GROUP-GRADED ALGEBRAS, EXTENSIONS OF INFINITESIMAL GROUPS, AND APPLICATIONS

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Abstract. Using results on algebras that are graded by p-groups, we study representations of infinitesimal groups $\mathcal{G}$ that possess a normal subgroup $\mathcal{N} \trianglelefteq \mathcal{G}$ with a diagonalizable factor group $\mathcal{G}/\mathcal{N}$. When combined with rank varieties, Auslander-Reiten theory and Premet's work on $\mathrm{SL}(2)_1$-modules, these techniques lead to the determination of the indecomposable modules of the infinitesimal groups of domestic representation type.

0. Introduction

Group-graded algebras occur in a number of contexts, including representations of finite groups, Lie theory and abstract representation theory. In the latter two disciplines, gradings relative to torsion-free groups usually are the most effective tools. The main motivation for studying algebras that are graded by p-groups in this paper derives from the determination of the indecomposable modules of the infinitesimal groups of domestic representation type. In view of the main results of [22, 23], such a group scheme $\mathcal{G}$ possesses a normal subgroup $\mathcal{N}$ such that the factor group $\mathcal{G}/\mathcal{N}$ is diagonalizable, and it is a natural question to study the interplay between the representations of $\mathcal{N}$, $\mathcal{G}$, and $\mathcal{G}/\mathcal{N}$. Aside from the abovementioned context, such extensions also occur in the contexts of trigonalizable groups and smooth reductive groups G. Given a simply connected covering $\tilde{G} \longrightarrow G$, the image of the r-th Frobenius kernel $\tilde{G}_r$ of $\tilde{G}$ is a normal subgroup of G_r with a diagonalizable factor group, see [36, (II.9.7)].

According to the philosophy underlying the trichotomy of finite-dimensional algebras defined by representation type, the classification of indecomposable modules for tame algebras should be a feasible task. The example of the group algebra of the quaternion group of order 8 at characteristic 2 illustrates the difficulties that may arise in practice. This circumstance has motivated the subdivision of the class of tame algebras into smaller classes, with the hope that their representation theory can be better controlled. By definition of tameness, all but finitely many indecomposable modules of any given dimension occur in a finite number of continuous one-parameter families. The number of parameters may grow rapidly with the dimension, so demanding the existence of an upper bound seems expedient. The resulting class of *domestic algebras* was introduced by Ringel [43]. For group algebras of characteristic two, only those blocks affording the Klein four group as its defect group are of this type (see for instance [15, (4.1)]). In the context of cocommutative Hopf algebras of odd characteristic, this class is also much better behaved, cf. [19, (7.4.3)].

Our article is organized as follows. In section 1 we provide a basic criterion for an algebra of characteristic $p > 0$, that is graded relative to a p-group, to be local. As an immediate consequence we retrieve Green's indecomposablity theorem for induced modules of group algebras. Our result is applied in section 2, where we investigate extensions of infinitesimal groups with the properties mentioned above. In particular, we show that the canonical restriction functor $\mathrm{res}_{\mathcal{N}} : \mathrm{mod}\,\mathcal{G} \longrightarrow \mathrm{mod}\,\mathcal{N}$ preserves indecomposables and provide some information on its fibers. In

Date: May 10, 2008.
2000 *Mathematics Subject Classification.* Primary 14L17,16G70, Secondary 16W50,17B50.

the context of Galois extensions we obtain detailed information concerning the induction functor $\mathrm{ind}_{\mathcal{N}}^{\mathcal{G}} : \mathrm{mod}\,\mathcal{N} \longrightarrow \mathrm{mod}\,\mathcal{G}$.

The point of view advocated in this article, the interplay between rank varieties, Auslander-Reiten theory and the classification of indecomposables, is taken up in section 3, where we study almost split sequences as well as the representation type of the algebra $\mathrm{Dist}(\mathcal{G})$ of distributions of $\mathcal{G}$. Although the restriction functor $\mathrm{res}_{\mathcal{N}} : \mathrm{mod}\,\mathcal{G} \longrightarrow \mathrm{mod}\,\mathcal{N}$ is shown to "commute" with Auslander-Reiten translations, it usually does not preserve almost split sequences. We define an invariant of stable Auslander-Reiten components of $\mathrm{mod}\,\mathcal{G}$, telling us exactly when $\mathrm{res}_{\mathcal{N}}$ locally induces a morphism of (non-valued) stable representation quivers. By way of illustration, rank varieties are employed to determine the infinitesimal groups with domestic principal blocks. General principles then ensure that the blocks in question are those associated to the products $\mathrm{SL}(2)_1T_r$ of the first Frobenius kernel of $\mathrm{SL}(2)$ with the r-th Frobenius kernel of its standard maximal torus $T \subseteq \mathrm{SL}(2)$ of diagonal matrices. The results of the first three sections in conjunction with Premet's work on the representation theory of $\mathrm{SL}(2)_1$ (cf. [40]) are applied in the concluding section to classify the indecomposables of this class of domestic groups. From the vantage point of abstract representation theory the module categories of the special biserial algebras $\mathrm{Dist}(\mathrm{SL}(2)_1T_r)$ are of course well understood. One benefit of finding realizations of their indecomposable modules resides in their rôles as a suitable testing ground for a better understanding of the behavior of the Jordan types appearing in [7, 27].

1. Group-Graded Algebras

Throughout, we assume that k is an algebraically closed field of characteristic $p > 0$. Unless mentioned otherwise, all algebras and modules are finite-dimensional k-vector spaces. We are going to study certain group-graded k-algebras

$$R = \bigoplus_{g\in G} R_g,$$

whose grading is defined by a finite group G. We let $\mathrm{Rad}(R)$ be the Jacobson radical of R and denote the group algebra of G by kG. By way of motivation, we record the following consequences of R being local.

Lemma 1.1. *Suppose that $R = \bigoplus_{g\in G} R_g$ is a local algebra. Then the following statements hold:*

(1) *The algebra R_1 is local.*

(2) *If $R_g \not\subseteq \mathrm{Rad}(R)$ for every $g \in G$, then G is a p-group.*

Proof. By assumption, there exists an algebra homomorphism $\varepsilon : R \longrightarrow k$ such that $\ker\varepsilon = \mathrm{Rad}(R)$.

(1) Since $\ker\varepsilon|_{R_1}$ is a nilpotent ideal of codimension 1, it follows that R_1 is local.

(2) Consider $N := \bigoplus_{g\in G}(\ker\varepsilon) \cap R_g$. Then N is a nilpotent ideal of R. Thus, ε induces an algebra homomorphism $\gamma : S \longrightarrow k$ of the local, G-graded algebra $S := R/N$. By virtue of our current assumption, we have $\dim_k S_g = 1$ for every $g \in G$, and for every $g \in G$ there exists a unique element $s_g \in S_g$ such that $\gamma(s_g) = 1$. Consequently,

$$s_g s_h = s_{gh} \quad \forall\ g, h \in G,$$

so that the map $G \longrightarrow S\ \ ;\ \ g \mapsto s_g$ induces a surjective algebra homomorphism $\zeta : kG \longrightarrow S$. By equality of dimensions, this map is bijective. As a result, the group algebra kG is local, forcing G to be a p-group. □

We turn to algebras that are graded by some p-group G, beginning with the case where G is abelian.

Lemma 1.2. *Let $R = \bigoplus_{g\in G} R_g$ be a group-graded k-algebra. Suppose that*
(a) *G is an abelian p-group, and*
(b) *$\dim R_g \leq 1$ for every $g \in G$, and*
(c) *the elements of $R_g \setminus \{0\}$ are invertible for every $g \in G$.*
Then there exists a subgroup $H \subseteq G$ with $R \cong kH$.

Proof. In view of (c), $H := \{h \in G\ ;\ R_h \neq (0)\}$ is a subgroup of G, and $R = \bigoplus_{h\in H} R_h$. By general theory, the group H is a direct sum of cyclic groups with generators $h_1, \ldots, h_\ell$ of orders $p^{n_1}, \ldots, p^{n_\ell}$, say. Pick $r_i \in R_{h_i}$ with $r_i^{p^{n_i}} = 1$. Given $i, j \in \{1, \ldots, \ell\}$, there exists $\alpha_{ij} \in k$ such that

$$r_i r_j r_i^{-1} = \alpha_{ij} r_j$$

Thus,

$$r_j = r_i^{p^{n_i}} r_j r_i^{p^{-n_i}} = \alpha_{ij}^{p^{n_i}} r_j,$$

so that $\alpha_{ij} = 1$. Consequently, the elements $r_1, \ldots, r_\ell$ commute with each other. Since the subalgebra generated by these elements contains all homogeneous parts of R, we see that R is commutative. By the same token, the map $T_i \mapsto r_i$ defines an isomorphism

$$k[T_1, \ldots, T_r]/(T_1^{p^{n_1}} - 1, \ldots, T_r^{p^{n_r}} - 1) \xrightarrow{\sim} R,$$

with the truncated polynomial ring being isomorphic to kH. □

The interested reader may compare the following result to [28, (3.1)], where Artin algebras that are graded by torsion-free groups are considered.

Theorem 1.3. *Let G be a p-group, $R = \bigoplus_{g\in G} R_g$ be a G-graded algebra. If R_1 is local, then R is local.*

Proof. We first assume that G is abelian and write G additively. Since R_0 is local, there exists a linear map $\alpha : R_0 \longrightarrow k$ such that

$$\ker \alpha = \{r \in R_0\ ;\ r \text{ is nilpotent}\}.$$

Given $g \in G$, we set

$$N_g := \{r \in R_g\ ;\ r \text{ is nilpotent}\}.$$

Suppose that $\mathrm{ord}(G) = p^m$. For $g \in G$ and $r \in R_g$, we have $r^{p^m} \in R_{p^m g} = R_0$. By the above, we can write

$$(*) \qquad r^{p^m} = \alpha(r^{p^m})1 + x$$

for some nilpotent element $x \in N_0$. It follows that

$$\psi_g : R_g \longrightarrow k \ \ ; \ \ r \mapsto \alpha(r^{p^m})$$

is a homogeneous polynomial function of degree p^m, whose zero locus $\mathcal{Z}(\psi_g)$ is N_g. Since R_g and k are irreducible varieties and ψ_g is a morphism, it follows from standard results on morphisms (cf. [31, (I.4.1)]) that $\dim N_g \geq \dim_k R_g - 1$.

By property $(*)$, a homogeneous element $r \in R$ is either nilpotent or invertible. Given $r \in N_g$ and $s \in R_h$, we have $rs \in R_{g+h}$. If rs is invertible, then left multiplication by r is surjective, which contradicts the nilpotence of r. Hence $rs \in N_{g+h}$, and a similar argument shows that $sr \in N_{g+h}$. Consequently, $N := \bigcup_{g\in G} N_g$ is a nil weakly closed subset of R in the sense of [32, (II.1)]. Jacobson's Theorem [32, (II.2)] now implies that the associative algebra $\mathrm{alg}_k(N)$ without identity generated by N is nilpotent. In particular, N_g is a subspace of R_g, which, by our earlier observation has codimension ≤ 1. By the above, $J = \bigoplus_{g\in G} N_g$ is a nilpotent ideal of R, such that the factor algebra $S := R/J$ is G-graded with the following properties:

(a) $\dim_k S_g \leq 1$ for every $g \in G$, and
(b) every element of $S_g \setminus \{0\}$ is invertible.

Consequently, Lemma 1.2 provides a subgroup $H \subseteq G$ such that $S \cong kH$. In particular, S is local and the algebra R thus enjoys the same property.

In the general case, that is, when G is not necessarily abelian, we proceed by induction on the order of G. The p-group G has a non-trivial center $C(G)$. We set $G' := G/C(G)$ and denote by $\pi : G \longrightarrow G'$ the canonical projection. We endow R with the structure of a G'-graded ring by defining

$$R'_x := \bigoplus_{g\in\pi^{-1}(x)} R_g \qquad \forall\ x \in G'.$$

Since the k-algebra $R'_1 = \bigoplus_{g\in C(G)} R_g$ is graded with respect to the abelian p-group $C(G)$, the first part of the proof shows that R'_1 is local. By inductive hypothesis, the algebra R is also local. □

Corollary 1.4. *Let $R = \bigoplus_{g\in G} R_g$ be a group-graded algebra. If $H \trianglelefteq G$ is a normal subgroup of index a p-power such that the subalgebra $\bigoplus_{h\in H} R_h$ is local, then R is local.*

Proof. Let $\pi : G \longrightarrow G/H$ be the canonical projection. Setting $R'_x := \bigoplus_{g\in\pi^{-1}(x)} R_g$ for every $x \in G/H$, we endow R with a structure of a (G/H)-graded algebra such that $R'_1 = \bigoplus_{h\in H} R_h$. Hence (1.3) implies our assertion. □

If G is a finite group that acts on a k-algebra Λ via automorphisms, then $\Lambda{*}G$ denotes the *skew group algebra* of G with coefficients in Λ. Skew group algebras are examples of *strongly graded* algebras $R = \bigoplus_{g\in G} R_g$, which, by definition, satisfy the conditions $R_{gh} = R_gR_h$ for all $g, h \in G$. As a first consequence of Theorem 1.3, we record the following generalization of Green's Indecomposablity Theorem, see [1, (8.8)].

Corollary 1.5. *Let G be a finite group that operates on an algebra Λ via automorphisms, and suppose that $H \trianglelefteq G$ is a normal subgroup of index a power of p. If M is an indecomposable $\Lambda{*}H$-module, then the induced module $\Lambda{*}G \otimes_{\Lambda*H} M$ is indecomposable.*

Proof. The skew group algebra $\Lambda{*}G$ is strongly graded relative to the p-group G/H, with one-component $(\Lambda{*}G)_1 = \Lambda{*}H$. Moreover, the induced module $\Lambda{*}G \otimes_{\Lambda*H} M$ and its endomorphism ring are also G/H-graded, and [10, (4.8)] provides an isomorphism

$$\mathrm{End}_{\Lambda*G}(\Lambda{*}G \otimes_{\Lambda*H} M)_1 \cong \mathrm{End}_{\Lambda*H}(M)$$

of rings. Our assumption on M in conjunction with Theorem 1.3 now guarantees that the k-algebra $\mathrm{End}_{\Lambda*G}(\Lambda{*}G \otimes_{\Lambda*H} M)$ is local. Consequently, the module $\Lambda{*}G \otimes_{\Lambda*H} M$ is indecomposble. □

2. Extensions of Infinitesimal Groups

Given a k-algebra Λ, we let $\mathrm{mod}\,\Lambda$ be the category of finite-dimensional left Λ-modules. If $\varphi : \Lambda \longrightarrow \Lambda$ is an automorphism and $M \in \mathrm{mod}\,\Lambda$, then $M^{(\varphi)}$ denotes the Λ-module with underlying k-space M and action defined by

$$a.m := \varphi^{-1}(a)m \qquad \forall\ a \in \Lambda,\ m \in M.$$

In this fashion we obtain an action of the automorphism group $\mathrm{Aut}_k(\Lambda)$ on the set of isoclasses of indecomposable Λ-modules.

We refer the reader to [12, 36, 48] for basic facts on algebraic group schemes. Given an infinitesimal k-group $\mathcal{G}$, we let $\mathrm{Dist}(\mathcal{G})$ be its *algebra of distributions*. By definition, $\mathrm{Dist}(\mathcal{G}) = k[\mathcal{G}]^*$ is the dual Hopf algebra of the coordinate ring $k[\mathcal{G}]$ of $\mathcal{G}$. By definition, $\mathrm{Dist}(\mathcal{G})$ is a cocommutative Hopf algebra, whose comultiplication, co-unit and antipode will be denoted $\Delta : \mathrm{Dist}(\mathcal{G}) \longrightarrow \mathrm{Dist}(\mathcal{G}) \otimes_k \mathrm{Dist}(\mathcal{G})$, $\varepsilon : \mathrm{Dist}(\mathcal{G}) \longrightarrow k$ and $\eta : \mathrm{Dist}(\mathcal{G}) \longrightarrow \mathrm{Dist}(\mathcal{G})$, respectively. When working with these structure maps, we shall employ the Heyneman-Sweedler notation

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)} \qquad \forall\ h \in \mathrm{Dist}(\mathcal{G}).$$

Since Δ is cocommutative, we have

$$\Delta(h) = \sum_{(h)} h_{(2)} \otimes h_{(1)} \qquad \forall\ h \in \mathrm{Dist}(\mathcal{G})$$

as well as $\eta^2 = \mathrm{id}_{\mathrm{Dist}(\mathcal{G})}$.

The group $X(\mathcal{G}) := \mathrm{Alg}_k(\mathrm{Dist}(\mathcal{G}), k)$ of k-algebra homomorphisms from $\mathrm{Dist}(\mathcal{G})$ to the ground field k is called the *character group* of $\mathcal{G}$. Its product is given by the convolution

$$(\lambda * \mu)(h) := \sum_{(h)} \lambda(h_{(1)})\mu(h_{(2)})$$

for all $\lambda, \mu \in X(\mathcal{G})$ and $h \in \mathrm{Dist}(\mathcal{G})$. Since $\mathcal{G}$ is infinitesimal, $X(\mathcal{G})$ is an abelian p-group. If $\mathcal{N} \trianglelefteq \mathcal{G}$ is a normal subgroup of $\mathcal{G}$, then we identify $X(\mathcal{G}/\mathcal{N})$ with the subgroup of $X(\mathcal{G})$ consisting of those maps that annihilate the augmentation ideal $\mathrm{Dist}(\mathcal{N})^\dagger := \ker\varepsilon$ of $\mathrm{Dist}(\mathcal{N})$. Recall that the convolution product induces an action of $X(\mathcal{G})$ on $\mathrm{Dist}(\mathcal{G})$ via automorphisms:

$$\lambda.h := \sum_{(h)} h_{(1)}\lambda(h_{(2)}) \qquad \forall\ \lambda \in X(\mathcal{G}),\ h \in \mathrm{Dist}(\mathcal{G}).$$

In other words,

$$\lambda.h = \psi_\lambda(h),$$

where

$$\psi_\lambda := \mathrm{id}_{\mathrm{Dist}(\mathcal{G})} * \lambda$$

denotes the convolution of $\mathrm{id}_{\mathrm{Dist}(\mathcal{G})}$ with λ.

