\text{ind}_A^d \subseteq \bigcup_{j=1}^{\ell(d)} \text{GL}_d(k) \cdot \text{im } f_{\zeta(j)} \cup \text{GL}_d(k) \cdot (\bigcup_{j=1}^{\ell(d)} R_j \cup F_d).$$

Hence all but finitely many isoclasses of d -dimensional indecomposable A -modules are of the form $[X_{\zeta(j)} \otimes_{k[T]} k_\lambda]$ for $j \in \{1, \dots, \ell(d)\}$ and $\lambda : k[T] \rightarrow k$. Consequently, $\mu_A(d) \leq |\text{im } \zeta| \leq \ell(d)$, so that $\ell = \mu_A$ is the growth function of the tame algebra A .

(2) \Rightarrow (1) Let $d > 0$ and consider the $(A, k[T])$ -bimodules $X_1, \dots, X_{\mu_A(d)}$ that are provided by the definition of tameness. If $f_i : \mathbb{A}^1 \rightarrow \text{mod}_A^d$ are the associated morphisms, then the family $V_i := \overline{\text{im } f_i}$ ($1 \leq i \leq \mu_A(d)$) together with some finite set $F_d \subseteq \text{mod}_A^d$ satisfies condition (a).

Suppose we also have

$$\text{ind}_A^d \subseteq \bigcup_{j=1}^{\ell(d)} \text{GL}_d(k) \cdot W_j \cup \text{GL}_d(k) \cdot F'_d,$$

with F'_d finite and W_j irreducible, closed and of dimension 1. The above arguments provide a function $\xi : \{1, \dots, \mu_A(d)\} \rightarrow \{1, \dots, \ell(d)\}$ such that

$$V_i = \overline{V_i \cap \text{GL}_d(k) \cdot W_{\xi(i)}}.$$

Consequently, ξ is injective, so that $\mu_A(d) \leq \ell(d)$. □

For future reference, we record the following basic result:

Lemma 1.5. *The map*

$$s_{m,n} : \text{mod}_A^m \times \text{mod}_A^n \longrightarrow \text{mod}_A^{m+n} \quad ; \quad (M, N) \mapsto M \oplus N$$

is a morphism of affine varieties. □

2. SPLIT EXTENSIONS

In the sequel, we are going to study the behavior of polynomial growth under split extensions. Throughout this section, $A:B$ denotes an extension of finite dimensional k -algebras.

We begin with the following technical subsidiary result:

Lemma 2.1. *Suppose that $A:B$ is a split extension, with A being of polynomial growth. Then there exists a function $\ell : \mathbb{N} \longrightarrow \mathbb{N}_0$ with the following properties:*

- (1) *For every $d > 0$, there exist a finite subset $F_d \subseteq \text{ind}_B^d$ and $(A, k[T])$ -bimodules $X_1, \dots, X_{\ell(d)}$ that are finitely generated free right $k[T]$ -modules such that for every $V \in \text{ind}_B^d \setminus \text{GL}_d(k) \cdot F_d$ there exist $j \in \{1, \dots, \ell(d)\}$ and $\lambda : k[T] \longrightarrow k$ with

 - (a) V being isomorphic to a direct summand of $(X_j \otimes_{k[T]} k_\lambda)|_B$, and
 - (b) $X_j \otimes_{k[T]} k_\lambda$ being isomorphic to a direct summand of $A \otimes_B V$.*
- (2) *The function ℓ has polynomial growth of rate $\gamma_\ell \leq \gamma_A + 1$.*
- (3) *Further, if such ℓ is chosen so that $\ell(d)$ is minimal subject to properties (a) and (b), then*

$$|\{X_j \otimes_{k[T]} k_\lambda ; \lambda \in k\} \cap \text{GL}_{r_j}(k) \cdot x| < \infty$$

for all $j \in \{1, \dots, \ell(d)\}$, $x \in \text{mod}_A^{r_j}$, $r_j := \text{rk}_{k[T]}(X_j)$.

Proof. (1) If M is an $(A, k[T])$ -bimodule, we shall write $M(\lambda) := M \otimes_{k[T]} k_\lambda$ for ease of notation. Since A is tame, we can find, for every $d > 0$, a finite set $G_d \subseteq \text{ind}_A^d$ and $(A, k[T])$ -bimodules $M_1^d, \dots, M_{\mu_A(d)}^d$ of $k[T]$ -rank d such that every $M \in \text{ind}_A^d \setminus \text{GL}_d(k) \cdot G_d$ is isomorphic to $M_i^d(\lambda)$ for some $i \in \{1, \dots, \mu_A(d)\}$ and some $\lambda : k[T] \longrightarrow k$.

Consider the finite set $G_{(d)} := \bigcup_{i=d}^{(\dim_k A)d} G_i$ as well as the finite-dimensional A -module $Q := \bigoplus_{M \in G_{(d)}} M$. By the Theorem of Krull-Remak-Schmidt, the set

$$U_d := \{V \in \text{ind}_B^d ; V \text{ is isomorphic to a direct summand of } Q|_B\}$$

is the union of finitely many $\text{GL}_d(k)$ -orbits. Hence there exists a finite set $F_d \subseteq \text{ind}_B^d$ with

$$U_d = \text{GL}_d(k) \cdot F_d.$$

Given $V \in \text{ind}_B^d \setminus \text{GL}_d(k) \cdot F_d$, the splitting property of $A:B$ provides a decomposition

$$(A \otimes_B V)|_B \cong V \oplus W$$

of B -modules. There exist indecomposable A -modules N_1, \dots, N_q such that $A \otimes_B V \cong N_1 \oplus \dots \oplus N_q$, and the Theorem of Krull-Remak-Schmidt provides $j \in \{1, \dots, q\}$ such that V is isomorphic to a direct summand of $N_j|_B$. Since $\dim_k A \otimes_B V \leq (\dim_k A)d$, we have $N_j \in \text{ind}_A^t$ for some $t \in \{d, \dots, (\dim_k A)d\}$. The assumption $N_j \in \text{GL}_t(k) \cdot G_t$ entails $V \in U_d$, a contradiction. Consequently, N_j is isomorphic to some $M_i^t(\lambda)$. As a result, the modules $M_1^d, \dots, M_{\mu_A(d)}^d, M_1^{d+1}, \dots, M_{\mu_A(d(\dim_k A))}^{d(\dim_k A)}$ have the requisite properties.

(2) Given $d > 0$, we write $n := \dim_k A$ and put $\{M_r^s ; 1 \leq s \leq nd, 1 \leq r \leq \mu_A(s)\} =: \{X_1, \dots, X_{\ell(d)}\}$. By assumption, there exists a constant $c' > 0$ such that $\mu_A(d) \leq c'd^{\gamma_A-1}$ for all $d \geq 1$. Part (1) implies

$$\ell(d) \leq \sum_{i=1}^{nd} \mu_A(i) \leq c' \sum_{i=1}^{nd} i^{\gamma_A-1} \leq c'nd(nd)^{\gamma_A-1} = c'n^{\gamma_A}d^{\gamma_A}.$$

Consequently, ℓ has polynomial growth of rate $\gamma_\ell \leq \gamma_A + 1$.

(3) If the assertion is false, then there exist $j \in \{1, \dots, \ell(d)\}$ and $x_0 \in \text{mod}_A^{r_j}$ such that the constructible set $\mathcal{V}_j := \{X_j \otimes_{k[T]} k_\lambda ; \lambda \in k\} \cap \text{GL}_{r_j}(k).x_0$ is cofinite in $\{X_j \otimes_{k[T]} k_\lambda ; \lambda \in k\}$. We write $\{X_j \otimes_{k[T]} k_\lambda ; \lambda \in k\} = \mathcal{V}_j \cup G_{r_j}$ for some finite set $G_{r_j} \subseteq \text{mod}_A^{r_j}$, so that

$$\{X_j \otimes_{k[T]} k_\lambda ; \lambda \in k\} \subseteq \text{GL}_{r_j}(k).(G_{r_j} \cup \{x_0\}).$$

For every $M \in G_{r_j} \cup \{x_0\}$, we fix a decomposition of $M|_B$ into indecomposables, and consider the finite subset $\mathcal{F}_d \subseteq \text{ind}_B^d$ of d -dimensional indecomposable constituents of $\bigoplus_{M \in G_{r_j} \cup \{x_0\}} M|_B$.

