POLYHEDRAL GROUPS, McKAY QUIVERS, AND THE FINITE ALGEBRAIC GROUPS WITH TAME PRINCIPAL BLOCKS

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Abstract. Given an algebraically closed field k of characteristic $p \geq 3$, we classify the finite algebraic k-groups whose algebras of measures afford a principal block of tame representation type. The structure of such a group $\mathcal{G}$ is largely determined by a linearly reductive subgroup scheme $\hat{\mathcal{G}}$ of SL(2), with the McKay quiver of $\hat{\mathcal{G}}$ relative to its standard module being the Gabriel quiver of the principal block $\mathcal{B}_0(\mathcal{G})$. The graphs underlying these quivers are extended Dynkin diagrams of type $\tilde{A}$, $\tilde{D}$ or $\tilde{E}$, and the tame blocks are Morita equivalent to generalizations of the trivial extensions of the radical square zero tame hereditary algebras of the corresponding type.

0. Introduction

In his seminal paper [40], J. McKay associated to a finite group G and a complex G-module T a quiver $\Upsilon_T(\mathcal{G})$, whose vertices are the simple complex G-modules and whose arrows are given by the transpose of the matrix describing the multiplication effected by T in the complex Grothendieck ring of G. Subsequently, these quivers have made an appearance in various contexts, ranging from abstract representation theory to rational singularities (cf. for instance [2, 3, 28, 38, 39]).

McKay's construction exploits in an essential fashion the presence of a tensor product within the category of G-modules. Tensor products occur in other contexts as well, and this paper elicits the rôle of McKay quivers within the representation theory of cocommutative Hopf algebras. More specifically, the McKay quivers associated to two-dimensional, self-dual representations of certain linearly reductive finite algebraic groups are shown to give rise to the Gabriel quivers of blocks of tame cocommutative Hopf algebras over algebraically closed fields of odd characteristic.

Let H be a finite-dimensional cocommutative Hopf algebra, defined over an algebraically closed field k. If $\mathrm{char}(k) = 0$, then Cartier's Theorem implies that $H = k[G]$ is the group algebra of a finite group G. Thus, H is semisimple and the module category of H is completely determined by the simple H-modules. By contrast, Nagata's classical theorem shows that cocommutative Hopf algebras over fields of positive characteristic are usually not semisimple. Accordingly, the representation type of H becomes an important measure of the complexity of its module category. By Drozd's fundamental result [16], an associative k-algebra Λ is either *representation-finite, tame, or wild*. In the first case Λ admits only finitely many isoclasses of indecomposable modules. A representation-infinite algebra is tame if all but finitely many isoclasses of the indecomposables of any given dimension occur in a finite number of continuous one-parameter families. If the module category of Λ is at least as complicated as that of any other k-algebra, then a classification of the indecomposable Λ-modules is deemed hopeless and Λ is said to be wild. We refer the reader to [17, (I.4)] for the precise definitions.

For group algebras of finite groups the representation type of the blocks $\mathcal{B} \subset k[G]$ is governed by the structure of their defect groups. A block is representation-finite if and only if the corresponding defect groups are cyclic, while tame blocks only occur for $\mathrm{char}(k) = 2$ and if the defect groups are dihedral, semidihedral, or generalized quaternion (cf. [9, 29]). Thanks to the work by K. Erdmann (cf. [17]) the basic algebras of the tame blocks are completely understood.

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Cocommutative Hopf algebras can also be interpreted as group algebras. More precisely, every such algebra is the algebra $H(\mathcal{G})$ of measures on a finite algebraic k-group $\mathcal{G}$. By general theory, the group scheme $\mathcal{G}$ can be written as a semidirect product

$$\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\rm red},$$

of an infinitesimal normal subgroup $\mathcal{G}^0$ and a reduced group $\mathcal{G}_{\rm red}$. Here $\mathcal{G}^0$ is the connected component of $\mathcal{G}$, and $\mathcal{G}_{\rm red}$ is represented by the largest separable subalgebra of the function algebra of $\mathcal{G}$ (cf. [64, (6.8)]). At the level of the corresponding cocommutative Hopf algebras the above decomposition induces an identification

$$H(\mathcal{G}) = H(\mathcal{G}^0)[\mathcal{G}(k)]$$

of $H(\mathcal{G})$ with the skew group algebra $H(\mathcal{G}^0)[\mathcal{G}(k)]$ relative to the finite group $\mathcal{G}(k)$ of rational points of the group scheme $\mathcal{G}$. In analogy with abstract groups the representation theories of $\mathcal{G}$ and $H(\mathcal{G})$ coincide, so that we will use the notions "$\mathcal{G}$-modules" and "$H(\mathcal{G})$-modules" interchangeably.

The foregoing analogy notwithstanding, the representation theory of finite group schemes differs significantly from its classical precursor concerning modular representations of finite groups, which in our context are the reduced finite group schemes. There is no analogue for the notion of a defect group, and the principal block is not necessarily the most complicated block of the Hopf algebra (cf. [20, §8]). By Nagata's aforementioned theorem (cf. [13, (IV,§3,3.6)]) the algebra $H(\mathcal{G})$ is semisimple if and only if the characteristic p of the ground field k does not divide $\mathrm{ord}(\mathcal{G}(k))$, and $\mathcal{G}^0$ is a multiplicative group scheme. The corresponding result for representation-finite Hopf algebras can be found in [23]. Recently, the tame infinitesimal groups of characteristic $p \geq 3$ were determined in [24, 20, 21] by combining geometric methods involving rank varieties and schemes of tori with those from abstract representation theory. The main objective of this paper is the classification of the finite algebraic groups $\mathcal{G}$ of odd characteristic, whose Hopf algebras $H(\mathcal{G})$ possess a tame principal block $\mathcal{B}_0(\mathcal{G})$. The structures of $\mathcal{G}$ and $\mathcal{B}_0(\mathcal{G})$ turn out to be largely determined by linearly reductive subgroup schemes of PSL(2). Products of these *polyhedral groups* with the first Frobenius kernel of SL(2), in the sequel referred to as *amalgamated polyhedral groups*, are shown to be basic building blocks of tame groups. Our main results, which are formulated in greater detail in Section 7, also provide a link between the structure of $\mathcal{B}_0(\mathcal{G})$ and linearly reductive subgroups of SL(2).

Theorem A. *Let $\mathcal{G}$ be a finite algebraic group of characteristic $p \geq 3$, $\mathcal{M} \subset \mathcal{G}^0$ the multiplicative center of the connected component $\mathcal{G}^0$, and denote by $\mathcal{N} := \mathrm{Cent}_{\mathcal{G}}(\mathcal{G}^0/\mathcal{M})$ the centralizer of $\mathcal{G}^0/\mathcal{M}$ in $\mathcal{G}$. If the principal block $\mathcal{B}_0(\mathcal{G})$ is tame, then $\mathcal{G}/\mathcal{N}$ is an amalgamated polyhedral group, $\mathcal{N}^0 \cong \mathcal{M} \times (\mathcal{W}_\ell)_1$ is the direct product of $\mathcal{M}$ with the first Frobenius kernel of a group $\mathcal{W}_\ell$ of Witt vectors of length ℓ, and $\mathcal{N}_{\rm red}$ is linearly reductive.*

In fact, modulo its largest linearly reductive subgroup $\mathcal{G}_{\rm lr}$, the group $\mathcal{G}$ is fairly completely determined by these data: except for infinitesimal groups of height 1, which correspond to the restricted Lie algebras studied in [20], one can explicitly write down the possible isomorphism types (cf. Theorem 7.1.5). Theorem A also enables us to associate to $\mathcal{G}$ a linearly reductive subgroup $\hat{\mathcal{G}} \subset \mathrm{SL}(2)$, whose McKay quiver largely governs the representation theory of the principal block $\mathcal{B}_0(\mathcal{G})$ of $H(\mathcal{G})$:

Theorem B. *Suppose that $p \geq 3$. Let $\mathcal{G}$ be a finite algebraic k-group with tame principal block. Then the Gabriel quiver of $\mathcal{B}_0(\mathcal{G})$ is isomorphic to the McKay quiver of the linearly reductive group $\hat{\mathcal{G}}$ relative to its two-dimensional, self-dual module. In particular, the underlying graph is an extended Dynkin diagram of type $\tilde{A}$, $\tilde{D}$, or $\tilde{E}$.*

Accordingly, the graphs underlying the Gabriel quivers of the principal blocks are exactly those occurring in the classification of the tame hereditary algebras (cf. [14]). In fact, the precise relationship with these hereditary algebras is provided via the notion of a trivial extension of an algebra by its dual bimodule: The tame blocks are certain generalizations of the trivial extensions of the tame hereditary radical square zero algebras. Path algebras of affine quivers belong to the particularly tractable class of domestic algebras, whose well understood representation theory [53] was connected to binary polyhedral groups by Lusztig [39] in an explicit way.

Our paper can roughly be divided into two parts. In the first three sections we collect our basic tools. Passage to factor groups defined by linearly reductive normal subgroups is shown to induce isomorphisms of the corresponding principal blocks. Tensor products appear in Sections 2 and 3, where we analyze their structure in case the constituents are simple or principal indecomposable modules. In particular, we extend in Section 3 the classification of the binary polyhedral groups to finite linearly reductive group schemes $\mathcal{G} \subset \mathrm{SL}(2)$ (cf. (3.3)).

A combination of these results with methods from abstract representation theory leads in Section 4 to the identification of the representation type in a number of important instances. In default of a general Brauer correspondence, we address in Section 5 the behaviour of tameness under passage between groups and subgroups. While subgroups of tame infinitesimal groups may be wild (cf. [20, §6]), principal blocks of normal subgroups containing the connected component do inherit the representation type of the ambient group. In Section 6 we turn to the classification of tame group schemes by investigating groups of automorphisms of tame infinitesimal groups. We show that reduced linearly reductive groups of automorphisms always occur as subgroups of the automorphism group $\mathrm{Aut}(\mathrm{SL}(2)_1T_r)$ of the product $\mathrm{SL}(2)_1T_r$ of the first Frobenius kernel of SL(2) with the r-th Frobenius kernel of the standard maximal torus $T \subset \mathrm{SL}(2)$. For $r \geq 2$ these groups are readily seen to be cyclic or dihedral; the more complicated case $r = 1$ follows from the classification of binary polyhedral groups. Using the results of the foregoing sections we prove the linear reductivity of the reduced parts of groups affording a tame principal block.

In the final two sections we combine methods from the structure theory of algebraic groups with those from abstract representation theory to obtain refinements of the results stated above. Since the Gabriel quivers are certain McKay quivers, the detailed information given in [28] provides complete control over the simple and principal indecomposable $\mathcal{B}_0(\mathcal{G})$-modules as well as the structure of the underlying basic algebra. For groups without non-trivial linearly reductive normal subgroups the block structure of $H(\mathcal{G})$ is completely understood. Contrary to infinitesimal groups, tame principal blocks of finite group schemes are usually not special biserial. In fact, special biseriality occurs precisely when a certain factor group of the finite group $\mathcal{G}(k)$ is cyclic.

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1. Passage to Factor Groups

Throughout this paper, we will be dealing with finite algebraic groups $\mathcal{G}$, defined over an algebraically closed field k of characteristic $p > 0$. We refer the reader to [35, 64] concerning general facts on affine group schemes and unexplained terminology. The connected component of $\mathcal{G}$ will be denoted $\mathcal{G}^0$. Observe that $\mathcal{G}^0$ is a closed, normal, infinitesimal subgroup of $\mathcal{G}$. The unique block $\mathcal{B}_0(\mathcal{G}) \subset H(\mathcal{G})$ on which the co-unit $\varepsilon : H(\mathcal{G}) \longrightarrow k$ does not vanish is called the *principal block* of $H(\mathcal{G})$. We denote by $H(\mathcal{G})^\dagger := \ker \varepsilon$ the augmentation ideal of $H(\mathcal{G})$.

Recall that $\dim_k H(\mathcal{G})$ is also referred to as the *order* $\mathrm{ord}(\mathcal{G})$ of $\mathcal{G}$. In case $\mathcal{G}$ is reduced, $\mathrm{ord}(\mathcal{G})$ coincides with the order of the finite group $\mathcal{G}(k)$.

A finite algebraic group is *linearly reductive* if its Hopf algebra is semisimple. Following Voigt [62, (I.2.37)], we let $\mathcal{G}_{\rm lr}$ be the unique largest linearly reductive normal subgroup of $\mathcal{G}$. For a reduced group $\mathcal{G}$ this subgroup coincides with the largest normal subgroup $O_{p'}(\mathcal{G})$ of order prime to p.

Our first auxiliary result extends [24, (1.1)].

Proposition 1.1. *Let $\mathcal{N} \lhd \mathcal{G}$ be a normal subgroup of $\mathcal{G}$.*

(1) *We have $\mathcal{N}^0 \lhd \mathcal{G}^0$ and $\mathcal{N}_{\rm red} \lhd \mathcal{G}_{\rm red}$, and there is an isomorphism $\mathcal{G}/\mathcal{N} \cong (\mathcal{G}^0/\mathcal{N}^0) \rtimes (\mathcal{G}_{\rm red}/\mathcal{N}_{\rm red})$.*

(2) *If $\mathcal{B}_0(\mathcal{G})$ is tame, then $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ is tame or representation-finite.*

(3) *The canonical projection $\pi : H(\mathcal{G}) \longrightarrow H(\mathcal{G}/\mathcal{G}_{\rm lr})$ induces an isomorphism*

$$\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{G}_{\rm lr}).$$

(4) *$\mathcal{B}_0(\mathcal{G})$ is isomorphic to the principal block of the augmented algebra $\mathcal{B}_0(\mathcal{G}^0)[\mathcal{G}(k)]$.*

Proof. (1) Let $\mathcal{O}(\mathcal{G})$ be the function algebra (coordinate ring) of the affine group scheme $\mathcal{G}$. According to [64, (6.8)] the Hopf algebra $\mathcal{O}(\mathcal{G})_{\rm red} := \mathcal{O}(\mathcal{G})/\sqrt{(0)}$ is the function algebra of $\mathcal{G}_{\rm red}$. This readily implies that the restriction of the canonical inclusion $\mathcal{N} \hookrightarrow \mathcal{G}$ furnishes a closed embedding $\mathcal{N}_{\rm red} \hookrightarrow \mathcal{G}_{\rm red}$. There results a commutative diagram

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\iota} & \mathcal{G} \\ {\scriptstyle \pi_{\mathcal{N}}}\big\downarrow & & {\scriptstyle \pi_{\mathcal{G}}}\big\downarrow \\ \mathcal{N}_{\rm red} & \xrightarrow{\iota} & \mathcal{G}_{\rm red} \end{array}$$

with injective horizontal arrows. Since $\mathcal{G}^0$ and $\mathcal{N}^0$ are the kernels of $\pi_{\mathcal{G}}$ and $\pi_{\mathcal{N}}$, respectively, we obtain $\mathcal{N}^0 = \mathcal{G}^0 \cap \mathcal{N}$. In particular, $\mathcal{N}^0$ is a normal subgroup of $\mathcal{G}^0$. In view of [64, (6.2)] closed subgroups of reduced finite algebraic groups are reduced. Consequently, $\mathcal{N} \cap \mathcal{G}_{\rm red} = \mathcal{N}_{\rm red}$ is a normal subgroup of $\mathcal{G}_{\rm red}$.

Owing to [13, (II,§5,1.1)] the connected component $\mathcal{N}^0$ is a characteristic subgroup of $\mathcal{N}$. Consequently, $\mathcal{N}^0$ is normal in $\mathcal{G}$, so that $\mathcal{G}_{\rm red}$ acts on the factor group $\mathcal{G}^0/\mathcal{N}^0$ via conjugation. Note that the canonical quotient map $\pi^0 : \mathcal{G}^0 \longrightarrow \mathcal{G}^0/\mathcal{N}^0$ is $\mathcal{G}_{\rm red}$-equivariant. Since $\mathcal{N}_{\rm red}$ operates trivially on $\mathcal{G}^0/\mathcal{N}^0$, the above action factors through $\mathcal{G}_{\rm red}/\mathcal{N}_{\rm red}$. Consequently, the canonical quotient maps $\pi^0 : \mathcal{G}^0 \longrightarrow \mathcal{G}^0/\mathcal{N}^0$ and $\pi_{\rm red} : \mathcal{G}_{\rm red} \longrightarrow \mathcal{G}_{\rm red}/\mathcal{N}_{\rm red}$ induce a quotient map

$$\hat{\pi} : \mathcal{G}^0 \rtimes \mathcal{G}_{\rm red} \longrightarrow (\mathcal{G}^0/\mathcal{N}^0) \rtimes (\mathcal{G}_{\rm red}/\mathcal{N}_{\rm red}) \quad ; \quad (g,h) \mapsto (\pi^0(g), \pi_{\rm red}(h)),$$

with kernel $\mathcal{N} = \mathcal{N}^0 \rtimes \mathcal{N}_{\rm red}$. By the universal property of quotient maps, $\hat{\pi}$ gives rise to the desired isomorphism

$$\mathcal{G}/\mathcal{N} \cong (\mathcal{G}^0/\mathcal{N}^0) \rtimes (\mathcal{G}_{\rm red}/\mathcal{N}_{\rm red}).$$

(2) This follows exactly as in [24, (1.1)].

(3) Let $\eta : H(\mathcal{G}) \longrightarrow H(\mathcal{G})$ be the antipode of the cocommutative Hopf algebra $H(\mathcal{G})$. We begin by showing that the two-sided ideal $H(\mathcal{G})H(\mathcal{G}_{\rm lr})^\dagger$ is a block ideal of $H(\mathcal{G})$. Since $\mathcal{G}_{\rm lr} \lhd \mathcal{G}$ is a normal subgroup, $H(\mathcal{G})$ acts on $H(\mathcal{G}_{\rm lr})$ via the adjoint representation

$$h.x := \sum_{(h)} h_{(1)} x \eta(h_{(2)}) \qquad \forall\ h \in H(\mathcal{G}),\ x \in H(\mathcal{G}_{\rm lr}).$$

This action satisfies

$$\varepsilon(h.x) = \sum_{(h)} \varepsilon(h_{(1)})\varepsilon(x)\varepsilon(\eta(h_{(2)})) = \sum_{(h)} \varepsilon(h_{(1)})\varepsilon(x)\varepsilon(h_{(2)}) = \varepsilon(h)\varepsilon(x) \quad \forall\ h \in H(\mathcal{G}),\ x \in H(\mathcal{G}_{\rm lr}).$$

By definition, the Hopf algebra $H(\mathcal{G}_{\rm lr})$ is semisimple. Thus, its principal block is one-dimensional, and there exists a central idempotent $e_0 \in H(\mathcal{G}_{\rm lr})$ such that

$$xe_0 = \varepsilon(x)e_0 = e_0 x \quad \forall\ x \in H(\mathcal{G}_{\rm lr}).$$

Note that $H(\mathcal{G}_{\rm lr})^\dagger = H(\mathcal{G}_{\rm lr})(1-e_0)$. Given $h \in H(\mathcal{G})$ and $x \in H(\mathcal{G}_{\rm lr})$, we have

$$\begin{aligned} x(h.e_0) &= x(\sum_{(h)} h_{(1)}e_0\eta(h_{(2)})) = \sum_{(h)} \varepsilon(h_{(1)})xh_{(2)}e_0\eta(h_{(3)}) = \sum_{(h)} h_{(1)}\eta(h_{(2)})xh_{(3)}e_0\eta(h_{(4)}) \\ &= \sum_{(h)} h_{(1)}(\eta(h_{(2)}).x)e_0\eta(h_{(3)}) = \sum_{(h)} h_{(1)}\varepsilon(\eta(h_{(2)}).x)e_0\eta(h_{(3)}) \\ &= \sum_{(h)} h_{(1)}\varepsilon(h_{(2)})\varepsilon(x)e_0\eta(h_{(3)}) = \sum_{(h)} h_{(1)}\varepsilon(x)e_0\eta(h_{(2)}) = \varepsilon(x)(h.e_0). \end{aligned}$$

Consequently, $h.e_0$ belongs to $\mathcal{B}_0(\mathcal{G}_{\rm lr})$ and there exists a linear map $\zeta : H(\mathcal{G}) \longrightarrow k$ such that

$$h.e_0 = \zeta(h)e_0 \qquad \forall\ h \in H(\mathcal{G}).$$

Since

$$\varepsilon(h) = \varepsilon(h)\varepsilon(e_0) = \varepsilon(h.e_0) = \varepsilon(\zeta(h)e_0) = \zeta(h) \quad \forall\ h \in H(\mathcal{G}),$$

we obtain for every element $h \in H(\mathcal{G})$

$$\begin{aligned} he_0 &= \sum_{(h)} h_{(1)}e_0\varepsilon(h_{(2)}) = \sum_{(h)} h_{(1)}e_0\eta(h_{(2)})h_{(3)} = \sum_{(h)} (h_{(1)}.e_0)h_{(2)} \\ &= \sum_{(h)} \varepsilon(h_{(1)})e_0h_{(2)} = e_0h. \end{aligned}$$

As a result, e_0 and $1-e_0$ are central idempotents of $H(\mathcal{G})$. By virtue of $H(\mathcal{G}_{\rm lr})^\dagger = H(\mathcal{G}_{\rm lr})(1-e_0)$, it now follows that

$$H(\mathcal{G})H(\mathcal{G}_{\rm lr})^\dagger = H(\mathcal{G})(1-e_0)$$

is a block ideal of $H(\mathcal{G})$.

