AUSLANDER-REITEN COMPONENTS FOR G_1T-MODULES

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Abstract. In this paper we study the Auslander-Reiten quiver of the highest weight category of finite-dimensional G_1T-modules, associated to a smooth reductive algebraic group G. By relating properties of stable Auslander-Reiten components to those of their rank varieties we show that there are at most three isomorphism types of these components. For the Frobenius kernels G_1T_r the maximal ranks of tubes are determined.

1. Introduction and Preliminaries

Let G be a connected smooth reductive group, defined over an algebraically closed field k of characteristic $p \geq 3$. We denote by G_1 the first Frobenius kernel of G and fix a maximal torus $T \subset G$. In the representation theory of the restricted Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$, the highest weight category $\mathrm{mod}\, G_1T$ of finite-dimensional G_1T-modules, introduced by Jantzen [27], has played an important rôle. This category is equivalent to the category of those finite-dimensional modules over the restricted enveloping algebra $U_0(\mathfrak{g})$ that possess a grading by the character group $X(T)$ which is compatible with the action of the maximal torus $\mathfrak{t} := \mathrm{Lie}(T)$ of $\mathfrak{g}$.

The purpose of this note is to study $\mathrm{mod}\, G_1T$ from the point of view of Auslander-Reiten theory. This module category affords almost split sequences, and structural properties of the connected components of its stable Auslander-Reiten quiver are related to features of certain T-invariant subvarieties of $\mathfrak{g}$, the so-called *rank varieties* (see (2.4), (3.2), (4.1) and (4.3)). In this fashion the AR-components of $\mathrm{mod}\, G_1T$ are shown to be isomorphic to $\mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty^\infty]$, or possibly $\mathbb{Z}[D_\infty]$. Our methods and results also apply in the context of the module categories $\mathrm{mod}\, G_1T_r$ of the r-th Frobenius kernels G_1T_r of G_1T: If a non-projective indecomposable $U_0(\mathfrak{g})$-module affords a G_1T-structure, so does every module belonging to the stable AR-component of M. An analogous result holds for modules possessing a filtration by baby Verma modules. The module categories of the groups $\mathrm{SL}(2)_1T_r$ figure prominently in the determination of the infinitesimal groups of tame representation type [14, 15]. The existence of AR-components of type $\mathbb{Z}[A_\infty]/(\tau^{p^{r-1}})$, established in [14] by computational means, is explained in the context of semi-simple, simply connected groups. In view of the Friedlander-Suslin Theorem [20], the stable Auslander-Reiten quiver of G_1T_r does not possess tubes of rank $> p^{r-1}$.

Our main tools are rank varieties, cohomological support varieties [5, 17] as well as the representation theory of $\mathbb{Z}$-graded Artin algebras, expounded in [21, 22]. As observed in [30], the results of [21] are valid for Artin algebras that are graded by finitely generated free abelian groups. These comments also apply to the main results of [22]. For our purposes the case of $\mathbb{Z}^n$-graded finite-dimensional k-algebras is relevant. Given such an algebra Λ, we let $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}_{\mathbb{Z}^n}\Lambda$ be the categories of finite-dimensional left Λ-modules and finite-dimensional $\mathbb{Z}^n$-graded left Λ-modules and degree zero homomorphisms, respectively. The forgetful functor will be denoted $\mathcal{F} : \mathrm{mod}_{\mathbb{Z}^n}\Lambda \longrightarrow \mathrm{mod}\,\Lambda$.

We refer the reader to [1] for standard terminology of Auslander-Reiten theory, and use [7, 29, 35, 37] as standard references for algebraic groups. Smooth (reduced) algebraic group schemes will be identified with their groups of rational points (cf. [29, (I.2.1),(I.2.8)]).

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For future reference we record the following crucial result [22, (3.5)].

Theorem 1.1. *Let Λ be a finite-dimensional $\mathbb{Z}^n$-graded algebra. Then the following statements hold:*

(1) *The category $\operatorname{mod}_{\mathbb{Z}^n} \Lambda$ has almost split sequences.*

(2) *If $(0) \longrightarrow X \xrightarrow{g} Y \xrightarrow{f} Z \longrightarrow (0)$ is an almost split sequence in $\operatorname{mod}_{\mathbb{Z}^n} \Lambda$, then $(0) \longrightarrow \mathcal{F}(X) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(Z) \longrightarrow (0)$ is an almost split sequence in $\operatorname{mod} \Lambda$.* □

2. The stable Auslander-Reiten quiver of G_1T-modules

Throughout this section G is assumed to be connected, smooth (not necessarily reductive) algebraic group scheme with maximal torus $T \subset G$. Given any algebraic k-group H, we denote by $\operatorname{mod} H$ the category of finite-dimensional H-modules. By general theory, $\operatorname{mod} G_1$ is equivalent to the category $\operatorname{mod} U_0(\mathfrak{g})$ of finite-dimensional $U_0(\mathfrak{g})$-modules (cf. [7, (II,§7,4.2)]). Here $U_0(\mathfrak{g})$ is the *restricted enveloping algebra* of the restricted Lie algebra $(\mathfrak{g}, [p])$ associated to G. By definition,

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\})$$

is the factor algebra of the ordinary enveloping algebra $U(\mathfrak{g})$ by the two-sided ideal generated by $\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\}$. The *character group* $X(T)$ of T is a free abelian group, whose rank coincides with that of G. Since the diagonalizable group T acts on $(\mathfrak{g}, [p])$ via automorphisms, the algebra $U_0(\mathfrak{g})$ inherits the $X(T)$-grading from $U(\mathfrak{g})$.

In the sequel, we shall study $\operatorname{mod} G_1T$ as well as $\operatorname{mod}(G_1 \rtimes T)$. The latter category coincides with the category $\operatorname{mod}_{X(T)} G_1$ of $X(T)$-graded G_1-modules and degree zero homomorphisms. When convenient, we shall identify this category with $\operatorname{mod}_{X(T)} U_0(\mathfrak{g})$. This point of view places us into the context covered by Theorem 1.1. We identify $X(T)$ with the subgroup of those characters $\lambda \in X(G_1 \rtimes T)$ that are trivial on G_1. Thus, every $\lambda \in X(T)$ defines a one-dimensional $(G_1 \rtimes T)$-module k_λ. It now follows directly from the definition that the shift functor $M \mapsto M\langle\lambda\rangle$ of $\operatorname{mod}_{X(T)} U_0(\mathfrak{g})$ (cf. [30, (1.2)]) corresponds to the auto-equivalence $M \mapsto M \otimes_k k_\lambda$.

The block decomposition of the algebraic group $G_1 \rtimes T$ given in [29, (II.7.1)] yields a direct sum decomposition

$$\operatorname{mod}(G_1 \rtimes T) = \bigoplus_{b \in \mathcal{B}(G_1 \rtimes T)} (\operatorname{mod}(G_1 \rtimes T))_b,$$

whose constituents are referred to as *blocks*.

Lemma 2.1. *The following statements hold:*

(1) *The category $\operatorname{mod} G_1T$ is a sum of blocks of $\operatorname{mod}(G_1 \rtimes T)$.*

(2) *The category $\operatorname{mod} G_1T$ has almost split sequences.*

(3) *The canonical restriction functor $\operatorname{mod} G_1T \longrightarrow \operatorname{mod} G_1$ sends indecomposables to indecomposables and almost split sequences to almost split sequences.*

Proof. (1) We consider the exact sequence

$$e_k \longrightarrow G_1 \cap T \longrightarrow G_1 \rtimes T \longrightarrow G_1T \longrightarrow e_k$$

of algebraic groups. Since $G_1 \cap T$ is a multiplicative normal subgroup of the connected affine group scheme $G_1 \rtimes T$, rigidity of tori [37, (7.7)] implies that $G_1 \cap T$ lies in the center of $G_1 \rtimes T$. Thus, the $(G_1 \cap T)$-weight spaces M_λ of a $(G_1 \rtimes T)$-module M are $(G_1 \rtimes T)$-submodules of M. As a result,

$G_1 \cap T_1 = T_1$ operates on an indecomposable $(G_1 \rtimes T)$-module M via a single character $\lambda \in X(T_1)$. In view of the Krull-Schmidt property of $\operatorname{mod}(G_1 \rtimes T)$ we obtain a decomposition

$$\operatorname{mod}(G_1 \rtimes T) = \bigoplus_{\lambda \in X(T_1)} \operatorname{mod}(G_1 \rtimes T)_\lambda$$

with $\operatorname{mod} G_1T = \operatorname{mod}(G_1 \rtimes T)_0$. If V and W are simple $(G_1 \rtimes T)$-modules giving rise to characters $\lambda \neq \mu \in X(T_1)$, then [29, (I.6.6)] yields $\operatorname{Ext}^1_{G_1 \rtimes T}(V, W) \cong H^1(G_1T, \operatorname{Hom}_{G_1 \cap T}(V, W)) \cong (0)$, so that the modules belonging to a single block afford the same character. This shows that $\operatorname{mod} G_1T$ is a sum of blocks of $\operatorname{mod}(G_1 \rtimes T)$.

(2) As noted earlier, the category $\operatorname{mod}(G_1 \rtimes T)$ coincides with the category $\operatorname{mod}_{X(T)} G_1$ of $X(T)$-graded G_1-modules and homomorphisms of degree 0. Thanks to Theorem 1.1, $\operatorname{mod}(G_1 \rtimes T)$ has almost split sequences. In view of (1) the category $\operatorname{mod} G_1T$ also enjoys this property.

(3) Identifying $\operatorname{mod} G_1T$ with $\operatorname{mod}(G_1 \rtimes T)_0$, we see that the canonical restriction functor is the restriction of $\mathcal{F}$ to this full subcategory. Our assertions thus follow from (1.1). □

Remark. The shifts sending $\operatorname{mod} G_1T$ onto itself are given by those $\lambda \in X(T)$ that vanish on T_1. Now [29, (II.3.7)] provides an exact sequence

$$(0) \longrightarrow pX(T) \stackrel{\text{can.}}{\longrightarrow} X(T) \stackrel{\text{res.}}{\longrightarrow} X(T_1) \longrightarrow (0),$$

so that the relevant shifts are those belonging to the subgroup $pX(T)$ of $X(T)$.