Let V be an element of $\text{ind}_B^d \setminus \text{GL}_d(k).(F_d \cup \mathcal{F}_d)$. Since $V \notin \text{GL}_d(k).F_d$, there exist $q \in \{1, \dots, \ell(d)\}$ and $\lambda : k[T] \rightarrow k$ such that

- (a) V is a direct summand of some $(X_q \otimes_{k[T]} k_\lambda)|_B$, and
- (b) $X_q \otimes_{k[T]} k_\lambda$ is a direct summand of $A \otimes_B V$.

In view of (a), the assumption $q = j$ implies $V \in \text{GL}_d(k).\mathcal{F}_d$, a contradiction. Thus, $q \neq j$, which contradicts the minimality of $\ell(d)$. \square

Under additional assumptions, the growth rate of the function $\ell : \mathbb{N} \rightarrow \mathbb{N}_0$ of Lemma 2.1 is bounded by γ_A . For instance, if $\gamma_A = 0$, then A is representation-finite and $\gamma_\ell = 0$. The following result provides conditions that apply in the context of skew group algebras.

Let M be an A -module. Given an automorphism $\zeta \in \text{Aut}(A)$, we denote by $M^{(\zeta)}$ the A -module with underlying k -space M and action

$$a.m := \zeta^{-1}(a).m \quad \forall a \in A, m \in M.$$

Below, we shall apply this twisting operation to bimodules, which we shall consider as modules over the enveloping algebra $A \otimes_k A^{\text{op}}$.

Corollary 2.2. *Let $A : B$ be an extension of k -algebras such that $A \cong \bigoplus_{i=1}^r A_i$ is an isomorphism of (B, B) -modules with $A_i \cong B^{(\text{id}_B \otimes \zeta_i)}$ for some $\zeta_i \in \text{Aut}(B)$. Suppose that A is of polynomial growth. Then there exists a function $\ell : \mathbb{N} \rightarrow \mathbb{N}_0$ with the following properties:*

- (1) *For every $d > 0$, there exist a finite subset $F_d \subseteq \text{ind}_B^d$ and $(A, k[T])$ -bimodules $X_1, \dots, X_{\ell(d)}$ that are finitely generated free right $k[T]$ -modules such that for every $V \in \text{ind}_B^d \setminus \text{GL}_d(k).F_d$ there exist $i \in \{1, \dots, \ell(d)\}$ and $\lambda : k[T] \rightarrow k$ such that*
 - (a) *V is isomorphic to a direct summand of $(X_i \otimes_{k[T]} k_\lambda)|_B$, and*
 - (b) *$X_i \otimes_{k[T]} k_\lambda$ is a direct summand of $A \otimes_B V$.*
- (2) *The function ℓ has polynomial growth of rate $\gamma_\ell \leq \gamma_A$.*

Proof. Let $V \in \text{ind}_B^d$. Our current assumption implies that

$$(A \otimes_B V)|_B \cong \bigoplus_{i=1}^r (A_i \otimes_B V)|_B \cong \bigoplus_{i=1}^r V^{(\zeta_i)}.$$

Returning to the proof of Lemma 2.1, we write $A \otimes_B V = \bigoplus_{i=1}^q N_i$ as a sum of indecomposable modules and obtain that $t_i := \dim_k N_i = a_{N_i}d$ for $1 \leq i \leq q$, where $1 \leq a_{N_i} \leq r$. The arguments of (2.1) now

imply that the modules $M_1^d, \dots, M_{\mu_A(d)}^d, M_1^{2d}, \dots, M_{\mu_A(2d)}^{2d}, \dots, M_{\mu_A(rd)}^{rd}$ have the requisite properties. Consequently,

$$\ell(d) \leq \sum_{i=1}^r \mu_A(id) \leq rc'(rd)^{\gamma_A-1} = r^{\gamma_A} c' d^{\gamma_A-1},$$

as desired. \square

Let V be a variety. Given $x \in V$, we denote by $\dim_x V$ the *local dimension* of V at x . By definition, $\dim_x V$ is the maximum dimension of all irreducible components of V containing x .

Theorem 2.3. *Suppose that $A : B$ is a split extension. If A is of polynomial growth, then B is of polynomial growth with $\gamma_B \leq \gamma_A + 1$.*

Proof. Since the extension is split, every indecomposable B -module is a direct summand of an indecomposable A -module. According to [26, (4.2)], the algebra B is therefore tame. Thus, for each $d > 0$, there are parametrizing families

$$f_i : \mathbb{A}^1 \longrightarrow \text{mod}_B^d \quad ; \quad 1 \leq i \leq \mu_B(d).$$

We set $V_i := \overline{\text{im } f_i}$, and apply Proposition 1.4 and Lemma 1.3(1) to obtain

$$(\dagger) \quad |\text{im } f_i \cap \text{GL}_d(k) \cdot x| < \infty \quad \forall x \in \text{mod}_B^d.$$

Now let $X_1, \dots, X_{\ell(d)}$, F_d be the bimodules and the finite subset of ind_B^d provided by Lemma 2.1, respectively. For $j \in \{1, \dots, \ell(d)\}$ let

$$g_j : \mathbb{A}^1 \longrightarrow \text{mod}_A^{r_j} \quad ; \quad r_j := \text{rk}_{k[T]}(X_j)$$

be the corresponding parametrizing morphisms. We put $W_j := \overline{\text{im } g_j}$ and choose $\ell(d)$ to be minimal subject to properties (a) and (b) of Lemma 2.1.

Chevalley's Theorem provides dense open subsets $O_i \subseteq V_i$ and $U_j \subseteq W_j$ with $O_i \subseteq \text{im } f_i$ and $U_j \subseteq \text{im } g_j$.

Given $(i, j) \in \{1, \dots, \mu_B(d)\} \times \{1, \dots, \ell(d)\}$, we let $\mathcal{S}(i, j) \subseteq \text{im } f_i \times \text{im } g_j$ be the set of all those pairs $(M, N) \in \text{mod}_B^d \times \text{mod}_A^{r_j}$ such that M is isomorphic to a direct summand of $N|_B$ and N is isomorphic to a direct summand of $A \otimes_B M$. In view of Lemma 1.5 and the arguments of [25, p.183], this set is constructible. We denote by $\pi_{(i,j)} : \overline{\mathcal{S}(i, j)} \longrightarrow V_i$ the restriction of the projection $V_i \times W_j \longrightarrow V_i$.

(a) We have $\dim \overline{\mathcal{S}(i, j)} \leq 1$.

Since $\dim V_i = 1 = \dim W_j$, we have $\dim \overline{\mathcal{S}(i, j)} \leq 2$. Suppose equality to hold. Then $\overline{\mathcal{S}(i, j)} = V_i \times W_j$ is an irreducible variety. Thus, $O_i \times U_j \subseteq \text{im } f_i \times \text{im } g_j$ is an open, and thus dense, subset of $\overline{\mathcal{S}(i, j)}$. As $\mathcal{S}(i, j)$ is constructible, it also contains a dense, open subset of the irreducible variety $\overline{\mathcal{S}(i, j)}$. Its non-empty intersection with $O_i \times U_j$ will be denoted O .

Let $(M, N) \in O$ and consider an element $(V, W) \in O \cap \pi_{(i,j)}^{-1}(\pi_{(i,j)}(M, N))$. Then $M = V$, M is a direct summand of $W|_B$, and W is direct summand of the A -module $A \otimes_B M$. The Theorem of Krull-Remak-Schmidt provides a finite subset $\mathcal{F} \subseteq \text{mod}_A^{r_j}$ such that

$$O \cap \pi_{(i,j)}^{-1}(\pi_{(i,j)}(M, N)) \subseteq \{M\} \times (U_j \cap \text{GL}_{r_j}(k) \cdot \mathcal{F}).$$

Thanks to Lemma 2.1(3), the set $U_j \cap \text{GL}_{r_j}(k) \cdot \mathcal{F}$ is finite, so that $O \cap \pi_{(i,j)}^{-1}(\pi_{(i,j)}(M, N))$ also has this property.

Let $Z \subseteq \pi_{(i,j)}^{-1}(\pi_{(i,j)}(M, N))$ be an irreducible component containing (M, N) . Then $(M, N) \in O \cap Z$, so that $O \cap Z$ is a dense, open subset of Z . Since $O \cap Z$ is finite, we obtain $\dim Z = \dim Z \cap O = 0$, implying

$$\dim_{(M,N)} \pi_{(i,j)}^{-1}(\pi_{(i,j)}(M, N)) = 0 \quad \forall (M, N) \in O.$$

Hence the generic fiber of the morphism $\pi_{(i,j)}$ is zero-dimensional, so that $\dim V_i \geq 2$, a contradiction cf. [23, (I.§8)]. \diamond

(b) We have $V_i = \bigcup_{j=1}^{\ell(d)} \overline{\pi_{(i,j)}(\mathcal{S}(i, j))}$.