Since the canonical projection $\pi : H(\mathcal{G}) \longrightarrow H(\mathcal{G}/\mathcal{G}_{\rm lr})$ has kernel $H(\mathcal{G})H(\mathcal{G}_{\rm lr})^\dagger$, it induces an isomorphism $H(\mathcal{G})e_0 \xrightarrow{\sim} H(\mathcal{G}/\mathcal{G}_{\rm lr})$ of augmented k-algebras. From the identity $\varepsilon(e_0) = 1$, we see that the block $\mathcal{B}_0(\mathcal{G})$ is contained in the block ideal $H(\mathcal{G})e_0$. Consequently, $\mathcal{B}_0(\mathcal{G})$ is mapped by $\pi|_{H(\mathcal{G})e_0}$ onto $\mathcal{B}_0(\mathcal{G}/\mathcal{G}_{\rm lr})$.

(4) The group $\mathcal{G}(k)$ of group-like elements of $H(\mathcal{G})$ operates on $H(\mathcal{G}^0)$ via automorphisms of Hopf algebras. This implies in particular that the central idempotent $e_0 \in H(\mathcal{G}^0)$ defining the principal block $\mathcal{B}_0(\mathcal{G}^0)$ is fixed by $\mathcal{G}(k)$. Hence we obtain a decomposition

$$H(\mathcal{G}^0) = \mathcal{B}_0(\mathcal{G}^0) \oplus H(\mathcal{G}^0)(1-e_0)$$

of $\mathcal{G}(k)$-invariant block ideals. There results a decomposition

$$H(\mathcal{G}) = H(\mathcal{G}^0)[\mathcal{G}(k)] = \mathcal{B}_0(\mathcal{G}^0)[\mathcal{G}(k)] \oplus (H(\mathcal{G}^0)(1-e_0))[\mathcal{G}(k)]$$

of $H(\mathcal{G})$ by two-sided ideals, where the first summand is not annihilated by the co-unit of $H(\mathcal{G})$. Consequently, $\mathcal{B}_0(\mathcal{G})$ is isomorphic to the principal block of the augmented algebra $\mathcal{B}_0(\mathcal{G}^0)[\mathcal{G}(k)]$. □

Owing to (1.1(3)) the structure of the principal block of $H(\mathcal{G})$ is not affected by the largest linearly reductive normal subgroup of $\mathcal{G}$. Thus, when dealing with principal blocks we may assume without loss of generality that $\mathcal{G}_{\rm lr}$ is trivial. It is therefore of interest to determine the infinitesimal and reduced parts of this group.

By Nagata's Theorem, the infinitesimal linearly reductive groups are the multiplicative groups. By definition, an algebraic k-group is *multiplicative* or *diagonalizable* if its function algebra is a group algebra. Given a finite algebraic k-group $\mathcal{G}$, we let $\mathcal{M} = \mathcal{M}(\mathcal{G}^0)$ be the unique largest multiplicative normal subgroup of $\mathcal{G}^0$, the so-called *multiplicative center* of $\mathcal{G}^0$. It follows from [13, (IV,§3,1.1)] that $\mathcal{M} \lhd \mathcal{G}^0$ is a characteristic subgroup of $\mathcal{G}^0$. In particular, $\mathcal{M} \lhd \mathcal{G}$ is a normal subgroup of $\mathcal{G}$. Consequently, $\mathcal{G}$ acts on $\mathcal{G}^0/\mathcal{M}$ and

$$\mathcal{C}_{\mathcal{G}} := \mathrm{Cent}_{\mathcal{G}_{\mathrm{red}}}(\mathcal{G}^0/\mathcal{M})$$

is a closed normal subgroup of $\mathcal{G}_{\mathrm{red}}$ (cf. [35, (I.2.6(9))]).

Proposition 1.2. *Let $\mathcal{G}$ be a finite algebraic k-group. Then we have*

$$\mathcal{G}_{\mathrm{lr}} = \mathcal{M} \cdot O_{p'}(\mathcal{C}_{\mathcal{G}}) \cong \mathcal{M} \rtimes O_{p'}(\mathcal{C}_{\mathcal{G}}).$$

Proof. Since $\mathcal{M}$ is a linearly reductive, normal subgroup of $\mathcal{G}$, we have $\mathcal{M} \subset \mathcal{G}_{\mathrm{lr}}$. On the other hand, Nagata's Theorem [13, (IV,§3,3.6)] implies that $(\mathcal{G}_{\mathrm{lr}})^0$ is a multiplicative subgroup of $\mathcal{G}_{\mathrm{lr}}$. As $(\mathcal{G}_{\mathrm{lr}})^0$ is a characteristic subgroup of $\mathcal{G}_{\mathrm{lr}}$ [13, (II,§5,1.1)], it is normal in $\mathcal{G}^0$, so that $(\mathcal{G}_{\mathrm{lr}})^0 = \mathcal{M}$.

We consider the factor group $\mathcal{G}' := \mathcal{G}/\mathcal{M} \cong (\mathcal{G}^0/\mathcal{M}) \rtimes \mathcal{G}_{\mathrm{red}}$ (cf. (1.1(1))). The inverse image $\mathcal{N}$ of $\mathcal{G}'_{\mathrm{lr}}$ under the quotient map $\mathcal{G} \longrightarrow \mathcal{G}'$ is a normal subgroup of $\mathcal{G}$ such that $\mathcal{N}/\mathcal{M} \cong \mathcal{G}'_{\mathrm{lr}}$. Thus, $\mathcal{N}$ is a linearly reductive normal subgroup of $\mathcal{G}$, whence $\mathcal{N} \subset \mathcal{G}_{\mathrm{lr}}$. On the other hand, the isomorphism $\mathcal{G}/\mathcal{N} \cong \mathcal{G}'/\mathcal{G}'_{\mathrm{lr}}$ [13, (III,§3,3.7)] yields $\mathcal{G}_{\mathrm{lr}}/\mathcal{N} = e_k$. We conclude that $\mathcal{N} = \mathcal{G}_{\mathrm{lr}}$ and $\mathcal{G}'_{\mathrm{lr}} \cong \mathcal{G}_{\mathrm{lr}}/(\mathcal{G}_{\mathrm{lr}})^0 \cong (\mathcal{G}_{\mathrm{lr}})_{\mathrm{red}}$. In particular, $\mathcal{G}'_{\mathrm{lr}}$ is a reduced, linearly reductive normal subgroup of $\mathcal{G}'$. Consequently, $\mathcal{G}'^0 \cap \mathcal{G}'_{\mathrm{lr}} = e_k$, so that the group $(\mathcal{G}_{\mathrm{lr}})_{\mathrm{red}}$ centralizes the group $\mathcal{G}^0/\mathcal{M}$. Nagata's Theorem now shows $(\mathcal{G}_{\mathrm{lr}})_{\mathrm{red}} \subset O_{p'}(\mathcal{C}_{\mathcal{G}})$, whence $\mathcal{G}_{\mathrm{lr}} = \mathcal{G}^0_{\mathrm{lr}}(\mathcal{G}_{\mathrm{lr}})_{\mathrm{red}} \subset \mathcal{M}O_{p'}(\mathcal{C}_{\mathcal{G}})$. By the same token, $\mathcal{M}O_{p'}(\mathcal{C}_{\mathcal{G}})$ is a linearly reductive, normal subgroup of $\mathcal{G}$, so that

$$\mathcal{G}_{\mathrm{lr}} = \mathcal{M}O_{p'}(\mathcal{C}_{\mathcal{G}}) \cong \mathcal{M} \rtimes O_{p'}(\mathcal{C}_{\mathcal{G}}).$$

□

2. Tensor Products of Simple and Principal Indecomposable Modules

Let $\mathcal{G}$ be a finite algebraic k-group, $\mathcal{N} \lhd \mathcal{G}$ a closed, normal subgroup. Under favourable circumstances, which will be seen to obtain in all cases of interest, the simple and principal indecomposable $H(\mathcal{G})$-modules are given by tensor products of the corresponding modules for $H(\mathcal{N})$ and $H(\mathcal{G}/\mathcal{N})$. In particular, the Gabriel quiver of $H(\mathcal{G})$ turns out to be computable from the structure of tensor products of simple $H(\mathcal{G}/\mathcal{N})$-modules.

In the sequel we shall often view $(\mathcal{G}/\mathcal{N})$-modules as $\mathcal{G}$-modules via pull-back along the canonical quotient map $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$. We begin by studying the underlying set of vertices of the Gabriel quiver of $H(\mathcal{G})$ in a setting that will turn out to be appropriate for our purposes.

Lemma 2.1. *Let $\mathcal{N} \lhd \mathcal{G}$ be a normal subgroup of the finite algebraic k-group $\mathcal{G}$. If $L_1, \ldots, L_n$ are simple $\mathcal{G}$-modules such that $\{L_n|_{\mathcal{N}}, \ldots, L_n|_{\mathcal{N}}\}$ is a complete set of representatives of the isoclassess of simple $\mathcal{N}$-modules, then every simple $\mathcal{G}$-module S is of the form*

$$S \cong L_i \otimes_k M.$$

for a unique $i \in \{1, \ldots, n\}$ and a unique (up to isomorphism) simple $(\mathcal{G}/\mathcal{N})$-module M.

Proof. Let $L_1, \ldots, L_n$ and $M_1, \ldots, M_m$ be complete sets of representatives of the isoclasses of the simple $\mathcal{N}$-modules and simple $(\mathcal{G}/\mathcal{N})$-modules, respectively. By our current assumption each $\mathcal{N}$-module L_i is the restriction of a simple $\mathcal{G}$-module, which we will also denote by L_i. Given a simple $\mathcal{G}$-module S, there exists $i \in \{1, \ldots, n\}$ such that $L_i \hookrightarrow S|_{\mathcal{N}}$. Consequently, the $\mathcal{G}$-linear map

$$\varphi : \mathrm{Hom}_{\mathcal{N}}(L_i, S) \otimes_k L_i \longrightarrow S \quad ; \quad f \otimes x \mapsto f(x)$$

is surjective. The image $\operatorname{im}\varphi$ is the L_i-isotypical component S_{L_i} of the module $S|_{\mathcal{N}}$. By Schur's Lemma the $\mathcal{N}$-module S_{L_i} has dimension $\dim_k \operatorname{Hom}_{\mathcal{N}}(L_i, S) \otimes_k L_i$ (cf. [35, (2.14(3))]). As a result, φ is also injective, and

$$S \cong \operatorname{Hom}_{\mathcal{N}}(L_i, S) \otimes_k L_i \cong L_i \otimes_k \operatorname{Hom}_{\mathcal{N}}(L_i, S).$$

Note that $\operatorname{Hom}_{\mathcal{N}}(L_i, S)$ has the structure of a $(\mathcal{G}/\mathcal{N})$-module, which is necessarily simple.

Next, we consider the $\mathcal{G}$-module $S := L_i \otimes_k M_r$. By the above, S contains a simple submodule $T \cong L_j \otimes_k M_s$. Upon restriction to $\mathcal{N}$ we obtain

$$(\dim_k M_s)L_j \cong T|_{\mathcal{N}} \hookrightarrow S|_{\mathcal{N}} \cong (\dim_k M_r)L_i,$$

so that $i = j$. Moreover, we have homomorphisms

$$M_s \cong \operatorname{Hom}_{\mathcal{N}}(L_i, L_i \otimes_k M_s) \cong \operatorname{Hom}_{\mathcal{N}}(L_i, T) \hookrightarrow \operatorname{Hom}_{\mathcal{N}}(L_i, S) \cong M_r$$

of $(\mathcal{G}/\mathcal{N})$-modules, so that $r = s$. As a result, the module $S = L_i \otimes_k M_r$ is simple and the pair (i, r) is uniquely determined by S. □

The technical condition of (2.1) is known to hold for Frobenius kernels of semi-simple, simply connected groups (cf. [35, (II3.15)]). In our projected applications the normal subgroup $\mathcal{N}$ will be the first Frobenius kernel of the special linear group SL(2). The simple modules of this infinitesimal group do not afford non-trivial self-extensions. The following subsidiary result shows how this fact can be exploited in the computation of the Gabriel quiver of $H(\mathcal{G})$.

Lemma 2.2. *Let $\mathcal{N} \lhd \mathcal{G}$ be a normal subgroup of a finite algebraic k-group $\mathcal{G}$. Suppose that $\operatorname{Ext}^1_{\mathcal{N}}(S, S) = (0)$ for every simple $\mathcal{N}$-module S. If L_1, L_2 and M_1, M_2 are simple $\mathcal{G}$-modules and $(\mathcal{G}/\mathcal{N})$-modules, respectively, such that each $L_i|_{\mathcal{N}}$ is simple, then we have*

$$\operatorname{Ext}^1_{\mathcal{G}}(L_1 \otimes_k M_1, L_2 \otimes_k M_2) \cong \begin{cases} \operatorname{Ext}^1_{\mathcal{G}/\mathcal{N}}(M_1, M_2) & L_1 \cong L_2 \\ \operatorname{Hom}_{\mathcal{G}/\mathcal{N}}(M_1, \operatorname{Ext}^1_{\mathcal{N}}(L_1, L_2) \otimes_k M_2) & L_1 \not\cong L_2 \end{cases}$$

Proof. The five term sequence associated to the spectral sequence given in [35, (I.6.6(1))] yields the following exact sequence

$$\begin{aligned} (0) \longrightarrow\ & \operatorname{Ext}^1_{\mathcal{G}/\mathcal{N}}(M_1, \operatorname{Hom}_{\mathcal{N}}(L_1, L_2 \otimes_k M_2)) \longrightarrow \operatorname{Ext}^1_{\mathcal{G}}(L_1 \otimes_k M_1, L_2 \otimes_k M_2) \\ \longrightarrow\ & \operatorname{Hom}_{\mathcal{G}/\mathcal{N}}(M_1, \operatorname{Ext}^1_{\mathcal{N}}(L_1, L_2 \otimes_k M_2)) \longrightarrow \operatorname{Ext}^2_{\mathcal{G}/\mathcal{N}}(M_1, \operatorname{Hom}_{\mathcal{N}}(L_1, L_2 \otimes_k M_2)). \end{aligned}$$

Let X be a $\mathcal{G}$-module, N a $(\mathcal{G}/\mathcal{N})$-module. Direct computation shows that the functors

$$Y \mapsto \operatorname{Hom}_{\mathcal{N}}(X, Y \otimes_k N) \text{ and } Y \mapsto \operatorname{Hom}_{\mathcal{N}}(X, Y) \otimes_k N$$

from $\operatorname{mod}\mathcal{G}$ to $\operatorname{mod}(\mathcal{G}/\mathcal{N})$ are naturally equivalent. Since tensoring over k is exact, we obtain isomorphisms

$$\operatorname{Ext}^\ell_{\mathcal{N}}(X, Y \otimes_k N) \cong \operatorname{Ext}^\ell_{\mathcal{N}}(X, Y) \otimes_k N \qquad \ell \geq 0$$

of $(\mathcal{G}/\mathcal{N})$-modules.

If $L_1 \not\cong L_2$, then $\operatorname{Hom}_{\mathcal{N}}(L_1, L_2 \otimes_k M_2) = (0)$, and there results an isomorphism

$$\operatorname{Ext}^1_{\mathcal{G}}(L_1 \otimes_k M_1, L_2 \otimes_k M_2) \cong \operatorname{Hom}_{\mathcal{G}/\mathcal{N}}(M_1, \operatorname{Ext}^1_{\mathcal{N}}(L_1, L_2) \otimes_k M_2).$$

Alternatively, $L_1 \cong L_2$, and our general assumption implies

$$\operatorname{Ext}^1_{\mathcal{N}}(L_1, L_2 \otimes_k M_2) \cong \operatorname{Ext}^1_{\mathcal{N}}(L_1, L_1) \otimes_k M_2 = (0),$$

so that we get an isomorphism

$$\operatorname{Ext}^1_{\mathcal{G}}(L_1 \otimes_k M_1, L_2 \otimes_k M_2) \cong \operatorname{Ext}^1_{\mathcal{G}/\mathcal{N}}(M_1, \operatorname{Hom}_{\mathcal{N}}(L_1, L_2) \otimes_k M_2).$$

Owing to Schur's Lemma, the spaces $\mathrm{Hom}_{\mathcal{N}}(L_1, L_1)$ and $\mathrm{Hom}_{\mathcal{G}}(L_1, L_1)$ are one-dimensional. Hence they are equal and $\mathrm{Hom}_{\mathcal{N}}(L_1, L_1)$ is the trivial $(\mathcal{G}/\mathcal{N})$-module. Consequently, the right-hand side is isomorphic to $\mathrm{Ext}^1_{\mathcal{G}/\mathcal{N}}(M_1, M_2)$. □

Our final result of this section provides a criterion for the construction of principal indecomposable modules. For large p, condition (b) is known to hold in the classical context of Frobenius kernels [5, 34].

