SUBGROUP VARIETIES AND REPRESENTATIONS OF INFINITESIMAL GROUPS

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1. Motivation

In these talks we shall be concerned with those aspects of the representation theory of certain finite dimensional algebras that are motiviated by the problem of classifying indecomposable modules. Since the determination of the simple modules is often already difficult enough, one can in general not hope to solve this problem in a naive sense. However, the classification problem has resulted in an important subdivision of the category of algebras, which will be the general theme underlying these lectures.

Throughout, we will be working over an algebraically closed field k. Unless mentioned otherwise, all algebras and modules are assumed to be finite-dimensional over k. Let Λ be an associative algebra. According to a fundamental theorem by Drozd, Λ belongs to one of the following three classes:

Definition. The algebra Λ is
(1) representation-finite, if it affords only finitely many isoclasses of indecomposable modules,
(2) tame, if it is not representation-finite, and if for every $d > 0$ all but finitely many isoclasses of indecomposable modules of dimension d occur in finitely many one-parameter families,
(3) wild, if the module category of Λ is at least as complicated as that of the free algebra $k\langle X, Y\rangle$.

In the latter case, the module category of Λ is at least as complicated as that of any other algebra, so that a classification of indecomposables is considered hopeless.

In view of Drozd's result one may ask what this subdivision looks like for certain classes of algebras. As the representation type of an algebra is an invariant of its Morita equivalence class, the criteria one is looking for are often given in terms of the basic algebras. For instance, if Λ is hereditary, then Λ is Morita equivalent to the path algebra $k[Q]$ of its Ext-quiver Q, and a connected algebra $k[Q]$ is representation-finite or tame according as Q is a simply-laced Dynkin diagram or a simply-laced extended Dynkin diagram. In either case, the indecomposable modules can be classifield, with the positive roots of the associated root system corresponding to the indecomposables for representation-finite Λ.

While results of this type are very satisfactory from the point of view of abstract representation theory, they do rely on the knowledge of the quiver and the relations of the given algebra. However, even if an algebra is basic to begin with, the given presentation may not be suitable for our purposes. Let me illustrate this point by considering an easy example.

Example. Let $\mathrm{char}(k) = p > 0$, and consider the algebra given by

$$\Lambda = k\langle t, x\rangle/(tx - xt - x, t^p - t, x^p).$$

This is the natural presentation of the restricted enveloping algebra of the two-dimensional, non-abelian Lie algebra. The bound quiver presentation we are looking for is

$$\Lambda \cong k[\tilde{A}_{p-1}]/\mathrm{rad}(k[\tilde{A}_{p-1}])^p,$$

where the quiver $\tilde{A}_{p-1}$ is the clockwise oriented circle with p vertices.

In these lectures we will see how a combination of geometric and representation theoretic methods affords the transition to such a more complicated presentation for certain Hopf algebras of positive characteristic.

The classical examples of Hopf algebras are of course the group algebras of finite groups. Here we have the following situation:

- $k[G]$ group algebra of a finite group G, $\mathrm{char}(k) = p > 0$.
- $\mathcal{B} \subset k[G]$ block, $\mathcal{B} \rightsquigarrow D_{\mathcal{B}} \subset G$ defect group, a p-group.
- $\mathcal{B}$ is representation-finite $\Leftrightarrow D_{\mathcal{B}}$ cyclic.
- $\mathcal{B}$ is tame $\Leftrightarrow p = 2$, and $D_{\mathcal{B}}$ is dihedral, semidihedral, or generalized quaternion.

The defect groups thus provide a measure of the complexity of the module category of a block. Since the defect group of the principal block is a Sylow p-subgroup, this block is generally thought of as the most complicated block of $k[G]$. Moreover, the basic algebras of the representation-finite and tame blocks are completely understood. The representation-finite blocks were determined in the late sixties. Almost 20 years later, Erdmann classified blocks of tame representation type via the stable Auslander-Reiten quiver.

The module category of $k[G]$ enjoys a special feature, namely the presence of a tensor product. This has the effect that the blocks of the associative algebra $k[G]$ are not completely independent. For arbitrary Hopf algebras, the impact of the tensor product is in general not clear, but we will see some examples where one has a better understanding. We will focus on cocommutative Hopf algebras, because these may be interpreted as "group algebras" of finite algebraic k-groups.

2. Infinitesimal Groups and their Hopf algebras

We let M_k and Gr be the categories of not necessarily finite dimensional commutative k-algebras and groups, respectively. A representable functor

$$\mathcal{G} : M_k \longrightarrow \mathrm{Gr} \quad ; \quad R \mapsto \mathcal{G}(R)$$

is called an affine group scheme. Hence there exists a commutative k-algebra $\mathcal{O}(\mathcal{G})$ such that $\mathcal{G}(R)$ is the set of algebra homomorphisms $\mathcal{O}(\mathcal{G}) \longrightarrow k$ for every $R \in M_k$. By Yoneda's Lemma, the group functor structure of $\mathcal{G}$ corresponds to a Hopf algebra structure of $\mathcal{O}(\mathcal{G})$, which makes $\mathcal{O}(\mathcal{G})$ a commutative Hopf algebra.

We say that $\mathcal{G}$ is an algebraic group if the representing object $\mathcal{O}(\mathcal{G})$ is finitely generated. If $\mathcal{O}(\mathcal{G})$ is finite dimensional, then $\mathcal{G}$ is referred to as a finite algebraic group. Suppose that $\mathcal{G}$ is an algebraic group. Since k is algebraically closed, we have

$$\mathcal{G} \text{ is a finite algebraic group} \;\Leftrightarrow\; \mathcal{G}(k) \text{ is a finite group.}$$

If $\mathcal{G}$ is finite, then

$$H(\mathcal{G}) := \mathcal{O}(\mathcal{G})^*$$

is a finite dimensional, cocommutative Hopf algebra, the so-called algebra of measures on $\mathcal{G}$. In fact, the correspondence

$$\mathcal{G} \mapsto H(\mathcal{G})$$

provides an equivalence between the categories of finite algebraic groups and finite dimensional cocommutative Hopf algebras. In this equivalence, the group algebras of finite groups correspond to the reduced finite algebraic groups. An algebraic group $\mathcal{G}$ is called reduced or smooth if its function algebra $\mathcal{O}(\mathcal{G})$ does not possess any non-trivial nilpotent elements. If $\mathrm{char}(k) = 0$, then Cartier's Theorem asserts that any algebraic group is reduced, thus all cocommutative Hopf algebras are semisimple in this case. We shall therefore henceforth assume that $\mathrm{char}(k) = p > 0$.

For fields of positive characteristic, we also have the so-called infinitesimal groups. By definition a finite algebraic k-group $\mathcal{G}$ is infinitesimal if $\mathcal{O}(\mathcal{G})$ is local. Let $\mathcal{G}$ be an algebraic group. Then we have

$$\mathcal{G} \text{ is an infinitesimal group } \Leftrightarrow \mathcal{G}(k) = \{1\}.$$

Thus, we will be studying cocommutative Hopf algebras, whose dual algebras are local. Since every finite algebraic group is a semi-direct product of a reduced group operating on an infinitesimal group, our class of Hopf algebras complements the group algebras of finite groups.