Thanks to (†), the set $O_i \cap \mathrm{GL}_d(k) \cdot F_d$ is finite. Thus, $O_i \setminus (O_i \cap \mathrm{GL}_d(k) \cdot F_d)$ is a dense open subset of the one-dimensional irreducible variety V_i , and Lemma 1.3(2) implies that $O'_i := (O_i \setminus (O_i \cap \mathrm{GL}_d(k) \cdot F_d)) \cap (V_i \cap \mathrm{ind}_A^d)$ enjoys the same property. Owing to Lemma 2.1(1), we have

$$O'_i \subseteq \bigcup_{j=1}^{\ell(d)} \pi_{(i,j)}(\mathcal{S}(i, j)),$$

whence

$$V_i = \overline{O'_i} \subseteq \bigcup_{j=1}^{\ell(d)} \overline{\pi_{(i,j)}(\mathcal{S}(i, j))} \subseteq V_i,$$

as desired. \diamond

Let $f : X \rightarrow Y$ be a continuous map between topological spaces, $Q \subseteq X$ be a subset. Since $f(\overline{Q}) \subseteq \overline{f(Q)}$, we have

$$\overline{f(Q)} = f(\overline{Q}).$$

Accordingly, a morphism $\gamma : \overline{\mathcal{S}(i, j)} \rightarrow V_i$ is dominant if and only if $\overline{\gamma(\mathcal{S}(i, j))} = V_i$.

(c) Let $\sigma_{(i,j)} : \overline{\mathcal{S}(i, j)} \rightarrow W_j$ be induced by the projection $V_i \times W_j \rightarrow W_j$. If $Z \subseteq \overline{\mathcal{S}(i, j)}$ is an irreducible component such that $\pi_{(i,j)} : Z \rightarrow V_i$ is a dominant morphism, so is $\sigma_{(i,j)} : Z \rightarrow W_j$.

Suppose the contrary, so that $\sigma_{(i,j)} : Z \rightarrow W_j$ has a finite image. Thanks to (a) and the choice of Z , we have $\dim Z = 1$. Setting $r := \min\{\dim_z \sigma_{(i,j)}^{-1}(\sigma_{(i,j)}(z)) ; z \in Z\}$, we conclude from upper semicontinuity of fiber dimension [23, (I.§8)] that

$$O := \{z \in Z ; \dim_z \sigma_{(i,j)}^{-1}(\sigma_{(i,j)}(z)) = r\}$$

is a dense, open subset of the one-dimensional variety Z . On the other hand, the generic fiber dimension theorem implies $\dim_z \sigma_{(i,j)}^{-1}(\sigma_{(i,j)}(z)) = \dim Z - \dim \overline{\sigma_{(i,j)}(Z)} = 1$ on a non-empty, open subset of Z , so that $r = 1$.

Since the morphism $\pi_{(i,j)} : Z \rightarrow V_i$ is dominant, its image intersects the open subset $O_i \subseteq \mathrm{im} f_i \subseteq V_i$. Thus, $\pi_{(i,j)}^{-1}(O_i)$ is a dense, open subset of Z . The constructible set $\mathcal{S}(i, j)$ contains a dense, open subset U of $\overline{\mathcal{S}(i, j)}$ (cf. [7, (AG.1.3)]). The assumption $U \cap Z = \emptyset$ implies that U is contained in the union of the irreducible components of $\overline{\mathcal{S}(i, j)}$ different from Z . Thus, $\overline{U} \subsetneq \overline{\mathcal{S}(i, j)}$, a contradiction. As a result,

$$\tilde{O} := \pi_{(i,j)}^{-1}(O_i) \cap U \cap O$$

is a dense, open subset of Z .

Let $x := (M, N)$ be an element of \tilde{O} . If $(P, Q) \in \tilde{O} \cap \sigma_{(i,j)}^{-1}(\sigma_{(i,j)}(M, N))$, then we have $Q = N$ and P is a direct summand of $N|_B$. Consequently, the Theorem of Krull-Remak-Schmidt provides a finite subset $\mathcal{F} \subseteq \mathrm{mod}_B^d$ such that

$$\tilde{O} \cap \sigma_{(i,j)}^{-1}(\sigma_{(i,j)}(M, N)) \subseteq (O_i \cap \mathrm{GL}_d(k) \cdot \mathcal{F}) \times \{N\}.$$

Thanks to property (†) the right-hand set is finite. Since $x \in \tilde{O}$, the set $\tilde{O} \cap \sigma_{(i,j)}^{-1}(\sigma_{(i,j)}(x))$ intersects every irreducible component X of $\sigma_{(i,j)}^{-1}(\sigma_{(i,j)}(x))$ containing x non-trivially. Let \mathcal{A} be the set of these components. Consequently,

$$\dim_x \sigma_{(i,j)}^{-1}(\sigma_{(i,j)}(x)) = \max_{X \in \mathcal{A}} \dim X = \max_{X \in \mathcal{A}} \dim X \cap \tilde{O} = 0,$$

which contradicts the fact that $x \in O$. \diamond

(d) Let $\varphi : V \rightarrow W$ be a dominant morphism of one-dimensional irreducible varieties. Then φ is open.

Let $\emptyset \neq O \subseteq V$ be an open set. Then $\varphi(O)$ is a constructible subset of a one-dimensional irreducible variety and thus by Lemma 1.2 finite or cofinite. In the former case, we have

$$W = \overline{\varphi(V)} = \overline{\varphi(O)} \subseteq \overline{\varphi(O)} = \varphi(O),$$

a contradiction. Thus, $\varphi(O)$ is cofinite and in particular open. \diamond

(e) Given $j \in \{1, \dots, \ell(d)\}$, the set

$$\Gamma_j := \{i \in \{1, \dots, \mu_B(d)\} ; V_i = \overline{\pi_{(i,j)}(\mathcal{S}(i, j))}\}$$

has cardinality $|\Gamma_j| \leq \dim_k A$.

Let $r \in \Gamma_j$. By assumption, there exists an irreducible component $Z_r \subseteq \overline{\mathcal{S}(r, j)}$ such that the morphism $\pi_{(r,j)} : Z_r \rightarrow V_r$ is dominant. Thanks to (c), we also have a dominant morphism $\sigma_{(r,j)} : Z_r \rightarrow W_j$.

In view of Lemma 1.3(2), $\text{ind}_B^d \cap V_r$ is a dense, open subset of V_r , so that $\emptyset \neq \pi_{(r,j)}^{-1}(\text{ind}_B^d \cap V_r) \subseteq Z_r$ enjoys the same property. By constructibility of $\mathcal{S}(r, j)$, there exists a dense open subset $\tilde{U}_r \subseteq \mathcal{S}(r, j)$ of $\overline{\mathcal{S}(r, j)}$. As Z_r is a component of $\overline{\mathcal{S}(r, j)}$, this set intersects Z_r non-trivially. Put $U_r := V_r \cap \text{ind}_B^d \setminus (\bigcup_{\ell \neq r} \text{GL}_d(k) \cdot V_\ell)$. In view of (d) and Lemma 1.3(3),

$$O_r := \sigma_{(r,j)}(\tilde{U}_r \cap \pi_{(r,j)}^{-1}(U_r)) \subseteq \sigma_{(r,j)}(\mathcal{S}(r, j))$$

is a dense, open subset of W_j . Thus, $\tilde{O} := \bigcap_{r \in \Gamma_j} O_r$ also has these properties.

We fix an element $s \in \Gamma_j$. Since $\sigma_{(s,j)} : Z_s \rightarrow W_j$ is dominant, we have $\sigma_{(s,j)}^{-1}(\tilde{O}) \neq \emptyset$. As \tilde{U}_s meets Z_s , it follows that $U'_s := \tilde{U}_s \cap \sigma_{(s,j)}^{-1}(\tilde{O}) \subseteq \mathcal{S}(s, j)$ is a dense, open subset of the irreducible variety Z_s . In view of (a) and (d), $O'_s := \pi_{(s,j)}(U'_s)$ is a dense open subset of V_s .