By general theory [36, (I.8.6)], the categories of $\mathcal{G}$-modules and $\mathrm{Dist}(\mathcal{G})$-modules are naturally equivalent. We will write $\mathrm{mod}\,\mathcal{G}$ instead of $\mathrm{mod}\,\mathrm{Dist}(\mathcal{G})$ and use the notions "$\mathcal{G}$-module" and "$\mathrm{Dist}(\mathcal{G})$-module" interchangeably.

Example. Although $\mathrm{Dist}(\mathcal{G})$ plays a rôle analogous to that of the group algebra of a finite group, there are differences as regards the gradings. While the canonical basis G of the group algebra kG of a finite group G renders kG a strongly graded algebra, the PBW-bases of algebras of distributions may not have this property: Let $\mathcal{G} = \mathrm{SL}(2)_1T_r$ be the product of the first Frobenius kernel of $\mathrm{SL}(2)$ and the r-th Frobenius kernel of its standard maximal torus $T \subseteq \mathrm{SL}(2)$. If $\{e, h, f\}$ denotes the standard basis of the Lie algebra $\mathfrak{sl}(2)$, then

$$\{e^i \binom{h}{j} f^\ell \ ;\ 0 \le i, \ell \le p-1, 0 \le j \le p^r - 1\}$$

is a basis of $\mathrm{Dist}(\mathcal{G})$ over k, cf. [36, (II.3.3)]. Relative to the standard $\mathbb{Z}/(p^r)$-grading, we have $e^i\binom{h}{j}f^\ell \in \mathrm{Dist}(\mathcal{G})_{2(i-\ell)}$, whence

$$\mathrm{Dist}(\mathcal{G})_{2(p-1)} = e^{p-1}\,\mathrm{Dist}(T_r) \quad \text{and} \quad \mathrm{Dist}(\mathcal{G})_{-2(p-1)} = \mathrm{Dist}(T_r)f^{p-1}.$$

As a result, $\mathrm{Dist}(\mathcal{G})_{2(p-1)}\,\mathrm{Dist}(\mathcal{G})_{-2(p-1)} = e^{p-1}\,\mathrm{Dist}(T_r)f^{p-1}$ does not contain 1, and [10, (1.6)] shows that $\mathrm{Dist}(\mathcal{G})$ is not strongly graded.

2.1. **Extensions with multiplicative factor groups.** An infinitesimal group scheme $\mathcal{M}$ is *multiplicative* or *diagonalizable* if its coordinate ring is the group algebra $kX(\mathcal{M})$ of its character group. Since $X(\mathcal{M})$ is a p-group, the results of Section 1 apply for $X(\mathcal{M})$-graded algebras.

By Nagata's Theorem (cf. [12, (IV,§3,3.6)], the multiplicative groups are precisely the linearly reductive infinitesimal group schemes. Every $\mathcal{M}$-module M decomposes into weight spaces

$$M = \bigoplus_{\lambda\in X(\mathcal{M})} M_\lambda,$$

where $M_\lambda := \{m \in M \ ;\ h.m = \lambda(h)m \ \ \forall\, h \in \mathrm{Dist}(\mathcal{M})\}$.

Let $\lambda \in X(\mathcal{G})$ be a character. If M is a $\mathcal{G}$-module, then the canonical map $M \otimes_k k_\lambda \xrightarrow{\sim} M$ provides an isomorphism between the tensor product $M \otimes_k k_\lambda$ of M with the one-dimensional $\mathcal{G}$-module defined by λ and the twisted module $M^{(\psi_\lambda^{-1})}$. Given two $\mathcal{G}$-modules M and N, the algebra $\mathrm{Dist}(\mathcal{G})$ acts on $\mathrm{Hom}_k(M,N)$ via

$$(h.\varphi)(m) := \sum_{(h)} h_{(1)}\varphi(\eta(h_{(2)})m) \qquad \forall\, h \in \mathrm{Dist}(\mathcal{G}),\ m \in M,\ \varphi \in \mathrm{Hom}_k(M,N).$$

Recall that $\mathrm{Dist}(\mathcal{G})$ acts on itself via the (left) *adjoint representation*

$$h.x := \sum_{(h)} h_{(1)}x\eta(h_{(2)}) \qquad \forall\, h, x \in \mathrm{Dist}(\mathcal{G}).$$

We denote the corresponding module by $\mathrm{Dist}(\mathcal{G})_{\mathrm{ad}}$. Let M be a $\mathcal{G}$-module. Direct computation shows that the multiplication

$$\mathrm{Dist}(\mathcal{G})_{\mathrm{ad}} \otimes_k M \longrightarrow M \ \ ; \ \ a \otimes m \mapsto am$$

is $\mathrm{Dist}(\mathcal{G})$-linear. For future reference we record the following basic facts:

Lemma 2.1.1. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be infinitesimal groups such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Given $\mathcal{G}$-modules M and N, the following statements hold:*

(1) *The canonical operation of* $\mathrm{Dist}(\mathcal{G})$ *on* $\mathrm{Hom}_k(M,N)$ *induces an action of* $\mathrm{Dist}(\mathcal{G}/\mathcal{N})$ *on* $\mathrm{Hom}_{\mathcal{N}}(M,N)$.

(2) *The weight space decomposition*

$$\mathrm{Hom}_{\mathcal{N}}(M,N) = \bigoplus_{\lambda\in X(\mathcal{G}/\mathcal{N})} \mathrm{Hom}_{\mathcal{N}}(M,N)_\lambda$$

endows $\mathrm{Hom}_{\mathcal{N}}(M,N)$ *with the structure of an $X(\mathcal{G}/\mathcal{N})$-graded* $\mathrm{End}_{\mathcal{N}}(N)$*-module.*

(3) *The canonical map $M\otimes_k k_\lambda \xrightarrow{\sim} M$; $m\otimes\alpha \mapsto \alpha m$ induces an isomorphism* $\mathrm{Hom}_{\mathcal{N}}(M,N)_\lambda \cong \mathrm{Hom}_{\mathcal{G}}(M \otimes_k k_\lambda, N)$.

Proof. (1) Since $\mathcal{N}$ is a normal subgroup of $\mathcal{G}$, $\mathrm{Dist}(\mathcal{N})_{\mathrm{ad}}$ is a $\mathrm{Dist}(\mathcal{G})$-submodule of $\mathrm{Dist}(\mathcal{G})_{\mathrm{ad}}$. Let f be an element of $\mathrm{Hom}_{\mathcal{N}}(M,N)$, $h \in \mathrm{Dist}(\mathcal{G})$. Observing the cocommutativity of $\mathrm{Dist}(\mathcal{G})$, we obtain for $x \in \mathrm{Dist}(\mathcal{N})$ and $m \in M$

$$\begin{aligned} x.(h\text{.}f)(m) &= \sum_{(h)} x\varepsilon(h_{(1)})h_{(2)}f(\eta(h_{(3)}).m) = \sum_{(h)} h_{(1)}\eta(h_{(2)})xh_{(3)}f(\eta(h_{(4)}).m) \\ &= \sum_{(h)} h_{(1)}(\eta(h_{(2)})\text{.}x)f(\eta(h_{(3)}).m) = \sum_{(h)} h_{(1)}f((\eta(h_{(2)})\text{.}x)\eta(h_{(3)}).m) \\ &= \sum_{(h)} h_{(1)}f(\eta(h_{(2)})x\varepsilon(h_{(3)}).m) = \sum_{(h)} h_{(1)}f(\eta(h_{(2)})x.m) = (h\text{.}f)(x.m) \end{aligned}$$

Consequently, $h\text{.}f$ is $\mathcal{N}$-linear. Given $h \in \mathrm{Dist}(\mathcal{N})^\dagger$ and $m \in M$, we obtain

$$(h\text{.}f)(m) = \sum_{(h)} h_{(1)}f(\eta(h_{(2)}).m) = \sum_{(h)} h_{(1)}\eta(h_{(2)}).f(m) = \varepsilon(h)f(m) = 0.$$

As a result, the action of $\mathrm{Dist}(\mathcal{G})$ on $\mathrm{Hom}_\mathcal{N}(M,N)$ factors through $\mathrm{Dist}(\mathcal{G}/\mathcal{N})$.

(2) A computation similar to the one of (2) yields

$$h\text{.}(g \circ f) = \sum_{(h)} (h_{(1)}\text{.}g) \circ (h_{(2)}\text{.}f)$$

for $g \in \mathrm{End}_\mathcal{N}(N)$ and $f \in \mathrm{Hom}_\mathcal{N}(M,N)$, whence $\mathrm{End}_\mathcal{N}(M)_\lambda \mathrm{Hom}_\mathcal{N}(M,N)_\mu \subseteq \mathrm{Hom}_\mathcal{N}(M,N)_{\lambda*\mu}$ for every $\lambda, \mu \in X(\mathcal{G}/\mathcal{N})$.

(3) We consider the map $\mathrm{Hom}_\mathcal{N}(M,N)_\lambda \longrightarrow \mathrm{Hom}_\mathcal{G}(M \otimes_k k_\lambda, N)$ sending φ onto

$$\tilde{\varphi} : M \otimes_k k_\lambda \longrightarrow N \ \ ; \ m \otimes \alpha \mapsto \alpha\varphi(m).$$

Let $\varphi \in \mathrm{Hom}_\mathcal{N}(M,N)_\lambda$. Given $h \in \mathrm{Dist}(\mathcal{G})$, we have

$$\begin{aligned} \tilde{\varphi}(h.(m \otimes 1)) &= \sum_{(h)} \lambda(h_{(2)})\varphi(h_{(1)}.m) = \sum_{(h)} (h_{(2)}\text{.}\varphi)(h_{(1)}.m) = \sum_{(h)} h_{(2)}\varphi(\eta(h_{(3)})h_{(1)}.m) \\ &= \sum_{(h)} h_{(1)}\varphi(\eta(h_{(2)})h_{(3)}.m) = \sum_{(h)} h_{(1)}\varphi(\varepsilon(h_{(2)}).m) = h.\varphi(m), \end{aligned}$$

so that $\tilde{\varphi} \in \mathrm{Hom}_\mathcal{G}(M \otimes_k k_\lambda, N)$.

Conversely, if $\psi : M \otimes_k k_\lambda \longrightarrow N$ is $\mathcal{G}$-linear, it follows that $\psi_0 : M \longrightarrow N \ ; \ m \mapsto \psi(m \otimes 1)$ belongs to $\mathrm{Hom}_\mathcal{N}(M,N)_\lambda$ with $\tilde{\psi_0} = \psi$. □

Our next result applies Theorem 1.3 within the context of restriction functors of certain extensions of infinitesimal groups.

Theorem 2.1.2. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be infinitesimal groups such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Then the restriction functor*

$$\mathrm{res}_\mathcal{N} : \mathrm{mod}\,\mathcal{G} \longrightarrow \mathrm{mod}\,\mathcal{N} \ \ ; \ \ M \mapsto M|_\mathcal{N}$$

sends indecomposables to indecomposables.

Proof. Let $M \in \mathrm{mod}\,\mathcal{G}$ be indecomposable and consider the k-algebra

$$R := \mathrm{End}_\mathcal{N}(M).$$

According to Lemma 2.1.1 the weight space decomposition

$$R = \bigoplus_{\lambda \in X(\mathcal{G}/\mathcal{N})} R_\lambda$$

renders R a graded algebra relative to the abelian p-group $X(\mathcal{G}/\mathcal{N})$. Since M is indecomposable, the subalgebra

$$R_0 = \mathrm{End}_{\mathcal{N}}(M)^{\mathcal{G}/\mathcal{N}} = \mathrm{End}_{\mathcal{G}}(M)$$

of $(\mathcal{G}/\mathcal{N})$-invariants is local (cf. (2.1.1(3))), and (1.3) ensures that R inherits this property. Consequently, $M|_{\mathcal{N}}$ is indecomposable. □

In the following, we investigate the image and the fibres of the restriction functor.

Lemma 2.1.3. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup of the infinitesimal group $\mathcal{G}$ such that $\mathcal{G}/\mathcal{N}$ is multiplicative. If M and N are $\mathcal{G}$-modules such that $M|_{\mathcal{N}} \cong N|_{\mathcal{N}}$ is indecomposable, then there exists an element $\lambda \in X(\mathcal{G}/\mathcal{N})$ such that $N \cong M \otimes_k k_\lambda$.*

Proof. Consider the $\mathcal{G}/\mathcal{N}$-module $\mathrm{Hom}_{\mathcal{N}}(M, N) = \mathrm{Hom}_k(M, N)^{\mathcal{N}}$. Thanks to Lemma 2.1.1, this space decomposes into a direct sum of $\mathcal{G}$-modules:

$$\mathrm{Hom}_{\mathcal{N}}(M, N) = \bigoplus_{\lambda \in X(\mathcal{G}/\mathcal{N})} \mathrm{Hom}_{\mathcal{N}}(M, N)_\lambda.$$

By assumption, the $\mathrm{End}_{\mathcal{N}}(N)$-module $\mathrm{Hom}_{\mathcal{N}}(M, N) \cong \mathrm{End}_{\mathcal{N}}(N)$ is local with radical J, say. Thus, $\mathrm{Hom}_{\mathcal{N}}(M, N) \setminus J$ is the set of invertible homomorphisms. Let $\lambda \in X(\mathcal{G}/\mathcal{N})$ be a character such that $\mathrm{Hom}_{\mathcal{N}}(M, N)_\lambda \not\subseteq J$ and pick $f \in \mathrm{Hom}_{\mathcal{N}}(M, N)_\lambda \setminus J$. Then f is invertible, and we have

$$h.f = \lambda(h) f \qquad \forall\, h \in \mathrm{Dist}(\mathcal{G}).$$

According to Lemma 2.1.1 the invertible map

$$\tilde{f} : M \otimes_k k_\lambda \longrightarrow N \ \ ; \ \ m \otimes \alpha \mapsto \alpha f(m)$$

is $\mathrm{Dist}(\mathcal{G})$-linear, as desired. □

Given a $\mathcal{G}$-module M, the stabilizer

$$\mathrm{Stab}_{X(\mathcal{G}/\mathcal{N})}(M) := \{\lambda \in X(\mathcal{G}/\mathcal{N}) \ ; \ M \otimes_k k_\lambda \cong M\}$$

is a subgroup of $X(\mathcal{G}/\mathcal{N})$.

We continue by giving a criterion for the stabilizer of a module to be trivial. We call such a module *regular* (relative to $X(\mathcal{G}/\mathcal{N})$). In [11, Thm.4] it was shown that, for algebras of finite representation type, regularity of all simple modules implies this property to hold for all indecomposables. For $\mathbb{Z}$-graded Artin algebras all indecomposable graded modules are regular (see [28, (4.1)]). In our context, we obtain the following:

Lemma 2.1.4. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup of the infinitesimal group $\mathcal{G}$ with multiplicative factor group $\mathcal{G}/\mathcal{N}$. Then every indecomposable $\mathcal{G}$-module M with $\dim_k M \not\equiv 0 \mod(p)$ is regular.*

Proof. Let $\varrho : \mathrm{Dist}(\mathcal{G}) \longrightarrow \mathrm{End}_k(M)$ be the representation afforded by M. Recall that the Hopf algebra $\mathrm{Dist}(\mathcal{G})$ acts on $\mathrm{End}_k(M)$ via

$$h.f := \sum_{(h)} \varrho(h_{(1)}) \circ f \circ \varrho(\eta(h_{(2)})) \quad \forall\, h \in \mathrm{Dist}(\mathcal{G}),\ f \in \mathrm{End}_k(M).$$

In view of the cocommutativity of $\mathrm{Dist}(\mathcal{G})$, the traces of certain operators can be easily computed:

$$\begin{aligned}\mathrm{tr}(h\boldsymbol{.}f) &= \sum_{(h)} \mathrm{tr}(\varrho(h_{(1)}) \circ f \circ \varrho(\eta(h_{(2)}))) = \sum_{(h)} \mathrm{tr}(\varrho(\eta(h_{(2)})) \circ \varrho(h_{(1)}) \circ f) \\ &= \mathrm{tr}(\varrho(\sum_{(h)} \eta(h_{(2)})h_{(1)}) \circ f) = \mathrm{tr}(\varrho(\varepsilon(h)1) \circ f) = \varepsilon(h)\,\mathrm{tr}(f).\end{aligned}$$

Now suppose that $\lambda \in X(\mathcal{G}/\mathcal{N})$ is an element such that $M \otimes_k k_\lambda \xrightarrow{\sim} M$. Lemma 2.1.1 furnishes an invertible map $f \in \mathrm{End}_{\mathcal{N}}(M)_\lambda$. Thanks to (2.1.2), the ring $\mathrm{End}_{\mathcal{N}}(M)$ is local. As f is invertible, there exist $n \in \mathbb{N}$ and $\alpha \in k^\times$ such that $f^{p^n} = \alpha\,\mathrm{id}_M$. Thus,

$$\mathrm{tr}(f)^{p^n} = \mathrm{tr}(f^{p^n}) = \alpha \dim_k M,$$

so that the condition $\dim_k M \not\equiv 0 \mod(p)$ yields $\mathrm{tr}(f) \neq 0$. Since

$$\lambda(h)\,\mathrm{tr}(f) = \mathrm{tr}(h\boldsymbol{.}f) = \varepsilon(h)\,\mathrm{tr}(f)$$

for every $h \in \mathrm{Dist}(\mathcal{G})$, we obtain $\lambda = \varepsilon$. Consequently, the $\mathcal{G}$-module M is regular. □

Remark. If the dimension of the $\mathcal{G}$-module M is divisible by p, then the conclusion of (2.1.4) may or may not hold. Consider the group $\mathcal{G} = \mathrm{SL}(2)_1T_r$ along with its normal subgroup $\mathcal{N} = \mathrm{SL}(2)_1$. In view of [20, (5.1)], the Steinberg module $L(p-1)$ is a p-dimensional simple $\mathcal{G}$-module, which is regular. On the other hand, as we shall see in (4.2.4) below, the $X(\mathcal{G}/\mathcal{N})$-orbits of the indecomposable $\mathcal{G}$-modules belonging to homogeneous tubes of the stable Auslander-Reiten quiver of $\mathrm{Dist}(\mathcal{G})$ are singletons.