We turn to the description of the components of the stable Auslander-Reiten quiver $\Gamma_s(G_1T)$ of $\operatorname{mod} G_1T$. Recall from [29, (II.9.3)] that an object $P \in \operatorname{mod} G_1T$ is projective if and only if it is injective. Thus, $\operatorname{mod} G_1T$ is a Frobenius category (cf. [23, (I.2)]), and we can speak of the *stable Auslander-Reiten quiver* $\Gamma_s(G_1T)$ of G_1T. By definition, $\Gamma_s(G_1T)$ is obtained from the ordinary AR-quiver by removing all projective vertices and all arrows originating or terminating in them. Thus, $\Gamma_s(G_1T)$ is a stable translation quiver (cf. [1, p.252]). By work of Riedtmann [32] (cf. also [2, (4.15.6)]) the connected components of $\Gamma_s(G_1T)$ are largely determined by certain undirected trees, the so-called *tree classes*. We refer the reader to [2, (4.15)] for more details and undefined terminology.

Given a restricted Lie algebra $(\mathfrak{g}, [p])$, its *nullcone* is defined by

$$\mathcal{V}_{\mathfrak{g}} := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}.$$

Following Friedlander-Parshall [17, 18], we associate to a $U_0(\mathfrak{g})$-module M the conical affine variety

$$\mathcal{V}_{\mathfrak{g}}(M) := \{x \in \mathcal{V}_{\mathfrak{g}} \ ; \ M|_{U_0(kx)} \text{ is not projective}\} \cup \{0\},$$

called the *rank variety* of M. Its dimension $\dim \mathcal{V}_{\mathfrak{g}}(M)$ is known to coincide with the *complexity* $\operatorname{cx}_{\mathfrak{g}}(M)$ of the $U_0(\mathfrak{g})$-module M (cf. [17, (3.2)]). The latter number is the growth of a minimal projective resolution of M (cf. [3, Chap.5]).

Let $\mathcal{F} : \operatorname{mod} G_1T \longrightarrow \operatorname{mod} G_1$ be the canonical restriction functor and consider a component $\Theta \subset \Gamma_s(G_1T)$. In view of (2.1(3)) the arguments of [10, (5.2)] show that for $\mathfrak{g} := \operatorname{Lie}(G)$ we have

$$\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M)) = \mathcal{V}_{\mathfrak{g}}(\mathcal{F}(N)) \qquad \forall\ [M], [N] \in \Theta.$$

We shall therefore speak of the variety $\mathcal{V}_{\mathfrak{g}}(\Theta)$ of Θ.

Let τ_{G_1T} and τ_{G_1} be the Auslander-Reiten translations of $\operatorname{mod} G_1T$ and $\operatorname{mod} G_1$, respectively. Owing to (2.1) we obtain

$$\mathcal{F} \circ \tau^n_{G_1T} = \tau^n_{G_1} \circ \mathcal{F} \qquad \forall\ n \in \mathbb{Z}.$$

The stable Auslander-Reiten quiver of $\operatorname{mod} U_0(\mathfrak{g})$ will be denoted $\Gamma_s(G_1)$.

Lemma 2.2. *The following statements hold:*

(1) *The canonical restriction functor $\mathcal{F} : \operatorname{mod} G_1T \longrightarrow \operatorname{mod} G_1$ induces a homomorphism $\mathcal{F} : \Gamma_s(G_1T) \longrightarrow \Gamma_s(G_1)$ of stable translation quivers.*

(2) *If $\Theta \subset \Gamma_s(G_1T)$ is a component, then $\mathcal{F}(\Theta)$ is a component of $\Gamma_s(G_1)$.*

Proof. (1) Thanks to [29, (II.9.3)] an object $P \in \operatorname{mod} G_1T$ is projective if and only if its restriction $\mathcal{F}(P)$ enjoys this property. Since $\mathcal{F}$ commutes with direct sums, assertion (1) is thus an immediate consequence of (2.1) and [1, (V.5.3)].

(2) Since $\mathcal{F}$ is a homomorphism of stable translation quivers, there exists a unique component $\Psi \subset \Gamma_s(G_1)$ with $\mathcal{F}(\Theta) \subset \Psi$. As $\mathcal{F}(\Theta)$ is $\tau^n_{G_1}$-invariant for all $n \in \mathbb{Z}$, we only have to show that each neighbour of an element of $\mathcal{F}(\Theta)$ also belongs to $\mathcal{F}(\Theta)$. To that end, we consider an isoclasss $[M] \in \Theta$ as well as the almost split sequence

$$(0) \longrightarrow \tau_{G_1}(\mathcal{F}(M)) \longrightarrow E \longrightarrow \mathcal{F}(M) \longrightarrow (0).$$

We decompose $E = \bigoplus_{i=1}^n m_iE_i$ into indecomposable modules, so that the distinct isoclasses $[E_i]$ are the predecessors of $[\mathcal{F}(M)] \in \Psi$. We next consider the almost split sequence

$$(0) \longrightarrow \tau_{G_1T}(M) \longrightarrow X \longrightarrow M \longrightarrow (0)$$

terminating in M. Thanks to (2.1) the almost split sequence terminating in $\mathcal{F}(M)$ is isomorphic to

$$(0) \longrightarrow \tau_{G_1}(\mathcal{F}(M)) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}(M) \longrightarrow (0).$$

In particular, if $X = \bigoplus_{j=1}^\ell r_jX_j$ is the decomposition of X into its indecomposable constituents, then $E \cong \mathcal{F}(X) \cong \bigoplus_{j=1}^\ell r_j\mathcal{F}(X_j)$ is the corresponding decomposition of $\mathcal{F}(X)$. Thus, the Theorem of Krull-Remak-Schmidt implies $\{[E_1], \ldots, [E_n]\} = \{[\mathcal{F}(X_1)], \ldots, [\mathcal{F}(X_\ell)]\}$. Consequently, each isoclass $[E_i]$ belongs to $\mathcal{F}(\Theta)$, as desired. Using the bijectivity of τ_{G_1}, one proves the corresponding statement for the successors of vertices belonging to $\mathcal{F}(\Theta)$. □

Remarks. (1) By the proof of (2.2), the functor $\mathcal{F}$ maps the sets $[M]^+$ and $[M]^-$ of successors and predecessors of $[M] \in \Gamma_s(G_1T)$ *onto* the corresponding sets defined by $[\mathcal{F}(M)] \in \Gamma_s(G_1)$.

(2) The example of the reductive group SL(2) (cf. [14, p.503]) illustrates that the surjective morphism $\Theta \longrightarrow \Psi$ of stable translation quivers is not necessarily a covering in the sense of [2, (4.15)]. In the context of reductive groups, this phenomenon only occurs for components Θ, whose rank varieties have dimension ≤ 2 (cf. (3.3) below).

We next employ rank varieties in conjunction with a technique introduced in [9, §3] to obtain an analogue of Webb's classical result [38, Thm.A] for G_1T-modules. Thanks to [34, Lemma 3] the restricted enveloping algebra $U_0(\mathfrak{g})$ is a Frobenius algebra with Nakayama automorphism $\nu : U_0(\mathfrak{g}) \longrightarrow U_0(\mathfrak{g})$ given by

$$\nu(x) = x - \operatorname{tr}(\operatorname{ad} x)1 \quad \forall\, x \in \mathfrak{g}.$$

According to [1, (IV.3.7)] the Auslander-Reiten translation τ_{G_1} is the composite $\Omega^2_{\mathfrak{g}} \circ \mathcal{N} = \mathcal{N} \circ \Omega^2_{\mathfrak{g}}$ of the square of the Heller operator $\Omega^2_{\mathfrak{g}}$ with the Nakayama functor $\mathcal{N}$. By definition, $\mathcal{N}$ sends a $U_0(\mathfrak{g})$-module M to its twist by ν^{-1}. The validity of Webb's result for $\Gamma_s(G_1)$ was observed in [8, 10].

Proposition 2.3. *Let $\Theta \subset \Gamma_s(G_1T)$ be a component. Then the tree class T_Θ of Θ is a simply laced finite or infinite Dynkin diagram, a simply laced Euclidean diagram, or $\tilde{A}_{12}$.*

Proof. We retain the notation from our previous result. As $\mathcal{F}(\Theta)$ does not contain any projective modules (cf. [29, (II.9.3)]), we have $\mathcal{V}_{\mathfrak{g}}(\Theta) \neq \{0\}$. Let $x \in \mathcal{V}_{\mathfrak{g}}(\Theta) \setminus \{0\}$ and consider the induced $U_0(\mathfrak{g})$-module $V_x := U_0(\mathfrak{g}) \otimes_{U_0(kx)} k$ as well as the function

$$d_x : \Theta \longrightarrow \mathbb{N} \ ; \ \ [M] \mapsto \dim_k \operatorname{Ext}^1_{U_0(\mathfrak{g})}(V_x, \mathcal{F}(M)).$$

Since the restriction $\mathcal{F}(M)|_{U_0(kx)}$ of a vertex $[M] \in \Theta$ is not injective, Frobenius reciprocity yields

$$d_x([M]) = \dim_k \operatorname{Ext}^1_{U_0(kx)}(k, \mathcal{F}(M)) \neq 0,$$

so that d_x is well-defined. Moreover, from $\mathcal{F}(\tau_{G_1T}(M)) \cong (\Omega^2_{\mathfrak{g}} \circ \mathcal{N})(\mathcal{F}(M))$, $\mathcal{N}(V_x) \cong V_x$, and $\Omega^2_{\mathfrak{g}}(V_x) \cong V_x \oplus (\mathrm{proj})$, we obtain

$$\begin{aligned} d_x(\tau_{G_1T}([M])) &= \dim_k \operatorname{Ext}^1_{U_0(\mathfrak{g})}(V_x, (\Omega^2_{\mathfrak{g}} \circ \mathcal{N})(\mathcal{F}(M))) = \dim_k \operatorname{Ext}^1_{U_0(\mathfrak{g})}(\Omega^2_{\mathfrak{g}}(V_x), (\Omega^2_{\mathfrak{g}} \circ \mathcal{N})(\mathcal{F}(M))) \\ &= \dim_k \operatorname{Ext}^1_{U_0(\mathfrak{g})}(V_x, \mathcal{N}(\mathcal{F}(M))) = \dim_k \operatorname{Ext}^1_{U_0(\mathfrak{g})}(\mathcal{N}(V_x), \mathcal{N}(\mathcal{F}(M))) \\ &= d_x([M]), \end{aligned}$$

so that $d_x \circ \tau_{G_1T} = d_x$. As a result, the function d_x induces a subadditive function on the tree class T_Θ, and our assertion follows from [24, Thm.,p.286] in conjunction with k being algebraically closed. □

In the sequel we shall be concerned with the question as to which of the components allowed by (2.3) actually do occur. Broadly put, our objective is to relate structural features of components $\Theta \subset \Gamma_s(G_1T)$ to those of their rank varieties $\mathcal{V}_{\mathfrak{g}}(\Theta)$. Our results are based on homological techniques providing a finite morphism between rank varieties and cohomological support varieties (cf. [17, 18, 28]).