Examples. Let $r \in \mathbb{N}$.

(1) For $n\mathbb{N}$ let $\mathrm{GL}(n)_r : M_k \longrightarrow \mathrm{Gr}$ be given by

$$\mathrm{GL}(n)_r(R) := \{(\zeta_{ij}) \in \mathrm{GL}(n)(R) \ ; \ \zeta_{ij}^{p^r} = \delta_{ij}\}.$$

Every infinitesimal group $\mathcal{G}$ is a subgroup of a suitable $\mathrm{GL}(n)_r$.

(2) Consider $\mu_{p^r} := \mathrm{GL}(1)_r$, that is,

$$\mu_{p^r}(R) := \{x \in R^\times \ ; \ x^{p^r} = 1\} \subset R^\times.$$

Then we have

$$H(\mu_{p^r}) \cong k^{p^r}.$$

(3) Let $\alpha_{p^r} : M_k \longrightarrow \mathrm{Gr}$ be given by

$$\alpha_{p^r}(R) := \{x \in R \ ; \ x^{p^r} = 0\} \subset (R, +).$$

Then we have

$$H(\alpha_{p^r}) \cong k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p).$$

As an algebra $H(\alpha_{p^r})$ is the group algebra of an elementary abelian p-group of rank r. In particular, we have

$$H(\alpha_{p^r}) \text{ is representation} - \text{finite} \Leftrightarrow r = 1 \ ; \ H(\alpha_{p^r}) \text{ is tame} \Leftrightarrow p = 2 \text{ and } r = 2.$$

(4) For $m = np^r$ with $(n,p) = 1$ we consider

$$\mathcal{Q}_{(m)}(R) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)(R) \ ; \ a^m = 1 = d^m \ , \ b^p = 0 = c^p\}.$$

Then we have $\mathcal{Q}_{(m)}(k) = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \; ; \; a^n = 1\}$. Thus, $\mathcal{Q}_{(m)}$ is a finite algebraic group, which is infinitesimal if and only if $n = 1$. The infinitesimal groups $\mathcal{Q}_{(p^r)}$ will play an important rôle in the sequel, and we write $\mathcal{Q}_{[r]} := \mathcal{Q}_{(p^r)}$. The experts know that $\mathcal{Q}_{[r]} = \mathrm{SL}(2)_1 T_r$ is the product of the first Frobenius kernel of SL(2) with the r-th Frobenius kernel of its standard maximal torus T.

The first three examples differ from $\mathcal{Q}_{[r]}$ in that for $r \geq 2$ the latter group is not a Frobenius kernel of a smooth (reduced) algebraic group. To define Frobenius kernels, we recall that every algebraic group may be embedded into a suitable $\mathrm{GL}(n)$.

Let $\mathcal{G} \subset \mathrm{GL}(n)$ be an algebraic group, $r \in \mathbb{N}$. Then

$$\mathcal{G}_r := \mathcal{G} \cap \mathrm{GL}(n)_r$$

is the r-th Frobenius kernel of $\mathcal{G}$. Thus, $\mathcal{G}_r$ is an infinitesimal group.

If $\mathcal{G}$ is infinitesimal, then there exists $r \in \mathbb{N}$ with $\mathcal{G} = \mathcal{G}_r$ and

$$\mathrm{ht}(\mathcal{G}) := \min\{r \; ; \; \mathcal{G}_r = \mathcal{G}\}$$

is called the height of $\mathcal{G}$.

The Hopf algebra $H(\mathcal{G})$ possesses a co-unit $\varepsilon : H(\mathcal{G}) \longrightarrow k$. The unique block $\mathcal{B}_0(\mathcal{G}) \subset H(\mathcal{G})$ with $\varepsilon(\mathcal{B}_0(\mathcal{G})) \neq (0)$ is called the principal block of $H(\mathcal{G})$.

Problem. Let $\mathcal{G}$ be an infinitesimal group. When is $\mathcal{B}_0(\mathcal{G})$ representation-finite or tame?

Our problem has two interrelated aspects, namely the determination of the group $\mathcal{G}$, that is, the possible Hopf structures, as well as the structure of the basic algebra of $\mathcal{B}_0(\mathcal{G})$. In dealing with this problem one has to be aware of some special features of the representation theory of infinitesimal groups.

(a) The powerful tools from the modular representation theory of finite groups do not work in our context. In particular, there is no analogue of the Mackey decomposition theorem, and we do not have defect groups at our disposal.

(b) We will have to deal with arbitrary infinitesimal groups rather than Frobenius kernels of smooth groups. The failure of the Lie-Kolchin Theorem in this general context illustrates the complications that can arise. In fact, we will see later that tame principal blocks rarely occur for Frobenius kernels of smooth groups.

Roughly speaking, we shall pursue the following strategy. Using geometric tools we reduce the problem to the consideration of small examples that are amenable to the methods from abstract representation theory. The latter will enable us to see which of the examples have the desired representation type and what their quivers and relations are. Moreover, representation theoretic methods ensure that the examples are of a generic nature.

Let me conclude this section by stating the analogue of Maschke's Theorem in the context infinitesimal groups. Since the tensor product of a module with a projective module is projective, a Hopf algebra $H(\mathcal{G})$ is semisimple if and only if its principal block is simple.

Theorem 2.1 (Nagata). *Let $\mathcal{G}$ be an infinitesimal group. Then $H(\mathcal{G})$ is semisimple if and only if $\mathcal{G}$ contains no subgroup of type α_p.*

The groups satisfying Nagata's theorem are the so-called diagonalizable or multiplicative infinitesimal groups. By general theory, these are direct products of suitable μ_{p^r}'s. In particular, such groups are commutative, and their Hopf algebras are just of the form k^{p^n}.

A subgroup $\alpha_p \subset \mathcal{G}$ is automatically contained in the first Frobenius kernel $\mathcal{G}_1$. Thus, Nagata's Theorem states that $H(\mathcal{G})$ is semisimple if and only if $H(\mathcal{G}_1)$ enjoys this property. One might hope in general that the representation type of an infinitesimal group $\mathcal{G}$ can be decided by looking at $\mathcal{G}_1$ and by studying subgroups of type α_p. This is not quite true, as one also needs to consider subgroups of type α_{p^2}. However, the variety of subgroups of type α_p will provide valuable information.

3. Restricted Lie Algebras

3.1. **General facts.** Nagata's Theorem suggests that we take a closer look at infinitesimal groups of height 1. It turns out that this is equivalent to studying restricted Lie algebras. Given an infinitesimal group $\mathcal{G}$, we let $\Delta : H(\mathcal{G}) \longrightarrow H(\mathcal{G}) \otimes_k H(\mathcal{G})$ denote the comultiplication of $H(\mathcal{G})$. Then

$$\mathrm{Lie}(\mathcal{G}) := \{x \in H(\mathcal{G}) \ ; \ \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

is called the Lie algebra of $\mathcal{G}$. Writing $[x, y] = xy - yx$, we have

(a) $[x, y] \in \mathrm{Lie}(\mathcal{G})$ for every $x, y \in \mathrm{Lie}(\mathcal{G})$, and

(b) $x^p \in \mathrm{Lie}(\mathcal{G})$ for every $x \in \mathrm{Lie}(\mathcal{G})$.