Turning to irregular modules, we denote by $U(R)$ and $U_{\mathrm{gr}}(R)$ the groups of units and graded units of a group-graded k-algebra $R = \bigoplus_{g\in G} R_g$, respectively. As noted in [10, (5.3)], the degree homomorphism $\deg : U_{\mathrm{gr}}(R) \longrightarrow G$ induces a sequence

$$1 \longrightarrow U(R_1) \longrightarrow U_{\mathrm{gr}}(R) \longrightarrow G \longrightarrow 1$$

of group homomorphisms which is exact at the two left-hand terms. In what follows we shall use [10, 37] as general references and recall that $R = R_1 *_\gamma G$ is a *crossed product* if the above sequence is exact. In view of [37, (2.1.3)], crossed products correspond to crossed systems involving a "2-cocycle" $\gamma : G \times G \longrightarrow U(R_1)$.

Lemma 2.1.5. *Let M be a $\mathcal{G}$-module satisfying $M \otimes_k k_\lambda \cong M$ for all $\lambda \in X(\mathcal{G}/\mathcal{N})$. Then there exists a 2-cocycle $\gamma : X(\mathcal{G}/\mathcal{N}) \times X(\mathcal{G}/\mathcal{N}) \longrightarrow U(\mathrm{End}_{\mathcal{G}}(M))$ such that*

$$\mathrm{End}_{\mathcal{N}}(M) \cong \mathrm{End}_{\mathcal{G}}(M) *_\gamma X(\mathcal{G}/\mathcal{N})$$

is a crossed product.

Proof. By Lemma 2.1.1 the endomorphism ring $\mathrm{End}_{\mathcal{N}}(M)$ is $X(\mathcal{G}/\mathcal{N})$-graded, with $\mathrm{End}_{\mathcal{N}}(M)_\lambda$ corresponding to the space of $\mathcal{G}$-linear maps $M \otimes_k k_\lambda \longrightarrow M$. Thus, the condition $M \otimes_k k_\lambda \cong M$ for all $\lambda \in X(\mathcal{G}/\mathcal{N})$ implies that each homogeneous component $\mathrm{End}_{\mathcal{N}}(M)_\lambda$ contains an invertible element. Observing $\mathrm{End}_{\mathcal{N}}(M)_0 = \mathrm{End}_{\mathcal{G}}(M)$, we thus obtain the exactness of the sequence

$$(*) \qquad 1 \longrightarrow U(\mathrm{End}_{\mathcal{G}}(M)) \longrightarrow U_{\mathrm{gr}}(\mathrm{End}_{\mathcal{N}}(M)) \xrightarrow{\deg} X(\mathcal{G}/\mathcal{N}) \longrightarrow 1.$$

Our assertion is now a direct consequence of [37, (2.1.3)]. □

Given an $\mathcal{N}$-module N, we let

$$\mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} N := \mathrm{Dist}(\mathcal{G}) \otimes_{\mathrm{Dist}(\mathcal{N})} N$$

be the $\mathcal{G}$-module induced by N. The tensor identity (cf. [36, (I.3.6)]) shows that induced modules $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} N$ satisfy the technical condition of the foregoing result. For such a module, the map

$$\Gamma : X(\mathcal{G}/\mathcal{N}) \longrightarrow U_{\mathrm{gr}}(\mathrm{End}_{\mathcal{N}}(\mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} N)) \ \ ; \ \ \Gamma(\lambda)(a \otimes n) = \lambda.a \otimes n$$

provides a splitting for the sequence $(*)$, so that $\mathrm{End}_{\mathcal{N}}(\mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} N) \cong \mathrm{End}_{\mathcal{G}}(\mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} N) * X(\mathcal{G}/\mathcal{N})$ is actually a skew group algebra (cf. [37, (2.1.5)]). We shall see below that under additional hypotheses this property characterizes induced modules.

2.2. **Split extensions.** We turn to the case where $\mathcal{G} = \mathcal{N}\mathcal{M}$ is a product of a normal subgroup $\mathcal{N}$ and a multiplicative subgroup $\mathcal{M}$. In this particular situation, which will be seen to be sufficiently general for our purposes, we can identify the image of the restriction functor.

Since $\mathcal{N}$ is normal in $\mathcal{G}$, $\mathrm{Dist}(\mathcal{N})$ is a submodule of $\mathrm{Dist}(\mathcal{G})_{\mathrm{ad}}$. Thus, the restriction of the adjoint representation to $\mathrm{Dist}(\mathcal{M})$ endows $\mathrm{Dist}(\mathcal{N})$ with the structure of an $X(\mathcal{M})$-graded algebra. We denote by $\mathrm{mod}_{X(\mathcal{M})}^{\mathcal{M}\cap\mathcal{N}} \mathcal{N}$ the category of those $X(\mathcal{M})$-graded $\mathcal{N}$-modules $M = \bigoplus_{\lambda \in X(\mathcal{M})} M_\lambda$ and $\mathcal{N}$-linear maps of degree 0 satisfying

$$h.m = \lambda(h)m \qquad \forall\ h \in \mathrm{Dist}(\mathcal{M} \cap \mathcal{N}),\ m \in M_\lambda.$$

Our definition is motivated by Jantzen's category of u_n-T-modules, see [33].

We also consider the full subcategory $\mathrm{mod}_{\mathrm{gr}} \mathcal{N} \subseteq \mathrm{mod}\, \mathcal{N}$ of $X(\mathcal{M})$-*gradable* $\mathcal{N}$-modules. By definition, its objects are the images of the $X(\mathcal{M})$-graded $\mathcal{N}$-modules under the forgetful functor. The following Lemma shows in particular that the category $\mathrm{mod}_{X(\mathcal{M})} \mathcal{N}$ of $X(\mathcal{M})$-graded $\mathcal{N}$-modules coincides with $\mathrm{mod}(\mathcal{N} \rtimes \mathcal{M})$. In this interpretation, the usual degree shift of a module by some element $\lambda \in X(\mathcal{M})$ is readily seen to correspond to tensoring M with the one-dimensional $(\mathcal{N} \rtimes \mathcal{M})$-module k_λ, on which $\mathcal{N}$ acts trivially.

As before, we denote by $\mathrm{res}_{\mathcal{N}} : \mathrm{mod}\, \mathcal{G} \longrightarrow \mathrm{mod}\, \mathcal{N}$ the canonical restriction functor.

Lemma 2.2.1. *Let $\mathcal{G} = \mathcal{N}\mathcal{M}$ be a product of a normal subgroup $\mathcal{N}$ and a multiplicative subgroup $\mathcal{M}$. Then the following statements hold:*

(1) *We have $\mathrm{Im}(\mathrm{res}_{\mathcal{N}}) \cong \mathrm{mod}_{X(\mathcal{M})}^{\mathcal{M}\cap\mathcal{N}} \mathcal{N}$.*

(2) *Every indecomposable $X(\mathcal{M})$-gradable $\mathcal{N}$-module M is isomorphic to a restriction of a $\mathcal{G}$-module.*

Proof. (1) Let M be a $\mathcal{G}$-module. Since

$$\mathrm{Dist}(\mathcal{G})_{\mathrm{ad}} \otimes_k M \longrightarrow M \ \ ; \ \ a \otimes m \mapsto am$$

is $\mathrm{Dist}(\mathcal{M})$-linear, we see that $M|_{\mathcal{N}}$ is an $X(\mathcal{M})$-graded $\mathrm{Dist}(\mathcal{N})$-module which satisfies the compatibility condition

$$h.m = \lambda(h)m \qquad \forall\ h \in \mathrm{Dist}(\mathcal{M} \cap \mathcal{N}),\ m \in M_\lambda.$$

Moreover, any $\mathcal{G}$-linear map $\varphi : M \longrightarrow N$ respects the weight space decompositions and is thus $\mathcal{N}$-linear and of degree 0. Conversely, any such map is $\mathcal{G}$-linear.

Now let M be an object of $\mathrm{mod}_{X(\mathcal{M})} \mathcal{N}$, and write $M = \bigoplus_{\lambda \in X(\mathcal{M})} M_\lambda$. Then $\mathrm{Dist}(\mathcal{M})$ acts on M via

$$h.(\sum_{\lambda \in X(\mathcal{M})} m_\lambda) := \sum_{\lambda \in X(\mathcal{M})} \lambda(h) m_\lambda \qquad \forall\ h \in \mathrm{Dist}(\mathcal{M}).$$

We consider the smash product $\mathrm{Dist}(\mathcal{N}) \sharp \mathrm{Dist}(\mathcal{M})$ of the $\mathrm{Dist}(\mathcal{M})$-module algebra $\mathrm{Dist}(\mathcal{N})_{\mathrm{ad}}$ with $\mathrm{Dist}(\mathcal{M})$ (cf. [38, Chap.4]). Direct computation shows that

$$(u \sharp h).m := u(h.m) \qquad \forall\ u \in \mathrm{Dist}(\mathcal{N}),\ h \in \mathrm{Dist}(\mathcal{M}),\ m \in M$$

endows M with the structure of a $(\mathrm{Dist}(\mathcal{N})\sharp\,\mathrm{Dist}(\mathcal{M}))$-module. The quotient map $\mathcal{N} \rtimes \mathcal{M} \longrightarrow \mathcal{G}$ induces a surjective homomorphism

$$\mu : \mathrm{Dist}(\mathcal{N})\sharp\,\mathrm{Dist}(\mathcal{M}) \longrightarrow \mathrm{Dist}(\mathcal{G}) \quad ; \quad u\sharp h \mapsto uh,$$

whose kernel is generated by the augmentation ideal of the Hopf algebra $\mathrm{Dist}(\mathcal{N} \cap \mathcal{M})$. The latter is embedded into the smash product via

$$f : \mathrm{Dist}(\mathcal{N} \cap \mathcal{M}) \longrightarrow \mathrm{Dist}(\mathcal{N})\sharp\,\mathrm{Dist}(\mathcal{M}) \quad ; \quad a \mapsto \sum_{(a)} \eta(a_{(1)})\sharp a_{(2)}.$$

Let $m \in M_\lambda$. The compatibility condition yields

$$f(a)m = \sum_{(a)} \lambda(\eta(a_{(1)}))\lambda(a_{(2)})m = ((\lambda \circ \eta) * \lambda)(a)m = \varepsilon(a)m$$

for every $a \in \mathrm{Dist}(\mathcal{N}\cap\mathcal{M})$. Consequently, $\ker \mu$ is contained in the annihilator of M, and M obtains the structure of a $\mathrm{Dist}(\mathcal{G})$-module. By construction, this structure extends the given $\mathrm{Dist}(\mathcal{N})$-structure of M.

(2) Consider the exact sequence

$$e_k \longrightarrow \mathcal{N} \cap \mathcal{M} \stackrel{\iota}{\longrightarrow} \mathcal{N} \rtimes \mathcal{M} \longrightarrow \mathcal{G} \longrightarrow e_k$$

of infinitesimal group schemes, where ι is given by $g \mapsto (g^{-1}, g)$. By part (1), the $X(\mathcal{M})$-graded $\mathcal{N}$-modules are precisely the $(\mathcal{N} \rtimes \mathcal{M})$-modules.

Let M be an indecomposable $(\mathcal{N} \rtimes \mathcal{M})$-module. Since $(\mathcal{N} \cap \mathcal{M}) \trianglelefteq (\mathcal{N} \rtimes \mathcal{M})$ is a multiplicative, normal subgroup, rigidity of tori (cf. [48, (7.7)]) shows that it lies centrally in $\mathcal{N} \rtimes \mathcal{M}$. Hence it acts on the indecomposable $(\mathcal{N} \rtimes \mathcal{M})$-module M via a single character $\gamma \in X(\mathcal{N} \cap \mathcal{M})$. Since the composite $\mathrm{pr} \circ \iota$ of the canonical projection $\mathrm{pr} : \mathcal{N} \rtimes \mathcal{M} \longrightarrow \mathcal{M}$ with ι is the canonical inclusion $\mathcal{N} \cap \mathcal{M} \hookrightarrow \mathcal{M}$, and $X(\mathcal{M}) \longrightarrow X(\mathcal{N} \cap \mathcal{M})$ is surjective (see [48, (2.2)]), there exists a character $\lambda \in X(\mathcal{N} \rtimes \mathcal{M})$ with $\mathcal{N} \subseteq \ker \lambda$ and

$$\gamma(g) = \lambda(g^{-1}, g) \quad \forall\, g \in \mathcal{G}.$$

Accordingly, $N := M \otimes_k k_{-\lambda}$ is an indecomposable $(\mathcal{N} \rtimes \mathcal{M})$-module on which $\mathcal{N} \cap \mathcal{M}$ operates trivially. Thus, N is a $\mathcal{G}$-module with $\mathrm{res}_{\mathcal{N}}(N) \cong M$. □

In view of (2.1.4), indecomposable modules of dimension prime to p have trivial stabilizers. In our present context, the following result provides a stronger statement.

In the sequel, we will employ the standard identification of characters $X(\mathcal{G})$ with homomorphisms $\mathcal{G} \longrightarrow G_m$ from $\mathcal{G}$ into the multiplicative group scheme $G_m := \mathrm{Spec}_k(k[T]_T)$: An element of $X(\mathcal{G})$ corresponds to a group-like element of the representing Hopf algebra $k[\mathcal{G}]$. Such an element in turn determines a homomorphism $k[T]_T \longrightarrow k[\mathcal{G}]$ of Hopf algebras, which gives rise to the desired homomorphism (cf. [48, (2.1)] for more details). In view of the anti-equivalence between multiplicative groups and abelian groups (cf. [48, (2.2)]), a subgroup $G \subseteq X(\mathcal{M})$ of the character group associated to a multiplicative group $\mathcal{M}$ determines a normal subgroup $\mathcal{M}_G := \bigcap_{\lambda \in G} \ker \lambda$ of $\mathcal{M}$ such that $X(\mathcal{M}/\mathcal{M}_G) = G$.

Theorem 2.2.2. *Let $\mathcal{G} = \mathcal{N}\mathcal{M}$ be a product of a normal subgroup $\mathcal{N}$ and a multiplicative subgroup $\mathcal{M}$. If M is a $\mathcal{G}$-module, then*

$$\dim_k M \equiv 0 \qquad \mathrm{mod}(|\mathrm{Stab}_{X(\mathcal{G}/\mathcal{N})}(M)|).$$

Proof. We proceed in several steps and begin by considering the case where $\mathrm{Stab}_{X(\mathcal{G}/\mathcal{N})}(M) = X(\mathcal{G}/\mathcal{N})$. The canonical inclusion $\mathcal{M} \subseteq \mathcal{G}$ induces an isomorphism $\mathcal{M}/(\mathcal{M}\cap\mathcal{N}) \cong \mathcal{G}/\mathcal{N}$ (cf. [12, (III,§3,3.7c)]), so that $X(\mathcal{G}/\mathcal{N}) \cong X(\mathcal{M}/(\mathcal{M}\cap\mathcal{N})) \subseteq X(\mathcal{M})$. Since $\mathcal{M}$ is multiplicative, the $\mathcal{G}$-module M decomposes into its weight spaces $M = \bigoplus_{\lambda\in X(\mathcal{M})} M_\lambda$. Using the above identification, the $\mathrm{End}_{\mathcal{N}}(M)$-module M satisfies

$$\mathrm{End}_{\mathcal{N}}(M)_\lambda . M_\mu \subseteq M_{\lambda*\mu} \qquad \forall\ \lambda \in X(\mathcal{G}/\mathcal{N}),\ \mu \in X(\mathcal{M}).$$

We decompose $X(\mathcal{M}) = \bigcup_{i=1}^{\ell} X(\mathcal{G}/\mathcal{N})*\mu_i$ into its right cosets relative to $X(\mathcal{G}/\mathcal{N})$ and set $M_{[i]} := \sum_{\lambda\in X(\mathcal{G}/\mathcal{N})} M_{\lambda*\mu_i}$. Then each $M_{[i]}$ is a $X(\mathcal{G}/\mathcal{N})$-graded $\mathrm{End}_{\mathcal{N}}(M)$-module, and $M = \bigoplus_{i=1}^{\ell} M_{[i]}$. According to (2.1.1(2)) the algebra $\mathrm{End}_{\mathcal{N}}(M)$ is $X(\mathcal{G}/\mathcal{N})$-graded, while (2.1.1(3)) in conjunction with $\mathrm{Stab}_{X(\mathcal{G}/\mathcal{N})}(M) = X(\mathcal{G}/\mathcal{N})$ ensures that it is strongly $X(\mathcal{G}/\mathcal{N})$-graded. Thanks to [10, (2.8)], there thus exist $\mathrm{End}_{\mathcal{G}}(M)$-modules $V_{[i]}$ such that

$$M_{[i]} \cong \mathrm{End}_{\mathcal{N}}(M) \otimes_{\mathrm{End}_{\mathcal{G}}(M)} V_{[i]}.$$

Owing to (2.1.5), the algebra $\mathrm{End}_{\mathcal{N}}(M) \cong \mathrm{End}_{\mathcal{G}}(M)*_\gamma X(\mathcal{G}/\mathcal{N})$ is a free right $\mathrm{End}_{\mathcal{G}}(M)$-module of rank $|X(\mathcal{G}/\mathcal{N})|$ (cf. [37, (2.1.3)]), whence

$$\dim_k M = \sum_{i=1}^{\ell} \dim_k \mathrm{End}_{\mathcal{N}}(M) \otimes_{\mathrm{End}_{\mathcal{G}}(M)} V_{[i]} \equiv 0 \quad \mod(|X(\mathcal{G}/\mathcal{N})|).$$

In the general case, we note that $\mathcal{H}_M := \bigcap_{\lambda\in \mathrm{Stab}_{X(\mathcal{G}/\mathcal{N})}(M)} \ker\lambda$ is a normal subgroup of $\mathcal{G}/\mathcal{N}$. Thus, the inverse image $\mathcal{N}_M$ of $\mathcal{H}_M$ under the canonical quotient map $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$ is a normal subgroup of $\mathcal{G}$ containing $\mathcal{N}$ such that $\mathcal{G}/\mathcal{N}_M \cong (\mathcal{G}/\mathcal{N})/\mathcal{H}_M$. Consequently, $\mathcal{G} = \mathcal{N}_M\mathcal{M}$ and the factor group $\mathcal{G}/\mathcal{N}_M \cong \mathcal{M}/(\mathcal{N}_M\cap\mathcal{M})$ is multiplicative with character group $X(\mathcal{G}/\mathcal{N}_M) = X((\mathcal{G}/\mathcal{N})/\mathcal{H}_M) = \mathrm{Stab}_{X(\mathcal{G}/\mathcal{N})}(M)$. In particular, $M \otimes_k k_\lambda \cong M$ for all $\lambda \in X(\mathcal{G}/\mathcal{N}_M)$, so that the first part of the proof yields the assertion. □

2.3. **Galois extensions.** Let Λ be an associative k-algebra, G be a finite group that operates on Λ via automorphisms. We denote by $\Lambda*G$ and Λ^G the skew group algebra and the algebra of *G-invariants of G*, respectively. Following [3], we say that $\Lambda\!:\!\Lambda^G$ is a *Galois extension* (with Galois group G) if

(a) Λ is a projective $\Lambda*G$-generator, and
(b) for every simple Λ-module S, the restriction $S|_{\Lambda^G}$ is semi-simple.