Let H be an algebraic k-group, M an H-module. The H-module with underlying abelian group M and k-action $\alpha \cdot m := \alpha^{\frac{1}{p}} m$ is customarily denoted $M^{(1)}$ (cf. [29, (I.2.16)]). If H is defined over the Galois field $\mathbb{F}_p$, then M can be twisted by the Frobenius endomorphism of H. The resulting module $M^{[1]}$ is the *Frobenius twist* of M.

If T is a torus, then T is reduced and connected, and [37, (2.2)] provides an isomorphism $T \cong \mathrm{GL}(1)^m$ for some $m \geq 0$. Thus, T is defined over $\mathbb{F}_p$. Every T-module M is the direct sum $M = \bigoplus_{\lambda \in X(T)} M_\lambda$ of its weight spaces, each of which has an $\mathbb{F}_p$-structure. It follows that the representation of T on M is defined over $\mathbb{F}_p$, so that $M^{(1)} \cong M^{[1]}$ (cf. [29, (I.9.10)]). In particular, the weights of the T-module $M^{(1)}$ belong to the subgroup $pX(T)$.

Recall that G acts on the restricted enveloping algebra $U_0(\mathfrak{g})$ via the *adjoint representation* $G \longrightarrow \mathrm{Aut}_k(U_0(\mathfrak{g}))$. This action stabilizes the nullcone $\mathcal{V}_{\mathfrak{g}}$, and it is easily seen that the rank variety $\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M))$ of a G_1T-module M is T-stable. The adjoint action $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ gives rise to the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

of $\mathfrak{g}$ relative to T. Here $R \subset X(T) \setminus \{0\}$ is the set of roots and $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{t})$ denotes the centralizer of the maximal torus $\mathfrak{t} := \mathrm{Lie}(T)$ in $\mathfrak{g}$.

Recall the Nakayama functor

$$\mathcal{N} : \operatorname{mod} G_1 \longrightarrow \operatorname{mod} G_1 \ \ ; \ \ M \mapsto M \otimes_k k_\mu,$$

where $\mu(x) = x + \mathrm{tr}(\mathrm{ad}\, x)1$ for all $x \in \mathfrak{g}$. Since k_μ is the restriction of the one-dimensional G-module with action given by $g \mapsto \det(\mathrm{Ad}(g))$, we see that k_μ has the structure of a G_1T-module. Consequently, $M \mapsto M \otimes_k k_\mu$ defines a functor $\mathcal{N} : \operatorname{mod} G_1T \longrightarrow \operatorname{mod} G_1T$ such that $\mathcal{F} \circ \mathcal{N} = \mathcal{N} \circ \mathcal{F}$.

The following result elaborates on the well-known realization of closed, conical subsets of $\mathcal{V}_{\mathfrak{g}}$ as rank varieties of $U_0(\mathfrak{g})$-modules (cf. [18, (2.2)]).

Theorem 2.4. *The following statements hold:*

(1) *If $\mathcal{V} \subset \mathcal{V}_{\mathfrak{g}}$ is a conical, T-invariant closed subset, then there exists a finite-dimensional G_1T-module M such that $\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M)) = \mathcal{V}$.*

(2) *Let M be an indecomposable G_1T-module with $\mathrm{cx}_{\mathfrak{g}}(\mathcal{F}(M)) = 1$. Then there exists $\alpha_M \in R \cup \{0\}$ and $x \in \mathfrak{g}_{\alpha_M} \setminus \{0\}$ such that*

(a) *$\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M)) = kx$, and*

(b) *$\tau_{G_1T}(M) \cong \mathcal{N}(M) \otimes_k k_{p\alpha_M}$.*

Proof. We denote by $S(\mathfrak{g}^*)$ the symmetric algebra of $\mathfrak{g}^*$, that is, the algebra of polynomial functions on $\mathfrak{g}$. Since the Hochschild map [25, p.575]

$$(\mathfrak{g}^*)^{(1)} \longrightarrow H^2(U_0(\mathfrak{g}), k)$$

is natural in $\mathfrak{g}$, it is T-equivariant. If the elements of $\mathfrak{g}^*$ are given degree 2, then there results a homogeneous T-equivariant algebra homomorphism

$$\varphi : S(\mathfrak{g}^*)^{(1)} \longrightarrow H^{\mathrm{ev}}(U_0(\mathfrak{g}), k)$$

with values in the commutative k-algebra $H^{\mathrm{ev}}(U_0(\mathfrak{g}), k) := \bigoplus_{n\geq 0} H^{2n}(U_0(\mathfrak{g}), k)$. By Jantzen's Theorem [28, Satz] the variety $\mathcal{V}_{\mathfrak{g}} = Z(\ker \varphi)$ is the zero locus associated to the kernel of φ. In view of our foregoing observations, the T-module $S(\mathfrak{g}^*)^{(1)} \cong S(\mathfrak{g}^*)^{[1]}$ is the Frobenius twist of $S(\mathfrak{g}^*)$.

(1) Let $I \lhd S(\mathfrak{g}^*)$ be the vanishing ideal of $\mathcal{V}$. By assumption, I is a homogeneous T-invariant ideal containing $\ker \varphi$. It thus affords a weight space decomposition

$$I = \bigoplus_{\lambda \in pX(T)} I_\lambda,$$

providing homogeneous generators $f_1, \ldots, f_r$ of I such that $f_i \in I_{\lambda_i}$. Setting $n_i := \deg(f_i)$, we consider the G_1T-module $M_i := (\mathcal{N}^{-n_i} \circ \tau^{n_i}_{G_1T})(k)$, and observe that $\mathcal{F}(M_i) \cong \mathcal{N}^{-n_i} \circ \tau^{n_i}_{G_1}(k) \cong \Omega^{2n_i}_{\mathfrak{g}}(k)$. By general theory (cf. [2, (2.5.4)]), the cohomology class $\varphi(f_i) \in H^{2n_i}(U_0(\mathfrak{g}), k)_{\lambda_i}$ corresponds to a linear map $\zeta_i \in \mathrm{Hom}_{U_0(\mathfrak{g})}(\mathcal{F}(M_i), k)_{\lambda_i}$. Owing to [29, (I.6.9(5))] we have $\mathrm{Hom}_{U_0(\mathfrak{g})}(\mathcal{F}(M_i), k)_{\lambda_i} = \mathrm{Hom}_{G_1T}(M_i \otimes_k k_{\lambda_i}, k)$, and we denote by $L_{\zeta_i} \subset M_i \otimes_k k_{\lambda_i}$ the kernel of the G_1T-linear map ζ_i. In view of [18], Lemmas 4.1 and 4.2 of [17] hold in the context of arbitrary restricted Lie algebras. Thus, [17, (4.2)] ensures that $\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(L_{\zeta_i})) = Z(f_i)$ is the zero locus of the polynomial f_i. By virtue of [18, (2.1)] the G_1T-module $M := L_{\zeta_1} \otimes_k \cdots \otimes_k L_{\zeta_r}$ has rank variety

$$\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M)) = \mathcal{V}_{\mathfrak{g}}(\bigotimes_{i=1}^{r} \mathcal{F}(L_{\zeta_i})) = \bigcap_{i=1}^{r} \mathcal{V}_{\mathfrak{g}}(\mathcal{F}(L_{\zeta_i})) = \bigcap_{i=1}^{r} Z(f_i) = Z(I) = \mathcal{V},$$

as desired.

(2) Since M is a G_1T-module of complexity 1, the one-dimensional variety $\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M))$ is T-invariant. As $\mathcal{F}(M)$ is indecomposable, an application of [18, (2.2)] implies that $\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M)) = kx \neq (0)$ is a line. Consequently, $x \in \mathfrak{g}_{\alpha_M}$ for some $\alpha_M \in R \cup \{0\}$. It remains to show that

$$\tau_{G_1T}(M) \cong M \otimes_k k_{p\alpha_M}.$$

The root space decomposition induces a weight space decomposition $\mathfrak{g}^* = \bigoplus_{\alpha \in R\cup\{0\}} \mathfrak{g}^*_{-\alpha}$ with $\mathfrak{g}^*_{-\alpha} = \{f \in \mathfrak{g}^* \ ; \ f(\mathfrak{g}_\beta) = 0 \ \forall \ \beta \neq \alpha\} \cong (\mathfrak{g}_\alpha)^*$. By the above isomorphism, the weight space decomposition of $(\mathfrak{g}^*)^{(1)}$ has the form

$$(\mathfrak{g}^*)^{(1)} = \bigoplus_{\alpha \in R\cup\{0\}} (\mathfrak{g}^*)^{(1)}_{-p\alpha} \quad ; \quad (\mathfrak{g}^*)^{(1)}_{-p\alpha} \cong (\mathfrak{g}_\alpha)^*.$$