Proposition 2.3. *Let $\mathcal{G}$ be a finite algebraic group, $\mathcal{N} \lhd \mathcal{G}$ a normal subgroup. Suppose that*
(a) *every simple $\mathcal{N}$-module is the restriction of a $\mathcal{G}$-module, and*
(b) *every principal indecomposable $\mathcal{N}$-module is the restriction of a $\mathcal{G}$-module.*

Then the following statements hold:
(1) *Let X, Y be $\mathcal{G}$-modules, Q a projective $(\mathcal{G}/\mathcal{N})$-module. Then we have isomorphisms*

$$\mathrm{Ext}^\ell_{\mathcal{G}}(X \otimes_k Q, Y \otimes_k N) \cong \mathrm{Hom}_{\mathcal{G}/\mathcal{N}}(Q, \mathrm{Ext}^\ell_{\mathcal{N}}(X, Y) \otimes_k N) \qquad \forall\ \ell \geq 0$$

for any $(\mathcal{G}/\mathcal{N})$-module N.
(2) *If $P_1, \dots, P_n$ are $\mathcal{G}$-modules such that $\{P_1|_{\mathcal{N}}, \dots, P_n|_{\mathcal{N}}\}$ is a complete set of representatives of the isoclasses of principal indecomposable $\mathcal{N}$-modules, and $\{Q_1, \dots, Q_m\}$ is a complete set of representatives of the isoclasses of the principal indecomposable $(\mathcal{G}/\mathcal{N})$-modules, then the $\mathcal{G}$-modules $(P_i \otimes_k Q_r)_{1\leq i\leq n,\ 1\leq r\leq m}$ form a complete system of principal indecomposable $\mathcal{G}$-modules.*

Proof. Since Q is a projective $(\mathcal{G}/\mathcal{N})$-module, the spectral sequence $\mathrm{Ext}^m_{\mathcal{G}/\mathcal{N}}(Q, \mathrm{Ext}^n_{\mathcal{N}}(X, V)) \Rightarrow \mathrm{Ext}^{m+n}_{\mathcal{G}}(X \otimes_k Q, V)$ (cf. [35, (I.6.6(1))]) collapses to isomorphisms

$$(*) \qquad \mathrm{Ext}^\ell_{\mathcal{G}}(X \otimes_k Q, V) \cong \mathrm{Hom}_{\mathcal{G}/\mathcal{N}}(Q, \mathrm{Ext}^\ell_{\mathcal{N}}(X, V)) \qquad \forall\ \ell \geq 0$$

for any $\mathcal{G}$-module V.

(1) By the arguments of the proof of (2.2) we have isomorphisms

$$\mathrm{Ext}^\ell_{\mathcal{N}}(X, Y \otimes_k N) \cong \mathrm{Ext}^\ell_{\mathcal{N}}(X, Y) \otimes_k N \qquad \ell \geq 0$$

of $(\mathcal{G}/\mathcal{N})$-modules. Our assertion now follows from $(*)$.

(2) Setting $X := P_i$, $Q := Q_r$ and $N := k$ in (1) immediately yields the projectivity of the $\mathcal{G}$-module $P_i \otimes_k Q_r$. By assumption (a) there exist simple $\mathcal{G}$-modules $L_1, \dots, L_n$ such that

$$(**) \qquad \mathrm{Top}(P_i|_{\mathcal{N}}) \cong L_i|_{\mathcal{N}} \qquad 1 \leq i \leq n.$$

Setting $M_r := \mathrm{Top}(Q_r)$ for $1 \leq r \leq m$ we obtain a complete set $\{M_1, \dots, M_m\}$ of representatives of the isoclasses of the simple $(\mathcal{G}/\mathcal{N})$-modules. Thanks to (2.1) the modules $L_j \otimes_k M_s$ completely represent the isoclasses of the simple $\mathcal{G}$-modules.

A surjection $P_i \longrightarrow L_j \otimes_k M_r$ restricts to a surjection $P_i|_{\mathcal{N}} \longrightarrow (\dim_k M_r) L_j|_{\mathcal{N}}$. According to $(**)$, this can only happen if $i = j$ and $\dim_k M_r = 1$. Thus, replacing L_i by $L_i \otimes_k M_{r(i)}$ for a suitable $M_{r(i)}$ with $\dim_k M_{r(i)} = 1$, we have in addition to $(**)$ the condition

$$(*{*}*) \qquad \mathrm{Hom}_{\mathcal{G}}(P_i, L_i) \neq (0) \qquad 1 \leq i \leq n.$$

Now (1) implies

$$\mathrm{Hom}_{\mathcal{G}}(P_i \otimes_k Q_r, L_j \otimes_k M_s) \cong \mathrm{Hom}_{\mathcal{G}/\mathcal{N}}(Q_r, \mathrm{Hom}_{\mathcal{N}}(P_i|_{\mathcal{N}}, L_j|_{\mathcal{N}}) \otimes_k M_s).$$

Since $P_i|_{\mathcal{N}}$ is a principal indecomposable $\mathcal{N}$-module, the right-hand space is trivial unless $i = j$. In that case, the $\mathcal{G}$-module $\mathrm{Hom}_{\mathcal{N}}(P_i|_{\mathcal{N}}, L_i|_{\mathcal{N}}) \cong \mathrm{Hom}_{\mathcal{N}}(L_i|_{\mathcal{N}}, L_i|_{\mathcal{N}})$ is one-dimensional and contains the by $(*{*}*)$ non-zero space $\mathrm{Hom}_{\mathcal{G}}(P_i, L_i)$. Consequently, $\mathrm{Hom}_{\mathcal{N}}(P_i|_{\mathcal{N}}, L_i|_{\mathcal{N}}) = \mathrm{Hom}_{\mathcal{G}}(P_i, L_i)$ is the one-dimensional trivial $\mathcal{G}$-module, so that

$$\dim_k \mathrm{Hom}_{\mathcal{G}/\mathcal{N}}(Q_r, \mathrm{Hom}_{\mathcal{N}}(P_i|_{\mathcal{N}}, L_j|_{\mathcal{N}}) \otimes_k M_s) = \delta_{ij}\delta_{rs}.$$

Thus, $P_i \otimes_k Q_r$ is the projective cover of the simple $\mathcal{G}$-module $L_i \otimes_k M_r$. □

3. Linearly Reductive Subgroups of SL(2)

In this section we consider linearly reductive, closed, finite subgroup schemes $\mathcal{G}$ of the group scheme SL(2). If $\mathcal{G}$ is reduced, then this amounts to studying the finite subgroups of $\mathrm{SL}(2)(k)$, whose orders are not divisible by the prime p. In the classical situation where $k = \mathbb{C}$ is the field of complex numbers, these are just the binary polyhedral groups (cf. [15, (26.1)]), and it is easy to see that the proof given there transfers verbatim to our situation. For our purposes it will be most expedient to follow the approach expounded by Happel-Preiser-Ringel [28], in which the binary polyhedral groups are characterized by means of their McKay graphs relative to a two-dimensional, faithful, self-dual module. For the case of not necessarily self-dual modules we refer to [3, Thm.1].

Let $\mathcal{G}$ be a linearly reductive finite algebraic group, $M_1, \ldots, M_n$ a complete set of representatives for the isoclasses of the simple $\mathcal{G}$-modules. Since the algebra $H(\mathcal{G})$ is semisimple, the tensor product $L \otimes_k M_j$ of M_j with a fixed $\mathcal{G}$-module L decomposes into a direct sum of simple modules, and we obtain an integral $(n \times n)$-matrix $A = (a_{ij})$ given by

$$L \otimes_k M_j = \bigoplus_{i=1}^{n} a_{ij} M_i.$$

The quiver with underlying vertex set $\{M_1, \ldots, M_n\}$ and a_{ij} arrows from M_i to M_j is called the *McKay quiver* $\Upsilon_L(\mathcal{G})$ of $\mathcal{G}$ relative to L. In fact, our definition differs from McKay's original one, who considers the opposite quiver of $\Upsilon_L(\mathcal{G})$ (cf. [40, 41]). As all our matrices will be symmetric, there will result no confusion in the formulation of our main results. We begin by extending a well-known fact from the representation theory of finite groups to our setting.

Lemma 3.1. *Suppose that L is a faithful $\mathcal{G}$-module. Then the McKay quiver $\Upsilon_L(\mathcal{G})$ is connected.*

Proof. Let $\varrho : H(\mathcal{G}) \longrightarrow \mathrm{End}_k(L)$ be the representation of the Hopf algebra $H(\mathcal{G})$ associated to the $\mathcal{G}$-module L. If $I \subset \ker \varrho$ is a Hopf ideal, then ϱ factors through to a representation of the cocommutative Hopf algebra $H(\mathcal{G})/I$. By general theory (cf. [64, (1.4)]), there exists a finite algebraic group $\mathcal{G}'$ and a homomorphism $\pi : \mathcal{G} \longrightarrow \mathcal{G}'$ which induces the canonical projection $H(\mathcal{G}) \longrightarrow H(\mathcal{G}')$. This implies that π is a quotient map, whose kernel we denote by $\mathcal{N}$. It follows that $\mathcal{N}$ is contained in the kernel of the representation $\mathcal{G} \longrightarrow \mathrm{GL}(L)$ of the $\mathcal{G}$-module L. As this representation is faithful, we obtain $\mathcal{N} = e_k$ as well as

$$I = H(\mathcal{G})H(\mathcal{N})^\dagger = (0).$$

We may now apply [51, Cor.1] or [45, Cor.10] to see that for every $j \in \{1, \ldots, n\}$ there exists $m_j \in \mathbb{N}$ such that M_j is a direct summand of $L^{\otimes m_j}$. Since the connected component of $\Upsilon_L(\mathcal{G})$ containing the trivial $\mathcal{G}$-module k contains the simple direct summands of all tensor powers of L, our assertion follows. □

If L is two-dimensional and self-dual, then the matrix A is symmetric, and $C := 2I_n - A$ is a *generalized Cartan matrix* in the sense of [28]. Writing $d_i := \dim_k M_i$ and $C = (c_{ij})$ we obtain

$$\sum_{i=1}^{n} d_i c_{ij} = 0$$

for every $j \in \{1, \ldots, n\}$, so that

$$d : \{1, \ldots, n\} \longrightarrow \mathbb{N} \ ; \ \ i \mapsto d_i$$

is an *additive function* for C. Thanks to Lemma 3.1 the valued graph $\bar{\Upsilon}_L(\mathcal{G})$ associated to C, the so-called *McKay graph* of $\mathcal{G}$, is connected whenever L is a faithful $\mathcal{G}$-module. As was shown in [28, Theorem 2] the graph $\bar{\Upsilon}_L(\mathcal{G})$ is a generalized Euclidean diagram in that case (see also Theorem 3.3 below). We refer the reader to [4, (VII.3)] for more details concerning Cartan matrices and additive functions.

We now turn to binary polyhedral groups and describe the linearly reductive finite algebraic subgroups of the smooth group scheme SL(2). As before, T denotes standard maximal torus of diagonal matrices. For $m \geq 1$ we let

$$D_m := \langle x, y \mid x^m = 1 = y^2 \, ; \, yxy^{-1} = x^{-1} \rangle$$

and

$$Q_m := \langle x, y \mid x^m = y^2 \, ; \, yxy^{-1} = x^{-1} \rangle$$

be the *dihedral group* of order $2m$ and the *generalized quaternion group* of order $4m$, respectively. In our context Q_m will occur as the subgroup of SL(2)(k) generated by the matrices

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } x(\zeta_{2m}) := \begin{pmatrix} \zeta_{2m} & 0 \\ 0 & \zeta_{2m}^{-1} \end{pmatrix},$$

where $\zeta_{2m} \in k$ is a primitive $2m$-th root of unity and $(p, 2m) = 1$. We let $\hat{T}$, $\hat{O}$ and $\hat{I}$ be the *binary tetrahedral group*, the *binary octahedral group*, and the *binary icosahedral group* of orders 24, 48 and 120, respectively (cf. [56, (4.4)]). These groups can be realized as follows

$$\hat{T} := \langle \omega, x(\zeta_4), y(\zeta_4) \rangle \quad ; \quad \hat{O} := \langle \omega, x(\zeta_8), y(\zeta_4) \rangle \quad ; \quad \hat{I} := \langle \omega, x(\zeta_5), y(\zeta_5) \rangle,$$

where

$$y(\zeta_4) := \frac{1}{\zeta_4 - 1} \begin{pmatrix} 1 & 1 \\ \zeta_4 & -\zeta_4 \end{pmatrix} \text{ and } y(\zeta_5) := \frac{1}{\zeta_5^2 - \zeta_5^3} \begin{pmatrix} \zeta_5 + \zeta_5^4 & 1 \\ 1 & -(\zeta_5 + \zeta_5^4) \end{pmatrix}.$$

When considering these groups we tacitly assume $p \neq 2, 3$ for $\hat{T}$, $\hat{O}$ as well as $p \neq 2, 3, 5$ for $\hat{I}$. In the sequel we will repeatedly use the following basic facts:

(†): If $g \in \mathrm{SL}(2)(k) \setminus T(k)$ satisfies $gtg^{-1} \in T(k)$ for some $t \in T(k) \setminus \{I_2, -I_2\}$, then there exists $\alpha \in k^\times$ such that

$$g = \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \omega.$$

(‡): Let $\beta \in k^\times$. Then we have

$$\begin{pmatrix} 0 & \beta^2 \\ -\beta^{-2} & 0 \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \omega \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{pmatrix}.$$

For $m \in \mathbb{N}$ we consider the closed subgroup $T_{(m)} \subset T$, given by

$$T_{(m)}(R) := \{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \; ; \; x \in \mu_m(R) \}$$

for every commutative k-algebra R. Note that $T_{(m)}$ is the unique closed subgroup of T of order m.

Let $h := \begin{pmatrix} \zeta_4 & 1 \\ 1 & \zeta_4 \end{pmatrix} \in \mathrm{GL}(2)(k)$. Then $\mathcal{H}_4 := h^{-1} T_{(4)} h$ is a reduced, closed subgroup of SL(2) such that $\mathcal{H}_4(k) = \langle \omega \rangle \subset \mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k))$. Since T is reduced, an application of [35, (I.2.6(11))] shows that the latter group coincides with $\mathrm{Nor}_{\mathrm{SL}(2)}(T)(k)$. As $\mathcal{H}_4$ is reduced, this implies $\mathcal{H}_4 \subset \mathrm{Nor}_{\mathrm{SL}(2)}(T)$. Thus, $\mathcal{H}_4$ normalizes every closed subgroup of T, and we define

$$N_{(m)} := T_{(m)} \mathcal{H}_4$$

for $m \geq 2$. The construction of the other subgroups of SL(2) requires a few preparatory remarks.

When dealing with reduced algebraic group schemes we shall often make use of the fact that such a group $\mathcal{G}$ is uniquely determined by its group of rational points. In fact

$$\mathcal{G} \mapsto \mathcal{G}(k)$$

defines an equivalence between the category of reduced algebraic group schemes and the category of algebraic groups in the sense of [57]. Upon restriction to reduced finite algebraic groups, the above functor provides an equivalence with the category of finite groups (cf. [62, (0.16),(0.17)] for more details). Given a finite group G, we let G_k be the reduced finite algebraic group such that $G_k(k) = G$.

Lemma 3.2. *Let $\mathcal{G}$ and $\mathcal{H}$ be algebraic groups. Suppose that $\mathcal{G}$ is reduced and that there exists a closed embedding $\varphi : \mathcal{G}(k) \hookrightarrow \mathcal{H}(k)$. Then there exists a closed embedding $\psi : \mathcal{G} \hookrightarrow \mathcal{H}$ of group schemes such that $\psi_k = \varphi$.*

Proof. We let $\mathcal{O}(\mathcal{G})$ and $\mathcal{O}(\mathcal{H})$ denote the coordinate rings of $\mathcal{G}$ and $\mathcal{H}$, respectively. Given a commutative k-algebra R, we let R_{red} be the factor algebra of R by its nilpotent radical. Thus, $\mathcal{O}(\mathcal{G})_{\mathrm{red}}$ and $\mathcal{O}(\mathcal{H})_{\mathrm{red}}$ are the coordinate rings of the varieties $\mathcal{G}(k)$ and $\mathcal{H}(k)$, respectively.

By assumption, the morphism φ induces a surjective comorphism $\varphi^* : \mathcal{O}(\mathcal{H})_{\mathrm{red}} \longrightarrow \mathcal{O}(\mathcal{G})_{\mathrm{red}}$. As $\mathcal{G}$ is reduced, we have $\mathcal{O}(\mathcal{G})_{\mathrm{red}} = \mathcal{O}(\mathcal{G})$, and the composite $\psi^* = \varphi^* \circ \pi$ of φ^* with the canonical map $\pi : \mathcal{O}(\mathcal{H}) \longrightarrow \mathcal{O}(\mathcal{H})_{\mathrm{red}}$ defines the desired closed embedding $\psi : \mathcal{G} \hookrightarrow \mathcal{H}$. □

According to (3.2) there exist uniquely determined reduced subgroups $(\hat{T})_k$, $(\hat{O})_k$ and $(\hat{I})_k$ of SL(2) satisfying $(\hat{T})_k(k) = \hat{T}$, $(\hat{O})_k(k) = \hat{O}$ and $(\hat{I})_k(k) = \hat{I}$, respectively. The proof of Theorem 3.3 below, which extends [28, Thm.1] to our context, shows in particular that $N_{(m)} \cong (Q_m)_k$ whenever $(p, 2m) = 1$.

Let $\mathcal{G} := \mathrm{Spec}_k(A)$ be an affine k-group. For $r \geq 0$ we let $\mathcal{G}^{(r)}$ be the affine k-group with function algebra $A^{(r)}$, whose addition and multiplication coincide with that of A, and whose k-space structure is given by

$$\alpha \cdot a := \alpha^{p^{-r}} a \qquad \forall\, \alpha \in k,\ a \in A.$$

The homomorphism $F^r : \mathcal{G} \longrightarrow \mathcal{G}^{(r)}$ given by

$$F^r_R(\lambda)(a) = \lambda(a)^{p^r}$$

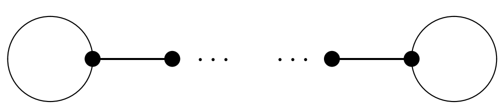
for every commutative k-algebra R, $\lambda \in \mathcal{G}(R)$ and $a \in A$ is called the *r-th Frobenius homomorphism* of $\mathcal{G}$. The normal subgroup

$$\mathcal{G}_r := \ker F^r$$

is the *r-th Frobenius kernel* of $\mathcal{G}$. If $\mathcal{G}$ is infinitesimal, then we have $\mathcal{G} = \mathcal{G}_r$ for some r, and the minimal nonnegative integer with this property is called the *height* $\mathrm{ht}(\mathcal{G})$ of $\mathcal{G}$.

Given $r \geq 1$, we denote by $\alpha_{p^r} := \mathrm{Spec}_k(k[X]/(X^{p^r}))$ and $\mu_{p^r} := \mathrm{Spec}_k(k[X]/(X^{p^r} - 1))$ the r-th Frobenius kernels of the additive group $\alpha_k := \mathrm{Spec}_k(k[X])$ and the multiplicative group $\mu_k := \mathrm{Spec}_k(k[X, X^{-1}])$, respectively.

Let $m \geq 0$. Following [28] we denote by $\tilde{L}_m$ the graph



with $m + 1$ vertices. In particular, $\tilde{L}_0$ is the point with two loops.

Theorem 3.3. *Let $\mathcal{G} \subset \mathrm{SL}(2)$ be a closed, linearly reductive subgroup of characteristic $p \geq 3$. Then there exists $g \in \mathrm{SL}(2)(k)$ such that $g\mathcal{G}g^{-1}$ and its McKay graph relative to its standard module L belong to the following list:*

$g\mathcal{G}g^{-1}$	$\bar{\Upsilon}_L(\mathcal{G})$
e_k	$\tilde{L}_0$
$T_{(np^r)}$	$\tilde{A}_{np^r-1}$
$N_{(np^r)}$	$\tilde{D}_{np^r+2}$
$(\hat{T})_k$	$\tilde{E}_6$
$(\hat{O})_k$	$\tilde{E}_7$
$(\hat{I})_k$	$\tilde{E}_8$,

where $(n,p) = 1$, $r := \mathrm{ht}(\mathcal{G}^0)$, and $n + r \neq 1$.