A subspace $\mathfrak{g} \subset \Lambda$ of an associative k-algebra Λ satisfying (a) and (b) is called a restricted Lie algebra. These algebras may also be defined axiomatically. Then a restricted Lie algebra is a pair $(\mathfrak{g}, [p])$ consisting of an abstract Lie algebra $\mathfrak{g}$ and an operator $\mathfrak{g} \longrightarrow \mathfrak{g} \ ; \ x \mapsto x^{[p]}$ that satisfies the formal properties of an associative p-power.

Given such a restricted Lie algebra $(\mathfrak{g}, [p])$ with universal enveloping algebra $U(\mathfrak{g})$, one defines the restricted enveloping algebra via

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\}).$$

The algebra $U_0(\mathfrak{g})$ inherits the Hopf algebra structure from $U(\mathfrak{g})$. Thus, it is a Frobenius algebra of dimension $p^{\dim_k \mathfrak{g}}$, whose Nakayama automorphism $\mu : U_0(\mathfrak{g}) \longrightarrow U_0(\mathfrak{g})$ is given by

$$\mu(x) = x - \mathrm{tr}(\mathrm{ad}\, x)1 \quad \forall \ x \in \mathfrak{g},$$

where $\mathrm{ad}\, x : \mathfrak{g} \longrightarrow \mathfrak{g} \ ; \ y \mapsto [x, y]$ is the left multiplication effected by x. Hence $U_0(\mathfrak{g})$ is symmetric if and only if $\mathrm{tr}(\mathrm{ad}\, x) = 0$ for every $x \in \mathfrak{g}$. The connection with infinitesimal groups of height 1 is given by:

Proposition 3.1. *Let $\mathcal{G}$ be an infinitesimal group of height* 1*. Then there exists an isomorphism*

$$H(\mathcal{G}) \cong U_0(\mathrm{Lie}(\mathcal{G}))$$

of Hopf algebras.

The subgroups $\alpha_p \subset \mathcal{G}$ correspond to the subspaces $(0) \neq kx \subset \mathfrak{g}$ such that $x^{[p]} = 0$.

3.2. **Important examples.** Many of our results to follow will depend on the following basic examples involving solvable and simple restricted Lie algebras.

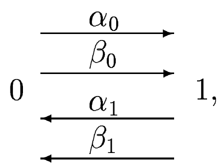
Examples. (1) Let V be a k-vector space, $t : V \longrightarrow V$ a linear transformation satisfying $t^p = t$. Then $\mathfrak{g}(t, V) := kt \oplus V$ obtains the structure of a restricted Lie algebra via

$$[(\alpha t, v), (\beta t, w)] := (0, \alpha t(w) - \beta t(v)) \ \ ; \ \ (\alpha t, v)^{[p]} = (\alpha^p t, \alpha^{p-1} t^{p-1}(v)).$$

For the corresponding restricted enveloping algebra one can compute the Gabriel quiver and the relations. Abstract representation theory then shows:

- $U_0(\mathfrak{g}(t, V))$ is representation-finite $\Leftrightarrow \dim_k V \leq 1$.
- $U_0(\mathfrak{g}(t, V))$ is tame $\Leftrightarrow \dim_k V = 2$ and $p = 2$.

(2) Let $\mathfrak{g} := \mathfrak{sl}(2)$ be the restricted Lie algebra of trace zero (2×2)-matrices. The restricted enveloping algebra $U_0(\mathfrak{sl}(2))$ possesses exactly p simple modules $L(0), \ldots, L(p-1)$ with $\dim_k L(i) = i+1$. In the early 1980's Fischer and Rudakov independently computed the quiver and the relations of $U_0(\mathfrak{sl}(2))$. For $p \geq 3$, the algebra $U_0(\mathfrak{sl}(2))$ has blocks $\mathcal{B}_0, \ldots, \mathcal{B}_{\frac{p-3}{2}}$ each possessing two simple modules $L(i)$ and $L(p-2-i)$. There is one additional simple block $\mathcal{B}_{p-1}$ belonging to the Steinberg module $L(p-1)$. The non-simple blocks have bound quiver presentation given by the quiver Δ_1:



and relations defining the ideal $J \subset k[\Delta_1]$ generated by

$$\{\beta_{i+1}\alpha_i - \alpha_{i+1}\beta_i, \ \ \alpha_{i+1}\alpha_i, \ \ \beta_{i+1}\beta_i \ \ ; \ \ i \in \mathbb{Z}/(2)\}.$$

These examples will turn out to be of major importance for our determination of the tame infinitesimal groups of odd characteristic. The first example is essentially the reason for the validity of the following result:

Proposition 3.2. *Suppose that $p \geq 3$, and let $\mathcal{G}$ be a solvable infinitesimal group. Then $\mathcal{B}_0(\mathcal{G})$ is either representation-finite or wild.*

Turning to the second example, we observe that the algebra $k[\Delta_1]/J$ is tame. In fact, our algebra belongs to an important class of tame algebras, the so-called special biserial algebras, which we will discuss again later. The uniformity of the presentation of these blocks is not accidental; it is a consequence of the so-called translation principle, which affords the passage between certain blocks. Roughly speaking, one proceeds as follows: Given two blocks $\mathcal{B}, \mathcal{C}$ of $U_0(\mathfrak{g})$ and a simple module S, one considers the functor

$$\mathrm{Tr} : \mathrm{mod}\,\mathcal{B} \longrightarrow \mathrm{mod}\,\mathcal{C} \ \ ; \ \ M \mapsto e_{\mathcal{C}} \cdot (S \otimes_k M).$$

Here $e_{\mathcal{C}} \in U_0(\mathfrak{g})$ is the central idempotent belonging to the block $\mathcal{C}$. Under certain compatibility conditions on $\mathcal{B}, \mathcal{C}$ and S, this functor is in fact a Morita equivalence. The easiest instance of the translation principle is given by one-dimensional modules. In particular, all blocks of basic cocommutative Hopf algebras (i.e., those corresponding to the so-called trigonalizable group schemes) are isomorphic.

4. Nullcones

The aforementioned geometric techniques rest on varieties that are defined by classes of unipotent and multiplicative subgroups of infinitesimal groups.

Our first tool is given by nullcones. These are special cases of the so-called rank varieties that were first studied for finite groups in connection with cohomological support varieties. In that context, rank varieties are defined by means of elementary abelian p-groups; for infinitesimal groups one uses the groups α_{p^r}. We are going to simplify matters by considering only the case $r = 1$, which turns out to be sufficient for most of our purposes. Accordingly, we confine our attention to restricted Lie algebras.

Definition. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. The conical variety

$$\mathcal{V}_{\mathfrak{g}} := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}$$

is called the nullcone of $\mathfrak{g}$.