Let $f_x \in (\mathfrak{g}^*)^{(1)}_{-p\alpha_M}$ be a linear form such that $f_x(x) = 1$. As before, the G_1T-module $N := (\mathcal{N}^{-1} \circ \tau_{G_1T})(k)$ satisfies $\mathcal{F}(N) = \Omega^2_{\mathfrak{g}}(k)$ and the cohomology class $\varphi(f_x) \in H^2(U_0(\mathfrak{g}), k)_{-p\alpha_M}$ corresponds to a G_1T-linear map $\varphi_x \in \mathrm{Hom}_{G_1T}(N \otimes_k k_{-p\alpha_M}, k)$. There results an exact sequence

$$(0) \longrightarrow L_x \longrightarrow N \otimes_k k_{-p\alpha_M} \xrightarrow{\varphi_x} k \longrightarrow (0)$$

of G_1T-modules. Tensoring with M yields an exact sequence

$$(0) \longrightarrow L_x \otimes_k M \longrightarrow N \otimes_k (M \otimes_k k_{-p\alpha_M}) \longrightarrow M \longrightarrow (0)$$

of G_1T-modules. Now [17, (4.2)] gives $\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(L_x)) = \ker f_x$ and [18, (2.1)] implies

$$\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(L_x \otimes_k M)) = \mathcal{V}_{\mathfrak{g}}(\mathcal{F}(L_x) \otimes_k \mathcal{F}(M)) = \ker f_x \cap \mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M)) = \{0\}.$$

As a result, $L_x \otimes_k M$ is an injective G_1T-module (cf. [29, (II.9.3)]), and the above sequence splits. Consequently,

$$N \otimes_k (M \otimes_k k_{-p\alpha_M}) \cong (L_x \otimes_k M) \oplus M,$$

and an application of the Nakayama functor yields

$$\tau_{G_1T}(k) \otimes_k (M \otimes_k k_{-p\alpha_M}) \cong \mathcal{N}(M) \oplus (\mathrm{proj}).$$

Thanks to [22, (1.3)] and (2.1) the category $\mathrm{mod}\, G_1T$ has projective covers. The arguments of [1, (IV.3.7)] show that $\tau_{G_1T}(X)$ is computable from a minimal projective presentation of X (see also the discussion in [30, §1]). This implies $\tau_{G_1T}(k) \otimes_k X \cong \tau_{G_1T}(X) \oplus (\mathrm{proj})$. By the Krull-Schmidt property, the non-projective summands in the above decompostion are isomorphic, whence $\tau_{G_1T}(M \otimes_k k_{-p\alpha_M}) \cong \mathcal{N}(M)$. Consequently,

$$\tau_{G_1T}(M) \cong \mathcal{N}(M) \otimes_k k_{p\alpha_M},$$

as desired. □

3. G_1T-Components for Reductive Groups

Throughout the remainder of the paper, our group scheme G is assumed to be smooth, connected and reductive. In this situation the root space decomposition of $\mathfrak{g}$ relative to T takes the form

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

with $\dim_k \mathfrak{g}_\alpha = 1$ for every root $\alpha \in R$ (cf. [35, Chap.9]). From the presentation $G = (G, G)C(G)$ of G (cf. [35, (9.4)]) as an almost direct product of its derived group and its center, we obtain $\det(\mathrm{Ad}(g)) = 1$ for all $g \in G$. Differentiation then yields $\mathrm{tr}(\mathrm{ad}\, x) = 0 \;\; \forall\, x \in \mathfrak{g}$. Consequently, the Nakayama functors on $\mathrm{mod}\, G_1T$ and $\mathrm{mod}\, G_1$ coincide with the respective identities, and $U_0(\mathfrak{g})$ is a symmetric algebra.

An indecomposable G_1T-module M is referred to as *τ_{G_1T}-periodic* if there exists a natural number $n \in \mathbb{N}$ with $\tau^n_{G_1T}(M) \cong M$.

Proposition 3.1. *Let M be a non-projective, indecomposable G_1T-module. Then M is not τ_{G_1T}-periodic.*

Proof. If M is τ_{G_1T}-periodic, then there exists a number $n > 0$ such that

$$\tau^n_{G_1T}(M) \cong M.$$

Application of the functor $\mathcal{F}$ yields $\Omega^{2n}_{\mathfrak{g}}(\mathcal{F}(M)) \cong \mathcal{F}(M)$, so that $\mathrm{cx}_{\mathfrak{g}}(\mathcal{F}(M)) = 1$. As $\mathfrak{g}_0 = \mathfrak{t}$, the variety of $\mathcal{F}(M)$ intersects $\mathfrak{g}_0$ trivially and (2.4(2)) provides a root $\alpha_M \in R$ with $\tau_{G_1T}(M) \cong M \otimes_k k_{p\alpha_M}$. Thus, $M \cong M \otimes_k k_{np\alpha_M}$, and [21, (4.1)] gives $n = 0$, a contradiction. □

Theorem 3.2. *Let $\Theta \subset \Gamma_s(G_1T)$ be a component. If $\dim \mathcal{V}_{\mathfrak{g}}(\Theta) \neq 2$, then $\Theta \cong \mathbb{Z}[A_\infty]$.*

Proof. Given $x \in \mathcal{V}_{\mathfrak{g}}(\Theta) \setminus \{0\}$, we consider the subadditive function

$$d_x : \Theta \longrightarrow \mathbb{N} \ \ ; \ \ [M] \mapsto \dim_k \operatorname{Ext}^1_{U_0(\mathfrak{g})}(V_x, \mathcal{F}(M)).$$

By virtue of (2.2) the component $\Psi = \mathcal{F}(\Theta)$ of $\Gamma_s(G_1)$ shares this subadditive function with Θ. More precisely, the subadditive function

$$\delta_x : \Psi \longrightarrow \mathbb{N} \ \ ; \ \ [W] \mapsto \dim_k \operatorname{Ext}^1_{U_0(\mathfrak{g})}(V_x, W)$$

satisfies

$$\delta_x \circ \mathcal{F} = d_x.$$

If $\dim \mathcal{V}_{\mathfrak{g}}(\Psi) = \dim \mathcal{V}_{\mathfrak{g}}(\Theta) \geq 3$, then [11, (2.1)] implies $\Psi \cong \mathbb{Z}[A_\infty]$. In particular, Θ has infinitely many τ_{G_1T}-orbits, so that (2.3) implies $T_\Theta = A_\infty$, D_∞, A_∞^∞. Passage to orbit graphs yields a surjective morphism $\hat{\mathcal{F}} : T_\Theta \longrightarrow A_\infty$ and subadditive functions $\hat{\delta}_x : A_\infty \longrightarrow \mathbb{N}$, $\hat{d}_x : T_\Theta \longrightarrow \mathbb{N}$ such that $\hat{\delta}_x \circ \hat{\mathcal{F}} = \hat{d}_x$. In the latter two cases [1, (VII.3.4,VII.3.5)] ensure that $\hat{d}_x$ is a bounded, additive function. Then $\hat{\delta}_x$ is readily shown to enjoy the same properties. As $T_\Psi \cong A_\infty$ affords no such functions, we have reached a contradiction, so that $T_\Theta \cong A_\infty$. The Riedtmann structure theorem [2, (4.15.6)] provides an admissible subgroup $\Pi \subset \operatorname{Aut}(\mathbb{Z}[A_\infty])$ such that $\mathbb{Z}[A_\infty]/\Pi \cong \Theta$. A non-trivial admissible subgroup Π of $\operatorname{Aut}(\mathbb{Z}[A_\infty])$ is readily seen to contain a power τ^n of the translation $\tau : \mathbb{Z}[A_\infty] \longrightarrow \mathbb{Z}[A_\infty]$ for some $n > 0$. Thus, every vertex of Θ is periodic. As this contradicts (3.1), we obtain $\Theta \cong \mathbb{Z}[A_\infty]$.

It remains to consider the case, where $\dim \mathcal{V}_{\mathfrak{g}}(\Theta) = 1$. Owing to [10, (2.5),(5.3)] we have $\Psi \cong \mathbb{Z}[A_\infty]/(\tau^n_{G_1})$ for some $n > 0$. If $f : \mathbb{Z}[T_\Theta] \longrightarrow \Psi$ denotes the compositive of $\mathcal{F}$ with the covering $\mathbb{Z}[T_\Theta] \longrightarrow \Theta$, then passage to the orbit graphs provides a surjection $T_\Theta \longrightarrow A_\infty$. Thus, the above arguments yield $\Theta \cong \mathbb{Z}[A_\infty]$. □

Proposition 3.3. *Let $\Theta \subset \Gamma_s(G_1T)$ be a component. Suppose there exist $[M] \in \Theta$ and $\lambda \in pX(T) \setminus \{0\}$ such that $[M \otimes_k k_\lambda] \in \Theta$. Then either $\dim \mathcal{V}_{\mathfrak{g}}(\Theta) = 1$ and $\Theta \cong \mathbb{Z}[A_\infty]$, or $\Theta \cong \mathbb{Z}[A_\infty^\infty]$ and $\dim \mathcal{V}_{\mathfrak{g}}(\Theta) = 2$.*

Proof. We write $X^{(\lambda)} := X \otimes_k k_\lambda$ and recall that $X \mapsto X^{(\lambda)}$ is an auto-equivalence of $\operatorname{mod} G_1T$. Consequently, there is an associated isomorphism $[X] \mapsto [X^{(\lambda)}]$ of $\Gamma_s(G_1T)$, and $\Theta^{(\lambda)}$ is an AR-component. Thus, we either have $\Theta = \Theta^{(\lambda)}$ or $\Theta \cap \Theta^{(\lambda)} = \emptyset$. In view of our current assumption, the former alternative applies, and $[X] \mapsto [X^{(\lambda)}]$ defines an automorphism of Θ. Moreover, this map also induces a bijection on the set $\Theta/\langle \tau_{G_1T} \rangle$ of τ_{G_1T}-orbits of Θ.