Proof. According to [35, (II.2.5)] the two-dimensional, simple $\mathrm{SL}(2)$-module $L = L(1)$ is self-dual. Consequently, L is also a faithful, self-dual $\mathcal{G}$-module.

Recall that $\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}}$ is a semidirect product of an infinitesimal group $\mathcal{G}^0$ and a reduced group $\mathcal{G}_{\mathrm{red}}$. Thanks to Nagata's Theorem (cf. [13, (IV,§3,3.6)]), the connected component $\mathcal{G}^0$ is multiplicative and p does not divide the order of $\mathcal{G}(k) = \mathcal{G}_{\mathrm{red}}(k)$. Since $\mathcal{G}^0$ is diagonalizable, there exists an element $g \in \mathrm{SL}(2)(k)$ such that $g\mathcal{G}^0g^{-1} \subset T$. If $\mathrm{ht}(\mathcal{G}^0) = r$, then we have $g\mathcal{G}^0g^{-1} \subset T_r \cong \mu_{p^r}$. On the other hand, $\mathrm{ord}(\mathcal{G}^0) \geq p^r$ (cf. [64, (2.2)]), so that $g\mathcal{G}^0g^{-1} = T_r$. Thus, replacing $\mathcal{G}$ by a suitable conjugate group, we may assume that $\mathcal{G}^0 = T_r = T_{(p^r)}$.

If $r = 0$, then the group $\mathcal{G}$ is reduced, and $\mathcal{G}$ is completely determined by the binary polyhedral group $\mathcal{G}(k) \subset \mathrm{SL}(2)(k)$. Directly from [56, (4.4)] we infer that $\mathcal{G}(k)$ is conjugate to e_k, $T_{(n)}(k)$ for $n \geq 2$, $N_{(n)}(k)$ for $n \geq 2$, $\hat{T}$, $\hat{O}$, or $\hat{I}$. As all groups involved are associated to reduced group schemes, we obtain the left-hand column of our list. Thanks to [28, Thm.1] the McKay graphs of these groups have the asserted structure.

If $r \geq 1$, then the identity $\mathcal{G}^0 = T_r$ in conjunction with $p \neq 2$ and [35, (I.2.6(11))] implies that

$$\mathcal{G}(k) \subset \mathrm{Nor}_{\mathrm{SL}(2)}(T_r)(k) = T(k)\langle\omega\rangle = \mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k)) = \mathrm{Nor}_{\mathrm{SL}(2)}(T)(k).$$

Consequently, $\mathcal{G}_{\mathrm{red}} \subset \mathrm{Nor}_{\mathrm{SL}(2)}(T)$, so that $\mathcal{G} \subset \mathrm{Nor}_{\mathrm{SL}(2)}(T)$. There results an embedding $\mathcal{G}/(\mathcal{G} \cap T) \hookrightarrow \mathrm{Nor}_{\mathrm{SL}(2)}(T)/T$, with the latter group being the *Weyl group* of $\mathrm{SL}(2)$. In particular, $\mathcal{G}/(\mathcal{G}\cap T)$ has order ≤ 2. The finite algebraic group $\mathcal{G} \cap T$ coincides with $T_{(np^r)}$ for some n not divisible by p. By our observations above, we have $\mathcal{G}^0 \subset \mathcal{G} \cap T$. If $\mathcal{G}(k) \subset T(k)$, then $\mathcal{G} = \mathcal{G} \cap T = T_{(np^r)}$. Alternatively, (†) and (‡) allow us to assume that $\mathcal{H}_4(k) \subset \mathcal{G}(k)$. As $\mathcal{H}_4$ is reduced, we conclude that $\mathcal{G} \cap T \subset N_{(np^r)} \subset \mathcal{G}$, so that $\mathcal{G} = N_{(np^r)}$.

It remains to identify the McKay graphs of $T_{(np^r)}$ and $N_{(np^r)}$. If $\mathcal{G} = T_{(np^r)}$, then $\mathcal{G}$ is abelian with np^r one-dimensional simple modules. Accordingly, [28, Thm.2] identifies $\tilde{A}_{np^r-1}$ and $\tilde{L}_{np^r-1}$ as the only possible McKay graphs of $\mathcal{G}$. As in the proof of [28, Thm.1] it follows that $np^r - 1 \leq 1$ in the latter case. Since $p \neq 2$, this implies $r = 0$, a contradiction.

Alternatively, $\mathcal{G} = N_{(np^r)}$ and every simple $\mathcal{G}$-module has dimension ≤ 2. Let k_λ be a one-dimensional $\mathcal{G}$-module, corresponding to the character $\lambda : \mathcal{G} \longrightarrow \mu_k$. Since $\lambda(t) = \lambda(\omega t \omega^{-1}) = \lambda(t^{-1}) = \lambda(t)^{-1}$ for every $t \in \mathcal{G} \cap T$, we see that $\lambda(t)^2 = 1$ for every $t \in \mathcal{G} \cap T$. For $t \in T_{(p^r)}$ we also have $\lambda(t)^{p^r} = 1$, which entails $\lambda(t) = 1$. Let x be a generator of $T_{(np^r)}(k)$. Then $1 = \lambda(x)^2 = \lambda(\omega)^2$, so that $\mathcal{G}$ possesses 4 one-dimensional modules. Wedderburn's Theorem now shows that $\mathcal{G}$ has $np^r - 1$ two-dimensional simple modules. By [28, Thm.2] this readily implies that the McKay graph of $\mathcal{G}$ is of the form $\tilde{D}_{np^r+2}$. □

4. Some Wild Groups

In this section we collect a few examples of wild finite algebraic groups that figure in the proofs of our main results. Each of these occurs as the final step of a reduction process, leading to groups that are amenable to the methods from abstract representation theory. *Throughout this section, we assume that* $p \geq 3$.

Lemma 4.1. *Let* $\mathcal{F} := (\alpha_p \rtimes \mu_{p^m}) \rtimes \mathcal{P}$ *be a finite algebraic group such that* $\mathcal{P} \cong (\mathbb{Z}/(p))_k$*, and with* μ_{p^m} *acting faithfully on* α_p*. Then* $\mathcal{B}_0(\mathcal{F})$ *is wild.*

Proof. The subgroup α_p is $\mathcal{P}(k)$-invariant, and in view of [35, (I.2.6(8))] normal in $\mathcal{F}$. As the automorphism group $\mathrm{Aut}(\alpha_p)$ coincides with μ_k, the conjugation action of $\mathcal{F}$ on α_p induces a homomorphism $\gamma : \mathcal{F} \longrightarrow \mu_k$ whose restriction to μ_{p^m} is injective. Since α_p is commutative, γ factors through to a homomorphism $\hat{\gamma} : \mathcal{F}/\alpha_p \longrightarrow \mu_k$. The action of $\mathcal{P}$ on $\alpha_p \rtimes \mu_{p^m}$ induces an operation of $\mathcal{P}$ on μ_{p^m} by automorphisms such that $\mathcal{F}/\alpha_p \cong \mu_{p^m} \rtimes \mathcal{P}$ (cf. (1.1)). Since the restriction of the quotient map $\pi : \mathcal{F} \longrightarrow \mathcal{F}/\alpha_p \cong \mu_{p^m} \rtimes \mathcal{P}$ to μ_{p^m} maps $\mu_{p^m} \subset \mathcal{F}$ bijectively onto μ_{p^m}, our assumption concerning the μ_{p^m}-action on α_p implies $\ker(\hat{\gamma}|_{\mu_{p^m}}) = e_k$. Given $r \in \mu_{p^m}$ and $g \in \mathcal{P}$, we have

$$\hat{\gamma}(r) = \hat{\gamma}(grg^{-1}) = \hat{\gamma}(g \cdot r),$$

so that $g \cdot r = r$. (By standard abuse of notation we will often write $g \in \mathcal{P}$ instead of $g \in \mathcal{P}(R)$ for some commutative k-algebra R). Consequently, $\mathcal{P}$ acts trivially on μ_{p^m}, and $\mu_{p^m} \rtimes \mathcal{P} = \mu_{p^m} \times \mathcal{P}$ is abelian. In particular, $\mathcal{P}$ is a normal subgroup of $\mu_{p^m} \times \mathcal{P} \cong \mathcal{F}/\alpha_p$, and there results a quotient map $\omega : \mathcal{F} \longrightarrow \mu_{p^m} \times \mathcal{P} \longrightarrow \mu_{p^m}$ with kernel $\mathcal{V} := \alpha_p \rtimes \mathcal{P}$. As $\mathcal{V}$ is unipotent, ω induces an embedding $\mu_{p^m} \hookrightarrow \mu_{p^m}$, which, by equality of orders, is an isomorphism. Consequently, we have

$$\mathcal{F} \cong \mathcal{V} \rtimes \mu_{p^m}.$$

Since $\mathrm{Aut}(\alpha_p) \cong \mu_k$, the group $\mathcal{P}$ acts trivially on α_p, so that the group scheme $\mathcal{V}$ is abelian and unipotent. We may now use the methods of [24, §2] to compute the basic algebra of $H(\mathcal{F})$: The vertex set is $\mathbb{Z}/(p^m)$, an there exists $r \in \{0, \ldots, p^m - 1\}$ such that the arrows and relations are given by

$$\alpha_i : i \to i+1 \quad ; \quad \beta_i : i \to i+r,$$

and

$$\beta_{i+1}\alpha_i - \alpha_{i+r}\beta_i \quad ; \quad \alpha_{i+(p-1)} \cdots \alpha_i \quad ; \quad \beta_{i+(p-1)r} \cdots \beta_i,$$

respectively. (One can actually prove $r = 0$.) As was shown in [24, (2.4)], Galois covering techniques then establish the wildness of the connected algebra $H(\mathcal{F})$. □

The remaining examples are given by closed, finite subgroup schemes of $\mathrm{SL}(2)$ containing the first Frobenius kernel $\mathrm{SL}(2)_1$. For $n \geq 0$, we let $L(n)$ be the simple $\mathrm{SL}(2)$-module with highest weight n. If $n \leq p-1$, then $L(n)$ is also a simple module for the first Frobenius kernel $\mathrm{SL}(2)_1$ (cf. [35, (II.3.15)]). The projective cover of the simple $H(\mathrm{SL}(2)_1)$-module $L(n)$ will be denoted $P(n)$.

Let M_k be the category of commutative k-algebras. We begin by considering the group scheme $\mathcal{L} : M_k \longrightarrow \mathrm{Gr}$ given by

$$\mathcal{L}(R) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)(R) \ ; \ a^p = 1 = d^p \ , \ b^{p^2} = b^p \ , \ c^p = 0\}$$

for every $R \in M_k$. Then $\mathcal{L}$ is a closed subgroup of $\mathrm{SL}(2)$ such that $\mathcal{L}(k) = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} ; \ b \in \mathbb{F}_p\} \cong \mathbb{Z}/(p)$. Moreover, since $\mathcal{L}_r = \mathcal{L}^0$ for some $r \geq 1$, we obtain $\mathcal{L}^0 = \mathrm{SL}(2)_1$. Consequently,

$$\mathcal{L} = \mathrm{SL}(2)_1 \rtimes \mathcal{U},$$

where, for every commutative k-algebra R, the R-points of $\mathcal{U}$ are the group

$$\mathcal{U}(R) = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \ ; \ b^p = b\}.$$

In the proof of the succeeding result we will identify modules of certain basic algebras with the corresponding representations of their bound quivers. We refer the reader to [4, III] for the details.

Lemma 4.2. *The principal block $\mathcal{B}_0(\mathcal{L})$ is wild.*

Proof. Setting $\mathcal{N} := \mathrm{SL}(2)_1$, we obtain that the factor group $\mathcal{L}/\mathcal{N} \cong \mathcal{U} \cong (\mathbb{Z}/(p))_k$ is unipotent. Thus, a consecutive application of [35, (II.3.15)] and (2.1) ensures that the restriction functor $M \mapsto M|_{\mathcal{N}}$ furnishes a bijection between the sets of isoclasses of simple $\mathcal{L}$-modules and simple $\mathcal{N}$-modules, with the latter being represented by the SL(2)-modules $L(0), \ldots, L(p-1)$. Owing to [48, Sect.1] and [25, Kap.1] the extension groups of the simple $\mathcal{N}$-modules satisfy

$$\dim_k \mathrm{Ext}^1_{\mathcal{N}}(L(i), L(j)) = \begin{cases} 2 & i+j = p-2 \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, (2.2) implies

$$\mathrm{Ext}^1_{\mathcal{L}}(L(i), L(j)) \cong \begin{cases} \mathrm{Ext}^1_{\mathcal{U}}(k,k) & i = j \\ \mathrm{Ext}^1_{\mathcal{N}}(L(i), L(p-2-i))^{\mathcal{U}} & 0 \leq i \leq p-2, j = p-2-i \\ (0) & \text{otherwise.} \end{cases}$$

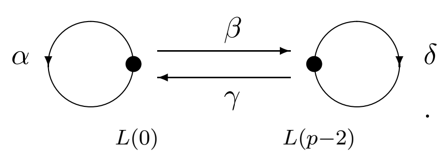
It follows that $L(0)$ and $L(p-2)$ are the only simple modules that can belong to $\mathcal{B}_0(\mathcal{L})$. In view of [47, Theorem] we have isomorphisms

$$\mathrm{Ext}^1_{\mathcal{N}}(L(0), L(p-2)) \cong L(1)^{[1]} \cong \mathrm{Ext}^1_{\mathcal{N}}(L(p-2), L(0))$$

of SL(2)-modules, where $L(1)^{[1]}$ is the Frobenius twist of the simple SL(2)-module $L(1)$ (cf. [35, (II.3.16)]). Since the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{L}(k)$ acts non-trivially on $L(1)^{[1]}$, we obtain

$$\dim_k \mathrm{Ext}^1_{\mathcal{L}}(L(0), L(p-2)) = 1 = \dim_k \mathrm{Ext}^1_{\mathcal{L}}(L(p-2), L(0)).$$

As a result, $L(0)$ and $L(p-2)$ are the only simple $\mathcal{B}_0(\mathcal{L})$-modules, and the Gabriel quiver $\Gamma(\mathcal{L})$ of $\mathcal{B}_0(\mathcal{L})$ is given by



By Gabriel's Theorem (cf. [6, (4.1.7)]) the basic algebra Λ of $\mathcal{B}_0(\mathcal{L})$ is isomorphic to the factor algebra $k[\Gamma(\mathcal{L})]/I$ of the path algebra $k[\Gamma(\mathcal{L})]$ by an ideal I which is generated by paths of length ≥ 2.

Since $p \geq 3$, the group $\mathcal{L}/\mathcal{N} \cong \mathcal{U}$ possesses a uniserial module of length 3 with $L(0) = k$ as its only composition factor. Accordingly, the algebra Λ enjoys the same property. This means that the corresponding Λ-module $M = M(0) \oplus M(p-2)$ (with $M(i)$ denoting the vector space attached to the vertex i) is concentrated in the vertex 0. Hence there exists $m \in M(0)$ such that $\alpha^2 m \neq 0$, while $\beta M(0) = (0)$.

Suppose that the path α^2 is a summand of a generator of the ideal I. Then there exist paths $\pi_i \neq \alpha^2$ of length ≥ 2 from 0 to 0 and $a_i \in k$ such that

$$\alpha^2 \equiv \sum_{i=1}^{n} a_i \pi_i \qquad \mathrm{mod}(I).$$

Observe that either $\pi_i = \alpha^{n_i}$ for some $n_i \geq 3$, or $\pi_i = \omega_i \beta \alpha^{m_i}$ for some path ω_i and some $m_i \in \mathbb{N}_0$. Since M has length 3, paths of the first kind annihilate m. As β operates on M via 0, paths of the second kind do so as well. Accordingly, the above congruence yields

$$0 \neq \alpha^2 m = 0,$$

a contradiction. It follows that α^2 is not a summand of a relation.

Now consider the projective cover $P(p-2)$ of the simple $H(\mathrm{SL}(2)_1)$-module $L(p-2)$. According to [36] the principal indecomposable module $P(p-2)$ has an SL(2)-structure which extends the $\mathrm{SL}(2)_1$-structure. Restriction to $\mathcal{L}$ provides a $\mathcal{L}$-structure with the same property. Owing to [30, Thm.3] the SL(2)-module $P(p-2)$ has Loewy series $L(p-2)$, $L(p)$, $L(p-2)$. Consequently, $\mathrm{rad}(P(p-2))$ is an SL(2)-module of length 2 with Loewy series $L(p)$, $L(p-2)$. Steinberg's tensor product theorem (cf. [35, (II.3.17)]) provides an isomorphism $L(p) \cong L(1)^{[1]}$ of SL(2)-modules. In particular, $\mathcal{N} = \mathrm{SL}(2)_1$ operates trivially on $L(p)$, and $L(p)|_{\mathcal{L}}$ is a uniserial $\mathcal{L}$-module of length 2. As $\mathrm{Soc}_{\mathcal{B}_0(\mathcal{L})}(P(p-2)) = \mathrm{Soc}_{H(\mathcal{N})}(P(p-2)) = L(p-2)$, the $\mathcal{L}$-module $\mathrm{rad}(P(p-2))$ is uniserial with composition factors (from top to bottom) $L(0)$, $L(0)$, $L(p-2)$. Accordingly, the basic algebra Λ also has such a module, which will be denoted $N = N(0) \oplus N(p-2)$. Note that $\dim_k N(0) = 2$ and $\dim_k N(p-2) = 1$.

Since $N(p-2) = \mathrm{Soc}_\Lambda(N)$ the arrows δ and γ operate trivially on N. Consider the two-dimensional submodule $N' := \alpha N(0) \oplus N(p-2)$ of N. Since N' is indecomposable, we have $\beta\alpha N(0) \neq (0)$. Hence there exists a basis $\{e_1, e_2\}$ of $N(0)$ with $\alpha e_1 = e_2$, $\alpha e_2 = 0$ and $\beta e_2 \neq 0$. We write $N(p-2) = ke_3$ with $e_3 = \beta e_2$ and obtain $\beta e_1 = a e_3$ for some $a \in k$. The element $n_0 := e_1 - a e_2 \in N(0)$ satisfies

(i) $\beta\alpha n_0 \neq 0$, and
(ii) $\beta n_0 = 0$.

Assume that $\beta\alpha$ is a summand of a relation of Λ. As before, we write

$$\beta\alpha \equiv \sum_{i=1}^{n} a_i \pi_i \qquad \mathrm{mod}(I),$$

where $a_i \in k$ and $\pi_i \neq \beta\alpha$ is a path of length ≥ 2 from 0 to $p-2$. For each path π_i there is a path ω_i such that

$$\pi_i = \begin{cases} \omega_i \alpha^{j_i} & j_i \geq 2 \\ \omega_i \delta\beta\alpha & \\ \omega_i \gamma\beta\alpha & \\ \omega_i \beta. & \end{cases}$$

Since $\delta = 0 = \gamma$ on N, the above relation in conjunction with (i) and (ii) implies

$$0 \neq \beta\alpha n_0 = \sum_{i=1}^{n} a_i \pi_i n_0 = 0,$$

a contradiction.

As an upshot of the above, neither α^2 nor $\beta\alpha$ is a summand of a relation. We may now apply [17, (I.10.9)] to see that the algebra Λ is wild. Consequently, the block $\mathcal{B}_0(\mathcal{L})$ is also wild. □

Remark. For $p = 2$, (1.2) implies

$$\mathcal{L}_{\mathrm{lr}} = \mathcal{M} = \mathrm{Cent}(\mathrm{SL}(2)_1) \cong \mu_2.$$

Owing to (1.1), the block $\mathcal{B}_0(\mathcal{L})$ is isomorphic to the algebra of measures of the unipotent factor group $\mathcal{L}' := \mathcal{L}/\mathcal{L}_{\mathrm{lr}}$. Direct computation shows that this local symmetric algebra has a presentation

$$H(\mathcal{L}') \cong k\langle X, Y\rangle/(X^2, Y^2, (XY)^2 - (YX)^2),$$

so that [17, (III.1(c))] ensures the tameness of $\mathcal{B}_0(\mathcal{L})$. (In view of [17, (III.13)], our block is isomorphic to the group algebra $k[D_4]$ of the dihedral group of order 8).