Thanks to Lemma 1.3(2), $V_s \cap \text{ind}_B^d$ is a dense open subset of V_s . Its intersection with O'_s yields a dense open subset $O''_s \subseteq O'_s \cap \text{ind}_B^d$ of V_s . In particular, we have $V_s = O''_s \cup \mathcal{F}_s$ for some finite subset $\mathcal{F}_s \subseteq V_s$.

Let M be an element of O''_s . Then M is indecomposable and $M = \pi_{(s,j)}(M, N)$ with $(M, N) \in U'_s$. Thus, $(M, N) \in \mathcal{S}(s, j)$ and $N = \sigma_{(s,j)}(M, N) \in O_r \subseteq \sigma_{(r,j)}(\mathcal{S}(r, j))$ for every $r \in \Gamma_j$. Hence there exists, for every $r \in \Gamma_j$, a B -module $M_r \in U_r$ such that $(M_r, N) \in \mathcal{S}(r, j)$.

The B -modules $(M_r)_{r \in \Gamma_j}$ are indecomposable direct summands of $N|_B$. As each M_r belongs to U_r , the M_r are pairwise non-isomorphic. By the Theorem of Krull-Remak-Schmidt, the module $\bigoplus_{r \in \Gamma_j} M_r$ is also a direct summand of $N|_B$. Since $\dim_k N \leq \dim_k A \otimes_B M_s \leq (\dim_k A)d$, we obtain

$$d|\Gamma_j| \leq \dim_k N \leq d(\dim_k A),$$

so that $|\Gamma_j| \leq \dim_k A$. \diamond

Let $d > 0$. Since each V_i is irreducible, claim (b) provides a map $\omega : \{1, \dots, \mu_B(d)\} \longrightarrow \{1, \dots, \ell(d)\}$ such that $V_i = \overline{\pi_{(i, \omega(i))}(\mathcal{S}(i, \omega(i)))}$. We set $\Omega := \text{im } \omega$. There exists a finite set $F'_d \subseteq \text{mod}_B^d$ such that

$$\text{ind}_B^d \subseteq \bigcup_{t \in \Omega} \bigcup_{i \in \omega^{-1}(t)} \text{GL}_d(k) \cdot V_i \cup \text{GL}_d(k) \cdot F'_d \subseteq \bigcup_{t \in \Omega} \bigcup_{i \in \Gamma_t} \text{GL}_d(k) \cdot V_i \cup \text{GL}_d(k) \cdot F'_d.$$

Thanks to (e) and Lemma 2.1, there exists $c > 0$ such that the number $\mu_B(d)$ of one-parameter families is bounded by

$$|\Omega| (\max_{j \in \Omega} |\Gamma_j|) \leq \ell(d) (\dim_k A) \leq c (\dim_k A) d^{\gamma_A}.$$

Consequently, the algebra B has polynomial growth of rate $\gamma_B \leq \gamma_A + 1$. \square

The values of γ_A and γ_B may be arbitrarily far apart. Let A be a tame algebra. Then $A : k$ is a split extension with $\gamma_k = 0$.

The upper bound for γ_B depends on the knowledge of the indecomposable constituents of $A \otimes_B V$ containing V as a direct summand. The following result shows how a better understanding of induced modules leads to improved estimates.

Corollary 2.4. *Suppose that $A : B$ is an extension of k -algebras such that $A = \bigoplus_{i=1}^r A_i$ is a direct sum of (B, B) -modules with $A_i \cong B^{(\text{id}_B \otimes \zeta_i)}$ for some $\zeta_i \in \text{Aut}(B)$. If A is of polynomial growth, then B is of polynomial growth with $\gamma_B \leq \gamma_A$.*

Proof. Using Corollary 2.2 in the last paragraph of the above proof, we arrive at the stronger estimate. \square

Examples. (1) The classical example of a split extension is the trivial extension $A := B \oplus M$ of B by a (B, B) -bimodule M . In that case, B is also a factor algebra of A and our prefatory remarks show that B inherits polynomial growth from A along with $\gamma_B \leq \gamma_A$.

(2) Let G be a finite group, $H \subseteq G$ be a subgroup, $S \subseteq G$ be a set of right coset representatives of H in G containing 1. Then

$$kG = kH \oplus \left(\bigoplus_{s \neq 1} kHs \right)$$

is a direct sum of (kH, kH) -bimodules, so that the extension $kG : kH$ is split.

If $H \trianglelefteq G$ is normal in G , then

$$kG = \bigoplus_{s \in S} kHs$$

is a decomposition of kH -bimodules, with each summand being isomorphic to $kH^{(\text{id}_{kH} \otimes s^{-1})}$, where $s \cdot^{-1}$ denotes the conjugation by s^{-1} . Suppose that kG has polynomial growth. It now follows from Corollary 2.4 that kH has polynomial growth with growth rate $\gamma_{kH} \leq \gamma_{kG}$.

(3) Suppose that $\text{char}(k) = p > 0$. The following example shows that extensions defined by “group algebras” of infinitesimal group schemes tend to behave differently. Let $\mathfrak{b} \subseteq \mathfrak{sl}(2)$ be the Borel subalgebra of upper triangular (2×2) -matrices of the restricted Lie algebra $\mathfrak{sl}(2)$. The extension $U_0(\mathfrak{sl}(2)) : U_0(\mathfrak{b})$, given by the associated restricted enveloping algebras, is not split: Let $\lambda \in \mathfrak{b}^*$ be the Steinberg weight, so that $U_0(\mathfrak{sl}(2)) \otimes_{U_0(\mathfrak{b})} k_\lambda$ is the Steinberg module. This module is known to be projective, so that its restriction $(U_0(\mathfrak{sl}(2)) \otimes_{U_0(\mathfrak{b})} k_\lambda)|_{U_0(\mathfrak{b})}$ is also projective. If $U_0(\mathfrak{sl}(2)) : U_0(\mathfrak{b})$ was split, then k_λ would be a direct summand of $(U_0(\mathfrak{sl}(2)) \otimes_{U_0(\mathfrak{b})} k_\lambda)|_{U_0(\mathfrak{b})}$ and hence also projective. Since \mathfrak{b} is not a torus, this is impossible. Note that $\gamma_{U_0(\mathfrak{sl}(2))} = 1$, while $\gamma_{U_0(\mathfrak{b})} = 0$.

3. SEPARABLE EXTENSIONS

In this section we show that polynomial growth is preserved under ascent via separable extensions.

Lemma 3.1. *Let $A : B$ be a separable extension, and suppose that B is tame of polynomial growth. Then there exists a function $\ell : \mathbb{N} \rightarrow \mathbb{N}_0$ with the following properties:*

- (1) *For every $d > 0$ there exist a finite subset $F_d \subseteq \text{ind}_A^d$ and $(B, k[T])$ -bimodules $Y_1, \dots, Y_{\ell(d)}$ that are free $k[T]$ -modules of rank $\leq d$ such that for every $M \in \text{ind}_A^d \setminus \text{GL}_d(k).F_d$ there exist $i \in \{1, \dots, \ell(d)\}$ and $\lambda : k[T] \rightarrow k$ with

 - (a) M being isomorphic to a direct summand of $(A \otimes_B Y_i) \otimes_{k[T]} k_\lambda$, and
 - (b) $Y_i \otimes_{k[T]} k_\lambda$ being isomorphic to a direct summand of $M|_B$.*
- (2) *The function ℓ has polynomial growth of rate $\gamma_\ell \leq \gamma_B + 1$.*
- (3) *Further, if such ℓ is chosen so that $\ell(d)$ is minimal subject to (a) and (b), then*

$$|\{Y_i \otimes_{k[T]} k_\lambda ; \lambda \in k\} \cap \text{GL}_{r_i}(k).x| < \infty$$

for all $i \in \{1, \dots, \ell(d)\}$, $x \in \text{mod}_B^{r_i}$, $r_i := \text{rk}_{k[T]}(Y_i)$.

Proof. (1) Given $j > 0$, we let $X_1^j, \dots, X_{\mu_B(j)}^j$ be the parametrizing $(B, k[T])$ -modules and $G_j \subseteq \text{ind}_B^j$ be the finite set of exceptional modules.

Let $d > 0$, $M \in \text{ind}_A^d$. Since the extension $A : B$ is separable, we have $M | A \otimes_B M$, and there exists an indecomposable direct summand N of $M|_B$ such that $M | A \otimes_B N$. Hence $N \in \text{ind}_B^j$ for some $j \in \{1, \dots, d\}$.