We shall see below that the algebras of distributions associated to extensions $\mathcal{N} \trianglelefteq \mathcal{G}$ of infinitesimal groups with multiplicative factor groups $\mathcal{G}/\mathcal{N}$ frequently give rise to Galois extensions. One class of examples is given by trigonalizable group schemes and their unipotent radicals. The following example illustrates that not all extensions of the above type are Galois.

By general theory [12, (II,§7,4.4)], the infinitesimal groups of height ≤ 1 correspond to restricted Lie algebras. In particular, the algebras of distributions of these groups are just the restricted enveloping algebras of their Lie algebras. Given a restricted Lie algebra $(\mathfrak{g},[p])$, we denote the *restricted enveloping algebra* of $(\mathfrak{g},[p])$ by $U_0(\mathfrak{g})$. By definition, $U_0(\mathfrak{g})$ is the factor algebra

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/I_0$$

of the ordinary enveloping algebra $U(\mathfrak{g})$ by the ideal I_0 that is generated by the central elements $\{x^p - x^{[p]}\ ;\ x \in \mathfrak{g}\}$ of $U(\mathfrak{g})$. The reader may consult [45] for more details on restricted Lie algebras and their enveloping algebras.

Example. Consider the restricted Lie algebra $\mathfrak{g} := kt \oplus kx \oplus kz$, whose bracket and p-map are given by

$$[t, x] = x \ , \ [t, z] = 0 \ , \ [x, z] = 0 \ \ ; \ \ t^{[p]} = t \ , \ x^{[p]} = z \ , \ z^{[p]} = z.$$

We consider the abelian p-subalgebra $\mathfrak{n} := kx \oplus kz$ as well as the character $\lambda : U_0(\mathfrak{n}) \longrightarrow k$ satisfying $\lambda(x) = 1 = \lambda(z)$. Let ϱ be the representation afforded by the p-dimensional $U_0(\mathfrak{g})$-module

$$S := U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{n})} k_\lambda.$$

In view of $z.(a \otimes 1) = az \otimes 1 = a \otimes 1$ for all $a \in U_0(\mathfrak{g})$, the central element $z \in U_0(\mathfrak{g})$ acts on S via the identity. Since

$$\operatorname{tr}(\varrho(z)) = \operatorname{tr}(\varrho(x)^p) = \operatorname{tr}(\varrho(x))^p = \operatorname{tr}(\varrho([t, x]))^p = 0,$$

the $U_0(\mathfrak{g})$-module S is simple.

The element $x - z$ belongs to $\operatorname{Rad}(U_0(\mathfrak{n}))$ and thus annihilates every semi-simple $U_0(\mathfrak{n})$-module. Since

$$(x - z).(t \otimes 1) = [x - z, t] \otimes 1 + t \otimes (x - z).1 = [x, t] \otimes 1 = -x \otimes 1 = -1 \otimes 1 \neq 0,$$

the module $S|_{\mathfrak{n}}$ is not semi-simple. Accordingly, $U_0(\mathfrak{g}) : U_0(\mathfrak{n})$ is not a Galois extension.

We have seen in (2.1.2) that the canonical restriction functor $\operatorname{res}_{\mathcal{N}} : \operatorname{mod} \mathcal{G} \longrightarrow \operatorname{mod} \mathcal{N}$ preserves indecomposables. We shall now investigate extensions, where this functor also sends simples to simples. According to [23, (3.2)] and [3, (5.1)], this condition ensures that $\operatorname{Dist}(\mathcal{G}) : \operatorname{Dist}(\mathcal{N})$ is a Galois extension with Galois group $X(\mathcal{G}/\mathcal{N})$.

Lemma 2.3.1. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup of the infinitesimal group $\mathcal{G}$ such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Suppose that the restriction $S|_{\mathcal{N}}$ of every simple $\mathcal{G}$-module S is simple. If T is a simple $\mathcal{N}$-module, then there exists a simple $\mathcal{G}$-module S such that*

$$\operatorname{Dist}(\mathcal{G}) \otimes_{\operatorname{Dist}(\mathcal{N})} T \cong \bigoplus_{\lambda \in X(\mathcal{G}/\mathcal{N})} S \otimes_k k_\lambda \cong S \otimes_k \operatorname{Dist}(\mathcal{G}/\mathcal{N}).$$

In particular, $S|_{\mathcal{N}} = T$.

Proof. Since $\operatorname{Dist}(\mathcal{G}) : \operatorname{Dist}(\mathcal{N})$ is a Galois extension with Galois group $X(\mathcal{G}/\mathcal{N})$, [3, (1.2)] ensures that the functor

$$\operatorname{Dist}(\mathcal{G}) \otimes_{\operatorname{Dist}(\mathcal{N})} - : \operatorname{mod} \operatorname{Dist}(\mathcal{N}) \longrightarrow \operatorname{mod} \operatorname{Dist}(\mathcal{G}) * X(\mathcal{G}/\mathcal{N})$$

is an equivalence, so that $V := \operatorname{Dist}(\mathcal{G}) \otimes_{\operatorname{Dist}(\mathcal{N})} T$ is a simple $\operatorname{Dist}(\mathcal{G}) * X(\mathcal{G}/\mathcal{N})$-module. Let $S \subseteq V$ be a simple $\mathcal{G}$-submodule. Clifford theory yields $V|_{\mathcal{G}} = \sum_{\lambda \in X(\mathcal{G}/\mathcal{N})} \lambda.S$, and since $X(\mathcal{G}/\mathcal{N})$ acts freely on the set of simple $\mathcal{G}$-modules (cf. [23, (2.2)]), the summands are pairwise non-isomorphic. As a result, the sum is direct and $V|_{\mathcal{G}} \cong \bigoplus_{\lambda \in X(\mathcal{G}/\mathcal{N})} S \otimes_k k_\lambda$. Since the $\mathcal{G}$-module $\operatorname{Dist}(\mathcal{G}/\mathcal{N})$ is isomorphic to $\bigoplus_{\lambda \in X(\mathcal{G}/\mathcal{N})} k_\lambda$, the right-hand isomorphism follows. By assumption, the module $V|_{\mathcal{N}} \cong |X(\mathcal{G}/\mathcal{N})| S|_{\mathcal{N}}$ is semi-simple with $S|_{\mathcal{N}}$ being the sole constituent. The canonical embedding $T \hookrightarrow V|_{\mathcal{N}}$ thus induces an isomorphism $T \cong S|_{\mathcal{N}}$. □

In the following, we denote by $\ell\ell_{\mathcal{G}}(M)$ the *Loewy length* of the $\mathcal{G}$-module M. Recall that $\ell\ell_{\mathcal{G}}(M)$ is the length of the *socle series* $(\operatorname{Soc}^n_{\mathcal{G}}(M))_{n \geq 1}$ of M (cf. [2, (V.1.3)]).

Proposition 2.3.2. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup of the infinitesimal group $\mathcal{G}$ such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Suppose that the restriction $S|_{\mathcal{N}}$ of every simple $\mathcal{G}$-module S is simple. Given a $\mathcal{G}$-module M, the canonical inclusions $\operatorname{Dist}(\mathcal{G}) \otimes_{\operatorname{Dist}(\mathcal{N})} \operatorname{Soc}^n_{\mathcal{N}}(M) \hookrightarrow \operatorname{Dist}(\mathcal{G}) \otimes_{\operatorname{Dist}(\mathcal{N})} M$ induce isomorphisms*

$$\operatorname{Dist}(\mathcal{G}) \otimes_{\operatorname{Dist}(\mathcal{N})} \operatorname{Soc}^n_{\mathcal{N}}(M) \xrightarrow{\sim} \operatorname{Soc}^n_{\mathcal{G}}(\operatorname{Dist}(\mathcal{G}) \otimes_{\operatorname{Dist}(\mathcal{N})} M) \qquad \forall\, n \geq 1.$$

In particular, we have $\ell\ell_{\mathcal{G}}(\operatorname{Dist}(\mathcal{G}) \otimes_{\operatorname{Dist}(\mathcal{N})} M) = \ell\ell_{\mathcal{N}}(M)$.

Proof. We denote the induction functor $\operatorname{Dist}(\mathcal{G}) \otimes_{\operatorname{Dist}(\mathcal{N})} -$ by $\operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} -$ and put $V := \operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} M$. Thanks to (2.3.1) the induced module $\operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} \operatorname{Soc}_{\mathcal{N}}(M)$ is semi-simple, so that the canonical inclusion yields an embedding

$$\operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} \operatorname{Soc}_{\mathcal{N}}(M) \hookrightarrow \operatorname{Soc}_{\mathcal{G}}(V).$$

Since $\operatorname{Soc}_{\mathcal{G}}(V)$ is an $X(\mathcal{G}/\mathcal{N})$-submodule of V, [3, (1.2)] provides an isomorphism $\operatorname{Soc}_{\mathcal{G}}(V) \cong \operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} \operatorname{Soc}_{\mathcal{G}}(V)^{X(\mathcal{G}/\mathcal{N})}$. By the same token, we have $\operatorname{Soc}_{\mathcal{G}}(V)^{X(\mathcal{G}/\mathcal{N})} \subseteq V^{X(\mathcal{G}/\mathcal{N})} \cong M|_{\mathcal{N}}$. By assumption, $\operatorname{Soc}_{\mathcal{G}}(V)|_{\mathcal{N}}$ is semi-simple, so that $\operatorname{Soc}_{\mathcal{G}}(V)^{X(\mathcal{G}/\mathcal{N})} \subseteq \operatorname{Soc}_{\mathcal{G}}(V)|_{\mathcal{N}}$ is a semi-simple submodule of $M|_{\mathcal{N}}$. Consequently, $\dim_k \operatorname{Soc}_{\mathcal{G}}(V) \leq \dim_k \operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} \operatorname{Soc}_{\mathcal{N}}(M)$, and the above embedding is an isomorphism.

Setting $V_n := \operatorname{Soc}^n_{\mathcal{G}}(\operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} M)$ as well as $M_n := \operatorname{Soc}^n_{\mathcal{N}}(M)$, we now prove our statement by induction. By the above, the canonical inclusion $\operatorname{Soc}_{\mathcal{N}}(M/M_{n-1}) \hookrightarrow M/M_{n-1}$ induces an isomorphism

$$\operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} \operatorname{Soc}_{\mathcal{N}}(M/M_{n-1}) \xrightarrow{\sim} \operatorname{Soc}_{\mathcal{G}}(\operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}}(M/M_{n-1})).$$

Accordingly, the image of the canonical inclusion $\operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} M_n \hookrightarrow V$ is contained in V_n, and we obtain a commutative diagram

$$\begin{array}{ccccccccc}
(0) & \longrightarrow & \operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} M_{n-1} & \longrightarrow & \operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} M_n & \longrightarrow & \operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} \operatorname{Soc}_{\mathcal{N}}(M/M_{n-1}) & \longrightarrow & (0) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(0) & \longrightarrow & V_{n-1} & \longrightarrow & V_n & \longrightarrow & \operatorname{Soc}_{\mathcal{G}}(V/V_{n-1}) & \longrightarrow & (0)
\end{array}$$

with exact rows (cf. [36, (I.8.16)]) and first and third vertical arrows being isomorphisms. Hence the middle arrow is also bijective, as desired.

In particular, V and M have the same Loewy length. □

Corollary 2.3.3. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup of the infinitesimal group $\mathcal{G}$ such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Suppose that the restriction $S|_{\mathcal{N}}$ of every simple $\mathcal{G}$-module S is simple. Then we have*

$$\operatorname{Soc}_{\mathcal{G}}(\operatorname{Dist}(\mathcal{G})) = \operatorname{Dist}(\mathcal{G}) \operatorname{Soc}_{\mathcal{N}}(\operatorname{Dist}(\mathcal{N})) = \operatorname{Soc}_{\mathcal{N}}(\operatorname{Dist}(\mathcal{N})) \operatorname{Dist}(\mathcal{G}).$$

Proof. Adopting our earlier notation, we consider the canonical isomorphism

$$\mu : \operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} \operatorname{Dist}(\mathcal{N}) \longrightarrow \operatorname{Dist}(\mathcal{G}) \;\; ; \;\; a \otimes b \mapsto ab.$$

According to (2.3.2) we have

$$\operatorname{Soc}_{\mathcal{G}}(\operatorname{Dist}(\mathcal{G})) = \mu(\operatorname{Soc}_{\mathcal{G}}(\operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} \operatorname{Dist}(\mathcal{N}))) = \mu(\operatorname{Ind}^{\mathcal{G}}_{\mathcal{N}} \operatorname{Soc}_{\mathcal{N}}(\operatorname{Dist}(\mathcal{N}))) = \operatorname{Dist}(\mathcal{G}) \operatorname{Soc}_{\mathcal{N}}(\operatorname{Dist}(\mathcal{N})).$$

Since $\operatorname{Dist}(\mathcal{G})$ and $\operatorname{Dist}(\mathcal{N})$ are self-injective algebras, their socles are also the sums of the minimal right ideals (cf. [9, (58.12)]). Hence both spaces are invariant under the antipode $\eta : \operatorname{Dist}(\mathcal{G}) \longrightarrow \operatorname{Dist}(\mathcal{G})$. As a result, we obtain

$$\begin{aligned}
\operatorname{Soc}_{\mathcal{G}}(\operatorname{Dist}(\mathcal{G})) &= \eta(\operatorname{Soc}_{\mathcal{G}}(\operatorname{Dist}(\mathcal{G}))) = \eta(\operatorname{Dist}(\mathcal{G}) \operatorname{Soc}_{\mathcal{N}}(\operatorname{Dist}(\mathcal{N}))) = \eta(\operatorname{Soc}_{\mathcal{N}}(\operatorname{Dist}(\mathcal{N})))\eta(\operatorname{Dist}(\mathcal{G})) \\
&= \operatorname{Soc}_{\mathcal{N}}(\operatorname{Dist}(\mathcal{N})) \operatorname{Dist}(\mathcal{G}),
\end{aligned}$$

as desired. □

We let $\Omega_{\mathcal{G}}$ be the Heller operator of $\operatorname{mod}\mathcal{G}$. By definition, the modules $\Omega^i_{\mathcal{G}}(M)$ are the syzygies of a minimal projective resolution of M, see [4, 5]. Our next result shows that the Heller operator commutes with the induction functor.

Corollary 2.3.4. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup of the infinitesimal group $\mathcal{G}$ such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Suppose that the restriction $S|_{\mathcal{N}}$ of every simple $\mathcal{G}$-module S is simple. If $M \in \operatorname{mod}\mathcal{N}$ is a non-projective, indecomposable $\mathcal{N}$-module, then we have*

$$\Omega^n_{\mathcal{G}}(\mathrm{Dist}(\mathcal{G}) \otimes_{\mathrm{Dist}(\mathcal{N})} M) \cong \mathrm{Dist}(\mathcal{G}) \otimes_{\mathrm{Dist}(\mathcal{N})} \Omega^n_{\mathcal{N}}(M)$$

for every $n \geq 1$.

Proof. We begin by showing that $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{N}} M$ has no non-zero projective direct summands. Identifying M with the $\mathcal{N}$-submodule $1 \otimes M \subseteq \mathrm{Ind}^{\mathcal{G}}_{\mathcal{N}} M$, we have

$$\mathrm{Ind}^{\mathcal{G}}_{\mathcal{N}} M = \mathrm{Dist}(\mathcal{G})M.$$

Thus, (2.3.3) implies

$$\mathrm{Soc}_{\mathcal{G}}(\mathrm{Dist}(\mathcal{G}))\,\mathrm{Ind}^{\mathcal{G}}_{\mathcal{N}} M = \mathrm{Soc}_{\mathcal{N}}(\mathrm{Dist}(\mathcal{N}))\,\mathrm{Dist}(\mathcal{G})M = \mathrm{Dist}(\mathcal{G})\,\mathrm{Soc}_{\mathcal{N}}(\mathrm{Dist}(\mathcal{N}))M.$$

As M is non-projective and indecomposable, the space $\mathrm{Soc}_{\mathcal{N}}(\mathrm{Dist}(\mathcal{N}))M$ vanishes. Consequently, $\mathrm{Soc}_{\mathcal{G}}(\mathrm{Dist}(\mathcal{G}))\,\mathrm{Ind}^{\mathcal{G}}_{\mathcal{N}} M = (0)$, proving that $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{N}} M$ does not contain any non-zero projective summands.

By general theory, we have

$$\Omega^n_{\mathcal{G}}(\mathrm{Ind}^{\mathcal{G}}_{\mathcal{N}} M) \oplus (\mathrm{proj.}) \cong \mathrm{Ind}^{\mathcal{G}}_{\mathcal{N}} \Omega^n_{\mathcal{N}}(M).$$

Since the indecomposable $\mathcal{N}$-module $\Omega^n_{\mathcal{N}}(M)$ (cf. [30]) is not projective, our above observations show that the projective summand of $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{N}} \Omega^n_{\mathcal{N}}(M)$ is trivial. □

We finally provide the announced characterization of induced modules.

Proposition 2.3.5. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup of the infinitesimal group $\mathcal{G}$ such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Suppose that the restriction $S|_{\mathcal{N}}$ of every simple $\mathcal{G}$-module S is simple. Let M be a $\mathcal{G}$-module such that $M \otimes_k k_\lambda \cong M$ for all $\lambda \in X(\mathcal{G}/\mathcal{N})$. Then the following statements are equivalent:*

(1) *There exists an $\mathcal{N}$-module N such that $M \cong \mathrm{Ind}^{\mathcal{G}}_{\mathcal{N}} N$.*
(2) *The exact sequence $1 \longrightarrow U(\mathrm{End}_{\mathcal{G}}(M)) \longrightarrow U_{\mathrm{gr}}(\mathrm{End}_{\mathcal{N}}(M)) \stackrel{\deg}{\longrightarrow} X(\mathcal{G}/\mathcal{N}) \longrightarrow 1$ splits.*
(3) *$\mathrm{End}_{\mathcal{N}}(M) \cong \mathrm{End}_{\mathcal{G}}(M) {*} X(\mathcal{G}/\mathcal{N})$ is a skew group algebra.*

Proof. (1) ⇒ (2). This was observed in Section 2.1.