If Θ affords only finitely many τ_{G_1T}-orbits, then there exists a natural number $n \in \mathbb{N}$ such that for every $[X] \in \Theta$ there is $m_X \in \mathbb{Z}$ with

$$X^{(n\lambda)} \cong \tau^{m_X}_{G_1T}(X).$$

Owing to [21, (4.1)] we have $X^{(n\lambda)} \ncong X$, so that $m_X \neq 0$. As

$$\mathcal{F}(X) = \mathcal{F}(X^{(n\lambda)}) \cong \mathcal{F}(\tau^{m_X}_{G_1T}(X)) \cong \Omega^{2m_X}_{\mathfrak{g}}(\mathcal{F}(X)),$$

the indecomposable $U_0(\mathfrak{g})$-module $\mathcal{F}(X)$ is $\Omega_{\mathfrak{g}}$-periodic. Consequently, $\operatorname{cx}_{\mathfrak{g}}(\mathcal{F}(X)) = 1$, and (2.4(2)) provides a root $\alpha_X \in R$ such that $\tau_{G_1T}(X) \cong X \otimes_k k_{p\alpha_X}$. Thus,

$$\dim_k \tau^\ell_{G_1T}(X) \leq \dim_k X \quad \ \forall \ \ell \in \mathbb{Z},$$

and the finiteness of $\Theta/\langle \tau_{G_1T} \rangle$ implies the existence of $n_0 \in \mathbb{N}$ with $\dim_k Z \leq n_0$ for all $[Z] \in \Theta$. The same holds for the AR-component $\mathcal{F}(\Theta)$ (cf. (2.2)) of $\Gamma_s(G_1)$. Thanks to [1, (VII.2.1)] the set $\mathcal{F}(\Theta)$ therefore consists of the isoclasses of non-projective indecomposable modules belonging to a non-simple, representation-finite block $\mathcal{B} \subset U_0(\mathfrak{g})$. By virtue of [12, (5.2)], however, the restricted enveloping algebra $U_0(\mathfrak{g})$ does not possess such blocks.

We conclude that Θ possesses infinitely many τ_{G_1T}-orbits, and apply (2.3) to obtain $T_\Theta = A_\infty$, D_∞, A_∞^∞. Owing to [2, (4.15.5)] the map $X \mapsto X^{(\lambda)}$ induces an automorphism of $\mathbb{Z}[T_\Theta]$ and hence on the orbit graph T_Θ. If $T_\Theta = A_\infty, D_\infty$, then each automorphism possesses a fixed point, implying the existence of a vertex $[X] \in \Theta$ and $m_X \in \mathbb{Z}$ with

$$X^{(\lambda)} \cong \tau_{G_1T}^{m_X}(X).$$

As before, we obtain $\dim \mathcal{V}_{\mathfrak{g}}(\Theta) = \mathrm{cx}_{\mathfrak{g}}(\mathcal{F}(X)) = 1$, and (3.2) yields $\Theta \cong \mathbb{Z}[A_\infty]$. Alternatively, $T_\Theta \cong A_\infty^\infty$ and Riedtmann's Theorem provides an admissible subgroup $\Pi \subset \mathrm{Aut}(\mathbb{Z}[A_\infty^\infty])$ with $\Theta \cong \mathbb{Z}[A_\infty^\infty]/\Pi$. Thanks to (3.1) the group Π intersects the subgroup $\langle\tau\rangle$ generated by the translation of $\mathbb{Z}[A_\infty^\infty]$ trivially. From the knowledge of the automorphism group of $\mathbb{Z}[A_\infty^\infty]$ one concludes that Θ has finitely many τ_{G_1T}-orbits unless $\Pi = \{1\}$. The first part of the proof now implies $\Theta \cong \mathbb{Z}[A_\infty^\infty]$. □

Example. Consider the group SL(2) together with its standard torus T of diagonal matrices. Given $i \geq 0$, we let $L(i)$ be the simple SL(2)-module of highest weight i (cf. [29, (II.2.7)]). According to [29, (II.3.15)] the restriction of each $L(i)$ to G_1 is simple whenever $i \in \{0, \dots, p-1\}$. For $i \in \{0, \dots, p-2\}$ we denote by $P(i)$ the principal indecomposable $U_0(\mathfrak{sl}(2))$-module with top $L(i)$. Then each $P(i)$ is an SL(2)-module with Loewy series $L(i)$, $L(2p-2-i)$, $L(i)$ (cf. [29, (II.3.15)], [26, Thm.3]). Let α in $X(T)$ be given by

$$\alpha(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) = t.$$

Steinberg's tensor product theorem [29, (II.3.17)] implies $L(2p-2-i) \cong L(p-2-i) \otimes_k L(1)^{[1]}$, so that we obtain an isomorphism

$$(*) \qquad L(2p-2-i) \cong (L(p-2-i) \otimes_k k_{p\alpha}) \oplus (L(p-2-i) \otimes_k k_{-p\alpha})$$

of $\mathrm{SL}(2)_1T$-modules.

Now let $\Theta \subset \Gamma_s(\mathrm{SL}(2)_1T)$ be a component such that $\dim \mathcal{V}_{\mathfrak{sl}(2)}(\Theta) = 2$. Then the component $\Psi := \mathcal{F}(\Theta) \subset \Gamma_s(G_1)$ has a two-dimensional rank variety and is thus isomorphic to $\mathbb{Z}[\tilde{A}_{12}]$ (cf. [12, p.62f]). By the analogue of [38, Thm.A] Ψ is attached to a principal indecomposable module $P(i)$. Consider the non-split sequence

$$(0) \longrightarrow \mathrm{Rad}(P(i)) \longrightarrow P(i) \oplus (\mathrm{Rad}(P(i))/\mathrm{Soc}(P(i))) \longrightarrow P(i)/\mathrm{Soc}(P(i)) \longrightarrow (0).$$

of $\mathrm{SL}(2)_1T$-modules. As observed in [30, (1.5)] the radicals and socles in $\mathrm{mod}\,\mathrm{SL}(2)_1T$ coincide with those taken with respect to $\mathrm{mod}\,\mathrm{SL}(2)_1$. Application of the exact functor $\mathcal{F}$ thus yields a standard almost split sequence of $\mathrm{SL}(2)_1$-modules. Hence the above sequence is almost split in $\mathrm{mod}(\mathrm{SL}(2)_1T)$. It now follows from $(*)$ that Θ contains a simple $\mathrm{SL}(2)_1T$-module S along with its shift $S \otimes_k k_{2p\alpha}$. Thanks to (3.3) we obtain $\Theta \cong \mathbb{Z}[A_\infty^\infty]$. In view of (3.2) the components of $\Gamma_s(\mathrm{SL}(2)_1T)$ with one-dimensional rank variety are isomorphic to $\mathbb{Z}[A_\infty]$.

Let $Z \cong \mu_{(2)}$ be the center of SL(2) and consider the group PSL(2) along with its maximal torus $T' := T/Z$. As $p \neq 2$, the center is étale and $\mathrm{PSL}(2)_1 \cong \mathrm{SL}(2)_1$. Thus, $\mathrm{mod}\,\mathrm{PSL}(2)_1T'$ is a sum of blocks of $\mathrm{mod}\,\mathrm{SL}(2)_1T$, and the foregoing observations also apply in this case.

Our final result of this section further narrows the possible shapes of components of $\Gamma_s(G_1T)$. Since any two maximal tori of G are conjugate by some element $g \in G(k)$ (see [35, (7.2.6)]), the structure of $\mathrm{mod}\,G_1T$ does not depend on the choice of T.

Since T is connected, the central primitive idempotents $\{e_1, \dots, e_s\}$ of $U_0(\mathfrak{g})$ are pointwise fixed by the adjoint representation. Consequently, each e_i is a homogeneous element of degree 0 and each block $\mathcal{B}_i := U_0(\mathfrak{g})e_i$ is a homogeneous subspace of $U_0(\mathfrak{g})$. There results a decomposition

$$\operatorname{mod} G_1T = \bigoplus_{i=1}^{s} \operatorname{mod} \mathcal{B}_i T,$$

where $\operatorname{mod} \mathcal{B}_i T$ is the category of finite-dimensional $X(T)$-graded $\mathcal{B}_i$-modules with $\mathfrak{t}$-compatible gradation. We let $\Gamma_s(\mathcal{B}_i T)$ be the stable Auslander-Reiten quiver of $\operatorname{mod} \mathcal{B}_i T$ and note that

$$\Gamma_s(G_1T) = \bigcup_{i=1}^{s} \Gamma_s(\mathcal{B}_i T)$$

is a disjoint union of translation subquivers.

Theorem 3.4. *Let $\Theta \subset \Gamma_s(G_1T)$ be a component of the stable Auslander-Reiten quiver of G_1T. Then Θ is isomorphic to $\mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty^\infty]$, or $\mathbb{Z}[D_\infty]$.*

Proof. By (2.2) the canonical restriction $\mathcal{F} : \operatorname{mod} G_1T \longrightarrow \operatorname{mod} G_1$ induces a homomorphism $\mathcal{F} : \Gamma_s(G_1T) \longrightarrow \Gamma_s(G_1)$ of stable translation quivers such that $\Psi := \mathcal{F}(\Theta)$ is a component of $\Gamma_s(G_1)$. According to (3.2) we may assume that

$$\dim \mathcal{V}_{\mathfrak{g}}(\Theta) = 2 = \dim \mathcal{V}_{\mathfrak{g}}(\Psi).$$

In view of [12, (5.4)] we then have $\Psi \cong \mathbb{Z}[T_\Psi]$, with $T_\Psi \in \{A_\infty, A_\infty^\infty, D_\infty, \tilde{A}_{12}\}$. Consequently, the surjective morphism $\mathcal{F} : \Theta \longrightarrow \Psi$ defines a surjection $\mathbb{Z}[T_\Theta] \longrightarrow \mathbb{Z}[T_\Psi]$, which in turn induces a surjection $T_\Theta \longrightarrow T_\Psi$.

If T_Ψ is infinite, then (2.3) implies $T_\Theta \cong A_\infty$, A_∞^∞, D_∞. Moreover, Ψ has infinitely many τ_{G_1}-orbits, so that Θ also has infinitely many τ_{G_1T}-orbits. If $T_\Theta = A_\infty$, D_∞, then any non-trivial admissible subgroup $\Pi \subset \operatorname{Aut}(\mathbb{Z}[T_\Theta])$ contains a positive power of the translation. By virtue of (3.1) we thus have $\Theta \cong \mathbb{Z}[T_\Theta]$. The arguments employed in the proof of (3.3) yield $\Theta \cong \mathbb{Z}[A_\infty^\infty]$ in the remaining case.