We consider the $(B, k[T])$ -bimodules $\{X_i^j ; 1 \leq j \leq d, 1 \leq i \leq \mu_B(j)\}$. Let $\ell(d)$ be the number of these modules, which we denote $Y_1, \dots, Y_{\ell(d)}$. By definition, each Y_i is a free $k[T]$ -module of rank $\text{rk}_{k[T]}(Y_i) \leq d$.

Consider the finite set

$$U_d := \{A \otimes_B N ; N \in \bigcup_{j=1}^d G_j\}.$$

For each $X \in U_d$ we pick a decomposition into indecomposable constituents. Let \mathcal{U}_d be the finite set of these constituents and put $F_d := \text{ind}_A^d \cap \mathcal{U}_d$.

Let $M \in \text{ind}_A^d \setminus \text{GL}_d(k).F_d$. By the above, there exist $j \in \{1, \dots, d\}$ and an indecomposable direct summand $N | (M|_B)$ with $\dim_k N = j$, such that M is isomorphic to a direct summand of $A \otimes_B N$. The assumption $N \in \text{GL}_j(k).G_j$ implies $M \in \text{GL}_d(k).F_d$, a contradiction. It follows that $N \cong Y_i \otimes_{k[T]} k_\lambda$ for some $i \in \{1, \dots, \ell(d)\}$.

- (2) Since B has polynomial growth, there exists $c > 0$ such that

$$\ell(d) = \sum_{j=1}^d \mu_B(j) \leq c \sum_{j=1}^d j^{\gamma_B - 1} \leq c d d^{\gamma_B - 1} \leq c d^{\gamma_B}.$$

As a result, we have $\gamma_\ell \leq \gamma_B + 1$.

- (3) Suppose there is $x_0 \in \text{mod}_B^{r_i}$ for some $i \in \{1, \dots, \ell(d)\}$ such that the set $\mathcal{V}_i := \{Y_i \otimes_{k[T]} k_\lambda ; \lambda \in k\} \cap \text{GL}_{r_i}(k).x_0$ is infinite. Lemma 1.2 then shows that \mathcal{V}_i is cofinite in $\{Y_i \otimes_{k[T]} k_\lambda ; \lambda \in k\}$. We write

$$\{Y_i \otimes_{k[T]} k_\lambda ; \lambda \in k\} = \mathcal{V}_i \cup G_{r_i}$$

for some finite set $G_{r_i} \subseteq \text{mod}_B^{r_i}$. For every $N \in G_{r_i} \cup \{x_0\}$, we fix a decomposition of $A \otimes_B N$ into indecomposables and consider the finite subset $\mathcal{F}_d \subseteq \text{ind}_A^d$ of d -dimensional indecomposable constituents.

Let M be an element of $\text{ind}_A^d \setminus \text{GL}_d(k).(F_d \cup \mathcal{F}_d)$. Since $M \notin \text{GL}_d(k).F_d$, M is a direct summand of some $(A \otimes_B Y_j) \otimes_{k[T]} k_\lambda$, while $M \notin \text{GL}_d(k).\mathcal{F}_d$ ensures that $j \neq i$. This, however, contradicts the minimality of $\ell(d)$. \square

Given a finite group G that acts on a k -algebra B via automorphisms, we denote by $B * G$ the *skew group algebra*, cf. [2, (III.4)]. For separable extensions of the form $B * G : B$, the estimate for the growth of the function ℓ may be strengthened.

In the following, we denote by $\lfloor q \rfloor$ the least integer greater than or equal to $q \in \mathbb{Q}$.

Lemma 3.2. *Let G be a finite group acting on a k -algebra B via automorphisms. Suppose that B has polynomial growth, and that $p := \text{char}(k) \nmid \text{ord}(G)$. Then there exists a function $\ell : \mathbb{N} \rightarrow \mathbb{N}_0$ with the following properties:*

- (1) *For every $d > 0$ there exist a finite subset $F_d \subseteq \text{ind}_{B * G}^d$ and $(B, k[T])$ -bimodules $Y_1, \dots, Y_{\ell(d)}$ that are free $k[T]$ -modules of rank $\leq d$ such that for every $M \in \text{ind}_{B * G}^d \setminus \text{GL}_d(k) \cdot F_d$ there exist $i \in \{1, \dots, \ell(d)\}$ and $\lambda : k[T] \rightarrow k$ such that*
 - (a) *M is isomorphic to a direct summand of $(B * G \otimes_B Y_i) \otimes_{k[T]} k_\lambda$, and*
 - (b) *$Y_i \otimes_{k[T]} k_\lambda$ is isomorphic to a direct summand of $M|_B$.*
- (2) *The function ℓ has polynomial growth of rate $\gamma_\ell \leq \gamma_B$.*
- (3) *Further, if such ℓ is chosen so that $\ell(d)$ is minimal subject to (a) and (b), then*

$$|\{Y_i \otimes_{k[T]} k_\lambda ; \lambda \in k\} \cap \text{GL}_{r_i}(k) \cdot x| < \infty$$

for all $i \in \{1, \dots, \ell(d)\}$, $x \in \text{mod}_B^{r_i}$, $r_i := \text{rk}_{k[T]}(Y_i)$.

Proof. We put $A := B * G$ for ease of notation. Since p does not divide $\text{ord}(G)$, the extension $A : B$ is separable (cf. Proposition 4.1.1 below).

Given $j > 0$, we let $X_1^j, \dots, X_{\mu_B(j)}^j$ be parametrizing $(B, k[T])$ -bimodules, $G_j \subseteq \text{ind}_B^j$ be the finite set of exceptional modules.

Let $d > 0$, $M \in \text{ind}_A^d$. Then there exists an indecomposable direct summand N of $M|_B$ such that M is a direct summand of $A \otimes_B N$. Since $(A \otimes_B N)|_B \cong \bigoplus_{g \in G} N^{(g)}$, there exists a subset $S_M \subseteq G$ with $M|_B \cong \bigoplus_{g \in S_M} N^{(g)}$. As a result $\dim_k M = |S_M| \dim_k N$, and we conclude that $N \in \text{ind}_B^j$, where $d = rj$ and $r \leq \text{ord}(G) =: |G|$. We consider the $(A, k[T])$ -bimodules

$$A \otimes_B X_i^{\lfloor d/r \rfloor} ; \quad 1 \leq r \leq |G|, \quad 1 \leq i \leq \mu_B(\lfloor d/r \rfloor).$$

Let $\ell(d)$ be the number of these modules. Since B has polynomial growth, there exists $c > 0$ such that

$$\ell(d) = \sum_{r=1}^{|G|} \mu_B(\lfloor d/r \rfloor) \leq c \sum_{r=1}^{|G|} \lfloor d/r \rfloor^{\gamma_B - 1} \leq c |G| d^{\gamma_B - 1}.$$

As a result, we have $\gamma_\ell \leq \gamma_B$. The remaining assertions follow as in Lemma 3.1. \square

Theorem 3.3. *Let $A : B$ be a separable extension. If B is of polynomial growth, then A has polynomial growth with $\gamma_A \leq \gamma_B + 1$.*

Proof. Since the extension $A : B$ is separable, every indecomposable A -module is a direct summand of $A \otimes_B N$ for some indecomposable B -module N . It thus follows from [26, (4.2)] that A is tame. The arguments of Theorem 2.3 now apply mutatis mutandis. We indicate the requisite changes.

Let $d > 0$. There are parametrizing morphisms

$$f_i : \mathbb{A}^1 \rightarrow \text{mod}_A^d ; \quad 1 \leq i \leq \mu_A(d).$$

We set $V_i := \overline{\text{im } f_i}$, and apply Proposition 1.4 and Lemma 1.3(1) to obtain

$$(\dagger) \quad |\text{im } f_i \cap \text{GL}_d(k) \cdot x| < \infty \quad \forall x \in \text{mod}_B^d.$$

Now let $Y_1, \dots, Y_{\ell(d)}, F_d$ be the $(B, k[T])$ -bimodules and the finite subset of ind_B^d provided by Lemma 3.1, respectively. For $j \in \{1, \dots, \ell(d)\}$ let

$$g_j : \mathbb{A}^1 \longrightarrow \text{mod}_B^{r_j} \quad ; \quad r_j := \text{rk}_{k[T]}(X_j) \leq d$$

be the corresponding parametrizing morphisms. We put $W_j := \overline{\text{im } g_j}$ and choose $\ell(d)$ to be minimal subject to properties (a) and (b) of Lemma 3.1.

Chevalley's Theorem provides dense open subsets $O_i \subseteq V_i$ and $U_j \subseteq W_j$ with $O_i \subseteq \text{im } f_i$ and $U_j \subseteq \text{im } g_j$.