(2) ⇒ (3). This is a direct consequence of [37, (2.1.5)].

(3) ⇒ (1). Owing to (3), there exists for each $\lambda \in X(\mathcal{G}/\mathcal{N})$ an invertible $\mathcal{N}$-linear maps $\varphi_\lambda \in \mathrm{End}_{\mathcal{N}}(M)_\lambda$ such that

$$\mathrm{End}_{\mathcal{N}}(M) = \bigoplus_{\lambda \in X(\mathcal{G}/\mathcal{N})} \mathrm{End}_{\mathcal{G}}(M)\varphi_\lambda.$$

Moreover, $\lambda \mapsto \varphi_\lambda$ is a homomorhism of groups. The module M thus carries the structure of an $X(\mathcal{G}/\mathcal{N})$-module by defining

$$\lambda.m := \varphi_\lambda^{-1}(m).$$

Since the elements of $\mathrm{End}_{\mathcal{N}}(M)_\lambda$ correspond to $\mathcal{G}$-linear maps $M \otimes_k k_\lambda \longrightarrow M$ (cf. (2.1.1)), we have

$$\varphi_\lambda(h.m) = (\lambda^{-1}.h)\varphi_\lambda(m).$$

Consequently,

$$\lambda.(h.m) = \varphi_{\lambda^{-1}}(h.m) = (\lambda.h)\varphi_{\lambda^{-1}}(m) = (\lambda.h)(\lambda.m),$$

so that M is a $\mathrm{Dist}(\mathcal{G}){*}X(\mathcal{G}/\mathcal{N})$-module. Since $\mathrm{Dist}(\mathcal{G})\!:\!\mathrm{Dist}(\mathcal{N})$ is a Galois extension with Galois group $X(\mathcal{G}/\mathcal{N})$, [3, (1.2)] implies that $M \cong \mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} M^{X(\mathcal{G}/\mathcal{N})}$, as desired. □

3. AR-Components and Representation Type

In Section 4, we shall apply the foregoing techniques to study indecomposable modules of the group schemes $\mathrm{SL}(2)_1T_r$. By definition, $\mathrm{SL}(2)_1T_r$ is the product of the first Frobenius kernel of $\mathrm{SL}(2)$ and the r-th Frobenius kernel of its standard maximal torus $T \subseteq \mathrm{SL}(2)$ of diagonal matrices. The twofold purpose of this section is to introduce the necessary tools from the theory of almost split sequences on the one hand, while explaining the relevance of the family $(\mathrm{SL}(2)_1T_r)_{r\geq 1}$ of infinitesimal group schemes on the other.

We begin by recalling a few facts from Auslander-Reiten theory. Let $\mathcal{G}$ be an infinitesimal group. The *stable Auslander-Reiten quiver* $\Gamma_s(\mathcal{G})$ is a directed graph, with vertices corresponding to the isoclasses of the non-projective indecomposable $\mathrm{Dist}(\mathcal{G})$-modules. The arrows are determined by the so-called *irreducible morphisms.* The graph $\Gamma_s(\mathcal{G})$ is fitted with an automorphism $\tau_{\mathcal{G}}$, the so-called *Auslander-Reiten translation.* It is well-known that

$$\tau_{\mathcal{G}}(M) = \Omega^2_{\mathcal{G}}(M^{(\nu)}) \quad \forall\ M \in \Gamma_s(\mathcal{G}).$$

Here ν denotes a Nakayama automorphism of $\mathrm{Dist}(\mathcal{G})$. We refer the reader to [4, 5] for further details.

The connected components of $\Gamma_s(\mathcal{G})$ are connected stable translation quivers. By work of Riedtmann [41, Struktursatz], the structure of such a quiver Θ is determined by a directed tree T_Θ and an *admissible group* $\Pi \subseteq \mathrm{Aut}_k(\mathbb{Z}[T_\Theta])$, giving rise to an isomorphism

$$\Theta \cong \mathbb{Z}[T_\Theta]/\Pi$$

of stable translation quivers. The underlying undirected tree $\bar{T}_\Theta$, the so-called *tree class* of Θ, is uniquely determined by Θ. We refer the reader to [5, (4.15.6)] for further details.

In view of [17, (1.3)], the tree classes are simply-laced finite or infinite Dynkin diagrams, or Euclidean diagrams of type $\tilde{A}_{12}, \tilde{D}_n$, or $\tilde{E}_{6,7,8}$. Components of tree class A_∞ allow for each module only one sectional path (see [4, (V.II)]) to the end of the component. Following [42, §3], we refer to the length of this path as the *quasi-length* $\mathrm{q}\ell(M)$ of M. The modules of quasi-length 1 are located at the end of their component (that is, they only have one predecessor in $\Gamma_s(\mathcal{G})$), they are the *quasi-simple* modules. Components of type $\mathbb{Z}[A_\infty]/(\tau)$ are often called *homogeneous tubes.* Such components possess for each $n \geq 1$ exactly one module of quasi-length n.

To each component of $\Gamma_s(\mathcal{G})$ one can associate various rank varieties. We will only require the varieties given by the first Frobenius kernel of $\mathcal{G}$. These correspond to the varieties of the restricted Lie algebra $\mathrm{Lie}(\mathcal{G})$.

Given a restricted Lie algebra $(\mathfrak{g}, [p])$, the affine variety

$$\mathcal{V}_{\mathfrak{g}} := \{x \in \mathfrak{g}\ ;\ x^{[p]} = 0\}$$

is called the *nullcone* of $\mathfrak{g}$. If M is a $U_0(\mathfrak{g})$-module, then

$$\mathcal{V}_{\mathfrak{g}}(M) := \{x \in \mathcal{V}_{\mathfrak{g}}\ ;\ M|_{U_0(kx)} \text{ is not projective}\} \cup \{0\}$$

is the *rank variety* of M. The dimension of this affine variety is known to coincide with the *complexity* $\mathrm{cx}_{\mathfrak{g}}(M)$ of the $U_0(\mathfrak{g})$-module M (cf. [34, 26]).

If $\mathcal{G}$ is an infinitesimal group with Lie algebra $\mathfrak{g}$, then the canonical embedding $U_0(\mathfrak{g}) \hookrightarrow \mathrm{Dist}(\mathcal{G})$ enables us to view $\mathcal{G}$-modules as $U_0(\mathfrak{g})$-modules. Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component. Either by appealing to [17, (1.1)], or by using the arguments of [16, (5.2)] one readily obtains

$$\mathcal{V}_{\mathfrak{g}}(M) = \mathcal{V}_{\mathfrak{g}}(N) \qquad \forall\ M, N \in \Theta.$$

We can therefore speak of the variety $\mathcal{V}_{\mathfrak{g}}(\Theta)$.

3.1. **Almost Split Sequences.** Let M be a non-projective indecomposable $\mathcal{G}$-module. Then there exists an *almost split sequence*

$$(0) \longrightarrow \tau_{\mathcal{G}}(M) \longrightarrow E \longrightarrow M \longrightarrow (0)$$

terminating in M, see [4, (V.1.15)]. Since $\mathrm{Dist}(\mathcal{G})$ is a Frobenius algebra, the Nakayama functor $\mathrm{mod}\,\mathcal{G} \longrightarrow \mathrm{mod}\,\mathcal{G}$ is the twist by a Nakayama automorphism $\nu : \mathrm{Dist}(\mathcal{G}) \longrightarrow \mathrm{Dist}(\mathcal{G})$. Owing to [25, (1.5)], one of these automorphisms is given by

$$\nu := \mathrm{id}_{\mathrm{Dist}(\mathcal{G})} * (\zeta \circ \eta),$$

where $\zeta : \mathrm{Dist}(\mathcal{G}) \longrightarrow k$ is the *right modular function*, which by definition, satisfies

$$xh = \zeta(h)x$$

for every $h \in \mathrm{Dist}(\mathcal{G})$ and every left integral $x \in \mathrm{Dist}(\mathcal{G})$.

Lemma 3.1.1. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Then we have*

$$\tau_{\mathcal{G}}(M)|_{\mathcal{N}} \cong \tau_{\mathcal{N}}(M|_{\mathcal{N}})$$

for every non-projective indecomposable $\mathcal{G}$-module M.

Proof. Suppose that $M|_{\mathcal{N}}$ is projective. Then [36, (I.6.9)] implies

$$\mathrm{Ext}^1_{\mathcal{G}}(M, N) \cong \mathrm{Ext}^1_{\mathcal{N}}(M, N)^{\mathcal{G}/\mathcal{N}} = (0),$$

for every $N \in \mathrm{mod}\,\mathcal{G}$, so that M is projective, a contradiction. According to Theorem 2.1.2, the module $M|_{\mathcal{N}}$ is indecomposable, so that $\tau_{\mathcal{N}}(M|_{\mathcal{N}})$ is well-defined.

By general theory, there are isomorphisms

$$\Omega^n_{\mathcal{G}}(M)|_{\mathcal{N}} \cong \Omega^n_{\mathcal{N}}(M|_{\mathcal{N}}) \oplus (\text{proj.}).$$

Since $\Omega^n_{\mathcal{G}}(M)$ is a non-projective indecomposable module, the foregoing observation in conjunction with Theorem 2.1.2 ensures that its restriction to $\mathcal{N}$ is projective-free, that is,

$$\Omega^n_{\mathcal{G}}(M)|_{\mathcal{N}} \cong \Omega^n_{\mathcal{N}}(M|_{\mathcal{N}}).$$

Let $\zeta_{\mathcal{G}}$ and $\zeta_{\mathcal{N}}$ be the left modular functions of $\mathcal{G}$ and $\mathcal{N}$, respectively. (Hence $\zeta_{\mathcal{G}} \circ \eta$ is the right modular function of $\mathcal{G}$.) We let $\mathrm{Dist}(\mathcal{G})$ act on itself via the adjoint representation and observe that $\mathrm{Dist}(\mathcal{N})$ is a $\mathrm{Dist}(\mathcal{G})$-submodule of $\mathrm{Dist}(\mathcal{G})_{\mathrm{ad}}$. Let $x_{\mathcal{N}} \in \mathrm{Dist}(\mathcal{N})$ be a left integral. According to [36, (I.8.19)], we have

$$h.x_{\mathcal{N}} = \zeta_{\mathcal{G}}(h)x_{\mathcal{N}} \qquad \forall\ h \in \mathrm{Dist}(\mathcal{G}).$$

Thus, if h is an element of $\mathrm{Dist}(\mathcal{N})$ we obtain, observing [36, (I.8.8.(6))],

$$\zeta_{\mathcal{G}}(h)x_{\mathcal{N}} = h.x_{\mathcal{N}} = \sum_{(h)} h_{(1)}x_{\mathcal{N}}\eta(h_{(2)}) = \sum_{(h)} \varepsilon(h_{(1)})x_{\mathcal{N}}\eta(h_{(2)}) = x_{\mathcal{N}}\eta(h) = \zeta_{\mathcal{N}}(h)x_{\mathcal{N}}.$$

Consequently, $\zeta_{\mathcal{G}}|_{\mathrm{Dist}(\mathcal{N})} = \zeta_{\mathcal{N}}$, and our above observations imply that the Nakayama functors $\nu_{\mathcal{G}}$ and $\nu_{\mathcal{N}}$ of $\mathrm{mod}\,\mathcal{G}$ and $\mathrm{mod}\,\mathcal{N}$ satisfy

$$\mathrm{res}_{\mathcal{N}} \circ \nu_{\mathcal{G}} = \nu_{\mathcal{N}} \circ \mathrm{res}_{\mathcal{N}}.$$

Consequently,

$$\tau_{\mathcal{G}}(M)|_{\mathcal{N}} \cong (\mathrm{res}_{\mathcal{N}} \circ \nu_{\mathcal{G}} \circ \Omega^2_{\mathcal{G}})(M) = (\nu_{\mathcal{N}} \circ \mathrm{res}_{\mathcal{N}} \circ \Omega^2_{\mathcal{G}})(M) \cong (\nu_{\mathcal{N}} \circ \Omega^2_{\mathcal{N}} \circ \mathrm{res}_{\mathcal{N}})(M) \cong \tau_{\mathcal{N}}(M|_{\mathcal{N}}),$$

as desired. □

Let M be a $\mathcal{G}$-module and recall that

$$\mathrm{Stab}_{X(\mathcal{G}/\mathcal{N})}(M) := \{\lambda \in X(\mathcal{G}/\mathcal{N}) \ ; \ M \otimes_k k_\lambda \cong M\}$$

is a subgroup of the abelian p-group $X(\mathcal{G}/\mathcal{N})$. Our next result shows that these groups usually provide invariants of Auslander-Reiten components.

Proposition 3.1.2. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Suppose that $p \geq 5$, and let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be an AR-component. Then we have*

$$\mathrm{Stab}_{X(\mathcal{G}/\mathcal{N})}(M) = \mathrm{Stab}_{X(\mathcal{G}/\mathcal{N})}(N)$$

for every $M, N \in \Theta$.

Proof. In view of [17, (1.3)], the tree class $\bar{T}_\Theta$ of Θ is either a finite Dynkin diagram of type A, D, E, an infinite Dynkin diagram A_∞, A_∞^∞, or one of the Euclidean diagrams $\tilde{A}_{12}$, $\tilde{D}_n$, $\tilde{E}_{6,7,8}$. By general theory, this implies in particular, that, for every $M \in \Theta$, the sets M^+ and M^- of successors and predecessors of M have at most 3 elements.

Given $\lambda \in X(\mathcal{G}/\mathcal{N})$, the functor $M \mapsto M \otimes_k k_\lambda$ is an auto-equivalence of $\mathrm{mod}\,\mathcal{G}$ that commutes with $\tau_{\mathcal{G}}$. Hence λ induces a quiver automorphism $t_\lambda : \Gamma_s(\mathcal{G}) \longrightarrow \Gamma_s(\mathcal{G})$ of order a power of p. Let $M \in \Theta$ and $\lambda \in \mathrm{Stab}_{X(\mathcal{G}/\mathcal{N})}(M)$. Since t_λ permutes the components of $\Gamma_s(\mathcal{G})$, we conclude that $t_\lambda(\Theta) = \Theta$.

The sets $\mathcal{X}(\lambda) := \{N \in \Theta \ ; \ t_\lambda(N) = N\}$ and $\mathcal{Y}(\lambda) := \{N \in \Theta \ ; \ t_\lambda(N) \neq N\}$ are $\tau_{\mathcal{G}}$-invariant. Suppose that N and N' are elements of $\mathcal{X}(\lambda)$ and $\mathcal{Y}(\lambda)$, respectively, such that $N' \in N^+$. Since $t_\lambda(N^+) = N^+$ is a set with at most 3 elements and t_λ has order a power of p, the assumption $p \geq 5$ implies $t_\lambda|_{N^+} = \mathrm{id}_{N^+}$, a contradiction. By the same token, $\mathcal{Y}(\lambda) \cap N^- = \emptyset$. Since Θ is connected, we obtain $\mathcal{X}(\lambda) = \Theta$, whence $\lambda \in \mathrm{Stab}_{X(\mathcal{G}/\mathcal{N})}(N)$ for all $N \in \Theta$. □

Corollary 3.1.3. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Suppose that $p \geq 5$, and let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component. If Θ contains a $\mathcal{G}$-module M with $\dim_k M \not\equiv 0 \mod(p)$, then every $\mathcal{G}$-module N belonging to Θ is regular.*

Proof. This is a direct consequence of (2.1.4) and (3.1.2). □

Remark. The foregoing results afford various refinements. For instance, if the vertices of $\Gamma_s(\mathcal{G})$ are known to have at most 2 successors (and hence at most 2 predecessors), Proposition 3.1.2 and Corollary 3.1.3 also hold for $p \geq 3$. This is the case for the groups $\mathcal{G} = \mathrm{SL}(2)_1 T_r$, to be discussed in Section 4 below.

We now investigate the compatibility of the restriction functor $\mathrm{res}_{\mathcal{N}} : \mathrm{mod}\,\mathcal{G} \longrightarrow \mathrm{mod}\,\mathcal{N}$ with almost split sequences. In light of [28, (4.1)], the following result can be viewed as an analogue of [29, (3.5)] for modules and algebras that are graded relative to a torsion group.

Proposition 3.1.4. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Let $(0) \longrightarrow \tau_{\mathcal{G}}(M) \longrightarrow E \longrightarrow M \longrightarrow (0)$ be the almost split sequence terminating in the indecomposable non-projective $\mathcal{G}$-module M. Then the following statements are equivalent:*

(1) *The $\mathcal{G}$-module M is regular.*
(2) *The sequence*

$$(0) \longrightarrow \tau_{\mathcal{G}}(M)|_{\mathcal{N}} \longrightarrow E|_{\mathcal{N}} \longrightarrow M|_{\mathcal{N}} \longrightarrow (0)$$

is the almost split sequence terminating in $M|_{\mathcal{N}}$.

Proof. Let $\gamma : E \longrightarrow M$ be the given right almost split epimorphism. We consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{G}}(M,E) & \xrightarrow{\gamma_*} & \mathrm{End}_{\mathcal{G}}(M) \\ {\scriptstyle \mathrm{res}_{\mathcal{N}}}\downarrow & & {\scriptstyle \mathrm{res}_{\mathcal{N}}}\downarrow \\ \mathrm{Hom}_{\mathcal{N}}(M|_{\mathcal{N}},E|_{\mathcal{N}}) & \xrightarrow{\mathrm{res}_{\mathcal{N}}(\gamma)_*} & \mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}}). \end{array}$$

Thanks to (2.1.1), the canonical $\mathcal{G}/\mathcal{N}$-actions on $\mathrm{Hom}_{\mathcal{N}}(M|_{\mathcal{N}}, E|_{\mathcal{N}})$ and $\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})$ endow both spaces with an $X(\mathcal{G}/\mathcal{N})$-grading, rendering the former an $X(\mathcal{G}/\mathcal{N})$-graded right module of the latter. Moreover, the maps γ_* and $\mathrm{res}_{\mathcal{N}}(\gamma)_*$ are $\mathrm{End}_{\mathcal{G}}(M)$-linear and $\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})$-linear, respectively, with the latter being a homomorphism of degree 0. Note that $\mathrm{res}_{\mathcal{N}}$ maps $\mathrm{Hom}_{\mathcal{G}}(M,E)$ and $\mathrm{End}_{\mathcal{G}}(M)$ isomorphically onto $\mathrm{Hom}_{\mathcal{N}}(M|_{\mathcal{N}}, E|_{\mathcal{N}})_0$ and $\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})_0$, respectively.