By our earlier observations, the projective variety $\mathrm{Proj}(\mathcal{V}_{\mathfrak{g}})$ is the variety of subgroups $\alpha_p \subset \mathcal{G}$.

Before giving examples, let me quote an important theorem, which I have tailored to our needs. The result is essentially based on work by Heller, Rickard, Friedlander-Parshall, and Jantzen.

Theorem 4.1. *Let $\mathcal{G}$ be an infinitesimal group with Lie algebra $\mathfrak{g}$.*
(1) *If $\mathcal{B}_0(\mathcal{G})$ is representation-finite, then* $\dim \mathcal{V}_{\mathfrak{g}} \leq 1$.
(2) *If $\mathcal{B}_0(\mathcal{G})$ is tame, then* $\dim \mathcal{V}_{\mathfrak{g}} \leq 2$.

The foregoing result provides in principle a connection between the representation type of $H(\mathcal{G})$ and the structure of the underlying group $\mathcal{G}$. However, these conditions are only necessary, so the classification of the relevant groups requires additional information.

Examples. (1) Let $\mathfrak{g}(t, V)$ be as before, then $\mathcal{V}_{\mathfrak{g}(t,V)} = V$, so that we retrieve part of our earlier result.

(2) Let $\mathcal{G} = \mathcal{Q}_{[r]}$, so that $\mathfrak{g} = \mathfrak{sl}(2)$. Then $\mathcal{V}_{\mathfrak{sl}(2)}$ is the variety of nilpotent (2×2)-matrices. Accordingly, we have $\dim \mathcal{V}_{\mathfrak{sl}(2)} = 2$. The algebra $\mathcal{B}_0(\mathcal{Q}_{[r]})$ is tame.

By way of illustration, let us take a quick look at the classical case of a Frobenius kernel $\mathcal{G}_r$ of a smooth connected, reductive algebraic group $\mathcal{G}$ of characteristic $p \geq 3$. The algebraic group $G := \mathcal{G}(k)$ acts on $\mathfrak{g} := \mathrm{Lie}(\mathcal{G}_r) = \mathrm{Lie}(\mathcal{G})$ via the adjoint representation

$$\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g}),$$

which for $G = \mathrm{GL}(n)(k)$ is just the conjugation of G on $\mathrm{Mat}_n(k)$. Since G operates via homomorphisms of restricted Lie algebras, the variety $\mathcal{V}_{\mathfrak{g}}$ is G-invariant.

If $\mathcal{B}_0(\mathcal{G}_r)$ is representation-finite, then $\dim \mathcal{V}_{\mathfrak{g}} \leq 1$, and each irreducible component of $\mathcal{V}_{\mathfrak{g}}$ is a G-invariant line kx. Thus, $kx \subset \mathcal{V}_{\mathfrak{g}}$ is a unipotent ideal of $\mathfrak{g}$, which implies $x = 0$. It now follows from Nagata's Theorem that $\mathcal{B}_0(\mathcal{G}_r) \cong k$.

By refining the above arguments somewhat one obtains the following result:

Proposition 4.2. *Let $p \geq 3$ and let $\mathcal{G}$ be a smooth, reductive algebraic group, $r \in \mathbb{N}$. Let $\mathcal{B} \subset H(\mathcal{G}_r)$ be a block.*
(1) *If $\mathcal{B}$ is representation-finite, then $\mathcal{B}$ is simple.*
(2) *If $\mathcal{B}$ is tame, then $\mathcal{B}$ is Morita equivalent to $k[\Delta_1]/J$.*

Let me conclude this section with another result, which underscores the utility of Theorem 4.1. From Nagata's Theorem we know that the assumption $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G})} = 0$ entails the diagonalizability of the infinitesimal group $\mathcal{G}$. The following result generalizes this to groups with one-dimensional nullcones.

Proposition 4.3. *Let $\mathcal{G}$ be an infinitesimal group. If $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G})} \leq 1$, then $\mathcal{G}$ is supersolvable.*

Thus, for $p \geq 3$, a combination of (4.1), (4.3), and (3.2) shows that the nullcones of infinitesimal groups with tame principal blocks are two-dimensional.

5. Varieties of Tori

We have seen in the preceding section that nullcones in conjunction with the adjoint representation are a useful tool for the determination of the representation type in case the underlying group is a Frobenius kernel of a smooth reductive group. For an arbitrary infinitesimal group, we do not have such an action as $\mathcal{G}(k) = \{1\}$. The problem is that arbitrary restricted Lie algebras do not behave as well as those associated to smooth groups, whose structure theory parallels that of the complex Lie algebras. One basic tool of the theory is the root space decomposition relative to a maximal torus. In contrast to Lie algebras of smooth groups, the maximal tori of arbitrary restricted Lie algebras need not be conjugate under their groups of automorphisms. Accordingly, the information encoded in the root space decomposition of a maximal torus is highly sensitive to its choice. We address this problem by studying generic properties of families of tori of a given isomorphism type.

A restricted Lie algebra $(\mathfrak{g}, [p])$ is a torus if $\mathcal{V}_{\mathfrak{g}} = \{0\}$. Tori are abelian restricted Lie algebras, whose representations decompose into direct sums of eigenspaces.

Definition. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a torus. Then

$$\mathcal{T}_{\mathfrak{g}} := \{\varphi \in \mathrm{Lie}_p(\mathfrak{t}, \mathfrak{g}) \ ; \ \varphi \text{ is injective}\}$$

is called the variety of embeddings from $\mathfrak{t}$ to $\mathfrak{g}$.

Due to the complexity of the defining equations one actually has to consider an affine algebraic scheme, whose rational points form the variety $\mathcal{T}_{\mathfrak{g}}$. I'll skip these technicalities here.

Theorem 5.1. (1) *$\mathcal{T}_{\mathfrak{g}}$ is a smooth, affine algebraic variety.*
(2) *If $\mathcal{X} \subset \mathcal{T}_{\mathfrak{g}}$ is an irreducible component, then*

$$\dim \mathcal{X} = \dim_k \mathfrak{g} - \dim_k C_{\mathfrak{g}}(\varphi(\mathfrak{t})) \qquad \forall\ \varphi \in \mathcal{X}.$$

Here

$$C_{\mathfrak{g}}(\varphi(\mathfrak{t})) := \{v \in \mathfrak{g} \ ; \ [v,w] = 0 \ \ \forall \ w \in \varphi(\mathfrak{t})\}$$

denotes the centralizer of $\varphi(\mathfrak{t})$ in $\mathfrak{g}$. In particular, we see that the dimensions of the centralizers of the points of $\mathcal{X}$ are constant.