It remains to consider the case, where $T_\Psi \cong \tilde{A}_{12}$. Thanks to [38, Thm.A] the component Ψ is attached to a principal indecomposable $U_0(\mathfrak{g})$-module P. By virtue of (2.1) there exists a projective indecomposable G_1T-module $\hat{P}$ with $\mathcal{F}(\hat{P}) = P$ as well as an almost split sequence

$$(0) \longrightarrow \operatorname{Rad}(\hat{P}) \longrightarrow \hat{P} \oplus (\operatorname{Rad}(\hat{P})/\operatorname{Soc}(\hat{P})) \longrightarrow \hat{P}/\operatorname{Soc}(\hat{P}) \longrightarrow (0)$$

such that the indecomposable summands of $\operatorname{Rad}(\hat{P})/\operatorname{Soc}(\hat{P})$ belong to Θ.

Moreover, the component $\Omega_{\mathfrak{g}}(\Psi)$ contains a simple $U_0(\mathfrak{g})$-module S, implying that the two-dimensional variety $\mathcal{V}_{\mathfrak{g}}(\Psi) = \mathcal{V}_{\mathfrak{g}}(\Omega_{\mathfrak{g}}(\Psi))$ is invariant under the adjoint representation. In this situation [12, (5.1)] provides a decomposition $G = HK$ of G into an almost direct product such that

(a) $\mathfrak{g} = \operatorname{Lie}(H) \oplus \operatorname{Lie}(K)$ with $\operatorname{Lie}(H) = \mathfrak{sl}(2)$, and

(b) $\mathcal{V}_{\mathfrak{g}}(M) = \mathcal{V}_{\operatorname{Lie}(H)}(M)$ and $M|_{U_0(\operatorname{Lie}(K))}$ is projective for every $[M] \in \Psi$.

Since H is an almost simple group of rank 1, it follows that the central subgroup $H \cap K \subset H$ is either trivial or isomorphic to $\mu_{(2)}$ (cf. [35, (8.2.4)]). Hence, if $H \cap K \neq e_k$, then there exists a character $\lambda \in X(H \cap K) \cong \mathbb{Z}/(2)$ such that $H \cap K$ acts on every vertex $[M] \in \Theta$ via λ. As $H \cap K$ is contained in the maximal torus T, we can find $\gamma \in X(T)$ with $\gamma|_{H\cap K} = \lambda$. In view of p being odd, we also have $p\gamma|_{H\cap K} = \lambda$. Consequently, $H \cap K$ acts trivially on every vertex of the component $\Theta^{(-p\gamma)} \cong \Theta$.

Let $G' := G/(H \cap K)$ and consider its maximal torus $T' := T/(H \cap K)$ (cf. [35, (7.2.7)]). Since $H \cap K$ is étale, there results an exact sequence

$$e_k \longrightarrow H \cap K \longrightarrow G_1T \longrightarrow G'_1T' \longrightarrow e_k$$

with $\operatorname{mod} G'_1T'$ being a sum of blocks of $\operatorname{mod} G_1T$. By the above observation, a suitable shift of Θ belongs to $\operatorname{mod}(G'_1T)$. Setting $H' := H/(H \cap K)$ and $K' := K/(H \cap K)$, we have $G' = H' \times K'$ while (a) and (b) continue to hold for H' and K'. As a result, we may assume in addition that

(c) $G = H \times K$.

By general theory, there exists maximal tori $T_H \subset H$ and $T_K \subset K$ such that $T = T_H \times T_K$. Consequently, the isomorphism

$$U_0(\mathfrak{g}) \cong U_0(\mathfrak{sl}(2)) \otimes_k U_0(\operatorname{Lie}(K))$$

induced by (a) is compatible with the T-action. Thus, the outer tensor product defines a functor

$$\operatorname{mod} H_1T_H \times \operatorname{mod} K_1T_K \longrightarrow \operatorname{mod} G_1T \ \ ; \ \ (M, N) \mapsto M \otimes_k N.$$

Let $\mathcal{B} \subset U_0(\mathfrak{g})$ be the block containing the simple $U_0(\mathfrak{g})$-module S, so that $\Theta \subset \Gamma_s(\mathcal{B}T)$. Owing to [12, (4.1)], the $U_0(\mathfrak{g})$-module S is an outer tensor product

$$S \cong S_1 \otimes_k S_2$$

with a simple projective $U_0(\operatorname{Lie}(K))$-module S_2. Letting $\mathcal{B}_1 \subset U_0(\mathfrak{sl}(2))$ be the block containing S_1, we obtain inverse equivalences

$$\operatorname{mod} \mathcal{B}_1 \longrightarrow \operatorname{mod} \mathcal{B} \ \ ; \ \ X \mapsto X \otimes_k S_2$$

and

$$\operatorname{mod} \mathcal{B} \longrightarrow \operatorname{mod} \mathcal{B}_1 \ \ ; \ \ Y \mapsto \operatorname{Hom}_{U_0(\operatorname{Lie}(K))}(S_2, Y),$$

so that the first functor induces an equivalence

$$\operatorname{mod} \mathcal{B}_1T_H \longrightarrow \operatorname{mod} \mathcal{B}T \ \ ; \ \ X \mapsto X \otimes_k S_2.$$

Thus, Θ is isomorphic to a component $\Theta_1 \subset \Gamma_s(H_1T_H)$, which, by (b), has a two-dimensional rank variety. Since the structure of $\operatorname{mod} H_1T_H$ does not depend on the choice of T_H, our above example provides isomorphisms $\Theta \cong \Theta_1 \cong \mathbb{Z}[A_\infty^\infty]$. □

Remark. It is not known whether components of type $\mathbb{Z}[D_\infty]$ actually occur.

4. Applications

We retain the general conventions of the foregoing section and turn to applications concerning gradable modules and Frobenius kernels of G_1T. Fix a Borel subgroup $B \subset G$ containing T and let $\mathfrak{b} := \operatorname{Lie}(B)$ be the corresponding Borel subalgebra of $\mathfrak{g} := \operatorname{Lie}(G)$. Given a character $\lambda \in X(T)$, we denote by $\hat{Z}(\lambda) := U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} k_\lambda$ the *baby Verma module* with highest weight λ. Since $U_0(\mathfrak{b})$ is stable under the adjoint action of B on $U_0(\mathfrak{g})$, $\hat{Z}(\lambda)$ is a G_1T-module. The varieties of the baby Verma modules $Z(\lambda) := \mathcal{F}(\hat{Z}(\lambda))$ are B-invariant subspaces of the unipotent radical $\mathfrak{u}$ of $\mathfrak{b}$ (cf. [13, (2.1)]). A G_1T-module M is said to have a *$\hat{Z}$-filtration* if there exist G_1T-submodules $(0) = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$ such that $M_i/M_{i-1} \cong \hat{Z}(\lambda_i)$ for suitable $\lambda_1, \ldots, \lambda_n \in X(T)$. Z-filtrations for $U_0(\mathfrak{g})$-modules are defined analogously. According to [29, (II.11.2)] the length of a $\hat{Z}$-filtration or a Z-filtration is an invariant of M.

Theorem 4.1. *Let M be an indecomposable G_1T-module, $\Theta \subset \Gamma_s(G_1T)$ and $\Psi \subset \Gamma_s(G_1)$ the stable AR-components containing M and $\mathcal{F}(M)$, respectively.*
(1) *Every vertex of Ψ has a G_1T-structure.*
(2) *If M affords a $\hat{Z}$-filtration, so does every indecomposable G_1T-module belonging to Θ.*
(3) *If $\mathcal{F}(M)$ affords a Z-filtration, so does every indecomposable $U_0(\mathfrak{g})$-module belonging to Ψ.*

Proof. Owing to (2.2) $\mathcal{F}(\Theta)$ is a component of $\Gamma_s(G_1)$ containing $\mathcal{F}(M)$. Thus, $\Psi = \mathcal{F}(\Theta)$, so that (1) follows.

Let $\mathfrak{b}^-$ be the Borel subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{b}$. Thanks to [29, (II.11.2)], a module $N \in \operatorname{mod} G_1T$ affords a $\hat{Z}$-filtration if and only if $\mathcal{F}(N)|_{U_0(\mathfrak{b}^-)}$ is injective. The latter condition is equivalent to $\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(N)) \cap \mathfrak{b}^- = \{0\}$. By standard properties of rank varieties (cf. [19, §7]), the presence of a $\hat{Z}$-filtration for M implies

$$\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(N)) \cap \mathfrak{b}^- = \mathcal{V}_{\mathfrak{g}}(\Theta) \cap \mathfrak{b}^- = \mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M)) \cap \mathfrak{b}^- \subset \mathfrak{u} \cap \mathfrak{b}^- = \{0\}$$

for every $[N] \in \Theta$. This yields our second assertion.

If $\mathcal{F}(M)$ affords a Z-filtration, then $\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M)) \cap \mathfrak{b}^- = \{0\}$ and M possesses a $\hat{Z}$-filtration. In view of $\Psi = \mathcal{F}(\Theta)$, assertion (3) is now a consequence of (2). □

Remarks. (1) As noted in [6, (3.3)], the category $\operatorname{mod} G_1T$ is a highest weight category, whose standard objects are the $\hat{Z}(\lambda)$. According to [6, (3.6)] highest weight categories with finitely many simple objects correspond to quasi-hereditary algebras. In that context, Ringel [33, Thm.2] proved the existence of relative almost split sequences within the subcategory of modules affording a filtration by standard modules.

(2) Let $\alpha \in R$ be a positive root. According to (2.4(1)) there exists a G_1T-module M such that $\mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M)) = \mathfrak{g}_\alpha$. As M is not projective, some indecomposable constituent of X of M also enjoys this property. The rank variety of the component $\Theta \subset \Gamma_s(G_1T)$ containing X satisfies

$$\mathcal{V}_{\mathfrak{g}}(\Theta) \cap \mathfrak{b}^- = \mathfrak{g}_\alpha \cap \mathfrak{b}^- = \{0\},$$

so that every vertex of Θ affords a $\hat{Z}$-filtration. Unless α is a highest root of $\mathfrak{g}$, the variety $\mathcal{V}_{\mathfrak{g}}(\Theta) = \mathfrak{g}_\alpha$ is not B-invariant, proving that Θ usually does not contain a baby Verma module (see [13, (3.3)] for a precise description of the exceptional cases).