(1) $\Rightarrow$ (2). Let λ be an element of $X(\mathcal{G}/\mathcal{N}) \setminus \{0\}$. Given $f \in \mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})_\lambda$, Lemma 2.1.1 shows that

$$\tilde{f} : M \otimes_k k_\lambda \longrightarrow M \quad ; \quad m \otimes \alpha \mapsto \alpha f(m)$$

is a homomorphism of $\mathcal{G}$-modules. Since M is regular, $\tilde{f}$ is not an isomorphism, and γ being right almost split implies the existence of a $\mathcal{G}$-linear map $\tilde{g} : M \otimes_k k_\lambda \longrightarrow E$ such that $\gamma \circ \tilde{g} = \tilde{f}$. Then $g : M \longrightarrow E \ ; \ m \mapsto \tilde{g}(m \otimes 1)$ belongs to $\mathrm{Hom}_{\mathcal{N}}(M,E)_\lambda$ and

$$\mathrm{res}_{\mathcal{N}}(\gamma)_*(g) = f.$$

In view of $\mathrm{Hom}_{\mathcal{N}}(M|_{\mathcal{N}}, E|_{\mathcal{N}})_0 = \mathrm{res}_{\mathcal{N}}(\mathrm{Hom}_{\mathcal{G}}(M,E))$, the above diagram now yields

$$\begin{aligned} \mathrm{im}\, \mathrm{res}_{\mathcal{N}}(\gamma)_* &= \mathrm{res}_{\mathcal{N}}(\gamma)_*(\mathrm{Hom}_{\mathcal{N}}(M|_{\mathcal{N}}, E|_{\mathcal{N}})_0) \oplus \bigoplus_{\lambda \in X(\mathcal{G}/\mathcal{N}) \setminus \{0\}} \mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})_\lambda \\ &= \mathrm{res}_{\mathcal{N}}(\gamma_*(\mathrm{Hom}_{\mathcal{G}}(M,E))) \oplus \bigoplus_{\lambda \in X(\mathcal{G}/\mathcal{N}) \setminus \{0\}} \mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})_\lambda. \end{aligned}$$

Since M is regular and $M|_{\mathcal{N}}$ is indecomposable (2.1.2), the elements of $\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})_\lambda$ belong to the radical $\mathrm{Rad}(\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}}))$ of the local algebra $\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})$ whenever $\lambda \neq 0$. According to [4, (V.2.2)], we have $\gamma_*(\mathrm{Hom}_{\mathcal{G}}(M,E)) = \mathrm{Rad}(\mathrm{End}_{\mathcal{G}}(M))$, so that

$$\mathrm{im}\, \mathrm{res}_{\mathcal{N}}(\gamma)_* = \mathrm{Rad}(\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})_0) \oplus \bigoplus_{\lambda \neq 0} \mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})_\lambda \subseteq \mathrm{Rad}(\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}}))$$

is a subspace of $\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})$ of codimension 1, whence

$$\mathrm{im}\, \mathrm{res}_{\mathcal{N}}(\gamma)_* = \mathrm{Rad}(\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})).$$

Thanks to Lemma 3.1.1, we may now apply [4, (V.2.2)] to obtain (2).

(2) $\Rightarrow$ (1). In view of the above diagram, a twofold application of [4, (V.2.2)] yields

$$\begin{aligned}
\mathrm{Rad}(\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})) &= \mathrm{im}\,\mathrm{res}_{\mathcal{N}}(\gamma)_* \\
&= \bigoplus_{\lambda\neq 0} \mathrm{res}_{\mathcal{N}}(\gamma)_*(\mathrm{Hom}_{\mathcal{N}}(M|_{\mathcal{N}}, E|_{\mathcal{N}})_\lambda) \oplus \mathrm{res}_{\mathcal{N}}(\gamma)_*(\mathrm{Hom}_{\mathcal{N}}(M|_{\mathcal{N}}, E|_{\mathcal{N}})_0) \\
&= \bigoplus_{\lambda\neq 0} \mathrm{res}_{\mathcal{N}}(\gamma)_*(\mathrm{Hom}_{\mathcal{N}}(M|_{\mathcal{N}}, E|_{\mathcal{N}})_\lambda) \oplus \mathrm{res}_{\mathcal{N}}(\mathrm{Rad}(\mathrm{End}_{\mathcal{G}}(M))) \\
&\subseteq \bigoplus_{\lambda\neq 0} \mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})_\lambda \oplus \mathrm{Rad}(\mathrm{End}_{\mathcal{N}}(M)_0).
\end{aligned}$$

Since the right-hand space has codimension 1 in $\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})$, we have equality throughout, proving that $\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}})_\lambda \subseteq \mathrm{Rad}(\mathrm{End}_{\mathcal{N}}(M|_{\mathcal{N}}))$ for all $\lambda \in X(\mathcal{G}/\mathcal{N}) \setminus \{0\}$. Consequently, none of these spaces contains an invertible element, and (2.1.1) shows that M is regular. □

Corollary 3.1.5. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup such that $\mathcal{G}/\mathcal{N}$ is multiplicative. Suppose that $p \geq 5$. Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component containing a regular $\mathcal{G}$-module. Then $\mathrm{res}_{\mathcal{N}}$ induces a morphism*

$$\mathrm{res}_{\mathcal{N}} : \Theta \longrightarrow \Gamma_s(\mathcal{N})$$

of stable translation quivers.

Proof. In virtue of Proposition 3.1.2, every module belonging to Θ is regular. Let M and N be elements of Θ such that there is an arrow $M \to N$. By general theory [4, (V.5.3)], M is a direct summand of the middle term E of the almost split sequence terminating in N. According to Proposition 3.1.4, the module $M|_{\mathcal{N}}$ is a direct summand of the middle term $E|_{\mathcal{N}}$ of the almost split sequence terminating in $N|_{\mathcal{N}}$, so that [4, (V.5.3)] furnishes an arrow $M|_{\mathcal{N}} \longrightarrow N|_{\mathcal{N}}$. In view of Lemma 3.1.1, the map $\mathrm{res}_{\mathcal{N}}$ satisfies the identity

$$\mathrm{res}_{\mathcal{N}} \circ \tau_{\mathcal{G}} = \tau_{\mathcal{N}} \circ \mathrm{res}_{\mathcal{N}},$$

so that $\mathrm{res}_{\mathcal{N}}$ is in fact a morphism of stable translation quivers. □

Remark. The map $\mathrm{res}_{\mathcal{N}}$ is usually *not* a morphism of valued stable translation quivers, see Section 4.

3.2. **Domestic Infinitesimal Groups.** Recall that a finite-dimensional k-algebra Λ of infinite representation type is referred to as *tame* if for each $d > 0$ all but finitely many indecomposable Λ-modules of dimension d occur in finitely many one-parameter families. If the number of parameters is uniformly bounded, then Λ is called *domestic*. The reader may consult [14] for the formal definition.

We retain our general conventions and consider infinitesimal groups that are defined over an algebraically closed field of characteristic $p \geq 3$. Let $\mathcal{B}_0(\mathcal{G})$ be the principal block of the algebra $\mathrm{Dist}(\mathcal{G})$ of distributions of $\mathcal{G}$. The *center* of $\mathcal{G}$ will be denoted $\mathcal{C}(\mathcal{G})$. By general theory (cf. [48, (9.5)]), the abelian group scheme $\mathcal{C}(\mathcal{G})$ can be written as a direct product

$$\mathcal{C}(\mathcal{G}) \cong \mathcal{M}(\mathcal{G}) \times \mathcal{U}(\mathcal{G})$$

of its *multiplicative center* $\mathcal{M} = \mathcal{M}(\mathcal{G})$ and a unipotent subgroup $\mathcal{U}(\mathcal{G})$.

The proof of the following result, which can also be established by an analysis of the relevant basic algebras (see [23, (6.1)]), emphasizes the interplay of AR-theory and rank varieties.

Proposition 3.2.1. *The following statements are equivalent:*
(1) *The principal block $\mathcal{B}_0(\mathcal{G})$ is domestic.*
(2) *$\mathcal{B}_0(\mathcal{G})$ is tame and $\mathcal{C}(\mathcal{G})$ is multiplicative.*
(3) *There exists $r \geq 1$ such that $\mathcal{G}/\mathcal{M} \cong \mathrm{SL}(2)_1T_r$.*

Proof. (1) ⇒ (2). We let $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ be the Lie algebra of $\mathcal{G}$. According [22, (7.1)], the principal block $\mathcal{B}_0(\mathcal{G})$ is a self-injective, special biserial algebra. Since $\mathcal{B}_0(\mathcal{G})$ is domestic, an application of [15, (2.1)] implies that each component $\Theta \subseteq \Gamma_s(\mathcal{B}_0(\mathcal{G}))$ of the stable Auslander-Reiten quiver of $\mathcal{B}_0(\mathcal{G})$, which is not a tube $\mathbb{Z}[A_\infty]/(\tau^\ell)$ is of Euclidean type $\mathbb{Z}[\tilde{A}_n]$ (for some orientation of the arrows of the quiver $\tilde{A}_n$ that is not relevant for our purposes). In view of [6, p.155], such a component Θ is attached to a principal indecomposable module, so that Θ contains the radical $\mathrm{Rad}(P)$ of a principal indecomposable $\mathcal{B}_0(\mathcal{G})$-module. Note that

$$\mathcal{V}_\mathfrak{g}(\Theta) = \mathcal{V}_\mathfrak{g}(\mathrm{Rad}(P)) = \mathcal{V}_\mathfrak{g}(\Omega_\mathcal{G}^{-1}(\mathrm{Rad}(P))),$$

with $\Omega_\mathcal{G}^{-1}(\mathrm{Rad}(P))$ being a simple $\mathcal{B}_0(\mathcal{G})$-module. Owing to [22, (7.1)] the dimensions of the simple $\mathcal{B}_0(\mathcal{G})$-modules are not divisible by p. Consequently, the rank variety $\mathcal{V}_\mathfrak{g}(\Theta)$ coincides with the nullcone $\mathcal{V}_\mathfrak{g}$ of $\mathfrak{g}$.

If the group $\mathcal{G}$ has height 1, then our result is a direct consequence of [21, (7.2)]. Alternatively, [22, (6.4)] yields a quotient map $\mathcal{G} \longrightarrow \mathrm{SL}(2)_1T_r$ as well as an isomorphism $\mathfrak{g} \cong \mathfrak{sl}(2)_s^n$ of restricted Lie algebras. Here n denotes the length of the unipotent group $\mathcal{U}(\mathcal{G})$. By definition, the Lie algebra

$$\mathfrak{sl}(2)_s^n = \mathfrak{sl}(2) \oplus \mathfrak{n}_n$$

is the central extension of $\mathfrak{sl}(2)$ by the n-dimensional nilcyclic Lie algebra $\mathfrak{n}_n := \bigoplus_{i=0}^{n-1} kv_0^{[p]^i}$ satisfying

$$f^{[p]} = 0 = e^{[p]} \quad , \quad h^{[p]} = v_0.$$

Direct computation shows that $\mathcal{V}_{\mathfrak{sl}(2)_s^n} = (ke \oplus kv_0^{[p]^{n-1}}) \cup (kf \oplus kv_0^{[p]^{n-1}})$. On the other hand, the baby Verma module $Z(0) = \mathrm{Dist}(\mathrm{SL}(2)_1) \otimes_{\mathrm{Dist}(B_1)} k$ is an indecomposable $\mathcal{B}_0(\mathcal{G})$-module, with rank variety $\mathcal{V}_{\mathfrak{sl}(2)_s^n}(Z(0)) = ke \oplus kv_0^{[p]^{n-1}}$. Thus, $Z(0)$ belongs to a tube, so that $\dim \mathcal{V}_\mathfrak{g}(Z(0)) = 1$. As a result, $n = 0$, and $\mathcal{C}(\mathcal{G})$ is multiplicative.

(2) ⇒ (3). This is a direct consequence of [22, (3.5)].

(3) ⇒ (1). By assumption, [20, (1.1)] implies $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{M}) \cong \mathcal{B}_0(\mathrm{SL}(2)_1T_r)$. According to [20, (5.6)], this algebra has domestic representation type. □

Remark. The algebra $\mathcal{B}_0(\mathrm{SL}(2)_1T_r)$ is Morita equivalent to the trivial extension of tame hereditary radical square zero algebra of type $\tilde{A}_{2p^{r-1}-1}$. By work of Tachikawa [46], such algebras are 2-*parametric*, that is, there are in each dimension at most two continuous 1-parameter families of indecomposable modules.

4. Indecomposable $\mathrm{SL}(2)_1T_r$-modules

Assuming $p \geq 3$, we employ our results to study the indecomposable $\mathrm{SL}(2)_1T_r$-modules. Aside from Premet's work on $\mathrm{SL}(2)_1$-modules (cf. [40]), the stable Auslander-Reiten quiver will play an important rôle.

4.1. **The stable Auslander-Reiten Quiver of** $\mathrm{SL}(2)_1$**.** We begin by integrating Premet's results [40] into the framework of Auslander-Reiten Theory. To that end, we consider the restricted Lie algebra $\mathfrak{sl}(2)$. The Hopf algebra $U_0(\mathfrak{sl}(2))$ is a symmetric algebra, whose stable Auslander-Reiten quiver will be denoted $\Gamma_s(\mathfrak{sl}(2))$.

The simple $U_0(\mathfrak{sl}(2))$-modules are labelled $L(0), \ldots, L(p-1)$ with $\dim_k L(i) = i+1$. Given $d \geq 0$, we let $V(d)$ be the *Weyl module* of highest weight d. This module is obtained from the d-th symmetric power of the standard module $L(1)$ by twisting its dual by the Cartan involution $(x \mapsto -x^{\mathrm{tr}})$. Thus, if $\{e, h, f\} \subseteq \mathfrak{sl}(2)$ is the standard basis of $\mathfrak{sl}(2)$, then $V(d) = \bigoplus_{i=0}^{d} kv_i$ and

$$e.v_i = (i+1)v_{i+1} \quad ; \quad f.v_i = (d-i+1)v_{i-1} \quad ; \quad h.v_i = (2i-d)v_i.$$

Hence we have $\dim_k V(d) = d+1$ as well as $V(d) = L(d)$ for $d \in \{0, \ldots, p-1\}$. By construction, each $V(d)$ is a rational module for the group scheme $\mathrm{SL}(2)$. For $d = sp + a$, where $s \geq 1$ and $a \in \{0, \ldots, p-2\}$, Premet defines the maximal $U_0(\mathfrak{g})$-submodule

$$W(d) := \bigoplus_{i=a+1}^{d} kv_i \subseteq V(d).$$

By definition, $W(d)$ is an sp-dimensional module that is stable under the standard Borel subgroup $B \subseteq \mathrm{SL}(2)$ of upper triangular matrices.

Given $g \in \mathrm{SL}(2)(k)$, the subspace $g.W(d) \subseteq V(d)$ is $U_0(\mathfrak{sl}(2))$-stable and isomorphic to the twist $W(d)^{(g)} := W(d)^{(\mathrm{Ad}(g))}$ of $W(d)$ by the adjoint representation $\mathrm{Ad}(g) \in \mathrm{Aut}(U_0(\mathfrak{sl}(2)))$. In [40] Premet shows that

$$\mathcal{V}_{\mathfrak{sl}(2)}(g.W(d)) = k\mathrm{Ad}(g)(e) \qquad \forall\, g \in \mathrm{SL}(2)(k).$$

The following result shows in particular that the modules $g.W(d)$ give rise to a complete list of the indecomposable $U_0(\mathfrak{sl}(2))$-modules with one-dimensional supports:

Theorem 4.1.1 ([40])**.** *The following statements hold*

(1) *Let* $C \subseteq \mathrm{SL}(2)$ *be a complete set of coset representatives of* $\mathrm{SL}(2)/B$. *Then any non-projective indecomposable* $U_0(\mathfrak{sl}(2))$*-module is isomorphic to exactly one of the modules of the following list:*

- $V(d), V(d)^*$ *for* $d \geq p, d \not\equiv -1 \mod(p)$,
- $V(r)$ *for* $0 \leq r \leq p-1$,
- $g.W(d)$ *for* $g \in C$ *and* $d = sp + a$ *with* $s \geq 1$ *and* $a \in \{0, \ldots, p-2\}$.

In particular, the modules appearing in the list are pairwise non-isomorphic.

(2) *Up to isomorphism, every indecomposable* $U_0(\mathfrak{sl}(2))$*-module* M *is uniquely determined by the triple* $(\dim_k M, \mathrm{Soc}_{\mathfrak{sl}(2)}(M), \mathcal{V}_{\mathfrak{sl}(2)}(M))$. □

By work of Drozd [13], Fischer [24], and Rudakov [44], each of the $\frac{p-1}{2}$ non-simple blocks of $U_0(\mathfrak{sl}(2))$ is Morita equivalent to the trivial extension $\mathrm{Kr}^* \rtimes \mathrm{Kr}$ of the path algebra Kr associated to the Kronecker quiver $\bullet \rightrightarrows \bullet$. Consequently, the stable Auslander-Reiten quiver $\Gamma_s(\mathfrak{sl}(2))$ is the disjoint union of $p-1$ components of type $\mathbb{Z}[\tilde{A}_{12}]$, and infinitely many homogeneous tubes.

For each $i \in \{0, \ldots, p-1\}$, we consider the *baby Verma module* $Z(i) := U_0(\mathfrak{sl}(2)) \otimes_{U_0(\mathfrak{b})} k_i$, where the Borel subalgebra $\mathfrak{b} := \mathrm{Lie}(B) = kh \oplus ke$ acts on k via $h.1 = i$ and $e.1 = 0$.

Lemma 4.1.2. *Suppose that* $d = sp + a$ *for* $s \geq 1$ *and* $0 \leq a \leq p-2$. *Let* $g \in \mathrm{SL}(2)(k)$. *Then the AR-component* $\Theta \subseteq \Gamma_s(\mathfrak{sl}(2))$ *containing* $g.W(d)$ *is a homogeneous tube with quasi-simple module* $Z(a)^{(g)}$. *Moreover, we have* $\mathrm{q}\ell(g.W(d)) = s$.