Since $\mathcal{T}_{\mathfrak{g}}$ is smooth, the irreducible components are the connected components. There thus exists exactly one component $\mathcal{X}_{\mathfrak{t}}$, such that the given embedding $\mathfrak{t} \hookrightarrow \mathfrak{g}$ belongs to $\mathcal{X}_{\mathfrak{t}}$. Since $\mathcal{X}_{\mathfrak{t}} \cong \mathrm{Spec}_k(A)(k)$ is affine, there is an embedding $j : \mathfrak{t} \hookrightarrow \mathfrak{g} \otimes_k A$ of restricted Lie algebras over k, the so-called universal embedding of $\mathcal{X}_{\mathfrak{t}}$, from which all elements of $\mathcal{X}_{\mathfrak{t}}$ can be obtained via specialization. Here $\mathfrak{g} \otimes_k A$ obtains the structure of a restricted Lie algebra via

$$(v \otimes a)^{[p]} := v^{[p]} \otimes a^p \qquad \forall \ v \in \mathfrak{g}, \ a \in A.$$

Every element $\varphi \in \mathcal{X}_{\mathfrak{t}}$ is given by

$$\varphi = (\mathrm{id}_{\mathfrak{g}} \otimes x) \circ j \quad , \quad x \in \mathrm{Spec}_k(A)(k) = \mathcal{X}_{\mathfrak{t}}.$$

We put $\tilde{\mathfrak{g}} := \mathfrak{g} \otimes_k A$, and let $\mathfrak{t}$ operate on $\tilde{\mathfrak{g}}$ via j. Since $\mathfrak{t}$ is a torus, every $\mathfrak{t}$-module is a direct sum of one-dimensional modules, and we obtain a root space decomposition

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_0 \oplus \bigoplus_{\alpha \in R} \tilde{\mathfrak{g}}_\alpha \quad ; \quad \tilde{\mathfrak{g}}_\alpha := \{v \in \tilde{\mathfrak{g}} \ ; \ [j(t), v] = \alpha(t) v \ \ \forall \ t \in \mathfrak{t}\}.$$

The set $R \subset \mathfrak{t}^* \setminus \{0\}$ is finite, and $\tilde{\mathfrak{g}}_\alpha$ is an A-submodule of $\tilde{\mathfrak{g}}$ for $\alpha \in R \cup \{0\}$.

For each A-direct summand $P \subset \tilde{\mathfrak{g}}$ and $x \in \mathrm{Spec}_k(A)(k)$ we put $P(x) := (\mathrm{id}_{\mathfrak{g}} \otimes x)(P) \subset \mathfrak{g}$ and obtain an algebraic family $(P(x))_{x \in \mathcal{X}_{\mathfrak{t}}}$ of subspaces of $\mathfrak{g}$. By varying the tori, the root spaces of $\tilde{\mathfrak{g}}$ provide algebraic families of root space decompositions of $\mathfrak{g}$.

Lemma 5.2. *Let* $P = \bigoplus_{\alpha \in S} \tilde{\mathfrak{g}}_\alpha$ *for some* $S \subset R \cup \{0\}$.

(1) *The map* $\mathcal{X}_{\mathfrak{t}} \longrightarrow \mathbb{N}_0 \ ; \ x \mapsto \dim P(x) \cap \mathcal{V}_{\mathfrak{g}}$ *is upper semicontinuous.*
(2) *If* $\mathrm{gen.}\dim P(x) \cap \mathcal{V}_{\mathfrak{g}} = \dim \mathcal{V}_{\mathfrak{g}}$, *then* $\dim \bigcap_{x \in \mathcal{X}_{\mathfrak{t}}} P(x) \cap \mathcal{V}_{\mathfrak{g}} = \dim \mathcal{V}_{\mathfrak{g}}$.
(3) $\bigcap_{x \in \mathcal{X}_{\mathfrak{t}}} P(x)$ *is an ideal of* $\mathfrak{g}$.

Let me illustrate these techniques by sketching a result that is somewhat weaker than Proposition 4.3.

Example. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra with representation-finite principal block $\mathcal{B}_0(\mathfrak{g}) \subset U_0(\mathfrak{g})$. Then $\mathfrak{g}$ is solvable. We use induction on $\dim_k \mathfrak{g}$.

- $\mathfrak{t} \subset \mathfrak{g}$ maximal torus; $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_0 \oplus \bigoplus_{\alpha \in R} \tilde{\mathfrak{g}}_\alpha$.
- Theorem 4.1 $\Rightarrow \dim \mathcal{V}_{\mathfrak{g}} \leq 1$.
- $\dim \mathcal{V}_{\mathfrak{g}} = 0 \Rightarrow \mathfrak{g}$ is a torus. Hence: $\dim \mathcal{V}_{\mathfrak{g}} = 1$.
- Let $P := \tilde{\mathfrak{g}}_0$, $d := \mathrm{gen.}\dim P(x) \cap \mathcal{V}_{\mathfrak{g}}$.
- $d = 1 \overset{(2),(3)}{\Rightarrow} \exists \ (0) \neq \mathfrak{n} \lhd_p \mathfrak{g}$ with $\mathfrak{n} \subset C_{\mathfrak{g}}(\mathfrak{t})$ nilpotent $\rightsquigarrow \mathfrak{g}/\mathfrak{n}$.
- If $d = 0$, then $C_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$.
- Let $\alpha \in R$, and put $P := \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_\alpha$. Then $\overset{(2),(3)}{\Rightarrow} \exists \ \mathfrak{n} \lhd_p \mathfrak{g}$ with $1 \leq \dim_k \mathfrak{n} \leq 2$.
- $\mathfrak{n}$ is solvable, so consider $\mathfrak{g}/\mathfrak{n}$.

6. Galois Extensions

When combined with the structure theory of solvable infinitesimal groups, the geometric techniques introduced so far essentially suffice for the treatment of the representation-finite principal blocks. However, it is more subtle to identify the tame groups among those having a two-dimensional nullcone. One major problem resides in the fact that, in contrast to the modular representation theory of finite groups, subgroups of tame infinitesimal groups are not necessarily tame. It turns out that descent to the first Frobenius kernel is better behaved, partly because the associated principal blocks are Galois extensions.

Let Λ be a k-algebra, G a finite group which acts on Λ via automorphisms. We denote by $\Lambda[G]$ and Λ^G the skew group algebra of G over Λ and the subalgebra of G-invariants, respectively. We say that $\Lambda : \Lambda^G$ is a Galois extension (with Galois group G) if

(a) Λ is a projective $\Lambda[G]$-generator, and

(b) for every simple Λ-module S, the restriction $S|_{\Lambda^G}$ is semisimple.

Condition (a) ensures that the algebras $\Lambda[G]$ and Λ^G are Morita equivalent. Thus, if Λ^G is tame, then Λ is tame or representation-finite.

The Galois group G naturally acts on the set $\mathcal{S}(\Lambda)$ of isoclasses of simple Λ-modules. If this action is free, then one has a particularly nice correspondence between the representation theories of Λ and Λ^G, given by a so-called Galois covering.