Our next example concerns a finite algebraic subgroup of SL(2) that is the semidirect product of $\mathrm{SL}(2)_1$ with the reduced subgroup of SL(2), whose group of rational points is the Chevalley group $\mathrm{SL}(2,p) = \mathrm{SL}(2)(\mathbb{F}_p)$.

Let

$$F : \mathrm{SL}(2) \longrightarrow \mathrm{SL}(2) \ \ ; \ \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$$

be the Frobenius endomorphism of SL(2). We consider the group scheme $\mathrm{SL}(2)_{2,1}$, which is given by

$$\mathrm{SL}(2)_{2,1}(R) := \{g \in \mathrm{SL}(2)(R) \ ; \ F_R^2(g) = F_R(g)\}$$

for every $R \in M_k$. Being the inverse image of the diagonal subgroup of $\mathrm{SL}(2) \times \mathrm{SL}(2)$ under the morphism $g \mapsto (F^2(g), F(g))$, $\mathrm{SL}(2)_{2,1}$ is a closed subgroup of SL(2) (cf. [35, (I.1.4)]).

By definition, we have $\mathrm{SL}(2)_1 \subset \mathrm{SL}(2)_{2,1}$, whence $\mathrm{SL}(2)_1 \subset \mathrm{SL}(2)_{2,1}^0$. Now let g be an element of $\mathrm{SL}(2)_{2,1}^0$. Since this group is infinitesimal, there exists $r \in \mathbb{N}$ such that $F^r(g) = 1$. As $g \in \mathrm{SL}(2)_{2,1}$, we have

$$F^r(g) = F(g),$$

so that $g \in (\mathrm{SL}(2)_{2,1})_1 \subset \mathrm{SL}(2)_1$. We thus obtain $\mathrm{SL}(2)_{2,1}^0 = \mathrm{SL}(2)_1$.

Let $g = (g_{ij})$ be an element of $\mathrm{SL}(2)_{2,1}(k)$, so that the entries of the matrix g satisfy

$$g_{ij}^{p^2} = g_{ij}^p.$$

Then we have

$$0 = g_{ij}^{p^2} - g_{ij}^p = (g_{ij}^p - g_{ij})^p,$$

so that $g_{ij} \in \mathbb{F}_p$. As a result, $g \in \mathrm{SL}(2,p)$, proving $\mathrm{SL}(2)_{2,1}(k) \subset \mathrm{SL}(2,p)$. The reverse inclusion is obvious.

Lemma 4.3. *The algebra* $\mathcal{B}_0(\mathrm{SL}(2)_{2,1})$ *is wild.*

Proof. Let $\mathcal{G} := \mathrm{SL}(2)_{2,1}$ and put $\mathcal{N} := \mathrm{SL}(2)_1$. As observed earlier, the modules $\{L(i)|_{\mathcal{N}} \ ; \ 0 \le i \le p-1\}$ form a ccomplete set of representatives of the simple $\mathcal{N}$-modules. From the representation theory of $\mathrm{SL}(2,p)$ (cf. [1, p.14f]) we see that $\mathcal{G}/\mathcal{N} \cong \mathrm{SL}(2,p)_k$ has simple modules $M(0), \ldots, M(p-1)$ with $\dim_k M(i) = i+1$. We may now apply (2.1) to see that $\{L(i) \otimes_k M(j) \ ; \ i,j \in \{0, \ldots, p-1\}\}$ is a complete set of representatives of the simple $\mathcal{G}$-modules. (By Steinberg's tensor product theorem, these are just the restrictions $L(i)|_{\mathcal{G}}$ for $0 \le i \le p^2 - 1$.)

Since the simple $\mathrm{SL}(2)_1$-modules do not possess non-trivial self-extensions, Lemma 2.2 readily implies that the simple $\mathcal{B}_0(\mathcal{G})$-modules belong to the set $\{L(i) \otimes_k M(j) \ ; \ i \in \{0, p-2\}, \ 0 \le j \le p-1\}$.

By earlier observations, we have isomorphisms

$$\mathrm{Ext}^1_{\mathcal{N}}(L(0), L(p-2)) \cong M(1) \cong \mathrm{Ext}^1_{\mathcal{N}}(L(p-2), L(0))$$

of $\mathrm{SL}(2,p)$-modules. Setting $M(-1) := \{0\}$, we apply [1, (7.5)] to see that the tensor products of simple $\mathrm{SL}(2,p)$-modules are given by

$$M(i) \otimes_k M(1) \cong M(i+1) \oplus M(i-1) \ \ \text{for} \ \ 0 \le i \le p-2,$$

while

$$M(p-1) \otimes_k M(1) \cong \tilde{P}(p-2)$$

is the projective cover of the $\mathrm{SL}(2,p)$-module $M(p-2)$ (see also [31, (7.2),(11.1)]). (The last identity follows from the fact that $\dim_k \tilde{P}(p-2) = 2p = \dim_k M(p-1) \otimes_k M(1)$, the projectivity of the tensor product, and $\mathrm{Hom}_{\mathrm{SL}(2,p)}(M(1)\otimes_k M(p-1), M(p-2)) \cong \mathrm{Hom}_{\mathrm{SL}(2,p)}(M(p-1), M(1)\otimes_k M(p-2)) \neq (0)$.) The extension groups of the simple $\mathrm{SL}(2,p)$-modules are computable from the structure of their projective covers, as provided in [1, §7]. Setting $G := \mathrm{SL}(2,p)$ we have

$$\dim_k \mathrm{Ext}^1_G(M(i), M(j)) = \begin{cases} 1 & i+j \in \{p-1, p-3\} \text{ and } 1 \leq i \leq p-3 \\ 1 & (i,j) \in \{(0,p-3),(p-2,1)\} \\ 0 & \text{otherwise.} \end{cases}$$

The tensor product decomposition gives rise to

$$\dim_k \mathrm{Hom}_G(M(i), M(1)\otimes_k M(j)) = \begin{cases} 1 & j \in \{i-1, i+1\} \text{ and } 0 \leq j \leq p-2 \\ 1 & (i,j) = (p-2, p-1) \\ 0 & \text{otherwise.} \end{cases}$$

These formulae in conjunction with Lemma 2.2 imply:

(a) The group algebra $k[G]$ has three blocks consisting of the simple modules $\{M(2i)\ ;\ 0 \leq i \leq \frac{p-3}{2}\}$, $\{M(2i+1)\ 0 \leq i \leq \frac{p-3}{2}\}$ and $\{M(p-1)\}$.

(b) If $L(i,j) := L(i) \otimes_k M(j)$ belongs to $\mathcal{B}_0(\mathcal{G})$, so does $L(i,\ell)$ for any ℓ such that $M(\ell)$ and $M(j)$ belong to the same block of the group algebra $k[G]$.

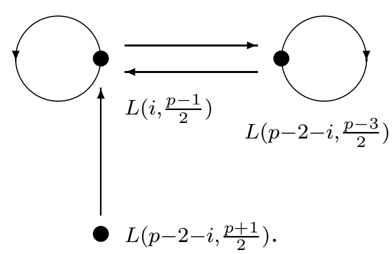
Let $i \in \{0, p-2\}$. Another application of Lemma 2.2 provides the following information:

(1) $\dim_k \mathrm{Ext}^1_{\mathcal{G}}(L(i,j), L(i,j)) = 1$ whenever $j \in \{\frac{p-3}{2}, \frac{p-1}{2}\}$.

(2) $\dim_k \mathrm{Ext}^1_{\mathcal{G}}(L(i, \frac{p-1}{2}), L(p-2-i, \frac{p-3}{2})) = 1 = \dim_k \mathrm{Ext}^1_{\mathcal{G}}(L(p-2-i, \frac{p-3}{2}), L(i, \frac{p-1}{2}))$.

(3) $\dim_k \mathrm{Ext}^1_{\mathcal{G}}(L(p-i-2, \frac{p+1}{2}), L(i, \frac{p-1}{2})) = 1$.

Thus, the Gabriel quiver of $H(\mathcal{G})$ has the following subquiver:



Thanks to [17, (I.10.8(iv))] the block containing this subquiver is wild. Since $L(0,0)$ and $L(p-2,1)$ both belong to the principal block, we see from (a) and (b) that $L(0, \frac{p-1}{2})$ belongs to $\mathcal{B}_0(\mathcal{G})$ if $\frac{p-1}{2}$ is even, while $L(p-2, \frac{p-1}{2})$ belongs to $\mathcal{B}_0(\mathcal{G})$ otherwise. Consequently, the block $\mathcal{B}_0(\mathcal{G})$ is wild. □

Lemma 4.4. *Suppose that $p=3$ and let $\mathcal{G} \subset \mathrm{SL}(2)$ be a closed subgroup such that*

(a) $\mathcal{G}^0 = \mathrm{SL}(2)_1$, *and*

(b) $\mathcal{G}(k) \cong \mathrm{SL}(2,5)$.

Then $\mathcal{B}_0(\mathcal{G})$ is wild.

Proof. Thanks to [56, (4.4)] the subgroup $G := \mathcal{G}(k) \subset \mathrm{SL}(2)(k)$ is conjugate to the binary icosahedral group. It follows that the $\mathrm{SL}(2)(k)$-modules $L(0)$, $L(1)$ and $L(2) \cong \mathfrak{sl}(2)$ remain simple upon restriction to G. Since G is not solvable, the Frobenius twist $L(1)^{[1]}$ is also a simple G-module, If $L(1)|_G \cong L(1)^{[1]}|_G$, then Lang's Theorem [57, (3.3.16)] provides an element $g_0 \in \mathrm{SL}(2)(k)$ such that $g_0 G g_0^{-1} \subset \mathrm{SL}(2,3)$, a contradiction. Consequently, $L(3) \cong L(1)^{[1]} \not\cong L(1)$ is another simple G-module. Recall that the factor group $G/C(G)$ of G by its center $C(G) \cong \mathbb{Z}/(2)$ is isomorphic to the alternating group A_5. Thus, the group G affords five 3-regular classes, and by Brauer's Theorem [1, §3,Thm.2], the group algebra $k[G]$ therefore has exactly 5 simple modules. Since the permutation module of A_5 has a 4-dimensional simple submodule, we conclude that the remaining simple G-module has dimension 4.

We claim that $L(4) \cong L(1) \otimes_k L(1)^{[1]} \cong L(1) \otimes_k L(3)$ is a simple G-module. Since the central element $c := -I_2 \in G$ acts on $L(i)$ via the scalar $(-1)^i$, it operates trivially on $L(4)$. Hence only $L(0)$ and $L(2)$ can possibly occur as proper composition factors of the G-module $L(4)$. Thanks to [35, (II.2.5)] every $L(i)$ is a self-dual $\mathrm{SL}(2)(k)$-module, so it is also a self-dual G-module. Thus, if $L(2)$ occurs as a composition factor, so does $L(0)$, and in that case $L(0)$ is a submodule of $L(4)$. Consequently, if $L(4)$ is reducible, then $L(0)$ is a submodule of $L(4)$, and

$$(0) \neq L(4)^G \cong (L(1)^* \otimes_k L(3))^G \cong \mathrm{Hom}_G(L(1), L(3)) = (0),$$

a contradiction. As a result, $L(4)$ is the remaining simple G-module.

Owing to [31, (10.2)] the principal indecomposable $\mathrm{SL}(2)_1$-modules $P(0)$, $P(1)$ and $P(2) = L(2)$ are projective $\mathrm{SL}(2,3)$-modules. Since a Sylow-3-subgroup $P \cong \mathbb{Z}/(3)$ of G is contained in $\mathrm{SL}(2,3)$, we may apply [1, §9, Cor.3] to see that the $P(i)$ are projective G-modules for $0 \le i \le 2$. According to [31, (11.1)] the composition factors of $P(i)$ are (from top to bottom) $L(i)$, $L(1-i) \otimes_k L(1)^{[1]}$, $L(i)$ for $i \in \{0, 1\}$. Since the dimension of every projective G-module is a multiple of 3, it follows that the $P(i)$ are in fact principal indecomposable G-modules. In particular, $L(4)$ belongs to the principal block of $k[G]$, and $\dim_k \mathrm{Ext}^1_G(L(0), L(4)) = 1$.

From general $\mathrm{SL}(2)$-theory one obtains $L(1) \otimes_k L(1) \cong L(0) \oplus L(2)$, so that

$$\mathrm{Hom}_G(L(1) \otimes_k L(2), L(1)) \cong \mathrm{Hom}_G(L(2), L(1) \otimes_k L(1)) \neq (0).$$

As $L(1) \otimes_k L(2)$ is a projective module of dimension 6, we obtain $L(1) \otimes_k L(2) \cong P(1)$. Consequently, there are isomorphisms

$$L(1) \otimes_k L(4) \cong (L(1) \otimes_k L(1)) \otimes_k L(3) \cong (L(0) \oplus L(2)) \otimes_k L(3) \cong L(3) \oplus (L(2) \otimes_k L(3)).$$

The module $L(2) \otimes_k L(3)$ is projective of dimension 6, with c acting via -1. Hence only $L(1)$ and $L(3)$ can occur as composition factors. Since

$$\mathrm{Hom}_G(L(1), L(2) \otimes_k L(3)) \cong \mathrm{Hom}_G(L(1) \otimes_k L(2), L(3)) \cong \mathrm{Hom}_G(P(1), L(3)) = (0),$$

it follows that $L(2) \otimes_k L(3)$ is the projective cover $P(3)$ of the G-module $L(3)$. Consequently,

$$L(1) \otimes_k L(4) \cong L(3) \oplus P(3).$$

Now let $M(0), \ldots, M(4)$ be the simple $\mathcal{G}_{\mathrm{red}}$-modules, which we view as $\mathcal{G}$-modules by letting $\mathcal{G}^0$ act trivially. Thus, [47, Theorem] yields

$$\mathrm{Ext}^1_{\mathcal{G}}(L(1), L(0)) \cong M(3) \cong \mathrm{Ext}^1_{\mathcal{G}}(L(0), L(1)).$$

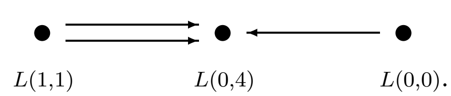
Adopting our earlier notation, we conclude from (2.1) that $\{L(i,j) \ ; \ 0 \le i \le 2 \ , 0 \le j \le 4\}$ are the simple $\mathcal{G}$-modules. The above computations in conjunction with (2.2) now imply that

$$\dim_k \mathrm{Ext}^1_{\mathcal{G}}(L(0,0), L(0,4)) = \dim_k \mathrm{Ext}^1_G(M(0), M(4)) = 1$$

as well as

$$\begin{aligned} \dim_k \mathrm{Ext}^1_{\mathcal{G}}(L(1,1), L(0,4)) &= \dim_k \mathrm{Hom}_G(M(1), M(3) \otimes_k M(4)) \\ &= \dim_k \mathrm{Hom}_G(M(3), M(1) \otimes_k M(4)) \\ &= \dim_k \mathrm{Hom}_G(M(3), M(3) \oplus P(3)) = 2. \end{aligned}$$

As a result, the Gabriel quiver of $\mathcal{B}_0(\mathcal{G})$ contains the subquiver



According to [17, (I.10.8(iii))] this entails the wildness of $\mathcal{B}_0(\mathcal{G})$. □

5. Passage between Groups and Subgroups

In this and the following Section we begin with the structural analysis of tame group schemes $\mathcal{G}$ by providing results on their connected and reduced parts and by studying the conjugation action of $\mathcal{G}_{\rm red}$ on $\mathcal{G}^0$.

5.1. **Block Descent.** In contrast to the theory of finite groups, where Brauer's Third Main Theorem implies that the representation type of principal blocks transfers well to subgroups, the results of [20] already indicate that infinitesimal groups may behave erratically. In this Section we consider the problem of descending to normal subgroups containing the connected component, which will turn out to be sufficient for our purposes.

In the following, a k-algebra is always meant to be a finite-dimensional associative algebra whose unit element acts on all modules via the identity operator. A k-algebra Λ is said to be *weakly tame* if for every $d \in \mathbb{N}$ there exist finitely generated $(\Lambda, k[T])$-bimodules $M_1, \ldots, M_{s(d)}$ such that each d-dimensional indecomposable Λ-module is a direct summand of $M_i \otimes_{k[T]} k_\lambda$ for some $i \in \{1, \ldots, s(d)\}$ and some algebra homomorphism $\lambda : k[T] \longrightarrow k$.

Our descent methods are based on the following subsidiary result, which is closely related to [9, Prop.2].

Lemma 5.1.1. *Let $\Gamma \subset \Lambda$ be k-algebras, $\mathcal{B}_\Gamma$ and $\mathcal{B}_\Lambda$ blocks of Γ and Λ, respectively. Suppose that $\mathcal{B}_\Gamma$ is isomorphic to a direct summand of the (Γ, Γ)-bimodule $\mathcal{B}_\Lambda$. If $\mathcal{B}_\Lambda$ is tame, then $\mathcal{B}_\Gamma$ is tame or representation-finite.*

Proof. By assumption, we have a decomposition

$$\mathcal{B}_\Lambda = X \oplus Y$$

of (Γ, Γ)-bimodules such that $X \cong \mathcal{B}_\Gamma$.

Let $d > 0$. Since $\mathcal{B}_\Lambda$ is tame, there exist $(\Lambda, k[T])$-bimodules $M_1, \ldots, M_s$ such that every indecomposable $\mathcal{B}_\Lambda$-module of V of dimension $\leq d \dim_k \Lambda$ is of the form $V \cong M_i \otimes_{k[T]} k_\lambda$ for some $\lambda : k[T] \longrightarrow k$. Let N be an indecomposable $\mathcal{B}_\Gamma$-module of dimension d. By the above, we have a decomposition

$$\mathcal{B}_\Lambda \otimes_\Gamma N \cong (X \otimes_\Gamma N) \oplus (Y \otimes_\Gamma N) \cong (\mathcal{B}_\Gamma \otimes_\Gamma N) \oplus (Y \otimes_\Gamma N)$$

of Γ-modules. Since N belongs to $\mathcal{B}_\Gamma$, the summand $\mathcal{B}_\Gamma \otimes_\Gamma N$ is isomorphic to N, so that N is a direct summand of $(\mathcal{B}_\Lambda \otimes_\Gamma N)|_\Gamma$. By the Theorem of Krull-Remak-Schmidt N is therefore also a constituent of some $(M_i|_\Gamma) \otimes_{k[T]} k_\lambda$. Accordingly, $\mathcal{B}_\Gamma$ is weakly tame, and [46, (I.2.3)] shows that Γ is tame or representation-finite. □

We now consider an augmented k-algebra (Λ, ε), and a finite group G operating on Λ via automorphisms such that $\varepsilon : \Lambda \longrightarrow k$ is G-equivariant. In this situation the map

$$\tilde{\varepsilon} : \Lambda[G] \longrightarrow k \quad ; \quad \sum_{g \in G} \lambda_g g \mapsto \sum_{g \in G} \varepsilon(\lambda_g)$$

is an augmentation of the skew group algebra $\Lambda[G]$, so that we can speak of the principal block $\mathcal{B}_0(G)$ of $\Lambda[G]$. Given a normal subgroup $N \lhd G$, we let $\Lambda[N]^e := \Lambda[N] \otimes_k \Lambda[N]^{\rm op}$ be the enveloping algebra of $\Lambda[N] \subset \Lambda[G]$. If $X_N \subset G$ is a complete set of representatives of the right N-cosets, then

$$(*) \qquad \Lambda[G] = \bigoplus_{g \in X_N} \Lambda[N] g$$

is a decomposition of the $\Lambda[N]^e$-module $\Lambda[G]$.