An indecomposable G_1T-module M is called *quasi-simple* if
(a) $[M]$ belongs to a component $\Theta \subset \Gamma_s(G_1T)$ of tree class A_∞, and
(b) $[M]$ has exactly one predecessor in $\Gamma_s(G_1T)$.

Corollary 4.2. *The following statements hold:*
(1) *If S is a non-projective, simple G_1T-module, then S is either quasi-simple or belongs to a component of type $\mathbb{Z}[A_\infty^\infty]$.*
(2) *Every non-projective baby Verma module $\hat{Z}(\lambda)$ is quasi-simple.*

Proof. (1) Thanks to [30, (1.5)] the $U_0(\mathfrak{g})$-module $\mathcal{F}(S)$ is simple. A consecutive application of [10, (3.2)] and [12, (5.2)] implies $\dim \mathcal{V}_{\mathfrak{g}}(\mathcal{F}(S)) \geq 2$. If $\dim \mathcal{V}_{\mathfrak{g}}(\mathcal{F}(S)) \geq 3$, then, by (3.2), the component $\Theta \subset \Gamma_s(G_1T)$ containing S is of type $\mathbb{Z}[A_\infty]$. Thanks to [11, (2.1),(4.1)] the module $\mathcal{F}(S)$ is quasi-simple. Thus, if S possesses two predecessors in Θ, then their images under $\mathcal{F}$ coincide, and [21, (4.1)] implies that Θ contains a module along with a non-trivial shift. This, however, contradicts (3.3).

In the remaining case, [12, (5.2(3))] implies that the component $\Psi := \mathcal{F}(\Theta)$ is of type $\mathbb{Z}[\tilde{A}_{12}]$. Hence every vertex in Ψ has exactly one predecessor. In view of (3.4) this does not hold in Θ, implying again that Θ satisfies the hypothesis of (3.3). Since $\dim \mathcal{V}_{\mathfrak{g}}(\Theta) = 2$, we obtain $\Theta \cong \mathbb{Z}[A_\infty^\infty]$.

(2) Let $\Theta(\lambda)$ be the component containing $\hat{Z}(\lambda)$. Then $\Psi(\lambda) := \mathcal{F}(\Theta(\lambda))$ is the component of $\Gamma_s(G_1)$ containing $Z(\lambda)$. According to [13, Theorem] we have $T_{\Psi(\lambda)} = A_\infty$ and the arguments of (3.2) yield $T_{\Theta(\lambda)} \cong A_\infty$. Owing to (4.1) the module $\hat{Z}(\lambda)$ is a vertex of $\Theta(\lambda)$ of minimal dimension. Since $\mathcal{V}_{\mathfrak{g}}(\Theta(\lambda)) \subset \mathfrak{u}$ is not G-stable, the component $\Theta(\lambda)$ is not attached to a principal indecomposable module. Consequently, the minimality of $\dim_k \hat{Z}(\lambda)$ implies that $\hat{Z}(\lambda)$ is quasi-simple. □

If V and W are simple G_1T-modules belonging to a component $\Theta \cong \mathbb{Z}[A_\infty]$, then $\mathcal{F}(V)$ and $\mathcal{F}(W)$ belong to the component $\mathcal{F}(\Theta) \cong \mathbb{Z}[A_\infty]$, and a consecutive application of [11, (4.1)], [21, (4.1)] and (3.3) implies $V \cong W$. Our next result provides the analogous property for components containing baby Verma modules. In view of (4.1), a baby Verma module is therefore the unique vertex of minimal dimension within its component.

Proposition 4.3. *Let $\lambda, \mu \in X(T)$.*

(1) If $\hat{Z}(\lambda)$ and $\hat{Z}(\mu)$ are non-projective baby Verma modules belonging to the same component of $\Gamma_s(G_1T)$, then $\lambda = \mu$.

(2) If $Z(\lambda)$ and $Z(\mu)$ are non-projective baby Verma modules belonging to the same component of $\Gamma_s(G_1)$, then $\lambda - \mu \in pX(T)$ and $Z(\lambda) \cong Z(\mu)$.

Proof. (1) We let W be the Weyl group of G, pick a simple system $\{\alpha_1, \ldots, \alpha_n\} \subset R$, and denote by $s_1, \ldots, s_n \in W$ the corresponding simple reflections. We consider the element

$$z_0 := \prod_{\alpha \in R^+} \frac{1 - e(-p\alpha)}{1 - e(-\alpha)}$$

of the group algebra $\mathbb{Z}[X(T)]$. Direct computation shows that, with respect to the canonical action of W on $\mathbb{Z}[X(T)]$, we have

$$(*) \qquad s_i.z_0 = e((p-1)\alpha_i)z_0 \qquad 1 \le i \le n.$$

Let Ω_{G_1T} be the Heller operator of the Frobenius category $\operatorname{mod} G_1T$. By the arguments of [1, (IV.3)] and [30, (1.7)] we have $\Omega^2_{G_1T} = \tau_{G_1T}$. Consequently, our assumption in conjunction with (4.2(2)) provides an exact sequence

$$(0) \longrightarrow \hat{Z}(\mu) \longrightarrow P_{2n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \hat{Z}(\lambda) \longrightarrow (0) \qquad (n \ge 0)$$

with projective G_1T-modules $P_0, \ldots, P_{2n-1}$. Passage to formal characters (see [29, (I.2.11)]for the definition) yields

$$\operatorname{ch}(\hat{Z}(\lambda)) - \operatorname{ch}(\hat{Z}(\mu)) = \sum_{i=0}^{2n-1} (-1)^i \operatorname{ch}(P_i).$$

Thanks to [29, (II.11.7)] the right-hand side is contained in the subalgebra $\mathbb{Z}[X(T)]^W$ of W-invariants of $\mathbb{Z}[X(T)]$. Observing $\operatorname{ch}(\hat{Z}(\gamma)) = e(\gamma)z_0$ for every $\gamma \in X(T)$ (cf. [29, (II.9.2)]), we obtain from $(*)$ the identity

$$(e(\lambda) - e(\mu))z_0 = s_i.[(e(\lambda) - e(\mu))z_0] = (e(s_i.\lambda) - e(s_i.\mu))e((p-1)\alpha_i)z_0$$

for every $i \in \{1, \ldots, n\}$. Since $\mathbb{Z}[X(T)]$ is an integral domain, we may cancel z_0 and the assumption $\lambda \neq \mu$ leads, by comparison of coefficients, to

$$\lambda = s_i.\lambda + (p-1)\alpha_i \qquad 1 \le i \le n.$$

Thus, the bilinear form $\langle\, , \rangle : X(T) \times Y(T) \longrightarrow \mathbb{Z}$ between $X(T)$ and the group $Y(T)$ of co-characters takes values

$$\langle \lambda, \alpha_i^\vee \rangle = p - 1 \qquad 1 \le i \le n.$$

In view of [29, (II.11.8)] these conditions render $\hat{Z}(\lambda)$ a projective G_1T-module, a contradiction. Consequently, $\lambda = \mu$, as desired.

(2) Let $\Psi \in \Gamma_s(G_1)$ be the component cotaining $Z(\lambda)$ and denote by $\Theta(\lambda)$ and $\Theta(\mu)$ the components of $\Gamma_s(G_1T)$ containing $\hat{Z}(\lambda)$ and $\hat{Z}(\mu)$, respectively. In view of (2.2) we have

$$\mathcal{F}(\Theta(\lambda)) = \Psi = \mathcal{F}(\Theta(\mu)),$$

so that there exists a G_1T-module M belonging to $\Theta(\lambda)$ such that $\mathcal{F}(M) = Z(\mu) = \mathcal{F}(\hat{Z}(\mu))$. Consequently, [21, (4.1)] provides $\gamma \in X(T)$ with $M \cong \hat{Z}(\mu) \otimes_k k_{p\gamma} \cong \hat{Z}(\mu + p\gamma)$ (see [29, (II.9.2)]). Now (1) implies $\lambda = \mu + p\gamma$, so that $Z(\lambda) = \mathcal{F}(\hat{Z}(\lambda)) = \mathcal{F}(\hat{Z}(\mu)) \cong Z(\mu)$. □

We turn to periodic Auslander-Reiten components of the infinitesimal groups G_1T_r for $r \geq 1$. Our aim is to show that all tubes of $\Gamma_s(G_1T_r)$ have rank p^i for some $i \in \{0, \ldots, r-1\}$, with the maximum being assumed by modules whose rank varieties are root spaces. The first assertion follows from a general result concerning the stable Auslander-Reiten quiver $\Gamma_s(\mathcal{G})$ of an arbitrary infinitesimal group scheme $\mathcal{G}$. In view of [29, (I.8.4),(I.8.6)] the module category $\operatorname{mod}\mathcal{G}$ coincides with the category $\operatorname{mod}\operatorname{Dist}(\mathcal{G})$ of finite-dimensional modules of the Hopf algebra $\operatorname{Dist}(\mathcal{G})$ of distributions associated to $\mathcal{G}$. With regard to periodicity, infinitesimal groups behave like p-groups with the height of $\mathcal{G}$ taking on a rôle similar to the minimal rank of the maximal p-elementary abelian subgroups (cf. [4, (2.2)]). By definition, the *height* of $\mathcal{G}$ is the minimal $r \geq 0$ such that $\mathcal{G}$ coincides with its r-th Frobenius kernel $\mathcal{G}_r$. The Heller operator and the Auslander-Reiten translation of the Frobenius algebra $\operatorname{Dist}(\mathcal{G})$ will be denoted $\Omega_{\mathcal{G}}$ and $\tau_{\mathcal{G}}$, respectively.