In our context, Galois extensions arise as follows: Let $\mathcal{G}$ be an infinitesimal group, $X(\mathcal{G}) := \mathrm{Alg}_k(H(\mathcal{G}), k)$ its character group, whose group structure is given by the convolution product

$$(\lambda * \mu)(h) := \sum_{i=1}^{n} \lambda(h_i)\mu(h_i') \qquad \forall\ \lambda, \mu \in X(\mathcal{G}),$$

where $\Delta(h) = \sum_{i=1}^{n} h_i \otimes h_i'$. The commutative p-group $X(\mathcal{G})$ acts on $H(\mathcal{G})$ via automorphisms of the associative algebra $H(\mathcal{G})$:

$$\lambda \cdot h := \sum_{i=1}^{n} \lambda(h_i)h_i' \qquad \text{for } \lambda \in X(\mathcal{G}),\ h \in H(\mathcal{G}).$$

If $\mathcal{N} \lhd \mathcal{G}$ is a normal subgroup, then

$$X(\mathcal{G}/\mathcal{N}) \cong \{\lambda \in X(\mathcal{G}) \ ;\ \lambda(H(\mathcal{N})^\dagger) = (0)\}$$

is the subgroup of characters that vanish on the augmentation ideal $H(\mathcal{N})^\dagger \subset H(\mathcal{N})$ of the Hopf algebra $H(\mathcal{N})$.

Proposition 6.1. *Let $\mathcal{N} \lhd \mathcal{G}$ be a normal subgroup such that*

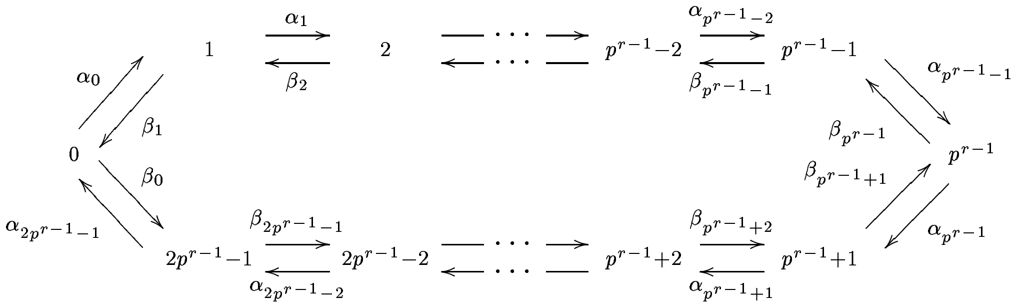
(a) *$\mathcal{G}/\mathcal{N}$ is multiplicative, and*

(b) *the restriction $S|_{H(\mathcal{N})}$ of every simple $H(\mathcal{G})$-module S is simple.*

Then $H(\mathcal{G}) : H(\mathcal{N})$ is a Galois extension, whose Galois group $X(\mathcal{G}/\mathcal{N})$ acts freely on $\mathcal{S}(H(\mathcal{G}))$.

We will apply this Proposition in case the principal block $\mathcal{B}_0(\mathcal{G})$ is tame. As we shall see later, the structure of $\mathcal{B}_0(\mathcal{G}_1)$ is well understood in that case, and the following result will enable us to lift this information to $\mathcal{B}_0(\mathcal{G})$.

For a natural number $r \geq 1$, we denote by Δ_r the quiver with underlying set of vertices $\mathbb{Z}/(2p^{r-1})$ and arrows $\alpha_i : i \to i+1 \quad ; \quad \beta_i : i \to i-1 \qquad \text{for} \quad i \in \mathbb{Z}/(2p^{r-1})$:



For $n \geq 0$, let $J_{r,n} \subset k[\Delta_r]$ be the ideal generated by

$$\{(\beta_{i+1}\alpha_i)^{p^n} - (\alpha_{i-1}\beta_i)^{p^n}, \quad \alpha_{i+1}\alpha_i, \quad \beta_i\beta_{i+1} \quad ; \quad i \in \mathbb{Z}/(2p^{r-1})\}.$$

The bound quiver algebras $\mathcal{N}^2(r,n) := k[\Delta_r]/J_{r,n}$ are examples of special biserial algebras, which are known to be tame or representation-finite. By definition, an algebra Λ is special biserial if it is Morita equivalent to a bound quiver algebra $k[Q]/I$ with the bound quiver (Q, I) satisfying

(a) there are at most two arrows starting and ending at each vertex, and

(b) for each arrow α, there is at most one arrow β and at most one arrow γ with $\beta\alpha,\ \alpha\gamma \notin I$.

If Λ is special biserial, then the heart $\mathrm{Rad}(P)/\mathrm{Soc}(P)$ of every principal indecomposable Λ-module P is a direct sum of at most two uniserial modules. In the modular representation theory of finite groups special biserial algebras occur as blocks with cyclic or dihedral defect groups.

Note that $\mathcal{N}^2(1,0)$ is the algebra $k[\Delta_1]/J$ that we have defined before. The following result provides the desired lifting property:

Proposition 6.2. *Let Λ be a k-algebra with Gabriel quiver Δ_r for some $r \geq 2$. Suppose that $G \subset \mathrm{Aut}_k(\Lambda)$ is a finite group of k-algebra automorphisms such that*

(a) *the induced action of G on the isoclasses of simple Λ-modules is free, and*

(b) *Λ^G is Morita equivalent to $\mathcal{N}^2(1,n)$.*

If Λ has a principal indecomposable module which is not sincere, then Λ is Morita equivalent to $\mathcal{N}^2(r,n)$.

Example. Consider the infinitesimal group $\mathcal{G} := \mathcal{Q}_{[r]} = \mathrm{SL}(2)_1 T_r$ as well as the normal subgroup $\mathcal{N} := \mathrm{SL}(2)_1$. Then $\mathcal{G}/\mathcal{N} \cong \mu_{p^{r-1}}$ is multiplicative, and $X(\mathcal{G}/\mathcal{N}) \cong \mathbb{Z}/(p^{r-1})$. One can verify (b) of Proposition 6.1 and show that the conditions of (6.2) obtain for $\Lambda := \mathcal{B}_0(\mathcal{G})$. Consequently, this block is Morita equivalent to $\mathcal{N}^2(r,0)$, and the translation principle implies that this also holds for all blocks not containing a simple module of dimension p.

7. Finite Representation Type

In this section we turn to infinitesimal groups with representation-finite principal block. Given an infinitesimal group $\mathcal{G}$, we let $\mathcal{M}(\mathcal{G})$ be the multiplicative center of $\mathcal{G}$. By definition, $\mathcal{M}(\mathcal{G})$ is the largest multiplicative normal subgroup of $\mathcal{G}$. One can show that the canonical quotient map $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{M}(\mathcal{G})$ induces an isomorphism $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{M}(\mathcal{G}))$. Accordingly, one can often assume that $\mathcal{M}(\mathcal{G}) = e_k$.

Recall from Section 3 that the assumption $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G})} \leq 1$ implies the supersolvabilty of $\mathcal{G}$. This means that $\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{U} \rtimes \mathcal{M}$ is a semidirect product of a unipotent, normal subgroup $\mathcal{U}$, and a multiplicative subgroup $\mathcal{M}$.