Given an element $g \in G$, the map

$$\gamma_g : \Lambda[N]^e \longrightarrow \Lambda[N]^e \quad ; \quad a \otimes b \mapsto a \otimes gbg^{-1}$$

is an automorphism of $\Lambda[N]^e$. For any $\Lambda[N]^e$-module V, we denote by $V^{(g)}$ the $\Lambda[N]^e$-module with underlying k-space V and action given by

$$\lambda.v := \gamma_g(\lambda)v \qquad \forall\, \lambda \in \Lambda[N]^e,\ v \in V.$$

Lemma 5.1.2. *If the principal block $\mathcal{B}_0(G) \subset \Lambda[G]$ is tame, then the principal block $\mathcal{B}_0(N) \subset \Lambda[N]$ is tame or representation-finite.*

Proof. We let G act on the algebra $\Lambda[N]$ via conjugation. Since $\tilde{\varepsilon}(gxg^{-1}) = \tilde{\varepsilon}(x)$ for $x \in \Lambda[N]$ and $g \in G$, it follows that this action, which permutes the blocks of $\Lambda[N]$, stabilizes $\mathcal{B}_0(N)$. Thus, the block decomposition of $\Lambda[N]$ gives rise to a decomposition $\Lambda[N] = \mathcal{B}_0(N) \oplus \mathcal{C}$ of G-invariant block ideals. Accordingly, $I := \bigoplus_{g\in X_N} \mathcal{B}_0(N)g$ and $J := \bigoplus_{g \in X_N} \mathcal{C}g$ are ideals of $\Lambda[G]$ such that

$$\Lambda[G] = I \oplus J.$$

Since $\tilde{\varepsilon}(J) = (0)$ the unicity of the block decomposition of $\Lambda[G]$ ensures that $\mathcal{B}_0(G)$ is a direct summand of the ideal I. In particular, $\mathcal{B}_0(G)|_{\Lambda[N]^e}$ is a direct summand of $I|_{\Lambda[N]^e} = \bigoplus_{g\in X_N} \mathcal{B}_0(N)g$. Direct computation shows that $\lambda \mapsto \lambda g$ furnishes an isomorphism $\mathcal{B}_0(N)^{(g)} \xrightarrow{\sim} \mathcal{B}_0(N)g$ of $\Lambda[N]^e$-modules for every $g \in G$. As $\mathcal{B}_0(N)$ is an indecomposable $\Lambda[N]^e$-module, the Theorem of Krull-Remak-Schmidt implies the existence of $g_0 \in X_N$ and a $\Lambda[N]^e$-module X such that

$$\mathcal{B}_0(G)|_{\Lambda[N]^e} \cong \mathcal{B}_0(N)g_0 \oplus X.$$

Since $\mathcal{B}_0(G)$ is an ideal of $\Lambda[G]$ we have $\mathcal{B}_0(G)g = \mathcal{B}_0(G)$, so that the above map provides an isomorphism $\mathcal{B}_0(G)^{(g)} \cong \mathcal{B}_0(G)$ of $\Lambda[N]^e$-modules. We now obtain

$$\mathcal{B}_0(G) \cong \mathcal{B}_0(G)^{(g_0^{-1})} \cong (\mathcal{B}_0(N)^{(g_0)} \oplus X)^{(g_0^{-1})} \cong \mathcal{B}_0(N) \oplus X^{(g_0^{-1})},$$

so that our result follows directly from (5.1.1). □

In the context of finite algebraic groups of odd characteristic the foregoing result may be sharpened. For normal subgroups containing the connected component, the tameness of the principal block is inherited by the corresponding block of the subgroup. Recall that the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$ of an algebraic group $\mathcal{G}$ is a finite-dimensional restricted Lie algebra. By definition a *restricted Lie algebra* is a pair $(\mathfrak{g}, [p])$ consisting of an ordinary Lie algebra $\mathfrak{g}$ and a p-map

$$\mathfrak{g} \longrightarrow \mathfrak{g} \quad ; \quad x \mapsto x^{[p]}$$

that satisfies the formal properties of a p-power operator of an associative k-algebra. Basic facts on restricted Lie algebras can be found in [33, 58]. In this paper we shall only be concerned with finite-dimensional restricted Lie algebras. The group $\mathcal{G}$ acts on $\mathrm{Lie}(\mathcal{G})$ via the *adjoint representation*, so that we have a homomorphism

$$\mathrm{Ad} : \mathcal{G} \longrightarrow \mathcal{AUT}(\mathrm{Lie}(\mathcal{G}))$$

from $\mathcal{G}$ to the automorphism scheme of $\mathrm{Lie}(\mathcal{G})$. The rational points of this scheme form the automorphism group $\mathrm{Aut}_p(\mathfrak{g})$ of the restricted Lie algebra $(\mathfrak{g}, [p])$.

Lemma 5.1.3. *Let $\mathcal{G}$ be a finite algebraic group such that $\mathrm{Cent}_{\mathcal{G}_{\mathrm{red}}}(\mathcal{G}^0) = e_k = \mathrm{Cent}(\mathcal{G}_1)$. Then the adjoint representation furnishes an embedding $\mathcal{G} \hookrightarrow \mathcal{AUT}(\mathrm{Lie}(\mathcal{G}))$.*

Proof. According to general theory (cf. [13, (II,§7,4.2)]) $\mathcal{N} := \mathrm{Cent}_{\mathcal{G}}(\mathcal{G}_1)$ is a normal subgroup of $\mathcal{G}$, which coincides with the kernel of the adjoint representation. Since

$$\mathcal{N}_1 \subset \mathrm{Cent}(\mathcal{G}_1) = e_k$$

we conclude that $\mathcal{N}^0 = e_k$. Accordingly, $\mathcal{N}$ is a reduced, normal subgroup of $\mathcal{G}$, whence

$$(\mathcal{G}^0, \mathcal{N}) \subset \mathcal{G}^0 \cap \mathcal{N} = e_k.$$

Consequently, (1.1) implies $\mathcal{N} \subset \mathrm{Cent}_{\mathcal{G}_{\mathrm{red}}}(\mathcal{G}^0) = e_k$, as desired. □

A commutative, unipotent infinitesimal k-group $\mathcal{U}$ is called *V-uniserial* if the Verschiebung $V_{\mathcal{U}}$ (cf. [13, (IV,§3,no.4)]) induces an exact sequence

$$\mathcal{U}^{(1)} \xrightarrow{V_{\mathcal{U}}} \mathcal{U} \longrightarrow \alpha_p \longrightarrow e_k.$$

According to [23, (2.6)] the V-uniserial groups are precisely the unipotent infinitesimal groups of finite representation type.

Theorem 5.1.4. *Let $\mathcal{G}$ be a finite algebraic group of odd characteristic and such that $\mathcal{B}_0(\mathcal{G})$ is tame. If $\mathcal{N} \lhd \mathcal{G}$ is a normal subgroup containing $\mathcal{G}^0$, then $\mathcal{B}_0(\mathcal{N})$ is tame.*

Proof. By assumption, we have $\mathcal{N} = \mathcal{G}^0 \rtimes \mathcal{N}_{\mathrm{red}}$, so that, writing $G := \mathcal{G}(k)$, $N := \mathcal{N}(k)$ and $\Lambda := H(\mathcal{G}^0)$, we obtain

$$H(\mathcal{G}) = \Lambda[G] \;\; ; \;\; H(\mathcal{N}) = \Lambda[N].$$

We may now apply (5.1.2) to see that the block $\mathcal{B}_0(\mathcal{N})$ is tame or representation-finite.

We first consider the case $\mathcal{N} = \mathcal{G}^0$, and suppose that $\mathcal{G}$ is a group of minimal order subject to $\mathcal{B}_0(\mathcal{G})$ being tame and $\mathcal{B}_0(\mathcal{G}^0)$ being representation-finite. Owing to (1.1) the group $\mathcal{G}' := \mathcal{G}/\mathcal{M} \cong (\mathcal{G}^0/\mathcal{M}) \rtimes \mathcal{G}_{\mathrm{red}}$ gives rise to principal blocks $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}')$ and $\mathcal{B}_0(\mathcal{G}'^0) \cong \mathcal{B}_0(\mathcal{G}^0)$, so that the minimality of $\mathrm{ord}(\mathcal{G})$ forces $\mathcal{M} = e_k$.

If p does not divide the order of G, then [23, (3.1)] implies that $\mathcal{B}_0(\mathcal{G})$ is representation-finite, a contradiction. As a result, the group G is not linearly reductive.

Thanks to [23, (2.7)] we have

$$\mathcal{G}^0 \cong \mathcal{U} \rtimes \mu_{p^m}$$

with μ_{p^m} acting faithfully on the V-uniserial group $\mathcal{U}$. The assumption $\mathcal{U} = e_k$ implies $\mathcal{G}^0 = \mu_{p^m} = e_k$. Thus, $\mathcal{B}_0(\mathcal{G}) = \mathcal{B}_0(G)$ is a tame principal block of a finite group of odd characteristic, a contradiction (cf. [9]). Since $\mathcal{U}' := V_{\mathcal{U}}(\mathcal{U}^{(p)})$ is a characteristic subgroup of $\mathcal{U}$ (cf. [13, (IV,§3,4.6,4.11)]) such that $\mathcal{U}/\mathcal{U}' \cong \alpha_p$, the minimality of $\mathrm{ord}(\mathcal{G})$ in conjunction with [23, (3.1)] implies $\mathcal{U}' = e_k$. Thus,

$$\mathcal{G}^0 \cong \alpha_p \rtimes \mu_{p^m}.$$

We now proceed in several steps.

(a) *Let $\mathcal{K} := \mathrm{Cent}_{\mathcal{G}_{\mathrm{red}}}(\mathcal{G}^0)$. Then $\mathcal{K} \neq e_k$ and $\mathcal{G}_{\mathrm{red}}/\mathcal{K}$ is linearly reductive.*

Suppose that $\mathcal{K} = e_k$. Since μ_{p^m} acts faithfully on α_p, the center of $\mathcal{G}_1$ is trivial and (5.1.3) furnishes an embedding

$$\mathcal{G} \hookrightarrow \mathcal{AUT}(\mathrm{Lie}(\mathcal{G})).$$

Observe that $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ is the two-dimensional non-abelian restricted Lie algebra. Its automorphism group $\mathrm{Aut}_p(\mathfrak{g})$ is readily seen to be isomorphic to $(k,+) \rtimes k^\times$ with $k^\times$ acting on $(k,+)$ via multiplication. As a result, $G \cong E \rtimes C$ is a semidirect product of a p-elementary abelian group E and a cyclic group C of order not divisible by p. Since $\mathcal{B}_0(G)$ is representation-finite (1.1), the Sylow-p-subgroups of G are cyclic (cf. [6, (6.3.5)]), so that $E \cong \mathbb{Z}/(p)$. Hence there exists a normal subgroup $\mathcal{P} \subset \mathcal{G}_{\rm red}$ with $\mathcal{P}(k) \cong \mathbb{Z}/(p)$. Owing to (5.1.2) the block $\mathcal{B}_0(\mathcal{G}^0 \rtimes \mathcal{P})$ is tame or representation-finite. We may now apply (4.1) to obtain a contradiction.

According to (1.1) the principal block $\mathcal{B}_0(\mathcal{G}^0 \rtimes (\mathcal{G}_{\rm red}/\mathcal{K}))$ of the factor group $\mathcal{G}/\mathcal{K}$ is tame or representation-finite. Since $\mathcal{K} \neq e_k$ the choice of $\mathcal{G}$ implies the validity of the latter alternative. The linear reductivity of $\mathcal{G}_{\rm red}/\mathcal{K}$ now follows directly from [23, (3.1)]. ◇

(b) *The group $\mathcal{G}_{\rm red}$ acts trivially on $\mathcal{G}^0$.*

Lemma 5.1.2, applied to the normal subgroup $(\mathcal{G}^0 \rtimes \mathcal{K}) \lhd \mathcal{G}$, ensures that $\mathcal{B}_0(\mathcal{G}^0 \rtimes \mathcal{K})$ is tame or representation-finite. Assume that $\mathcal{G}_{\rm red} \neq \mathcal{K}$. The minimality of $\mathrm{ord}(\mathcal{G})$ in conjunction with [23, (3.1)] implies the linear reductivity of $\mathcal{K}$. By (a) we conclude that $\mathcal{G}_{\rm red}$ is also linearly reductive, a contradiction. ◇

In view of (b) we have $H(\mathcal{G}) \cong H(\mathcal{G}^0) \otimes_k k[G]$. Thanks to [23, (2.4)] the algebra $H(\mathcal{G}^0)$ is connected, so that $\mathcal{B}_0(\mathcal{G}) \cong H(\mathcal{G}^0) \otimes_k \mathcal{B}_0(G)$. Since $\mathcal{B}_0(\mathcal{G})$ is not representation-finite, the group G is not linearly reductive, and there exists a closed subgroup $e_k \neq \mathcal{P} \subset \mathcal{G}_{\rm red}$ such that $P := \mathcal{P}(k)$ is a Sylow-p-subgroup of G. Setting $H := PC_G(P)$, we invoke [1, (16.1)] to see that $\mathcal{B}_0(G)$ is a Brauer correspondent of $\mathcal{B}_0(H)$. In particular, $\mathcal{B}_0(H)$ is a direct summand of $\mathcal{B}_0(G)|_{H\times H}$. Thus, $H(\mathcal{G}^0) \otimes_k \mathcal{B}_0(H)$ is a two-sided direct summand of $\mathcal{B}_0(\mathcal{G})$. In view of Lemma 5.1.1 the algebra $H(\mathcal{G}^0) \otimes_k \mathcal{B}_0(H)$ is tame or representation-finite. As [23, (3.1)] rules out the latter alternative, the minimality of the order of $\mathcal{G}$ implies $G = H = PC_G(P)$. In particular, P is a normal subgroup of G and the foregoing arguments yield $G = P$. Replacing P by a central subgroup of order p, we finally arrive at $G \cong \mathbb{Z}/(p)$. Owing to (4.1) the algebra $\mathcal{B}_0(\mathcal{G})$ is wild, and we have reached a contradiction.

We conclude that $\mathcal{B}_0(\mathcal{G}^0)$ is tame. In view of [23, (3.1)] the block $\mathcal{B}_0(\mathcal{N})$ is not representation-finite, so that we also obtain the tameness of $\mathcal{B}_0(\mathcal{N})$. □

We end this subsection with the following application:

Corollary 5.1.5. *Let H be a cocommutative Hopf algebra of characteristic $p \geq 3$. Then H does not possess any tame, basic blocks.*

Proof. Let $\mathcal{G}$ be the finite algebraic k-group associated to H, so that $H \cong H(\mathcal{G})$. Suppose that $\mathcal{B} \subset H(\mathcal{G})$ is a basic, tame block. The "translation principle" (cf. [24, (2.6)]) then implies that the principal block $\mathcal{B}_0(\mathcal{G})$ also has these properties. In view of (1.1(3)) the same holds for $\mathcal{B}_0(\mathcal{G}/\mathcal{G}_{\rm lr})$, so that we may assume $\mathcal{G}_{\rm lr} = e_k$.

Owing to [62, (I.2.37)] the group $\mathcal{G}$ is trigonalizable. Hence $\mathcal{G}^0$ is solvable, and [24, (2.4)] shows that $\mathcal{B}_0(\mathcal{G}^0)$ is not tame. This, however, contradicts (5.1.4). □

5.2. **Ascent.** We continue by recording another subsidiary result concerning the representation type of smash products. Let H be a finite dimensional Hopf algebra, Λ an H-module algebra. By definition, Λ is a left H-module such that

(a) $h.(ab) = \sum_{(h)}(h_{(1)}.a)(h_{(2)}.b) \quad \forall\, h \in H,\ a, b \in \Lambda$ and

(b) $h.1 = \varepsilon(h)1 \quad \forall\, h \in H$.

The smash product $\Lambda\sharp H$ will be denoted $\Lambda[H]$, see [42, (4.1)] for more details. Recall that H is referred to as *cosemisimple* if its dual algebra H^* is semisimple.

Lemma 5.2.1. *Let H be a finite dimensional Hopf algebra, Λ be a finite dimensional, tame H-module algebra.*

(1) *If H is semisimple, then $\Lambda[H]$ is tame or representation-finite.*

(2) *If H is semisimple and cosemisimple, then $\Lambda[H]$ is tame.*

Proof. Recall that the left H-comodule H obtains the structure of a right H^*-module via

$$h.\psi := \sum_{(h)} \psi(h_{(1)})h_{(2)} \quad \forall\ h \in H,\ \psi \in H^*.$$

Suppose that H is cosemisimple. Since $k1$ is a right H^*-submodule of H we have a decomposition

$$H = k1 \oplus V$$

of left H-comodules (see [42, (1.6.4)]). It follows that

$$\Lambda[H] \cong \Lambda \oplus (\Lambda \otimes_k V)$$

is a decomposition of the Λ-bimodule $\Lambda[H]$.

(1) Letting H^* act canonically on H and trivially on Λ, the smash product $\Lambda[H]$ obtains the structure of an H^*-module algebra. Thanks to Van den Bergh's Theorem [61] (see also [42, p.167]) we have

$$\Lambda[H][H^*] \cong \Lambda \otimes_k \operatorname{End}_k(H),$$

so that $\Lambda[H][H^*]$ and Λ are Morita equivalent. Consequently, the smash product $\Lambda[H][H^*]$ is tame. Since H is semisimple, the Hopf algebra H^* is cosemisimple, and the above remarks imply that $\Lambda[H]$ is a direct summand of the $\Lambda[H]$-bimodule $\Lambda[H][H^*]$. The arguments of (5.1.1) now show that $\Lambda[H]$ is weakly tame, and hence tame or representation-finite (cf. [46, (I.2.3)]).

(2) If H is also cosemisimple, then Λ is a direct summand of the Λ-bimodule $\Lambda[H]$. Thus, if $\Lambda[H]$ is representation-finite, so is Λ (cf. [49, (1.3)]), a contradiction. Consequently, $\Lambda[H]$ is tame. $\square$

Remark. Let G be a finite group acting on an algebra Λ via algebra homomorphisms. Then Λ is a $k[G]$-module algebra, and the smash product $\Lambda[k[G]]$ is just the skew group algebra $\Lambda[G]$. Since $k[G]^* \cong k^{|G|}$, the Hopf algebra $k[G]$ is cosemisimple. If $p := \operatorname{char}(k)$ does not divide the order of G, then Maschke's Theorem shows that $k[G]$ is semisimple. In that case, $\Lambda[G]$ is tame whenever Λ has this property (cf. also [9, Prop.2]).

The foregoing remark provides a simple recipe for the construction of finite algebraic groups, whose Hopf algebras are tame. In view of [23, (3.1)] the principal blocks associated to these so-called tame groups are also tame.

Examples. Let $p \geq 3$.