Given $(i, j) \in \{1, \dots, \mu_B(d)\} \times \{1, \dots, \ell(d)\}$, we denote by $\mathcal{S}(i, j) \subseteq V_i \times W_j$ the set of all pairs $(M, N) \in \text{im } f_i \times \text{im } g_j$ such that M is isomorphic to a direct summand of $A \otimes_B N$ and N is isomorphic to a direct summand of $M|_B$. In view of Lemma 1.5 and the arguments of [25, p.183], this set is constructible. We denote by $\pi_{(i,j)} : \overline{\mathcal{S}(i, j)} \longrightarrow V_i$ the restriction of the projection $V_i \times W_j \longrightarrow V_i$.

Assertions (a)-(d) of Theorem 2.3 can now be shown to hold in this context as well. In (e), we reach the conclusion that $\bigoplus_{r \in \Gamma_j} M_r$ is a direct summand of $A \otimes_B N$. Since $r_j \leq d$ this implies

$$d|\Gamma_j| \leq \dim_k A \otimes_B N \leq (\dim_k A)r_j \leq (\dim_k A)d.$$

The remaining arguments of Theorem 2.3 may be adopted verbatim. \square

4. APPLICATIONS

4.1. Smash products. Our first application concerns the smash product of Hopf algebras. We refer the reader to [22] for general facts concerning Hopf algebras. In the following, we shall write η and ε for the antipode and co-unit of a Hopf algebra and use Sweedler notation $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$ for calculations involving the comultiplication.

Let H be a finite-dimensional Hopf algebra, B be an H -module algebra. By definition, B is an H -module with action $(h, b) \mapsto h.b$ such that

- (a) $h.(ab) = \sum_{(h)} (h_{(1)}.a)(h_{(2)}.b)$ for all $h \in H, a, b \in B$, and
- (b) $h.1 = \varepsilon(h)1$ for all $h \in H$.

We consider the smash product $B \sharp H$, with underlying k -space $B \otimes_k H$ and multiplication

$$(a \otimes h).(b \otimes h') = \sum_{(h)} a(h_{(1)}.b) \otimes h_{(2)}h' \quad \forall h, h' \in H, a, b \in B.$$

Recall that the canonical maps $b \mapsto b \otimes 1$ and $h \mapsto 1 \otimes h$ are injective homomorphisms of algebras [22, Chap.4]. In this section we study the extension $B \sharp H : B$.

A Hopf algebra H is referred to as *cosemisimple* if its dual algebra H^* is semisimple. Recall that an extension $A : B$ of k -algebras is a *free Frobenius extension of first kind*, provided

- (a) A is a finitely generated free B -module, and
- (b) there exists an isomorphism $\varphi : A \longrightarrow \text{Hom}_B(A, B)$ of (A, B) -bimodules.

In this case, the B -module A is known to afford dual bases $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ such that $\varphi(x_i)(y_j) = \delta_{i,j}$. The map

$$c_{A:B} : A \longrightarrow A \quad ; \quad b \mapsto \sum_{i=1}^n y_i b x_i,$$

the so-called *Gaschütz-Ikeda operator*, does not depend on the choice of the bases.

Proposition 4.1.1. *Let B be an H -module algebra. Then the following statements hold:*

- (1) $B \sharp H : B$ is a free Frobenius extension of first kind.
- (2) If H is cosemisimple, then the extension $B \sharp H : B$ is split.

(3) If H is semisimple, then the extension $B\sharp H : B$ is separable.

Proof. (1) Let $\lambda \in H^*$ be a non-zero left integral of H^* , that is, a linear form satisfying

$$(*) \quad f * \lambda = f(1) \lambda \quad \forall f \in H^*,$$

where $f * g(h) = \sum_{(h)} f(h_{(1)})g(h_{(2)})$ denotes the convolution of $f, g \in H^*$. We define a k -linear map $\pi : B\sharp H \rightarrow B$ by means of

$$\pi(a \otimes h) := a \lambda(h) \quad \forall a \in B, h \in H.$$

Note that $\pi((a \otimes 1).(b \otimes h)) = \pi(ab \otimes h) = ab\lambda(h) = a\pi(b \otimes h)$ as well as

$$\begin{aligned} \pi((a \otimes h).(b \otimes 1)) &= \sum_{(h)} \pi(a(h_{(1)}.b) \otimes h_{(2)}1) = \sum_{(h)} a(h_{(1)}.b)\lambda(h_{(2)}) \\ &= a\left(\left(\sum_{(h)} h_{(1)}\lambda(h_{(2)})\right).b\right). \end{aligned}$$

Given $f \in H^*$ we obtain, observing (*),

$$f(\lambda(h)1) = f(1) \lambda(h) = \sum_{(h)} f(h_{(1)})\lambda(h_{(2)}) = f\left(\sum_{(h)} h_{(1)}\lambda(h_{(2)})\right),$$

so that the last term above equals

$$a \lambda(h)b = \pi(a \otimes h)b.$$

Since λ is a Frobenius homomorphism of the Frobenius algebra H (cf. [21]), there exist bases $\{x_1, \dots, x_n\}; \{y_1, \dots, y_n\}$ of H such that $\lambda(x_i y_j) = \delta_{i,j}$ for $1 \leq i, j \leq n$. Observing

$$\begin{aligned} \pi((1 \otimes x).(1 \otimes y)) &= \pi\left(\sum_{(x)} \varepsilon(x_{(1)})1_B \otimes x_{(2)}y\right) = \sum_{(x)} \varepsilon(x_{(1)})\lambda(x_{(2)}y) \\ &= \lambda\left(\sum_{(x)} \varepsilon(x_{(1)})x_{(2)}y\right) = \lambda(xy), \end{aligned}$$

we obtain $\pi((1 \otimes x_i).(1 \otimes y_j)) = \delta_{i,j}$ $1 \leq i, j \leq n$. By definition, the set $\{1 \otimes x_1, \dots, 1 \otimes x_n\}$ is a basis for the left B -module $B\sharp H$. Suppose that $0 = \sum_{i=1}^n (1 \otimes y_i).(b_i \otimes 1)$ for some $b_i \in B$. Then we have $0 = \pi((1 \otimes x_j).0) = \sum_{i=1}^n \pi((1 \otimes x_j)(1 \otimes y_i))b_i = b_j$ for $1 \leq j \leq n$. As a result, the k -vector space $\sum_{i=1}^n (1 \otimes y_i).B$ has dimension $\dim_k B\sharp H$, so that $\{1 \otimes y_1, \dots, 1 \otimes y_n\}$ generates $B\sharp H$ as a right B -module. The result now follows from [4, (1.2)].

(2) By assumption, the algebra H^* is semisimple. Recall that the left H -comodule H obtains the structure of a right H^* -module via

$$h.\psi := \sum_{(h)} \psi(h_{(1)})h_{(2)} \quad \forall h \in H, \psi \in H^*.$$

Consequently, $k1$ is an H^* -submodule of H and thus possesses a direct complement V . By general theory [22, (1.6.4)], V is a left subcomodule of H , so that $\Delta(V) \subseteq H \otimes_k V$. There results a vector space decomposition

$$B\sharp H = B \oplus (B \otimes_k V),$$

which obviously is a decomposition of left B -modules. It remains to show that $B \otimes_k V$ is a right B -submodule of $B\sharp H$. Given $a, b \in B$ and $v \in V$, we have

$$(a \otimes v).(b \otimes 1) = \sum_{(v)} a(v_{(1)}.b) \otimes v_{(2)} \in B \otimes_k V,$$

as desired.

(3) We denote by c_H and $c_{B\sharp H:B}$ the Gaschütz-Ikeda operators of the Frobenius extensions $H:k$ and $B\sharp H:B$, respectively. The arguments of (1) imply

$$c_{B\sharp H:B}(1\otimes h) = \sum_{i=1}^n (1\otimes y_i) \cdot (1\otimes h) \cdot (1\otimes x_i) = 1\otimes c_H(h)$$

for every $h \in H$. Since H is semisimple, the Frobenius extension $H:k$ is separable, and there exists an element z of the center of H such that $1 = c_H(z) = \sum_{i=1}^n y_i z x_i$, cf. [12, (2.6)]. Consequently,

$$1\otimes 1 = c_{B\sharp H:B}(1\otimes z) = (1\otimes z) \cdot c_{B\sharp H:B}(1\otimes 1) = c_{B\sharp H:B}(1\otimes 1) \cdot (1\otimes z).$$

As $c_{B\sharp H:B}(1\otimes 1)$ belongs to the centralizer $C_{B\sharp H:B}(B)$, the element $1\otimes z$ enjoys the same property, and [12, (2.6)] (cf. also [18, (2.18)]) shows that the extension is separable. \square

Proposition 4.1.2. *Let B be an H -module algebra. Then the following statements hold:*

- (1) *If H is cosemisimple and $B\sharp H$ is of polynomial growth, then B is of polynomial growth with $\gamma_B \leq \gamma_{B\sharp H} + 1$.*
- (2) *If H is semisimple and B is of polynomial growth, then $B\sharp H$ is of polynomial growth with $\gamma_{B\sharp H} \leq \gamma_B + 1$.*

Proof. (1) Thanks to Proposition 4.1.1(2), the extension $B\sharp H:B$ is split. Consequently, Theorem 2.3 implies our result.