Theorem 7.1. *Let $\mathcal{G}$ be an infinitesimal group. Then the following statements are equivalent:*
(1) $\mathcal{B}_0(\mathcal{G})$ *is representation-finite.*
(2) $\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{U} \rtimes \mu_{p^r}$ *with a V-uniserial normal subgroup* $\mathcal{U}$.
(3) $H(\mathcal{G})$ *is a Nakayama algebra.*
(4) $H(\mathcal{G}_2)$ *is a Nakayama algebra.*

The V-uniserial groups are precisely those unipotent group schemes, whose Hopf algebras are truncated polynomial rings of the form $k[X]/(X^{p^n})$, that is, local Nakayama algebras. These groups are commutative and they can be classified. For future reference we note that if $\mathcal{U}$ has height 1, then $\mathcal{U} \cong (\mathcal{W}_n)_1$ is the first Frobenius kernel of the group of Witt vectors of length n, whose Lie algebra $\mathrm{Lie}(\mathcal{W}_n) = \sum_{i=0}^{n-1} kx^{[p]^i}$, $x^{[p]^n} = 0 \neq x^{[p]^{n-1}}$ is just the n-dimensional nil-cyclic restricted Lie algebra.

In view of (3) all blocks of $H(\mathcal{G})$ are representation-finite, so that $\mathcal{B}_0(\mathcal{G})$ already determines the representation type of $H(\mathcal{G})$. Moreover, subgroups of representation-finite infinitesimal groups are representation-finite. Part (4) clarifies our earlier observation: finite representation type is decided on the second Frobenius kernel of $\mathcal{G}$.

One does in fact have much more information concerning the block structure of $H(\mathcal{G})$. Since $H(\mathcal{G})$ is a Frobenius algebra, results of Kupisch show that the isomorphism type of each block $\mathcal{B} \subset H(\mathcal{G})$ is determined by its Loewy length as well as the number and dimensions of its simple modules. In our case, all these numbers are p-powers and $\mathcal{B}$ is either primary, $\mathcal{B} \cong \mathrm{Mat}_{p^m}(k[X]/(X^{p^\ell}))$, or basic, $\mathcal{B} \cong k[\tilde{A}_{p^n-1}]/\mathrm{Rad}(k[\tilde{A}_{p^n-1}])^{p^\ell}$. One also has information relating the exponents to the structure of $\mathcal{G}$.

8. Tame Representation Type

In this final section we turn to the determination of the infinitesimal groups of tame respresentation type. We will proceed in several steps, beginning with restricted Lie algebras. Some of our methods work only for $p \geq 3$, which we are going to assume throughout.

8.1. **Groups of Height ≤ 1.** Recall that the algebra $H(\mathcal{G})$ of measures on $\mathcal{G}$ is isomorphic to the restricted enveloping algebra of $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$. We thus study the principal block $\mathcal{B}_0(\mathfrak{g})$ of the latter. We know from earlier work that we have $\dim \mathcal{V}_{\mathfrak{g}} = 2$ whenever this block is tame. Varieties of tori can then be used to show that a centerless restricted Lie algebra with self-centralizing maximal torus and two-dimensional nullcone is either solvable or isomorphic to $\mathfrak{sl}(2)$. But we have seen before that solvable groups do not afford tame principal blocks, so that only $\mathfrak{sl}(2)$ can occur in this context. By refining these arguments one arrives at the following result:

Proposition 8.1. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra such that $\mathcal{B}_0(\mathfrak{g})$ is tame. Then $\mathfrak{g}/C(\mathfrak{g}) \cong \mathfrak{sl}(2)$, and the center $C(\mathfrak{g}) := \{x \in \mathfrak{g} \ ; \ [x, \mathfrak{g}] = (0)\}$ has finite respresentation type.*

To obtain the structure of $\mathfrak{g}$ one first observes that the exact sequence

$$(0) \longrightarrow C(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{sl}(2) \longrightarrow (0)$$

splits when considered a sequence of ordinary Lie algebras. Thus, $\mathfrak{g} = \mathfrak{sl}(2) \oplus C(\mathfrak{g})$, and there exists a p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow C(\mathfrak{g})$ such that

$$(x, c)^{[p]} = (x^{[p]}, \psi(x) + c^{[p]}) \qquad \text{for } (x, c) \in \mathfrak{sl}(2) \oplus C(\mathfrak{g}).$$

Let us consider the case $C(\mathfrak{g}) = kv_0 \neq (0)$ and $v_0^{[p]} = 0$. The group $\mathrm{SL}(2)(k) \times k^{\times}$ operates on the space of p-semilinear forms such that the orbits of this action correspond to isomorphism classes of restricted Lie algebras. There are exactly three orbits, whose representatives are given in terms of the standard basis $\{e, h, f\} \subset \mathfrak{sl}(2)$ by

$$\psi_0 = 0 \ \ ; \ \ \psi_n(e) = 0 = \psi_n(h) \ , \ \psi_n(f) = v_0 \ \ ; \ \ \psi_s(e) = 0 = \psi_s(f) \ , \ \psi_s(h) = v_0.$$

We denote the corresponding central extensions by $\mathfrak{sl}(2)_0$, $\mathfrak{sl}(2)_n$, and $\mathfrak{sl}(2)_s$, respectively. Since $\dim \mathcal{V}_{\mathfrak{sl}(2)_0} = 3$, we only have to consider $\mathfrak{g} = \mathfrak{sl}(2)_n, \mathfrak{sl}(2)_s$.

The restricted enveloping algebras $U_0(\mathfrak{g})$ and $U_0(\mathfrak{sl}(2))$ have the same simple modules and almost the same Gabriel quiver. In particular, there is a correspondence between the blocks, so that each block has either one or two simple modules. It turns out that the representation type depends on the structure of the hearts of the principal indecomposables. These can be analyzed by means of filtrations by "Verma modules".

Let $J := \mathrm{Rad}(U_0(\mathfrak{g}))$ be the Jacobson radical of $U_0(\mathfrak{g})$.

Proposition 8.2. *Let P be a principal indecomposable $U_0(\mathfrak{g})$-module, belonging to a block $\mathcal{B} \subset U_0(\mathfrak{g})$ with two simple modules.*

(1) *If $\mathfrak{g} = \mathfrak{sl}(2)_n$, then JP/J^3P is indecomposable of length 4, and $\mathcal{B}$ is wild.*

(2) *If $\mathfrak{g} = \mathfrak{sl}(2)_s$, then JP/J^3P is a direct sum of two uniserial modules, and $\mathcal{B}$ is special biserial.*

The first part follows from the fact that a Galois covering of $\mathcal{B}/J^3\mathcal{B}$ contains as a subcategory the module category of a one-point extension of the path algebra of the Kronecker quiver given by a regular module of quasi-length 2 in a homogeneous tube.

The second part involves a delicate analysis by which one extends the information on $\mathcal{B}/J^3\mathcal{B}$ to $\mathcal{B}$. Here the fact that $\mathcal{B}$ is symmetric plays an important rôle. Recall that symmetry is given by the condition $\mathrm{tr}(\mathrm{ad}\, x) = 0$, which is easily seen to hold in our case. Our result has an interesting consequence.

- Let $\mathfrak{h} := ke \oplus kv_0 \subset \mathfrak{sl}(2)_s$. Then $U_0(\mathfrak{h}) \cong k[X, Y]/(X^p, Y^p)$ is wild.