Proof. By the above, we have $\mathrm{cx}_{\mathfrak{sl}(2)}(g.W(d)) = \dim \mathcal{V}_{\mathfrak{sl}(2)}(g.W(d)) = 1$, and [16, (2.5)] implies that $\Omega^2_{\mathfrak{sl}(2)}(g.W(d)) \cong g.W(d)$. As $U_0(\mathfrak{sl}(2))$ is symmetric, the Auslander-Reiten translation $\tau_{\mathfrak{sl}(2)}$ coincides with $\Omega^2_{\mathfrak{sl}(2)}$ (cf. [5, (4.12.8)]). Thanks to [16, (5.3)], the component $\Theta \cong \mathbb{Z}[A_\infty]/(\tau)$ is a homogeneous tube. Given X in Θ, we have an almost split sequence

$$(0) \longrightarrow X \longrightarrow E \longrightarrow X \longrightarrow (0).$$

In view of [16, (3.2)], Θ contains no simple modules, so that an application of [4, (V.3.2)] gives

$$\mathrm{Soc}_{\mathfrak{sl}(2)}(E) \cong 2\,\mathrm{Soc}_{\mathfrak{sl}(2)}(X).$$

As a result, the unique module $X_s \in \Theta$ of quasi-length s satisfies $\mathrm{Soc}_{\mathfrak{sl}(2)}(X_s) \cong s\,\mathrm{Soc}_{\mathfrak{sl}(2)}(X_1)$. Since $\mathrm{Soc}_{\mathfrak{sl}(2)}(Z(a)) = L(p-2-a)$ is simple, and $\mathcal{V}_{\mathfrak{sl}(2)}(Z(p-2-a)) = ke$, it follows that $Z(a)^{(g)}$ is contained in a component $\Psi \subseteq \Gamma_s(\mathfrak{sl}(2))$ with $\mathcal{V}_{\mathfrak{sl}(2)}(\Psi) = k\mathrm{Ad}(g)(e)$. In particular, every module Y belonging to Ψ is $U_0(\mathrm{Ad}(g)(f))$-projective, to that $\dim_k Z(a)^{(g)} = p$ is a divisor of $\dim_k Y$. This implies that $Z(a)^{(g)}$ is quasi-simple. Accordingly, the unique module $Y_s \in \Psi$ of quasi-length s satisfies

(a) $\dim_k Y_s = sp$, and
(b) $\mathrm{Soc}_{\mathfrak{sl}(2)}(Y_s) \cong sL(p-2-a)$, and
(c) $\mathcal{V}_{\mathfrak{sl}(2)}(Y_s) = k\mathrm{Ad}(g)(e)$.

By the proof of [40, (1.3)], the module $g.W(d)$ also enjoys these properties. Consequently, Theorem 4.1.1 implies $Y_s \cong g.W(d)$, so that $\Theta = \Psi$, and $Z(a)^{(g)}$ is the quasi-simple module of the component Θ. □

Each of the $p-1$ remaining Auslander-Reiten components is of type $\mathbb{Z}[\tilde{A}_{12}]$ and contains exactly one non-projective simple module $L(i)$. Moreover, the modules belonging to this component are the Heller shifts $\{\Omega^{2n}_{\mathfrak{sl}(2)}(L(i)), \Omega^{2n+1}_{\mathfrak{sl}(2)}(L(p-i))\ ;\ n \in \mathbb{Z}\}$.

4.2. **Tubes of $\Gamma_s(\mathrm{SL}(2)_1T_r)$.** Let $T \subseteq \mathrm{SL}(2)$ be the torus of diagonal matrices. For fixed $r \geq 2$ we consider the infinitesimal group $\mathrm{SL}(2)_1T_r$ along with its normal subgroup $\mathrm{SL}(2)_1$. Then $\mathrm{SL}(2)_1T_r/\mathrm{SL}(2)_1 \cong \mu_{(p^{r-1})}$ is the $(r-1)$-st Frobenius kernel of the multipicative group $G_m = \mathrm{Spec}_k(k[T]_T)$. Recall from [20, §5] that $\mathrm{Dist}(\mathrm{SL}(2)_1T_r)$ has $\frac{p-1}{2}$ non-simple blocks $\mathcal{B}_0, \ldots, \mathcal{B}_{\frac{p-3}{2}}$ with $\mathcal{B}_i$ having $2p^{r-1}$ simple modules $\{L(i) \otimes_k k_\lambda, L(p-2-i) \otimes_k k_\lambda\ ;\ \lambda \in X(\mu_{(p^{r-1})})\}$. The remaining p^{r-1} simple modules $L(p-1) \otimes_k k_\lambda$ are projective. In particular, the character group $X(\mu_{(p^{r-1})}) \cong \mathbb{Z}/(p^{r-1})$ acts freely on the simple $\mathcal{B}_i$-modules, and with 2 orbits. Moreover, the restriction $S|_{\mathrm{SL}(2)_1}$ of every simple $\mathrm{SL}(2)_1T_r$-module is simple, so that the general assumptions of Section 2 are valid in this context. In the following result, $\tilde{A}_{p^{r-1},p^{r-1}}$ denotes the quiver, whose underlying graph is a circle with $2p^{r-1}$ vertices and with exactly p^{r-1} consecutive arrows being clockwise oriented and the remaining arrows being counter-clockwise oriented.

Thanks to [20, (5.6)] the quiver $\Gamma_s(\mathrm{SL}(2)_1T_r)$ is the disjoint union of

(a) $p-1$ components of type $\mathbb{Z}[\tilde{A}_{p^{r-1},p^{r-1}}]$, and
(b) $2(p-1)$ components of type $\mathbb{Z}[A_\infty]/(\tau^{p^{r-1}})$ (exceptional tubes), and
(c) infinitely many components of type $\mathbb{Z}[A_\infty]/(\tau)$ (homogeneous tubes).

These components are "evenly distributed" over the non-simple blocks $\mathcal{B}_0, \ldots, \mathcal{B}_{\frac{p-3}{2}}$, and the proof of [20, (5.6)] shows that the 4 exceptional tubes belonging to the block $\mathcal{B}_i$ contain the baby Verma modules $Z(i)$ and $Z(p-2-i)$ as well as their twists by the adjoint representation of the standard generator

$$w_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of the Weyl group of SL(2). In particular, the rank variety $\mathcal{V}_{\mathfrak{sl}(2)}(\Theta)$ of an exceptional tube Θ is either ke or kf. Given $j \in \{i, p-2-i\}$, the spaces $W(sp+j)$ and $w_0.W(sp+j)$ are $\mathrm{SL}(2)_1T_r$-invariant.

Lemma 4.2.1. *Let M be an indecomposable $\mathcal{B}_i$-module belonging to an exceptional tube. Then there exist $j \in \{i, p-2-i\}$, $\ell \in \{0,1\}$, $s \geq 1$ and $\lambda \in X(\mu_{(p^{r-1})})$ such that $M \cong w_0^\ell.W(sp+j) \otimes_k k_\lambda$.*

Proof. By assumption, the rank variety $\mathcal{V}_{\mathfrak{sl}(2)}(M)$ coincides with ke or kf. In the former case, we apply (2.1.2) to see that $M|_{\mathrm{SL}(2)_1}$ belongs to a homogeneous tube with rank variety ke. By Theorem 4.1.1, we thus have $M|_{\mathrm{SL}(2)_1} \cong W(sp+j)$ for some $s \geq 1$ and $j \in \{i, p-2-i\}$, and our assertion now follows from (2.1.3). If $\mathcal{V}_{\mathfrak{sl}(2)}(M) = kf$, then the foregoing arguments may be applied to $M^{(w_0)}$. □

We now turn to the description of the modules belonging to homogeneous tubes. Our first result provides information concerning their tops and socles along with a restriction on their dimension. Given $i \in \{0, \dots, p-2\}$, we put $\tilde{L}(i) := \bigoplus_{\lambda \in X(\mu_{(p^{r-1})})} L(i) \otimes_k k_\lambda$. In view of (2.3.1) we have $\tilde{L}(i) \cong \mathrm{Dist}(\mathrm{SL}(2)_1T_r) \otimes_{\mathrm{Dist}(\mathrm{SL}(2)_1)} L(i)$.

Lemma 4.2.2. *Let M be an indecomposable $\mathrm{SL}(2)_1T_r$-module belonging to a homogeneous tube. Then there exist $\ell \in \mathbb{N}$ and $i \in \{0, \dots, p-2\}$ such that $\mathrm{Top}_{\mathrm{SL}(2)_1T_r}(M) \cong \ell\tilde{L}(i)$ and $\mathrm{Soc}_{\mathrm{SL}(2)_1T_r}(M) \cong \ell\tilde{L}(p-2-i)$. In particular, we have $\dim_k M = \ell p^r$.*

Proof. Setting $\mathcal{G} := \mathrm{SL}(2)_1T_r$ and $\mathcal{N} := \mathrm{SL}(2)_1$ we let $\Omega_{\mathcal{G}}$ and $\Omega_{\mathcal{N}}$ denote the Heller translates of $\mathrm{mod}\,\mathcal{G}$ and $\mathrm{mod}\,\mathcal{N}$, respectively. Since the algebra $\mathrm{Dist}(\mathcal{G})$ is symmetric (cf. [22, (7.2)]), the Auslander-Reiten translate of $\mathrm{mod}\,\mathcal{G}$ coincides with $\Omega^2_{\mathcal{G}}$.

Let N be an indecomposable $\mathcal{G}$-module. As $\mathrm{Dist}(\mathcal{G})$ is a projective $\mathrm{Dist}(\mathcal{N})$-module, general theory yields $\Omega^i_{\mathcal{N}}(N|_{\mathcal{N}}) \oplus (\mathrm{proj}) \cong \Omega^i_{\mathcal{G}}(N)|_{\mathcal{N}}$ for every $i \geq 1$. In the course of the proof of (3.1.1) we established isomorphisms

$$\Omega^i_{\mathcal{N}}(N|_{\mathcal{N}}) \cong \Omega^i_{\mathcal{G}}(N)|_{\mathcal{N}}$$

for all $i \geq 1$. As M belongs to a homogeneous tube, we have $\Omega^2_{\mathcal{G}}(M) \cong M$, whence $\Omega^2_{\mathcal{N}}(M|_{\mathcal{N}}) \cong M|_{\mathcal{N}}$.

Let $\mathcal{B}_i$ be the block of the module M. The non-projective, non-simple module $M|_{\mathcal{N}}$ has Loewy length 2 (cf. [39, Thm2]), and from [8, (3.3)] or the representation theory of the Kronecker quiver we obtain the existence of $n \in \mathbb{N}$ and $j \in \{i, p-2-i\}$ such that

$$(*) \qquad \mathrm{Top}_{\mathcal{N}}(M|_{\mathcal{N}}) \cong nL(j) \ \text{ and } \ \mathrm{Soc}_{\mathcal{N}}(M|_{\mathcal{N}}) \cong nL(p-2-j).$$

In view of (2.1.2), the restriction $P|_{\mathcal{N}}$ of a principal indecomposable $\mathcal{G}$-module P is a principal indecomposable $\mathcal{N}$-module. Moreover, the restriction $S|_{\mathcal{N}}$ of every simple $\mathcal{G}$-module S is simple. Thus, we have $\mathrm{Top}_{\mathcal{G}}(P)|_{\mathcal{N}} = \mathrm{Top}_{\mathcal{N}}(P|_{\mathcal{N}})$ for every principal indecomposable $\mathcal{G}$-module, and hence for every projective $\mathcal{G}$-module P. Using projective covers, we conclude $\mathrm{Top}_{\mathcal{G}}(X)|_{\mathcal{N}} \cong \mathrm{Top}_{\mathcal{N}}(X|_{\mathcal{N}})$ for every $X \in \mathrm{mod}\,\mathcal{G}$. Duality gives $\mathrm{Soc}_{\mathcal{G}}(X)|_{\mathcal{N}} \cong \mathrm{Soc}_{\mathcal{N}}(X|_{\mathcal{N}})$ for every $X \in \mathrm{mod}\,\mathcal{G}$. Thus, $(*)$ shows that the simple summands of $\mathrm{Top}_{\mathcal{G}}(M)$ and $\mathrm{Soc}_{\mathcal{G}}(M)$ belong to two different $X(\mathcal{G}/\mathcal{N})$-orbits.

We now label the simple $\mathcal{B}_i$-modules by $L_1, \dots, L_{2p^{r-1}}$, such that $L_s|_{\mathcal{N}} \cong L(j)$ for $s \equiv 0 \mod(2)$ and $L_s|_{\mathcal{N}} \cong L(p-2-j)$ for $s \equiv 1 \mod(2)$, and such that a generator of $X(\mathcal{G}/\mathcal{N}) \cong \mathbb{Z}/(p^{r-1})$ acts via $s \mapsto s+2 \ (\mathrm{mod}(2p^{r-1}))$. By assumption, the $\mathcal{G}$-module M possesses is a minimal projective resolution

$$(0) \longrightarrow M \longrightarrow Q \longrightarrow P \longrightarrow M \longrightarrow (0),$$

so that $\mathrm{Top}_{\mathcal{G}}(M) \cong \mathrm{Top}_{\mathcal{G}}(P)$ and $\mathrm{Soc}_{\mathcal{G}}(M) \cong \mathrm{Soc}_{\mathcal{G}}(Q)$. Letting P_s be the projective cover of L_s, we thus have decompositions

$$P = \bigoplus_{s\equiv 0} m_s P_s \quad \text{and} \quad Q = \bigoplus_{s\equiv 1} n_s P_s,$$

respectively. We define $m_s = 0$ for $s \equiv 1 \mod(2)$ and $n_s = 0$ for $s \equiv 0 \mod(2)$. Application of $\mathrm{Hom}_{\mathcal{G}}(P_t, -)$ to the above exact sequence yields

$$\begin{aligned}\sum_{s=1}^{2p^{r-1}} \dim_k \mathrm{Hom}_{\mathcal{G}}(P_t, P_s) m_s &= \dim_k \mathrm{Hom}_{\mathcal{G}}(P_t, P) = \dim_k \mathrm{Hom}_{\mathcal{G}}(P_t, Q) \\ &= \sum_{s=1}^{2p^{r-1}} \dim_k \mathrm{Hom}_{\mathcal{G}}(P_t, P_s) n_s,\end{aligned}$$

showing that the difference $m - n$ of the vectors $m = (m_s)$, $n = (n_s) \in \mathbb{Z}^{2p^{r-1}}$ belongs to the kernel of the Cartan matrix $C_{\mathcal{B}_i} \in \mathrm{Mat}_n(\mathbb{Z})$ of the block $\mathcal{B}_i$. According to [22, (7.1),(7.2)] we have

$$C_{\mathcal{B}_i} = \begin{pmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 2 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 1 & 2 \end{pmatrix}.$$

Since the first principal minor of $C_{\mathcal{B}_i}$ has determinant $3 \cdot 2^{2p^{r-1}-3}$, the kernel of $C_{\mathcal{B}_i}$ is spanned by the alternating sum $v := \sum_{s=1}^{2p^{r-1}} (-1)^s e_s$ of the canonical basis vectors $\{e_1, \ldots, e_{2p^{r-1}}\} \subseteq \mathbb{Q}^{2p^{r-1}}$. Hence there exists an element $\ell \in \mathbb{Q}$ with $m - n = \ell v$. The above then implies $m_s = \ell = n_t$ for $s \equiv 0$ and $t \equiv 1$. Thus,

$$\mathrm{Top}_{\mathcal{G}}(M) = \bigoplus_{s\equiv 0} \ell L_s \cong \ell \tilde{L}(j) \quad \text{and} \quad \mathrm{Soc}_{\mathcal{G}}(M) = \bigoplus_{s\equiv 1} \ell L_s \cong \tilde{L}(p-2-j).$$

Since $\dim_k L_s + \dim_k L_{s+1} = p$ for every s, it follows that $\dim_k M = \ell p^r$, as desired. □

Recall that an indecomposable $\mathcal{G}$-module M is referred to as *periodic* if there exists $n > 0$ such that $\Omega^n_{\mathcal{G}}(M) \cong M$. We next determine the quasi-simple modules of the homogeneous tubes.

Lemma 4.2.3. *Let $g \in \mathrm{SL}(2)(k)$ be an element such that $g \notin B \cup w_0 B$. For every $i \in \{0, \ldots, p-2\}$*

$$X(i,g) := \mathrm{Dist}(\mathrm{SL}(2)_1 T_r) \otimes_{\mathrm{Dist}(\mathrm{SL}(2)_1)} Z(i)^{(g)}$$

is an indecomposable, quasi-simple $\mathrm{SL}(2)_1 T_r$-module belonging to a homogeneous tube.

Proof. As before, we put $\mathcal{G} := \mathrm{SL}(2)_1 T_r$ and $\mathcal{N} := \mathrm{SL}(2)_1$ for ease of notation. Since $\Omega^2_{\mathcal{N}}(Z(i)^{(g)}) \cong Z(i)^{(g)}$ an application of (2.3.4) implies

$$X(i,g) \cong \Omega^2_{\mathcal{G}}(X(i,g)).$$

Consequently, every indecomposable direct summand of $X(i,g)$ is periodic.

To determine the period of the indecomposable constituents, we consider the rank variety $\mathcal{V}_{\mathfrak{g}}(X(i,g))$. Since the group $\mathcal{G}/\mathcal{N}$ is infinitesimal, we may apply [47, (9.6)] to see that $X(i,g)|_{\mathcal{N}}$

affords a filtration with $Z(i)^{(g)}$ as unique filtration factor. Standard properties of rank varieties then yield

$$\mathcal{V}_{\mathfrak{g}}(X(i,g)) \subseteq \mathcal{V}_{\mathfrak{g}}(Z(i)^{(g)}) = k\mathrm{Ad}(g)(e).$$

From (4.2.1) we know that the rank varieties of the exceptional tubes are of the form ke, kf. By choice of g we have $\mathrm{Ad}(g)(e) \notin ke \cup kf$. Consequently, each indecomposable direct summand of $X(i,g)$ belongs to a homogeneous tube. Since $\dim_k X(i,g) = p^r$, Lemma 4.2.2 now implies the indecomposabilty of $X(i,g)$. Thus, $X(i,g)$ belongs to a homogeneous tube and is quasi-simple for dimension reasons. □

Remark. Let Λ be a $\mathbb{Z}$-graded Artin algebra. According to [29, (4.2)] the presence of one gradable module within a given AR-component Θ of Λ entails that all modules of Θ are gradable. The following example shows that this result has no analogue within the context of $\mathbb{Z}/(r)$-graded modules.