(a) For a natural number $m \in \mathbb{N}$ we consider the finite algebraic groups

$$\mathcal{SC}_{(m)} := \mathrm{SL}(2)_1 T_{(m)} \quad \text{and} \quad \mathcal{SQ}_{(m)} := \mathrm{SL}(2)_1 N_{(m)}.$$

If $m = np^\ell$ with $(p,n) = 1$, then $(\mathcal{SC}_{(m)})^0 = \mathcal{SC}_{(p^r)} = (\mathcal{SQ}_{(m)})^0$ with $r := \max\{\ell, 1\}$, so that $\mathcal{M} = e_k$ in either case. Since $\mathcal{SC}_{(m)}(k) = T_{(n)}(k) \cong \mathbb{Z}/(n)$ and $\mathcal{SQ}_{(m)}(k) = T_{(n)}(k)\langle\omega\rangle \cong Q_n$, a consecutive application of [24, (5.5)] and Lemma 5.2.1 yields the tameness of the groups $\mathcal{SC}_{(m)}$ and $\mathcal{SQ}_{(m)}$.

(b) By the same token, the binary polyhedral groups $(\hat{T})_k$, $(\hat{O})_k$ and $(\hat{I})_k$ (cf. Section 3) define finite algebraic groups

$$\mathcal{S}\hat{T} := \mathrm{SL}(2)_1(\hat{T})_k \ \ , \ \ \mathcal{S}\hat{O} := \mathrm{SL}(2)_1(\hat{O})_k \ \text{ and } \ \mathcal{S}\hat{I} := \mathrm{SL}(2)_1(\hat{I})_k,$$

with $\mathcal{M} = e_k$ and such that $(\mathcal{G}/\mathcal{C}_\mathcal{G})(k) \cong A_4, S_4$, and A_5, respectively. By the above arguments these groups are tame.

6. Finite Algebraic Groups with Tame Principal Blocks

6.1. **Automorphisms of Tame Infinitesimal Groups.** *Throughout this section we assume that* $p \geq 3$. In view of (5.1.4) tameness of principal blocks is preserved under passage to the connected component of a finite algebraic group. By results of [21] the structure of tame infinitesimal groups is well understood, and we are led to the task of determining reduced subgroups of their automorphism groups.

We begin by considering the special case where the group has height 1. According to general theory (cf. [13, (II,§7,4.2)]), this amounts to studying finite-dimensional restricted Lie algebras. To a restricted Lie algebra $(\mathfrak{g}, [p])$ one associates its *restricted enveloping algebra*, which is defined to be the quotient

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\})$$

of the ordinary enveloping algebra $U(\mathfrak{g})$. We denote the principal block of the cocommutative Hopf algebra $U_0(\mathfrak{g})$ by $\mathcal{B}_0(\mathfrak{g})$.

If $(\mathfrak{g}, [p])$ is a restricted Lie algebra, then the maximal toral p-ideal of $\mathfrak{g}$ will be denoted $T(\mathfrak{g})$. This ideal is contained in the *center* $C(\mathfrak{g})$ of $\mathfrak{g}$ and has the property that the p-map possesses no non-trivial zeros on $T(\mathfrak{g})$. Given $n \in \mathbb{N}_0$, we let $\mathfrak{n}_n$ be the n-dimensional nil-cyclic Lie algebra, i.e.,

$$\mathfrak{n}_n := \bigoplus_{i=0}^{n-1} kx^{[p]^i} \quad ; \quad x^{[p]^{n-1}} \neq 0 = x^{[p]^n}.$$

For a p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow \mathfrak{n}_n$ we consider the central extension $\mathfrak{sl}(2)^n_\psi := \mathfrak{sl}(2) \oplus \mathfrak{n}_n$, whose bracket and p-map are given by

$$[(x,c),(y,d)] = ([x,y],0) \quad \text{and} \quad (x,c)^{[p]} = (x^{[p]}, \psi(x) + c^{[p]})$$

for all $x, y \in \mathfrak{sl}(2),\ c, d \in \mathfrak{n}_n$, respectively.

We let $\{e,h,f\}$ be the standard basis of $\mathfrak{sl}(2)$, and denote by $T \subset \mathrm{SL}(2)$ and $\mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k)) \subset \mathrm{SL}(2)(k)$ the standard maximal torus of diagonal matrices and the normalizer of its group of rational points, respectively.

Lemma 6.1.1. *Let* $(\mathfrak{g}, [p])$ *be a restricted Lie algebra such that* $\mathcal{B}_0(\mathfrak{g})$ *is tame and* $T(\mathfrak{g}) = (0)$. *Then the following statements hold:*

(1) *There exists an embedding* $\iota : \mathrm{Aut}_p(\mathfrak{g}) \hookrightarrow \mathrm{PSL}(2)(k)$.

(2) *If* $\mathfrak{g} \not\cong \mathfrak{sl}(2)$, *then* $\mathrm{im}\,\iota$ *is contained in the image of* $\mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k))$ *under the canonical projection* $\mathrm{SL}(2)(k) \longrightarrow \mathrm{PSL}(2)(k)$.

Proof. (1) Thanks to [20, (7.4)] there exists a p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow \mathfrak{n}_n$ with $\psi(h)$ generating the restricted Lie algebra $\mathfrak{n}_n$ and such that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2)^n_\psi$. By virtue of [21, (6.1)] every automorphism $g \in \mathrm{Aut}_p(\mathfrak{sl}(2)^n_\psi)$ is of the form

$$g(x,c) = (\eta(g)(x), \gamma(g)(c)),$$

where $\eta(g) \in \mathrm{Aut}_p(\mathfrak{sl}(2))$ and $\gamma(g) \in \mathrm{Aut}_p(\mathfrak{n}_n)$ are automorphisms satisfying

$$\psi = \gamma(g) \circ \psi \circ \eta(g)^{-1}.$$

In particular, we obtain a homomorphism

$$\mathrm{Aut}_p(\mathfrak{sl}(2)^n_\psi) \longrightarrow \mathrm{Aut}_p(\mathfrak{sl}(2)) \quad ; \quad g \mapsto \eta(g)$$

of groups. By the above, the assumption $\eta(g) = \mathrm{id}_{\mathfrak{sl}(2)}$ implies $\gamma(g) \circ \psi = \psi$. Since $\psi(h)$ generates the restricted Lie algebra $\mathfrak{n}_n$, we see that $\gamma(g) = \mathrm{id}_{\mathfrak{n}_n}$. Accordingly, we have an embedding

$$\mathrm{Aut}_p(\mathfrak{g}) \hookrightarrow \mathrm{Aut}_p(\mathfrak{sl}(2)) \cong \mathrm{PSL}(2)(k),$$

where the last isomorphism follows from [33, p.281ff].

(2) If $\mathfrak{sl}(2)^n_\psi \neq \mathfrak{sl}(2)$, then we consider the factor algebra $\mathfrak{sl}(2)^1_{\bar\psi}$ of $\mathfrak{sl}(2)^n_\psi$ which is given by $\mathfrak{n}_1 = \mathfrak{n}_n/(\mathfrak{n}_n)^{[p]}$ and the p-semilinear map $\bar\psi : \mathfrak{sl}(2) \longrightarrow \mathfrak{n}_1$ induced by ψ. Thanks to [20, (7.4)] we also have $\ker\bar\psi = ke \oplus kf$. Moreover, the homomorphism $\mathrm{Aut}_p(\mathfrak{g}) \hookrightarrow \mathrm{PSL}(2)(k)$ is readily seen to factor through $\mathrm{Aut}_p(\mathfrak{sl}(2)^1_{\bar\psi}) \longrightarrow \mathrm{PSL}(2)(k)$. If $g \in \mathrm{Aut}_p(\mathfrak{sl}(2)^1_{\bar\psi})$, then $\bar\gamma(g)\circ\bar\psi\circ\eta(g)^{-1} = \bar\psi$ implies that $\eta(g)$ leaves $\ker\bar\psi$ invariant. Since

$$kh = [\ker\bar\psi, \ker\bar\psi]$$

the automorphism $\eta(g)$ stabilizes the subspace kh. Consequently, a pre-image of $\eta(g)$ under $\mathrm{SL}(2)(k) \longrightarrow \mathrm{Aut}_p(\mathfrak{sl}(2))$ enjoys the same property and is thus contained in $\mathrm{Stab}_{\mathrm{SL}(2)(k)}(kh) = \mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k))$. □

We now turn to automorphism groups of tame infinitesimal groups of height ≥ 2. Given a finite algebraic k-group $\mathcal{G}$, we let $X(\mathcal{G})$ be its *character group*. By definition, $X(\mathcal{G})$ is the set of algebra homomorphisms $H(\mathcal{G}) \longrightarrow k$ endowed with the convolution product

$$(\lambda * \mu)(h) = \sum_{(h)} \lambda(h_{(1)})\mu(h_{(2)}) \quad \forall\ h \in H(\mathcal{G}).$$

When convenient, we will identify $X(\mathcal{G})$ with $\mathrm{Hom}(\mathcal{G}, \mu_k)$, the group of homomorphisms from $\mathcal{G}$ to μ_k. If $\mathcal{N} \lhd \mathcal{G}$ is a normal subgroup, then $X(\mathcal{G}/\mathcal{N})$ can be identified with the subgroup of $X(\mathcal{G})$ consisting of those algebra homomorphisms that annihilate the augmentation ideal $H(\mathcal{N})^\dagger$ of $H(\mathcal{N})$.

As before, we let $T \subset \mathrm{SL}(2)$ be the standard torus of diagonal matrices and consider, for $r \geq 1$, the infinitesimal group

$$\mathcal{SC}_{(p^r)} = \mathrm{SL}(2)_1 T_r \subset \mathrm{SL}(2).$$

According to [24, (5.5)] the groups $\mathcal{SC}_{(p^r)}$, which were denoted $\mathcal{Q}_{[r]}$ in [24, 21], are precisely the tame semisimple infinitesimal groups.

In our next result we shall use the following notational convention: If $\mathcal{G}$ is a group scheme and R is a commutative k-algebra, then the image of a rational point $g \in \mathcal{G}(k)$ under the canonical homomorphism $\mathcal{G}(k) \longrightarrow \mathcal{G}(R)$ will be denoted g_R. In this fashion every element g belonging to $\mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k))$ induces, via conjugation, an automorphism $\iota_g \in \mathrm{Aut}(\mathcal{SC}_{(p^r)})$.

Lemma 6.1.2. *Suppose that $r \geq 2$. Then the homomorphism*

$$\iota : \mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k)) \longrightarrow \mathrm{Aut}(\mathcal{SC}_{(p^r)}) \ \ ; \ \ g \mapsto \iota_g$$

is surjective.

Proof. We denote the two-dimensional standard $\mathcal{SC}_{(p^r)}$-module by V. Let $\varphi \in \mathrm{Aut}(\mathcal{SC}_{(p^r)})$ be an automorphism. We consider the module V^φ, whose underlying k-space is V and whose action is twisted by φ (cf. [35, (I.2.15)]). By the results of [24, §5] there exists a character $\lambda \in X(\mathcal{SC}_{(p^r)}/(\mathcal{SC}_{(p^r)})_1)$ such that $V^\varphi \cong V \otimes_k k_\lambda$. Hence there is $g' \in \mathrm{Mat}_2(k)$ such that

$$g'_R\varphi_R(q)v = q\lambda(q)g'_Rv \qquad \forall\ q \in \mathcal{SC}_{(p^r)}(R),\ v \in V\otimes_k R,\ R \in M_k,$$

where M_k is the category of commutative k-algebras. Consequently, we can find $g \in \mathrm{SL}(2)(k)$ such that

$$\varphi_R(q) = \lambda(q)g_Rqg_R^{-1} \qquad \forall\ q \in \mathcal{SC}_{(p^r)}(R),\ R \in M_k.$$

As $\mathcal{SC}_{(p^r)}/(\mathcal{SC}_{(p^r)})_1 \cong \mu_{p^{r-1}}$ we have

$$\lambda(q)^{p^{r-1}} = 1 = \det(\varphi_R(q)) = \lambda(q)^2$$

which, by $(2, p^{r-1}) = 1$, implies $\lambda(q) = 1$. It thus remains to show that $g \in \mathrm{SL}(2)(k)$ belongs to the normalizer $\mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k))$.

Let

$$F : \mathrm{SL}(2) \longrightarrow \mathrm{SL}(2) \;\; ; \;\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$$

be the Frobenius homomorphism of SL(2). Since

$$(F \circ \varphi)_R(q) = F_R(g_R)F_R(q)F_R(g_R^{-1}) \qquad \forall\, q \in \mathcal{SC}_{(p^r)}(R)$$

we see that the element $h := F_k(g) \in \mathrm{SL}(2)(k)$ satisfies

$$h_R F_R(q) h_R^{-1} \in T_{r-1}(R) \qquad \forall\, q \in \mathcal{SC}_{(p^r)}(R),\ R \in M_k.$$

Suppose that $h \notin T(k)$. Direct computation, using $p \neq 2$ and $r \geq 2$, yields

$$h = \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix}$$

for a suitably chosen $\beta \in k$. Hence there exists $\alpha \in k$ such that

$$g = \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus, $g \in \mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k))$, as desired. □

Having dealt with the special case $\mathcal{G} = \mathcal{SC}_{(p^r)}$, we now turn to infinitesimal groups with tame principal block and trivial multiplicative center. We shall see that the finite groups relevant for our purposes can be embedded into the automorphism group of $\mathcal{SC}_{(p^r)}$.

Lemma 6.1.3. *Let $\mathcal{G}$ be a finite algebraic k-group, $G \subset \mathrm{Aut}(\mathcal{G})$ a finite subgroup of order not divisible by p, whose elements fix a unipotent normal subgroup $\mathcal{U} \lhd \mathcal{G}$. Then the canonical quotient map $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{U}$ induces an embedding $G \hookrightarrow \mathrm{Aut}(\mathcal{G}/\mathcal{U})$.*

Proof. Let $\pi : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{U}$ be the canonical quotient map. By assumption, every element $\varphi \in G$ gives rise to an element $\Gamma(\varphi) \in \mathrm{Aut}(\mathcal{G}/\mathcal{U})$ such that

$$\Gamma(\varphi) \circ \pi = \pi \circ \varphi.$$

Let φ be an element of the kernel of the resulting homomorphism $\Gamma : G \longrightarrow \mathrm{Aut}(\mathcal{G}/\mathcal{U})$. Then

$$\sigma : \mathcal{G} \longrightarrow \mathcal{U} \;\; ; \;\; g \mapsto \varphi(g)g^{-1}$$

is a morphism of schemes such that

$$\varphi(g) = \sigma(g)g \quad \forall\, g \in \mathcal{G}.$$

As $\varphi|_{\mathcal{U}} = \mathrm{id}_{\mathcal{U}}$ we obtain

$$\varphi^n(g) = \sigma(g)^n g \quad \forall\, g \in \mathcal{G},\, n \in \mathbb{N}.$$

Since $\mathcal{U}$ is unipotent, it can be embedded into a group of strictly upper triangular matrices (cf. [13, (IV,§2,2.5)]). Hence there exists ℓ with $\sigma(g)^{p^\ell} = 1$ for every $g \in \mathcal{G}$. Consequently, $\varphi^{p^\ell} = \mathrm{id}_{\mathcal{G}}$, and as the order of φ is prime to p, it follows that $\varphi = \mathrm{id}_{\mathcal{G}}$, as desired. □

We let $\mathcal{W}_m$ be the commutative unipotent group of *Witt vectors of length m* (cf. [13, (V,§1,1.6)]). The *center* of an infinitesimal group $\mathcal{G}$ will be denoted $\mathrm{Cent}(\mathcal{G})$.

Proposition 6.1.4. *Suppose that $\mathcal{G}$ is an infinitesimal group with trivial multiplicative center and tame principal block. Let $G \subset \mathrm{Aut}(\mathcal{G})$ be a finite subgroup such that $p \not| \operatorname{ord}(G)$. Then there exists an embedding $G \hookrightarrow \mathrm{Aut}(\mathcal{SC}_{(p^r)})$, where $r := \mathrm{ht}(\mathcal{G}/\mathrm{Cent}(\mathcal{G})$.*

Proof. According to [21, (3.4)] there exist $r \geq 1$, $m \geq 0$ and an exact sequence

$$e_k \longrightarrow (\mathcal{W}_m)_1 \longrightarrow \mathcal{G} \xrightarrow{\pi} \mathcal{SC}_{(p^r)} \longrightarrow e_k,$$

with $(\mathcal{W}_m)_1 = \mathrm{Cent}(\mathcal{G})$. Thus, $(\mathcal{W}_m)_1$ is invariant under $\mathrm{Aut}(\mathcal{G})$, so that every $\varphi \in \mathrm{Aut}(\mathcal{G})$ gives rise to a unique $\Gamma(\varphi) \in \mathrm{Aut}(\mathcal{SC}_{(p^r)})$ such that

$$\Gamma(\varphi) \circ \pi = \pi \circ \varphi.$$

Consider the resulting homomorphism

$$\Gamma : G \longrightarrow \mathrm{Aut}(\mathcal{SC}_{(p^r)}),$$

and suppose that $\varphi \in \ker \Gamma$. Then $\varphi|_{\mathcal{G}_1}$ is an automorphism of $\mathcal{G}_1$ which induces the identity on $(\mathcal{SC}_{(p^r)})_1 = \mathrm{SL}(2)_1$. By virtue of [21, (3.4(1))], [13, (II,§7,4.3)] and the proof of (6.1.1) we thus have $\varphi|_{\mathcal{G}_1} = \mathrm{id}_{\mathcal{G}_1}$. Consequently, (6.1.3) applies to $H := \ker \Gamma$ and we conclude that $H = \{1\}$. □

6.2. **Linear Reductivity of Reduced Groups.** If $\mathcal{G}$ is a finite algebraic group with principal block of finite representation type and such that $\mathcal{B}_0(\mathcal{G}^0)$ is not simple, then [23, (3.1)] implies the linear reductivity of $\mathcal{G}_{\mathrm{red}}$. The example of the group scheme $\alpha_{2,1} \subset \alpha_k$ given by

$$\alpha_{2,1}(R) := \{x \in R \; ; \; x^{p^2} = x^p\}$$

for every $R \in M_k$ shows that for tame groups of characteristic $p = 2$ the analogous statement fails. We shall see in this subsection that the case $p = 2$ is special in this regard.

Let $\mathcal{G}$ be a finite algebraic group, M a $\mathcal{G}$-module. The *complexity* $\mathrm{cx}_{\mathcal{G}}(M)$ of M is the rate of growth of a minimal projective resolution $(P_n)_{n \geq 0}$ of M, i.e.,

$$\mathrm{cx}_{\mathcal{G}}(M) := \min\{c \in \mathbb{N}_0 \cup \{\infty\} \; ; \; \exists \, \lambda > 0 \text{ such that } \dim_k P_n \leq \lambda n^{c-1} \quad \forall \, n \geq 1\}.$$

We put $\mathrm{cx}_{\mathcal{G}} := \mathrm{cx}_{\mathcal{G}}(k)$ and observe that [44, (2.6)] implies $\mathrm{cx}_{\mathcal{H}} \leq \mathrm{cx}_{\mathcal{G}}$ for every closed subgroup $\mathcal{H} \subset \mathcal{G}$. The reader is referred to [7, §5] for further details concerning this notion.

Proposition 6.2.1. *Let $\mathcal{G}$ be a finite algebraic k-group of characteristic $p \geq 3$ such that $\mathcal{B}_0(\mathcal{G})$ is tame. Then $p \not| \operatorname{ord}(\mathcal{G}(k))$.*

Proof. We shall verify our result in a series of steps, assuming $\mathcal{G}$ to be a counter-example of minimal order. Since $\mathcal{B}_0(\mathcal{G})$ is tame, the principal block of the factor group $\mathcal{G}_{\mathrm{red}}$ is tame or representation-finite. As $H(\mathcal{G}_{\mathrm{red}}) \cong k[G]$ is the group algebra of the finite $G := \mathcal{G}(k)$ and $p \neq 2$, the results of [9] ensure that the principal block of $k[G]$ has finite representation type. In particular, the Sylow-p-subgroups of G are cyclic (cf. [6, (6.3.5)]).