In contrast to the modular representation theory of finite groups, there exist tame infinitesimal groups having wild subgroups. It turns out that the analysis of one-dimensional extensions already determines the entire picture:

Theorem 8.3. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then the following statements are equivalent:*
1. $\mathcal{B}_0(\mathfrak{g})$ *is tame.*
2. $\mathcal{B}_0(\mathfrak{g})$ *is Morita equivalent to* $\mathcal{N}^2(1, n)$.
3. $\mathfrak{g}/C(\mathfrak{g})^{[p]} \cong \mathfrak{sl}(2),\ \mathfrak{sl}(2)_s$.

This is, of course, only a recognition theorem, not a classification. The reason is that the group $\mathrm{SL}(2)(k) \times \mathrm{Aut}_p(C(\mathfrak{g}))$ operates on the space of p-semilinear forms with infinitely many orbits as soon as $C(\mathfrak{g})$ is unipotent of dimension ≥ 2. For the moment we just note that there are usually infinitely many restricted Lie algebras leading to the same Morita type.

The above result only provides information on the principal block. The study of the other blocks involves a detailed analysis of certain reduced enveloping algebras $U_\chi(\mathfrak{g})$ for Lie algebras occurring in (3). Fortunately, these algebras are well enough understood to ensure that no new tame blocks occur. However, it may happen that the principal block is tame and that some other blocks are wild. There are precise criteria on $\mathfrak{g}$ for this to happen, in particular wild blocks only appear when the multiplicative center of the corresponding infinitesimal group is not trivial. Thus, the principal block of $H(\mathcal{G})$ is not necessarily the most complicated block.

8.2. **Tame Infinitesimal Groups.** Until further notice, $\mathcal{G}$ denotes an infinitesimal group of characteristic $p \geq 3$ such that the principal block $\mathcal{B}_0(\mathcal{G}) \subset H(\mathcal{G})$ is tame.

If $\mathcal{G}$ is semisimple, then the geometric techniques imply $\mathrm{Lie}(\mathcal{G}) \cong \mathfrak{sl}(2)$. Thus, $\mathcal{G}$ is a closed subgroup of $\mathrm{SL}(2)$ that contains $\mathrm{SL}(2)_1$. This information, combined with the classification of the representation-finite infinitesimal groups implies that $\mathcal{G} \cong \mathcal{Q}_{[r]}$ for some $r \geq 1$. Thus, if $\mathcal{G}$ is arbitrary with solvable radical $\mathcal{R}(\mathcal{G})$, then $\mathcal{G}/\mathcal{R}(\mathcal{G}) \cong \mathcal{Q}_{[r]}$ for some $r \geq 1$.

One continues by analyzing the structure of $\mathcal{R}(\mathcal{G})$ and $\mathcal{G}_1$. This ultimately leads to the conclusion that $\mathcal{R}(\mathcal{G}) = \mathrm{Cent}(\mathcal{G})$ is the center of $\mathcal{G}$ and that $\mathcal{B}_0(\mathcal{G}_1)$ is also tame. Now assume that $\mathcal{M}(\mathcal{G}) = e_k$, so that $\mathrm{Cent}(\mathcal{G})$ is unipotent. One shows that $\mathrm{Cent}(\mathcal{G})$ is representation-finite of height 1, so that our earlier results provide a central extension

$$e_k \longrightarrow (\mathcal{W}_n)_1 \longrightarrow \mathcal{G} \longrightarrow \mathcal{Q}_{[r]} \longrightarrow e_k.$$

Moreover, most of the tame restricted Lie algebras listed in Theorem 8.3 cannot be those of tame infinitesimal groups of height ≥ 2.

Recall that $T \subset \mathrm{SL}(2)$ is the standard maximal torus of diagonal matrices. Given $n \geq 0$, there exists a distinguished infinitesimal group $\mathrm{SL}(2)_1^n$ of height 1 such that
(a) $\mathrm{SL}(2)_1^n$ is a central extension of $\mathrm{SL}(2)_1$ by $(\mathcal{W}_n)_1$,
(b) $\mathcal{B}_0(\mathrm{SL}(2)_1^n)$ is tame,
(c) T acts on $\mathrm{SL}(2)_1^n$ via automorphisms.

Theorem 8.4. *Suppose that $\mathcal{G}$ has height $r \geq 2$ and a unipotent center of length n. Then*

$$\mathcal{G} \cong (\mathrm{SL}(2)_1^n \rtimes T_r)/\mathcal{M}(\mathrm{SL}(2)_1^n \rtimes T_r).$$

Let me give you a consequence that illustrates how restricted the "classical case" of Frobenius kernels of smooth groups is. Let $\mathcal{G}$ be an arbitrary algebraic group of characteristic $p \geq 3$.

Corollary 8.5. *If $\mathcal{G}$ is smooth and connected, and $\mathcal{B}_0(\mathcal{G}_r)$ is tame, then $r = 1$, and there exists a torus $\mathcal{T}$ such that $\mathcal{G} \cong \mathrm{SL}(2)\mathcal{T}$, $\mathrm{PSL}(2)\mathcal{T}$ (an almost direct product). Moreover, $\mathcal{B}_0(\mathcal{G}_r)$ is Morita equivalent to $\mathcal{N}^2(1,0)$.*

Finally, we consider the basic algebras of infinitesimal groups with tame principal blocks. It turns out that the occurrence of wild blocks is essentially a phenomenon of groups of height 1. Given an infinitesimal group $\mathcal{G}$ we put

$$r(\mathcal{G}) := \mathrm{ht}(\mathcal{G}/\mathcal{M}(\mathcal{G})) \ \ ; \ \ n(\mathcal{G}) := \ell(\mathrm{Cent}(\mathcal{G})/\mathcal{M}(\mathcal{G})).$$

If $r(\mathcal{G}) = 1$ and $\mathcal{B}_0(\mathcal{G})$ is tame, then $H(\mathcal{G})$ and $U_0(\mathrm{Lie}(\mathcal{G}))$ have the same blocks, only with different multiplicities.

Using Proposition 6.2 and the structure of the blocks of $H(\mathcal{G}_1)$ one arrives at the following result:

Theorem 8.6. *Let $\mathcal{G}$ be an infinitesimal group with tame principal block, $r(\mathcal{G}) \geq 2$. Let $\mathcal{B} \subset H(\mathcal{G})$ be a block.*

(1) *Every simple $\mathcal{B}$-module has dimension $\leq p$.*

(2) *If $\mathcal{B}$ possesses a simple module of dimension p, then $\mathcal{B}$ is Morita equivalent to $k[X]/(X^{p^{n(\mathcal{G})}})$.*

(3) *If $\mathcal{B}$ possesses a simple module of dimension $\neq p$, then $\mathcal{B}$ is Morita equivalent to $\mathcal{N}^2(r(\mathcal{G}), n(\mathcal{G}))$.*

Contrary to finite groups, the number of simple modules of the tame blocks is not bounded. Moreover, the tame algebras occurring are symmetric, yet they differ from those of the modular representation theory of finite groups by the fact that their Cartan matrices are singular.