(2) In view of Proposition 4.1.1(3), the extension $B\sharp H:B$ is separable and the assertion follows from Theorem 3.3. \square

Our next result refines [25, (3.3)].

Corollary 4.1.3. *Let G be a finite group acting on a k -algebra B . Then the following statements hold:*

- (1) *If $B*G$ has polynomial growth, then B has polynomial growth with $\gamma_B \leq \gamma_{B*G}$.*
- (2) *Suppose that $p := \text{char}(k) \nmid \text{ord}(G)$. Then B has polynomial growth if and only if $B*G$ has polynomial growth with $\gamma_{B*G} = \gamma_B$.*

Proof. (1) Since $B*G = \bigoplus_{g \in G} Bg$ is a (B, B) -direct decomposition of $B*G$ with $Bg \cong B^{(\text{id}_B \otimes g^{-1})}$, the assertion is a direct consequence of Corollary 2.4.

(2) By replacing the reference to Lemma 3.1 by Lemma 3.2 in the proof of Theorem 3.3, we conclude that B being of polynomial growth implies that $B*G$ has polynomial growth with $\gamma_{B*G} \leq \gamma_B$. Part (1) provides the reverse direction as well as the reverse inequality. \square

A finite group scheme \mathcal{G} is said to be *linearly reductive*, provided its algebra of measures $k\mathcal{G}$ is semisimple. If B is an algebra, viewed as a functor $R \mapsto B \otimes_k R$ from commutative k -algebras to k -algebras, then there is the notion of \mathcal{G} acting on B via automorphisms. This is equivalent to B being a $k\mathcal{G}$ -module algebra. For the general theory of affine group schemes, we refer to [19, 31].

Corollary 4.1.4. *Let \mathcal{G} be a linearly reductive group scheme that operates on B via automorphisms. If B has polynomial growth, then $B\sharp k\mathcal{G}$ has polynomial growth of rate $\gamma_{B\sharp k\mathcal{G}} \leq \gamma_B$.*

Proof. By general theory [31, (6.8)], the group scheme \mathcal{G} is a semi-direct product

$$\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\text{red}},$$

with an infinitesimal, normal subgroup \mathcal{G}^0 and a reduced group scheme \mathcal{G}_{red} . This implies that

$$k\mathcal{G} = k\mathcal{G}^0 * \mathcal{G}(k)$$

is a skew group algebra. Since \mathcal{G} is linearly reductive, Nagata's Theorem [9, (IV.§3.3.6)] ensures that \mathcal{G}^0 is diagonalizable and $p \nmid \text{ord}(\mathcal{G}(k))$.

We first assume that \mathcal{G} is infinitesimal, so that $\mathcal{G} = \mathcal{D}$ is diagonalizable. By van den Bergh's Theorem [6] (see also [22, p.167]), we have

$$(B\sharp k\mathcal{D})\sharp k\mathcal{D}^* \cong B \otimes_k \text{End}_k(k\mathcal{D}),$$

so that the two-fold smash product is of polynomial growth with rate $\gamma_{(B\sharp k\mathcal{D})\sharp k\mathcal{D}^*} = \gamma_B$. Since \mathcal{D} is diagonalizable, $k\mathcal{D}^* = kX(\mathcal{D})$ is the group algebra of the character group $X(\mathcal{D})$ of \mathcal{D} . Thus, $(B\sharp k\mathcal{D})\sharp k\mathcal{D}^* = (B\sharp k\mathcal{D}) * X(\mathcal{D})$, and Corollary 4.1.3(1) shows that $B\sharp k\mathcal{D}$ has polynomial growth of rate $\gamma_{B\sharp k\mathcal{D}} \leq \gamma_B$.

In the general case, we write $\mathcal{G} = \mathcal{D} \rtimes G$, where \mathcal{D} is diagonalizable and G is finite with $p \nmid \text{ord}(G)$. By the first part, the algebra $C := B\sharp k\mathcal{D}$ is of polynomial growth with rate $\gamma_C \leq \gamma_B$. Since $B\sharp k\mathcal{G} \cong C * G$, the assertion now follows from Corollary 4.1.3(2). \square

4.2. Blocks of group algebras. In preparation for our discussion of finite group schemes, we briefly recall the relevant result for the "classical" case concerning finite groups. Recall that polynomial growth and domesticity are preserved under Morita equivalence.

Let G be a finite group, $\mathcal{B} \subseteq kG$ be a block of the group algebra kG . In the sequel, we denote by $D_{\mathcal{B}} \subseteq G$ the defect group of \mathcal{B} , a p -subgroup of G . If kG affords a tame block \mathcal{B} , then $p = 2$ and $D_{\mathcal{B}}$ is dihedral, semidihedral, or generalized quaternion, cf. [5, (4.4.4)]. We illustrate the use of Theorem 2.3 via the following well-known result, which underscores the scarcity of representation-infinite blocks of polynomial growth:

Theorem 4.2.1. *Suppose that $\text{char}(k) = 2$. Let $\mathcal{B} \subseteq kG$ be a representation-infinite block. Then the following statements are equivalent:*

- (1) *The block \mathcal{B} is of polynomial growth.*
- (2) *$D_{\mathcal{B}} \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2)$.*
- (3) *\mathcal{B} is Morita equivalent to $k(\mathbb{Z}/(2) \times \mathbb{Z}/(2))$, kA_4 , or to the principal block of kA_5 .*
- (4) *The block \mathcal{B} is domestic.*

Proof. (1) \Rightarrow (2) We put $V_4 := \mathbb{Z}/(2) \times \mathbb{Z}/(2)$. Let $C_G(D_{\mathcal{B}})$ be the centralizer of $D_{\mathcal{B}}$ in G and consider the subgroup $H := D_{\mathcal{B}}C_G(D_{\mathcal{B}})$. In view of Brauer's First Main Theorem (cf. [1, Thm.2,p.103] and its succeeding remark), there exists a block $\mathfrak{b} \subseteq \mathcal{B}$ of kH such that the extension $\mathcal{B} : \mathfrak{b}$ is split. Theorem 2.3 now shows that \mathfrak{b} is of polynomial growth. Thanks to [5, (6.4.6)], the algebras \mathfrak{b} and $kD_{\mathcal{B}}$ are Morita equivalent, so that $kD_{\mathcal{B}}$ is of polynomial growth. Since \mathcal{B} is representation-infinite, the group $D_{\mathcal{B}}$ is not cyclic, see [5, (6.5)]. Hence [29, Prop.1] implies $D_{\mathcal{B}} \cong V_4$.

(2) \Rightarrow (3) Let e be the inertial index of \mathcal{B} . General theory implies $e \in \{1, 3\}$. If $e = 1$, then [5, (6.6.1)] provides an isomorphism $\mathcal{B} \cong \text{Mat}_n(kV_4)$. If $e = 3$, then [5, (6.6.3)] shows that \mathcal{B} is Morita equivalent to the group algebra kA_4 of the alternating group A_4 on 4 letters, or to the principal block of kA_5 .

(3) \Rightarrow (4) Since V_4 is the Sylow-2-subgroup of V_4 , A_4 and A_5 , [5, (4.4.4)] implies that \mathcal{B} is domestic.

(4) \Rightarrow (1) Trivial. \square

4.3. Group schemes of domestic representation type. We shall use our results to determine (up to a linearly reductive normal subgroup) all finite k -group schemes \mathcal{G} over an algebraically closed field k of odd characteristic, whose algebra $k\mathcal{G}$ of measures is representation-infinite and domestic. As in the case of finite groups, non-domestic group schemes do not have polynomial growth.