Let g be an element of $\mathrm{SL}(2) \setminus (B \cup w_0B)$. In virtue of (4.2.3) and (2.1.2), the restriction $X(i,g)|_{\mathrm{SL}(2)_1}$ is an indecomposable $X(T_r)$-graded $\mathrm{SL}(2)_1$-module with rank variety $\mathcal{V}_{\mathfrak{sl}(2)}(X(i,g)) = \mathrm{Ad}(g)(e)$. Accordingly, $X(i,g)|_{\mathrm{SL}(2)_1}$ belongs to a homogeneous tube with quasi-simple module $Z(i)^{(g)}$.

If $Z(i)^{(g)}$ also affords an $X(T_r)$-grading, then (2.2.1) shows that $Z(i)^{(g)}$ has the structure of an $\mathrm{SL}(2)_1T_r$-module. Since $\mathcal{V}_{\mathfrak{sl}(2)}(Z(i)^{(g)}) \neq ke, kf$, the module $Z(i)^{(g)}$ belongs to a homogeneous tube and (4.2.2) provides a natural number ℓ with $\dim_k Z(i) = \ell p^r$, a contradiction.

Given $i \in \{0,\dots,p-2\}$ and $g \in \mathrm{SL}(2)(k) \setminus (B \cup w_0B)$, we denote by $\Theta(i,g)$ the AR-component containing $X(i,g)$. In view of (4.2.3), $\Theta(i,g)$ is a homogeneous tube with variety $\mathcal{V}_{\mathfrak{sl}(2)}(\Theta(i,g)) = k\mathrm{Ad}(g)(e)$ and quasi-simple module $X(i,g)$.

Lemma 4.2.4. *Let $\Theta \subseteq \Gamma_s(\mathrm{SL}(2)_1T_r)$ be a homogeneous tube. Then the following statements hold:*
(1) *There exist $i \in \{0,\dots,p-2\}$ and $g \in \mathrm{SL}(2)(k) \setminus (B \cup w_0B)$ such that $\Theta = \Theta(i,g)$.*
(2) *$M \otimes_k k_\lambda \cong M$ for every $M \in \Theta$ and $\lambda \in X(\mu_{(p^{r-1})})$.*

Proof. Writing $\mathcal{G} := \mathrm{SL}(2)_1T_r$ and $\mathcal{N} := \mathrm{SL}(2)_1$ we begin by showing that (2) holds for every vertex of a component $\Theta(i,g)$, where $g \in \mathrm{SL}(2)(k) \setminus (B \cup w_0B)$. Let λ be an element of $X(\mu_{(p^{r-1})})$. Since $k_\lambda|_{\mathcal{N}}$ is the trivial module, the tensor identity [5, (3.3.3)] yields

$$X(i,g) \otimes_k k_\lambda \cong \mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}}(Z(i)^{(g)} \otimes_k k_\lambda) \cong \mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} Z(i)^{(g)} \cong X(i,g).$$

Thus, the auto-equivalence $t_\lambda : \mathrm{mod}\,\mathcal{G} \longrightarrow \mathrm{mod}\,\mathcal{G}\ ;\ M \mapsto M \otimes_k k_\lambda$ induces an automorphism of the stable Auslander-Reiten quiver which fixes $X(i,g)$. Consequently, t_λ is an automorphism of the homogeneous tube $\Theta(i,g)$. As t_λ fixes the endpoint of that quiver, we obtain $t_\lambda|_{\Theta(i,g)} = \mathrm{id}_{\Theta(i,g)}$. Thus, (2) holds for all vertices of $\Theta(i,g)$.

Now let M be an element of Θ. Since $\Omega^2_{\mathcal{G}}(M) \cong M$, Lemma 4.2.2 provides $i \in \{0,\dots,p-2\}$ and $\ell \in \mathbb{N}$ such that
(a) $\mathrm{Top}_{\mathcal{G}}(M) = \ell\tilde{L}(i)$, $\mathrm{Soc}_{\mathcal{G}}(M) \cong \ell\tilde{L}(p-2-i)$, and
(b) $\dim_k M = \ell p^r$.

According to [26, (2.2)], the variety $\mathcal{V}_{\mathfrak{g}}(M)$ is a line. Hence there exists $g \in \mathrm{SL}(2)(k)$ with $\mathcal{V}_{\mathfrak{g}}(M) = k\mathrm{Ad}(g)(e)$. As $M|_{\mathcal{N}}$ is indecomposable, Theorem 4.1.1 yields $M|_{\mathcal{N}} \cong g.W(\ell p^r+i)$. If $g \in B \cup w_0B$, then $\mathcal{V}_{\mathfrak{g}}(M) = ke, kf$, and Theorem 4.1.1 implies $M|_{\mathcal{N}} \cong w_0^s.W(\ell p^r+i)$ for $s \in \{0,1\}$. Lemma 2.1.3 provides $\lambda \in X(\mu_{(p^{r-1})})$ with $M \cong w_0^\ell.W(\ell p^r+i) \otimes_k k_\lambda$. As a result, M belongs to an exceptional tube, a contradiction. Consequently, $g \in \mathrm{SL}(2)(k) \setminus (B \cup w_0B)$.

Frobenius reciprocity implies

$$\dim_k \operatorname{Hom}_{\mathcal{G}}(X(i,g), L(j) \otimes_k k_\lambda) = \dim_k \operatorname{Hom}_{\mathcal{N}}(Z(i)^{(g)}, L(j)) = \delta_{i,j}$$

for every $\lambda \in X(\mu_{(p^{r-1})})$ and $j \in \{0, \dots, p-1\}$, so that $\operatorname{Top}_{\mathcal{G}}(X(i,g)) \cong \tilde{L}(i)$. In view of (4.2.2) and (4.2.3), the module $X_\ell \in \Theta(i,g)$ of quasi-length ℓ satisfies (a) and (b) along with $\mathcal{V}_{\mathfrak{g}}(X_\ell) = \mathcal{V}_{\mathfrak{g}}(M)$. Thus, $M|_{\mathcal{N}} \cong X_\ell|_{\mathcal{N}}$, and (1.3) provides $\lambda \in X(\mu_{(p^{r-1})})$ such that $M \cong X_\ell \otimes_k k_\lambda$. By the first part of our proof, we thus have $M \cong X_\ell$, whence $\Theta = \Theta(i,g)$. □

4.3. **Classification of Indecomposables.** We shall use the above information on the Auslander-Reiten quiver to determine the indecomposable $\mathrm{SL}(2)T_r$-modules. In view of (4.2.4) there exists for each $\ell > 0$, $i \in \{0, \dots, p-2\}$ and $g \in \mathrm{SL}(2)(k) \setminus (B \cup w_0 B)$ a unique indecomposable $\mathrm{SL}(2)_1 T_r$-module $X(i,g,\ell)$ such that $X(i,g,\ell)|_{\mathrm{SL}(2)_1} \cong g.W(\ell p^r + i)$. We thus obtain:

Theorem 4.3.1. *Let $C \subseteq \mathrm{SL}(2)(k)$ be a complete set of representatives for $\mathrm{SL}(2)(k)/B$ containing 1 and w_0. If M is a non-projective indecomposable $\mathrm{SL}(2)_1 T_r$-module, then M is isomorphic to exactly one of the following:*

(a) $V(d) \otimes_k k_\lambda$, $V(d)^* \otimes_k k_\lambda$, $V(r) \otimes_k k_\lambda$ *with* $\lambda \in X(\mu_{(p^{r-1})})$ *and* $d \geq p$, $d \not\equiv -1 \mod(p), 0 \leq r \leq p-2$ (Modules belonging to Euclidean components),

(b) $W(sp+i) \otimes_k k_\lambda$, $w_0.W(sp+i) \otimes_k k_\lambda$ *with* $\lambda \in X(\mu_{(p^{r-1})})$, $i \in \{0, \dots, p-2\}$ *and* $s \geq 1$ (Modules belonging to exceptional tubes),

(c) $X(i,g,\ell)$ *with* $g \in C \setminus \{1, w_0\}$, $0 \leq i \leq p-2$, *and* $\ell \geq 1$ (Modules belonging to homogeneous tubes).

Proof. We determine M according to the dimension of its rank variety $\mathcal{V}_{\mathfrak{sl}(2)}(M)$. If $\dim \mathcal{V}_{\mathfrak{sl}(2)}(M) = 2$, then a consecutive application of (2.1.2) and Theorem 4.1.1 yields $M|_{\mathcal{N}} \cong V(d), V(d)^*, V(r)$ with d and r as given. Since all these modules are $\mathrm{SL}(2)_1 T_r$-modules, we conclude from (2.1.3) the existence of $\lambda \in X(\mu_{(p^{r-1})})$ such that $M \cong V(d) \otimes_k k_\lambda$, $V(d)^* \otimes_k k_\lambda, V(r) \otimes_k k_\lambda$.

If $\mathcal{V}_{\mathfrak{sl}(2)}(M)$ is one-dimensional, then we first consider the case where $\mathcal{V}_{\mathfrak{sl}(2)}(M) = ke$, kf. By virtue of (4.2.4), M belongs to an exceptional tube, and (4.2.1) yields $M \cong w_0^\ell.W(sp+i) \otimes_k k_\lambda$. Alternatively, we have $\Omega^2_{\mathcal{G}}(M) \cong M$, and (4.2.4) provides an isomorphism $M \cong X(i,h,\ell)$ for some $h \in \mathrm{SL}(2)(k) \setminus (B \cup w_0 B)$. We write $h = gb$ for $g \in C \setminus \{1, w_0\}$ and $b \in B$. Since $Z(i)^{(b)} \cong Z(i)$, we have

$$\begin{aligned} X(i,h) &\cong \operatorname{Dist}(\mathrm{SL}(2)_1 T_r) \otimes_{\operatorname{Dist}(\mathrm{SL}(2)_1)} Z(i)^{(h)} \cong \operatorname{Dist}(\mathrm{SL}(2)_1 T_r) \otimes_{\operatorname{Dist}(\mathrm{SL}(2)_1)} Z(i)^{(gb)} \\ &\cong \operatorname{Dist}(\mathrm{SL}(2)_1 T_r) \otimes_{\operatorname{Dist}(\mathrm{SL}(2)_1)} Z(i)^{(g)} \cong X(i,g). \end{aligned}$$

This implies $\Theta(i,h) = \Theta(i,g)$ and hence $X(i,h,\ell) \cong X(i,g,\ell)$.

It remains to show that the modules belonging to the list are pairwise non-isomorphic. Since the modules listed in (a), (b) and (c) have different varieties, isomorphisms can only occur within those subsets. If $X(i,g,\ell) \cong X(j,h,s)$, then the isomorphisms $g.W(\ell p^r + i) \cong X(i,g,\ell)|_{\mathrm{SL}(2)_1} \cong X(j,h,s)|_{\mathrm{SL}(2)_1} \cong h.W(sp^r + j)$ in conjunction with Theorem 4.1.1 imply $g = h$, $i = j$ and $s = \ell$, as desired.

For modules belonging to (b), it suffices to show that $W(sp+i) \cong W(sp+i) \otimes_k k_\lambda$ implies $\lambda = \varepsilon$. Since $\operatorname{Soc}_{\mathrm{SL}(2)_1 T_r}(W(sp+i)) \cong sL(p-2-i)$, the desired property follows from [20, (5.1)]. The same argument works for (a). □

Remarks. (1) The description of $X(i,g,1)$ as an induced module is exceptional in that no other vertex of $\Theta(i,g)$ is of this form. Set $\mathcal{G} := \mathrm{SL}(2)_1T_r$ and $\mathcal{N} := \mathrm{SL}(2)_1$. In view of (2.3.2), the assumption

$$X(i,g,\ell) \cong \mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} M$$

implies that M is an indecomposable $\mathcal{N}$-module of Loewy length 2, with $\mathrm{Soc}_{\mathcal{N}}(M) \cong \ell L(p-2-i)$, and with $\dim_k M \equiv 0 \mod(p)$. By Theorem 4.1.1, the latter condition implies that M belongs to a homogeneous tube of $\Gamma_s(\mathcal{N})$. Thus, $\dim_k \mathcal{V}_{\mathfrak{sl}(2)}(M) = 1$, and the inclusion $k\mathrm{Ad}(g)(e) = \mathcal{V}_{\mathfrak{sl}(2)}(X(i,1)) = \mathcal{V}_{\mathfrak{sl}(2)}(X(i,g,\ell)) \subseteq \mathcal{V}_{\mathfrak{sl}(2)}(M)$ in conjunction with [26, (2.2)] yields

$$\mathcal{V}_{\mathfrak{sl}(2)}(M) = k\mathrm{Ad}(g)(e).$$

Theorem 4.1.1 now provides an isomorphism $M \cong g.W(\ell p+i)$.

Let $\Theta \subseteq \Gamma_s(\Lambda)$ be a homogeneous tube of the stable Auslander-Reiten quiver of a self-injective algebra Λ. We denote by V_n the unique vertex of Θ of quasi-length n. If V_1 is a *brick*, that is, if $\mathrm{End}_\Lambda(V_1) \cong k$, then basic properties of almost split sequences [4, (V.1)] inductively imply the formula

$$\dim_k \mathrm{Hom}_\Lambda(V_m,V_n) = \min\{m,n\}.$$

Setting $Y_\ell := g.W(\ell p+i)$ and $X_\ell := X(i,g,\ell)$, we have $X_\ell|_{\mathcal{N}} \cong Y_{\ell p^{r-1}}$. Since $Y_1 \cong Z(i)^{(g)}$ is a brick, Frobenius reciprocity yields

$$\dim_k \mathrm{Hom}_{\mathcal{G}}(\mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} Y_\ell, X_m) = \dim_k \mathrm{Hom}_{\mathcal{N}}(Y_\ell, Y_{mp^{r-1}}) = \min\{\ell, mp^{r-1}\}.$$

Thanks to (4.2.3), we have $X_1 \cong \mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} Y_1$, so that X_1 is also a brick. The assumption $X_\ell \cong \mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} Y_\ell$ thus implies

$$\min\{\ell,p^{r-1}\} = \dim_k \mathrm{Hom}_{\mathcal{G}}(\mathrm{Ind}_{\mathcal{N}}^{\mathcal{G}} Y_\ell, X_1) = \dim_k \mathrm{Hom}_{\mathcal{G}}(X_\ell,X_1) = \min\{\ell,1\} = 1,$$

implying $\ell = 1$ whenever $r \geq 2$.

(2) Consider the restriction functor $\mathrm{res}_{\mathrm{SL}(2)_1} : \mathrm{mod}\,\mathrm{SL}(2)_1T_r \longrightarrow \mathrm{mod}\,\mathrm{SL}(2)_1$. As noted in the proof of Theorem 4.3.1, the modules belonging to Euclidean components or execptional tubes are regular, and (3.1.5) ensures that $\mathrm{res}_{\mathrm{SL}(2)_1}$ sends these components to Euclidean components and homogeneous tubes of $\Gamma_s(\mathrm{SL}(2)_1)$, respectively. According to (4.2.4) and (3.1.4) the restriction functor will not send almost split sequences of homogeneous tubes to almost split sequences. Let us illustrate this fact: Since $\Theta(i,g)$ is a homogeneous tube, we have, for each $\ell \geq 2$, an almost split sequence

$$(0) \longrightarrow X(i,g,\ell) \longrightarrow X(i,g,\ell+1) \oplus X(i,g,\ell-1) \longrightarrow X(i,g,\ell) \longrightarrow (0),$$

whose restriction is an exact sequence

$$(0) \longrightarrow g.W(\ell p^r+i) \longrightarrow g.W((\ell+1)p^r+i) \oplus g.W((\ell-1)p^r+i) \longrightarrow g.W(\ell p^r+i) \longrightarrow (0)$$

of $\mathrm{SL}(2)_1$-modules. In view of Lemma 4.1.2, these modules belong to a homogeneous tube of $\Gamma_s(\mathrm{SL}(2)_1)$, and we have $\mathrm{q}\ell(g.W(sp^r+i)) = sp^{r-1}$. Thus, for $r \geq 2$, the above sequences are not almost split.

In the theory of connected algebraic groups the Frobenius kernels are used to "approximate" the full group. The following Corollary, which can also be proved by combining Theorem 4.3.1 with [18, (2.4)], illustrates this process within our context.

Corollary 4.3.2. *Let M be a finite-dimensional non-projective indecomposable $\mathrm{SL}(2)_1T$-module. Then M is isomorphic to exactly one of the following modules:*

(a) $V(d)\otimes_k k_\lambda$, $V(d)^*\otimes_k k_\lambda$, $V(r)\otimes_k k_\lambda$ *with* $\lambda \in X(T)$ *and* $d \geq p$, $d \not\equiv -1 \mod(p), 0 \leq r \leq p-2$,

(b) $W(sp+i)\otimes_k k_\lambda$, $w_0.W(sp+i)\otimes_k k_\lambda$ *with* $\lambda \in X(T)$, $i \in \{0,\dots,p-2\}$ *and* $s \geq 1$.

Proof. Recall that $\mathrm{SL}(2)_1T_r = (\mathrm{SL}(2)_1T)_r$ is the r-th Frobenius kernel of the connected group $\mathcal{G} := \mathrm{SL}(2)_1T$. As M is finite-dimensional, an application of [36, (I.9.8(6))] provides $\ell \in \mathbb{N}$ such that

$$\mathrm{End}_{\mathcal{G}_r}(M) = \mathrm{End}_{\mathcal{G}}(M) \qquad \forall\ r \geq \ell.$$

In particular, $M|_{\mathcal{G}_r}$ is indecomposable and (4.2.2) implies that $M|_{\mathcal{G}_r}$ does not belong to a homogeneous tube for all $r \geq \max\{\dim_k M, \ell\}$. In view of (4.3.1), the restriction $M|_{\mathrm{SL}(2)_1}$ thus coincides with that of one of the $\mathcal{G}$-modules listed in (a) and (b). The result is now a direct consequence of [35, Thm.2] (see also [28, Thm.4.1]). □

Remark. The results of [18, §3] imply that the modules of type (a) and (b) belong to components of $\Gamma_s(\mathrm{SL}(2)_1T)$ of types $\mathbb{Z}[A_\infty^\infty]$ and $\mathbb{Z}[A_\infty]$, respectively.

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