Proposition 4.4. *Let $\mathcal{G}$ be an infinitesimal group of height r, M is an indecomposable, $\Omega_{\mathcal{G}}$-periodic $\mathcal{G}$-module belonging to a component $\Theta \subset \Gamma_s(\mathcal{G})$. Then the following statements hold:*
1. *The $\Omega_{\mathcal{G}}$-period of M divides $2p^{r-1}$.*
2. *If Θ has tree class A_∞, then $\Theta \cong \mathbb{Z}[A_\infty]/(p^i)$ for $i \in \{0, \ldots, r\}$.*
3. *If $\operatorname{Dist}(\mathcal{G})$ is symmetric and Θ has tree class A_∞, then $\Theta \cong \mathbb{Z}[A_\infty]/(p^i)$ for $i \in \{0, \ldots, r-1\}$.*

Proof. (1) By the Friedlander-Suslin Theorem [20, (1.1)], the cohomology ring $H^*(\mathcal{G}, k)$ is finitely generated. More precisely, [20, (1.5)] provides a subalgebra $A \subset H^{\mathrm{ev}}(\mathcal{G}, k)$, generated in degrees $2, 2p, \ldots, 2p^{r-1}$, such that $H^*(\mathcal{G}, k)$ is a finite A-module. According to [36, (7.5)] the arguments of [3, (5.10.6)] also apply in our context, so that the $\Omega_{\mathcal{G}}$-period of M is a divisor of $2p^{r-1}$.

(2),(3) Since $\mathcal{G}$ has height r, the modular function $\zeta \in X(\mathcal{G})$ of $\operatorname{Dist}(\mathcal{G})$ has order a divisor of p^r. Owing to [16, (1.5)], the Nakayama functor $\mathcal{N}$ of $\operatorname{mod}\mathcal{G}$ also has this property, and the identity

$$\tau_{\mathcal{G}} = \mathcal{N} \circ \Omega_{\mathcal{G}}^2$$

shows that M has $\tau_{\mathcal{G}}$-period p^i for some $i \leq r$. If $\operatorname{Dist}(\mathcal{G})$ is symmetric, then $\mathcal{N} = \mathrm{id}$, so that $i \leq r - 1$. □

Being the r-th Frobenius kernel of G_1T, the group G_1T_r is infinitesmal of height r. The category $\operatorname{mod} G_1T_r$ can be interpreted as the category of finite-dimensional $X(T_r)$-graded $U_0(\mathfrak{g})$-modules with t-compatible grading. Since the funcor $\mathcal{F} : \operatorname{mod} G_1T \longrightarrow \operatorname{mod} G_1$ factors through $\operatorname{mod} G_1T_r$, it readily follows from (2.1) that the restriction functor $\operatorname{mod} G_1T \longrightarrow \operatorname{mod} G_1T_r$ preserves indecomposables and almost split sequences.

The determination of the stable Auslander-Reiten quiver of the symmetric, special biserial algebra $\operatorname{Dist}(\mathrm{SL}(2)_1T_r)$ (cf. [14, (5.6)]) required the computation of the components containing the non-projective $\mathrm{SL}(2)_1T_r$-modules $\hat{Z}(\lambda)$. These were shown to be of type $\mathbb{Z}[A_\infty]/(\tau^{p^{r-1}})$, which (4.4(3)) predicts to be tubes of maximal rank. Our final result explains this phenomenon in terms of rank varieties.

Theorem 4.5. *Suppose that G is semi-simple and simply connected. Let M be an indecomposable G_1T_r-module such that $\mathcal{V}_{\mathfrak{g}}(M) = \mathfrak{g}_\alpha$ for some root $\alpha \in R$. Then M belongs to a stable AR-component of type $\mathbb{Z}[A_\infty]/(\tau^{p^{r-1}})$.*

Proof. We first show that $\mathrm{Dist}(G_1T_r)$ is a symmetric algebra. According to [16, (1.5)] the convolution $\zeta * \mathrm{id}_{\mathrm{Dist}(G_1T_r)}$ of the identity with the modular function $\zeta : \mathrm{Dist}(G_1T_r) \longrightarrow k$ is a Nakayama automorphism of the Frobenius algebra $\mathrm{Dist}(G_1T_r)$ (cf. also [29, (I.8.12)]). Owing to [29, (I.8.13)] the module k_ζ is the socle of the projective cover of the trivial $\mathrm{Dist}(G_1T_r)$-module k. Hence $\mathrm{Dist}(G_1T_r)$ is symmetric if and only if its principal block $\mathcal{B}_0(G_1T_r)$ enjoys this property. Since $G_1 \cap T_r$ is multiplicative, the canonical isomorphism $G_1T_r \cong (G_1 \rtimes T_r)/(G_1 \cap T_r)$ in conjunction with [14, (1.1)] implies

$$\mathcal{B}_0(G_1T_r) \cong \mathcal{B}_0(G_1 \rtimes T_r),$$

so that it suffices to verify the symmetry of the Hopf algebra $\mathrm{Dist}(G_1 \rtimes T_r)$. This algebra coincides with the smash product $\mathrm{Dist}(G_1)\sharp\,\mathrm{Dist}(T_r)$, defined via the (left) adjoint representation

$$t.x := \sum_{(t)} t_{(1)} x \eta(t_{(2)}) \qquad t \in \mathrm{Dist}(T_r),\ x \in \mathrm{Dist}(G_1)$$

of $\mathrm{Dist}(T_r)$ on $\mathrm{Dist}(G_1) \cong U_0(\mathfrak{g})$ (cf. [31, (4.1.3)]). (Here η denotes the antipode of $\mathrm{Dist}(T_r)$.) According to [29, (I.7.18(3))] this action is induced by the adjoint action of T on $\mathfrak{g}$. Let $\int_{G_1}$ and $\int_{T_r}$ be the one-dimensional spaces of left integrals of $\mathrm{Dist}(G_1)$ and $\mathrm{Dist}(T_r)$, respectively (cf. [29, (I.8.7)]). Since G is semi-simple, $t \mapsto \det(\mathrm{Ad}(t)))$ is the trivial character, and [29, (I.9.7)] shows that T acts trivially on $\int_{G_1}$. Thus, $\mathrm{Dist}(T_r)$ also acts trivially on $\int_{G_1}$, and direct computation identifies $\int_{G_1} \otimes_k \int_{T_r}$ as the space of left integrals of $\mathrm{Dist}(G_1)\sharp\,\mathrm{Dist}(T_r)$. Since the modular functions of $\mathrm{Dist}(G_1)$ and $\mathrm{Dist}(T_r)$ are trivial, the corresponding linear form for $\mathrm{Dist}(G_1)\sharp\,\mathrm{Dist}(T_r)$ is readily seen to be trivial as well. As a result, the algebra $\mathrm{Dist}(G_1)\sharp\,\mathrm{Dist}(T_r)$ is symmetric.

We put $\mathfrak{g}_\alpha = kx$ and recall from the proof of (2.4(2)) the exact sequence

$$(0) \longrightarrow L_x \longrightarrow N \otimes_k k_{-p\alpha} \longrightarrow k \longrightarrow (0)$$

of G_1T-modules. Since the restriction $\mathrm{mod}\, G_1T \longrightarrow \mathrm{mod}\, G_1T_r$ commutes with Auslander-Reiten translations, the symmetry of $\mathrm{Dist}(\mathcal{G}_1T_r)$ in conjunction with [1, (IV.3.8)] shows that the G_1T_r-module N is isomorphic to $\Omega^2_{G_1T_r}(k)$. Adopting the arguments of (2.4(2)) mutatis mutandis we obtain

$$\Omega^2_{G_1T_r}(M) \cong M \otimes_k k_{p\beta},$$

where $\beta \in X(T_r) \cong X(T)/p^rX(T)$ denotes the restriction of $\alpha \in X(T)$ to T_r (cf. [29, (II.3.7)]). Consequently, $p\beta$ has order p^ℓ for some $\ell \leq r-1$ and there exists $\gamma \in X(T)$ with $p^{\ell+1}\alpha = p^r\gamma$. Letting $\langle\, ,\, \rangle : X(T) \times Y(T) \longrightarrow \mathbb{Z}$ be the duality between $X(T)$ and the group $Y(T)$ of co-characters (cf. [35, (2.5.12)]), we obtain for the coroot $\alpha^\vee$ of α (cf. [35, (8.2.7)])

$$2p^{\ell+1} = \langle p^r\gamma, \alpha^\vee\rangle = p^r\langle\gamma, \alpha^\vee\rangle,$$

which, in view of $p \neq 2$, implies $\ell \geq r-1$. Thus, $p\beta \in X(T_r)$ has order p^{r-1}, and we conclude that

$$\tau^{p^{r-1}}_{G_1T_r}(M) \cong \Omega^{2p^{r-1}}_{G_1T_r}(M) \cong M \not\cong \tau^n_{G_1T_r}(M) \qquad 1 \leq n \leq p^{r-1}-1.$$

Hence the stable component Θ containing M possesses a periodic vertex. Moreover, $X \mapsto \dim_k X$ defines an subadditive function f on Θ.

If f is bounded, then [1, (VI.1.4)] provides a non-simple block $\mathcal{B} \subset \mathrm{Dist}(G_1T_r)$ such that Θ consists of the isoclasses of non-projective indecomposable $\mathcal{B}$-modules. Since G is semi-simple and simply connected, [29, (II.3.15)] shows that every simple G_1-module is the restriction of a simple G-module. The arguments of [14, (5.1)] now ensure the simplicity of the restriction $S|_{G_1}$ of any simple G_1T_r-module S. Since $\mathcal{B}$ is representation-finite and not simple, we obtain a simple G_1-module V such that $\mathrm{cx}_{\mathfrak{g}}(V) = 1$. According to [10, (3.2)] the block $\mathcal{C} \subset \mathrm{Dist}(G_1)$ containing V is representation-finite and not simple. This, however, contradicts [12, (5.2)].

Consequently, the function f is unbounded and [24, Thm., p.289] shows that $T_\Theta = A_\infty$. Since $\mathrm{Aut}(\mathbb{Z}[A_\infty]) = \langle \tau \rangle$ is generated by the translation τ, the above identity, which holds for every vertex $[N] \in \Theta$, implies $\Theta \cong \mathbb{Z}[A_\infty]/(\tau^{p^{r-1}})$. $\square$

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