Let (Λ, ε) be an augmented k -algebra. There is a unique block $\mathcal{B}_0(\Lambda)$ such that $\varepsilon(\mathcal{B}_0(\Lambda)) \neq (0)$. This block is referred to as the *principal block* of Λ . If the finite group G acts on Λ via automorphisms of augmented algebras, then $\Lambda * G$ is augmented with augmentation ε_G satisfying

$$\varepsilon_G(\lambda g) = \varepsilon(\lambda) \quad \forall \lambda \in \Lambda, g \in G.$$

Lemma 4.3.1. *There exists a (Λ, Λ) -bimodule X such that $\mathcal{B}_0(\Lambda * G) = \mathcal{B}_0(\Lambda) \oplus X$.*

Proof. This follows directly from the proof of [13, (5.1.2)]. \square

Let Λ be a k -algebra. The split extension $T(\Lambda) := \Lambda \ltimes \Lambda^*$ of Λ by its dual bimodule Λ^* is referred to as the *trivial extension* of Λ . We shall apply this construction to radical square zero tame hereditary algebras. By definition, such an algebra is of the form kQ , where the quiver Q is a Euclidean diagram of type \tilde{A}_n, \tilde{D}_n or $\tilde{E}_{6,7,8}$ with an orientation, such that there are no paths of length 2. (For \tilde{A}_n this is possible if and only if n is odd.)

We denote by \mathcal{Z} the center of the algebraic group scheme $\mathrm{SL}(2)$. Given a finite subgroup scheme $\mathcal{G} \subseteq \mathrm{SL}(2)$, we put $\mathbb{P}(\mathcal{G}) := \mathcal{G}/(\mathcal{G} \cap \mathcal{Z})$. The unique largest linearly reductive normal subgroup scheme of a given finite group scheme \mathcal{G} will be denoted $\mathcal{G}_{\mathrm{lr}}$.

Our final result characterizes the representation-infinite finite algebraic groups of domestic representation type, showing their close connection with binary polyhedral groups and tame hereditary algebras. Recall that a finite linearly reductive subgroup scheme of $\mathrm{SL}(2)$ is referred to as a *binary polyhedral group scheme*. We denote by $\mathrm{SL}(2)_1$ the first Frobenius kernel of $\mathrm{SL}(2)$.

Theorem 4.3.2. *Let \mathcal{G} be a finite group scheme over an algebraically closed field k of characteristic $p \geq 3$. Then the following statements are equivalent:*

- (1) *The principal block $\mathcal{B}_0(\mathcal{G})$ is representation-infinite and of polynomial growth.*
- (2) *There exists a radical square zero tame hereditary algebra Λ such that $\mathcal{B}_0(\mathcal{G})$ is Morita equivalent to $T(\Lambda)$.*
- (3) *The Hopf algebra $k\mathcal{G}$ is representation-infinite and domestic.*
- (4) *The principal block $\mathcal{B}_0(\mathcal{G})$ is representation-infinite and domestic.*
- (5) *There exists a binary polyhedral group scheme $\tilde{\mathcal{G}} \subseteq \mathrm{SL}(2)$ such that $\mathcal{G}/\mathcal{G}_{\mathrm{lr}} \cong \mathbb{P}(\mathrm{SL}(2)_1 \tilde{\mathcal{G}})$.*

Proof. Let $G := \mathcal{G}(k)$ be the finite group of k -rational points. By general theory, we have

$$k\mathcal{G} = k\mathcal{G}^0 \sharp kG = k\mathcal{G}^0 * G.$$

(1) \Rightarrow (2) In view of the above, Lemma 4.3.1 implies that the extension $\mathcal{B}_0(\mathcal{G}) : \mathcal{B}_0(\mathcal{G}^0)$ is split. Theorem 2.3 thus shows that the algebra $\mathcal{B}_0(\mathcal{G}^0)$ is of polynomial growth. Thanks to [14, (6.1)], $\mathcal{B}_0(\mathcal{G}^0)$ is domestic. By [15, (3.1)], $\mathcal{B}_0(\mathcal{G}^0)$ is not representation-finite and [14, (5.1)] implies that the center $\mathrm{Cent}(\mathcal{G}^0)$ is diagonalizable. The assertion now follows from [13, (7.3.1)].

(2) \Rightarrow (3) The algebra Λ is known to be representation-infinite and domestic [10, 24]. Thanks to [30], the trivial extension $T(\Lambda)$ also enjoys this property. It follows that the algebra $\mathcal{B}_0(\mathcal{G})$ is domestic and of infinite representation type. As before, we see that $\mathcal{B}_0(\mathcal{G}^0)$ also has these properties, and [14, (6.1)] implies the domesticity of $k\mathcal{G}^0$. Since $\mathcal{B}_0(\mathcal{G})$ is not representation-finite, [13, (6.2.1)] ensures that p does not divide $\mathrm{ord}(G)$. It now follows from Corollary 4.1.3(2) that $k\mathcal{G} = k\mathcal{G}^0 * G$ is of polynomial growth with rate $\gamma_{k\mathcal{G}} = \gamma_{k\mathcal{G}^0} \leq 1$. Hence $k\mathcal{G}$ is domestic and of infinite representation type.

(3) \Rightarrow (4) This follows from [15, (3.1)].

(4) \Rightarrow (5) Let $T \subseteq \mathrm{SL}(2)$ be the standard torus of diagonal (2×2) -matrices of determinant 1 and put $\mathcal{G}' := \mathcal{G}/\mathcal{G}_{\mathrm{lr}}$. Then we have $\mathcal{G}'_{\mathrm{lr}} = e_k$, while [13, (1.1)] implies that $\mathcal{B}_0(\mathcal{G}') \cong \mathcal{B}_0(\mathcal{G})$ is domestic. Since $\mathcal{G}'_{\mathrm{lr}} = e_k$, a consecutive application of [13, (1.2)] and [14, (6.1)] provides an isomorphism $\mathcal{G}'^0 \cong \mathrm{SL}(2)_1 T_r$ for some $r \geq 1$. According to [13, (6.2.1)], the prime p does not divide $\mathrm{ord}(\mathcal{G}'(k))$, so that [13, (1.2)] yields $C_{\mathcal{G}'} := \mathrm{Cent}_{\mathcal{G}'_{\mathrm{red}}}(\mathcal{G}'^0) = e_k$. Consequently, $\mathrm{Cent}_{\mathcal{G}'}(\mathcal{G}'^0) = e_k$, and our assertion follows from [13, (7.1.2)].

(5) \Rightarrow (1) Let $\tilde{\mathcal{G}} \subseteq \mathrm{SL}(2)$ be a binary polyhedral group scheme. We consider the semidirect product

$$\mathcal{G} := \mathrm{SL}(2)_1 \rtimes \tilde{\mathcal{G}}.$$

Since $\tilde{\mathcal{G}}$ is linearly reductive, Corollary 4.1.4 guarantees that $k\mathcal{G} \cong k\mathrm{SL}(2)_1 \# k\tilde{\mathcal{G}}$ is domestic. It follows that the factor algebra $k\mathrm{SL}(2)_1 \tilde{\mathcal{G}}$ of $k\mathcal{G}$ is domestic. Consequently, $\mathcal{B}_0(\mathbb{P}(\mathrm{SL}(2)_1 \tilde{\mathcal{G}})) \cong \mathcal{B}_0(\mathrm{SL}(2)_1 \tilde{\mathcal{G}})$ enjoys the same property. Let $\mathrm{cx}_{\mathcal{G}}(k)$ be the *complexity* of the trivial \mathcal{G} -module, that is, the polynomial rate of growth of a minimal projective resolution of k . If $\mathcal{B}_0(\mathrm{SL}(2)_1 \tilde{\mathcal{G}})$ is representation-finite, then $1 \geq \mathrm{cx}_{\mathrm{SL}(2)_1 \tilde{\mathcal{G}}}(k) \geq \mathrm{cx}_{\mathrm{SL}(2)_1}(k) = 2$, a contradiction. As a result, the principal block $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{G}_{\mathrm{lr}}) \cong \mathcal{B}_0(\mathbb{P}(\mathrm{SL}(2)_1 \tilde{\mathcal{G}}))$ (cf. [13, (1.1)]) is of polynomial growth. \square

Remark. The binary polyhedral group schemes are completely understood, see [13, (3.3)]. They are determined by their McKay quivers, which in turn give rise to the Ext-quivers of the principal blocks $\mathcal{B}_0(\mathbb{P}(\mathrm{SL}(2)_1 \tilde{\mathcal{G}}))$.

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