(a) *We have an embedding $\mathcal{G} \hookrightarrow \mathrm{PSL}(2)$ with $\mathcal{G}^0 \cong \mathrm{SL}(2)_1$.*

Let $\mathcal{R}(\mathcal{G}^0)$ be the solvable radical of the infinitesimal group $\mathcal{G}^0$. Since $g\mathcal{R}(\mathcal{G}^0)g^{-1} = \mathcal{R}(\mathcal{G}^0)$ for every $g \in \mathcal{G}(k)$, an application of [35, (I.2.6(8))] shows that the reduced group $\mathcal{G}_{\mathrm{red}}$ normalizes $\mathcal{R}(\mathcal{G}^0)$. Consequently, $\mathcal{R}(\mathcal{G}^0)$ is a normal subgroup of $\mathcal{G}$. If $\mathcal{R}(\mathcal{G}^0) \neq e_k$, then the minimality condition implies that the principal block of the group $\mathcal{G}/\mathcal{R}(\mathcal{G}^0) \cong (\mathcal{G}^0/\mathcal{R}(\mathcal{G}^0)) \rtimes \mathcal{G}_{\mathrm{red}}$ is representation-finite. Thus, a consecutive application of [23, (3.1)] and [23, (2.2)] yields the solvability of $\mathcal{G}^0/\mathcal{R}(\mathcal{G}^0)$. Consequently, $\mathcal{G}^0 \cong \mathcal{R}(\mathcal{G}^0)$ is solvable, and by [24, (2.4)] the principal block $\mathcal{B}_0(\mathcal{G}^0)$ is not tame. Now (5.1.4) gives a contradiction.

As a result, $\mathcal{R}(\mathcal{G}^0) = e_k$ and $\mathcal{G}^0$ is a semisimple, infinitesimal group with tame principal block (cf. (5.1.4)). Thanks to [24, (5.4)]

$$\mathcal{G}^0 \cong \mathrm{SL}(2)_1 T_r = \mathcal{SC}_{(p^r)}$$

is a product of the first Frobenius kernel of SL(2) and the r-th Frobenius kernel of the standard maximal torus $T \subset \mathrm{SL}(2)$ for some $r \geq 1$.

If $r \geq 2$, then (6.1.2) shows that the Sylow-p-subgroups of G act trivially on $\mathcal{G}^0$. Hence there exists a closed subgroup $\mathcal{P} \subset \mathcal{G}_{\mathrm{red}}$ such that $\mathcal{P} \cong (\mathbb{Z}/(p))_k$ and with $\mathcal{G}^0 \times \mathcal{P}$ being a closed subgroup of $\mathcal{G}$. By the Künneth Formula we have $\mathrm{cx}_{\mathcal{G}^0\times\mathcal{P}} = \mathrm{cx}_{\mathcal{G}^0} + \mathrm{cx}_{\mathcal{P}} = 3$, so that Rickard's Theorem [50, Thm.2] implies the wildness of $\mathcal{B}_0(\mathcal{G})$.

Consequently, $\mathcal{G}^0 \cong \mathrm{SL}(2)_1$, and the conjugation action defines a homomorphism

$$\mathcal{G} \longrightarrow \mathcal{AUT}(\mathrm{SL}(2)_1) \cong \mathrm{PSL}(2).$$

Let $\mathcal{C} := \mathrm{Cent}_{\mathcal{G}}(\mathrm{SL}(2)_1)$ be the kernel of this action. Then $\mathcal{C}^0 \subset \mathcal{G}^0$ is a normal subgroup of $\mathrm{SL}(2)_1$. As the latter group has trivial center, we obtain $\mathcal{C}^0 = e_k$, so that $\mathcal{C} = \mathcal{C}_{\mathcal{G}}$ is reduced. If $p|\mathrm{ord}(\mathcal{C}(k))$, then the above arguments provide a subgroup of $\mathcal{G}$ of complexity 3, a contradiction. Thus, $\mathcal{C} = O_{p'}(\mathcal{C}_{\mathcal{G}}) = \mathcal{G}_{\mathrm{lr}}$, so that

$$\mathcal{B}_0(\mathcal{G}/\mathcal{C}) \cong \mathcal{B}_0(\mathcal{G})$$

is tame (1.1). Since $p|\mathrm{ord}((\mathcal{G}/\mathcal{C})(k))$ the minimality condition yields $\mathcal{C} = e_k$. Consequently, the conjugation action induces an embedding $\mathcal{G} \hookrightarrow \mathrm{PSL}(2)$ that sends $\mathcal{G}^0$ onto $\mathrm{PSL}(2)_1 \cong \mathrm{SL}(2)_1$. ◇

(b) *The Sylow-p-subgroups of $G = \mathcal{G}(k)$ are isomorphic to $\mathbb{Z}/(p)$.*

Let $P \subset G$ be a Sylow-p-subgroup. We let $\hat{G} \subset \mathrm{SL}(2)(k)$ denote the double cover of $G \subset \mathrm{PSL}(2)(k)$ and consider the pre-image $\hat{P} \subset \mathrm{SL}(2)(k)$ of P under the canonical projection. Then $\hat{P} \cong \mathbb{Z}/(2)\times P$, and P is conjugate to a subgroup of strictly upper triangular matrices. Hence P is p-elementary abelian. On the other hand, P is cyclic, whence $P \cong \mathbb{Z}/(p)$. ◇

(c) *If $N \lhd G$ is a proper normal subgroup, then $N \subset O_{p'}(G)$.*

By general theory, there exists a proper normal subgroup $\mathcal{N} \lhd \mathcal{G}$ containing $\mathcal{G}^0$ and such that $N = \mathcal{N}(k)$. Thanks to (5.1.4) the principal block $\mathcal{B}_0(\mathcal{N})$ is tame, so that the minimality of $\mathcal{G}$ forces $N \subset O_{p'}(G)$. ◇

Let $\hat{\mathcal{G}} \subset \mathrm{SL}(2)$ be a covering of $\mathcal{G} \subset \mathrm{PSL}(2)$. We now apply Dickson's Theorem [59, (III.6.17)] to the subgroup $\hat{G} := \hat{\mathcal{G}}(k) \subset \mathrm{SL}(2)(k)$. Thanks to (b) and (c) only the following three cases arise:

(1) $\hat{G} \cong \mathbb{Z}/(p)$, $\mathbb{Z}/(p) \times \mathbb{Z}/(2)$, or
(2) $\hat{G} \cong \mathrm{SL}(2,p)$, or
(3) $p = 3$ and $\hat{G} \cong \mathrm{SL}(2,5)$.

Since a subgroup of $\mathrm{SL}(2)(k)$ of order p is conjugate to $\mathcal{L}(k)$, alternative (1) yields that $\mathcal{L}$ is a closed normal subgroup of a group that is conjugate to $\hat{\mathcal{G}}$. According to (5.1.4) this implies the tameness of $\mathcal{B}_0(\mathcal{L})$, which contradicts (4.2). If $\hat{G} \cong \mathrm{SL}(2,p)$, then $L(1)$ is the only two-dimensional simple $\hat{G}$-module. Since $\hat{G}$ is not trigonalizable, the Frobenius twist $L(1)^{[1]}$ is also a simple $\hat{G}$-module, whence $L(1)|_{\hat{G}} \cong L(1)^{[1]}|_{\hat{G}}$. Lang's Theorem [57, (3.3.16)] then provides an element $g_0 \in \mathrm{SL}(2)(k)$ such that $g_0\hat{G}g_0^{-1} = \mathrm{SL}(2,p)$. It now follows that $g_0\hat{\mathcal{G}}g_0^{-1} \subset \mathrm{SL}(2)_{2,1}$. By equality of orders these two groups coincide, so that (4.3) ensures the wildness of

$$\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\hat{\mathcal{G}}) \cong \mathcal{B}_0(\mathrm{SL}(2)_{2,1}).$$

As this contradicts (5.1.4) it remains to consider the case where $p = 3$ and $\hat{G} \cong \mathrm{SL}(2,5)$, which can be disposed of by invoking (4.4). □

As a first consequence, we demonstrate that the representation type of a finite algebraic group depends only on the structure of its reduced group and its second Frobenius kernel (see [23, (2.7), (3.1)] for finite representation type).

Corollary 6.2.2. *Let $\mathcal{G}$ be a finite algebraic group of characteristic $p \geq 3$. Then $\mathcal{B}_0(\mathcal{G})$ is tame if and only if $\mathcal{B}_0(\mathcal{G}_1)$ is tame and $(\mathcal{G}_2/\mathcal{G}_1) \rtimes \mathcal{G}_{\mathrm{red}}$ is linearly reductive.*

Proof. Suppose that $\mathcal{B}_0(\mathcal{G})$ is tame. According to (5.1.4) the principal block $\mathcal{B}_0(\mathcal{G}^0)$ is tame. A consecutive application of [21, (5.4)] and [21, (5.3)] now shows that $\mathcal{B}_0(\mathcal{G}_1)$ is tame and $\mathcal{G}_2/\mathcal{G}_1$ is multiplicative. In virtue of (6.2.1) the group $\mathcal{G}_{\mathrm{red}}$ is linearly reductive, and our assertion now follows from Nagata's Theorem [13, (IV,§3,3.6)].

For the converse we observe that p does not divide the order of $\mathcal{G}(k)$, so that (1.1(4)) and (5.2.1) reduce our task to verifying the tameness of $\mathcal{B}_0(\mathcal{G}^0)$. In view of [21, (5.4)] it suffices to show this for $\mathcal{B}_0(\mathcal{G}_2)$. By Nagata's Theorem, the group $\mathcal{G}_2/\mathcal{G}_1$ is multiplicative, so that [21, (5.3)] yields the desired result. □

7. Classification

In this section we shall prove and elaborate on the main results announced in the Introduction. Both, the group structure of finite algebraic groups with tame principal blocks as well as the bound quiver presentations of the blocks themselves rest on our earlier determination of the binary polyhedral group schemes.

7.1. **Group Structure.** In the following, we let $C(G)$ be the *center* of an abstract group G. Recall that $\mathcal{M}$ and $\mathcal{C}_{\mathcal{G}}$ denote the multiplicative center and the centralizer $\mathrm{Cent}_{\mathcal{G}_{\mathrm{red}}}(\mathcal{G}^0/\mathcal{M})$ associated to a finite algebraic group $\mathcal{G}$, respectively.

With (6.2.1) in hand, we are now in a position to establish the following recognition criterion.

Proposition 7.1.1. *Let $\mathcal{G}$ be a finite algebraic group of characteristic $p \geq 3$. Then the following statements are equivalent:*

(1) *The principal block $\mathcal{B}_0(\mathcal{G})$ is tame.*

(2) *The principal block $\mathcal{B}_0(\mathcal{G}^0)$ is tame, $p \not| \mathrm{ord}(\mathcal{G}(k))$, and the finite group $\mathcal{G}(k)/C_{\mathcal{G}}(k)$ is either cyclic, dihedral, or isomorphic to A_4, S_4, or A_5, with the latter three groups occurring only if $\mathcal{G}^0/\mathcal{M} \cong \mathrm{SL}(2)_1$.*

In either case, we have $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0((\mathcal{G}^0/\mathcal{M}) \rtimes (\mathcal{G}_{\mathrm{red}}/\mathcal{C}_{\mathcal{G}}))$.

Proof. (1) ⇒ (2) Owing to (6.2.1) the order of the finite group $\mathcal{G}(k)$ is not divisible by p, and the tameness of $\mathcal{B}_0(\mathcal{G}^0)$ follows directly from (5.1.4). We apply (1.1(1),(2)) to the normal subgroup $\mathcal{M}\mathcal{C}_{\mathcal{G}}$ of $\mathcal{G}$ to see that the principal block of the group

$$(\mathcal{G}^0/\mathcal{M}) \rtimes (\mathcal{G}_{\mathrm{red}}/\mathcal{C}_{\mathcal{G}})$$

is tame or representation-finite. In the latter case, [23, (3.1)] and [23, (2.7)] show that $\mathcal{B}(\mathcal{G}^0)$ is representation-finite, a contradiction. Thanks to [13, (III,§1,1.15)] we have $(\mathcal{G}_{\mathrm{red}}/\mathcal{C}_{\mathcal{G}})(k) \cong \mathcal{G}(k)/C_{\mathcal{G}}(k)$, so that it suffices to verify (2) under the additional assumption $\mathcal{M} = e_k = \mathcal{C}_{\mathcal{G}}$. In particular, the finite group $\mathcal{G}(k)$ may be considered a subgroup of the automorphism group $\mathrm{Aut}(\mathcal{G}^0)$.

As $\mathcal{B}_0(\mathcal{G}^0)$ is tame, we have $\mathcal{G}^0 \neq e_k$, whence $\mathrm{ht}(\mathcal{G}^0) \geq 1$. Suppose first that $\mathrm{ht}(\mathcal{G}^0) \geq 2$. A consecutive application of (6.1.4) and (6.1.2) provides a finite subgroup $G \subset \mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k))$ containing $Z := \{I_2, -I_2\}$ such that $G/Z \cong \mathcal{G}(k)$. Since $\mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k)) = T(k)\langle\omega\rangle$ (see Section 3), it follows that $G \cap T(k)$ is a normal subgroup of G of index ≤ 2. Suppose that G is not cyclic.

Then there exists an element $g \in G \setminus G \cap T(k)$. If $G \cap T(k) = Z$, then G' has order 4 and is thus cyclic, a contradiction. Alternatively, properties (†) and (‡) from Section 3 show that, upon conjugation by some element of $T(k)$, we may assume $\omega \in G$. Consequently, $G = (G \cap T(k))\langle\omega\rangle$ is generalized quaternion. It now follows that $\mathcal{G}(k)$ is either cyclic or dihedral.

If $\mathrm{ht}(\mathcal{G}^0) = 1$, then passage to the Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathcal{G}^0)$ (cf. [13, (II,§7,n^{0}4)]) yields an embedding $\mathcal{G}(k) \hookrightarrow \mathrm{Aut}_p(\mathfrak{g})$ which, by virtue of (6.1.1), gives rise to an injective homomorphism $\varrho : \mathcal{G}_{\mathrm{red}} \hookrightarrow \mathrm{PSL}(2)$. Let $\mathcal{G}' \subset \mathrm{SL}(2)$ be the inverse image of $\varrho(\mathcal{G}_{\mathrm{red}})$ under the canonical quotient map $\mathrm{SL}(2) \longrightarrow \mathrm{PSL}(2)$. Then $\mathcal{G}'$ satisfies the hypotheses of (3.3), so that $\mathcal{G}'(k)$ is cyclic, generalized quaternion, or $\mathcal{G}'(k)/C(\mathcal{G}'(k)) \cong A_4, S_4, A_5$ (cf. [15, (26.1)]). If $\mathcal{G}'(k)$ is cyclic, then $\mathcal{G}(k) \cong \mathcal{G}'(k)/C(\mathrm{SL}(2)(k))$ enjoys the same property. Alternatively, we have $C(\mathcal{G}'(k)) = C(\mathrm{SL}(2)(k)) = \{I_2, -I_2\}$, and passage to $\mathcal{G}(k) \cong \mathcal{G}'(k)/C(\mathcal{G}'(k))$ (cf. [13, (III,§1,1.15)]) shows that $\mathcal{G}(k)$ is dihedral or isomorphic to A_4, S_4 or A_5.

(2) ⇒ (1) According to (1.1(4)) and (5.2.1), $\mathcal{B}_0(\mathcal{G})$ is a block of the tame algebra $\mathcal{B}_0(\mathcal{G}^0)[\mathcal{G}(k)]$. Thus, it is tame or representation-finite, and (5.1.4) rules out the latter alternative.

Finally, the isomorphism $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0((\mathcal{G}^0/\mathcal{M}) \rtimes (\mathcal{G}_{\mathrm{red}}/\mathcal{C}_{\mathcal{G}}))$ is an immediate consequence of (1.1(3)) and (1.2). □

Given a finite closed subgroup scheme $\mathcal{G} \subset \mathrm{SL}(2)$, we consider the corresponding factor group $P\mathcal{G} := \mathcal{G}/(\mathcal{G}\cap\mathcal{Z})$, where $\mathcal{Z} := \mathrm{Cent}(\mathrm{SL}(2))$ denotes the center of $\mathrm{SL}(2)$. Since $\mathcal{Z}(R) = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \ ; \ a^2 = 1\}$ for every commutative k-algebra R, $\mathcal{Z}$ is a reduced group of order 2. Consequently, we either have $\mathcal{G} \cap \mathcal{Z} = e_k$ or $\mathcal{Z} \subset \mathcal{G}$. As $\mathcal{Z}$ is reduced, the latter case gives rise to $\mathcal{Z} \subset \mathcal{G}_{\mathrm{red}}$. Lemma 1.1 now shows that $P\mathcal{G} \cong \mathcal{G}$ or $P\mathcal{G} \cong \mathcal{G}^0 \rtimes (\mathcal{G}_{\mathrm{red}}/\mathcal{Z})$. As $\mathcal{Z}$ is linearly reductive, the arguments of (1.1(3)) imply $\mathcal{B}_0(P\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G})$.

We apply this to the groups defined in §5.2 and obtain tame finite algebraic groups

$$P\mathcal{SC}_{(m)},\ P\mathcal{SQ}_{(m)},\ P\mathcal{S}\hat{T},\ P\mathcal{S}\hat{O},\ \text{and}\ P\mathcal{S}\hat{I}$$

which will be referred to as *amalgamated polyhedral groups*. The following result provides an abstract characterization of these groups.

Proposition 7.1.2. *Let $\mathcal{G}$ be a finite algebraic k-group of characteristic $p \geq 3$ such that $\mathrm{Cent}_{\mathcal{G}}(\mathcal{G}^0) = e_k$. Then $\mathcal{B}_0(\mathcal{G})$ is tame if and only if $\mathcal{G}$ is isomorphic to an amalgamated polyhedral group.*

Proof. One direction being clear, we suppose that $\mathcal{B}_0(\mathcal{G})$ is tame. By our general assumption and (7.1.1), $\mathcal{G}^0$ is an infinitesimal group with trivial center and tame principal block. Thanks to [21, (3.4)] we thus obtain an isomorphism $\mathcal{G}^0 \cong \mathcal{SC}_{(p^r)}$ for a suitable $r \geq 1$, so that $\mathcal{G}_1 \cong \mathrm{SL}(2)_1$.

We consider the closed normal subgroup $\mathcal{H} := \mathrm{Cent}_{\mathcal{G}}(\mathcal{G}_1) \subset \mathcal{G}$. Since $\mathcal{H}_1 = \mathcal{G}_1 \cap \mathcal{H} = \mathrm{Cent}(\mathcal{G}_1) \cong \mathrm{Cent}(\mathrm{SL}(2)_1) = e_k$, we obtain $\mathcal{H}^0 = e_k$, so that $\mathcal{H} = \mathcal{H}_{\mathrm{red}}$ is reduced. As $\mathrm{Cent}_{\mathcal{G}}(\mathcal{G}^0) = e_k$, the conjugation action induces an embedding

$$\mathcal{H}(k) \hookrightarrow \mathrm{Aut}(\mathcal{SC}_{(p^r)}).$$

Let h be an element of $\mathcal{H}(k)$ and suppose that $r \geq 2$. According to (6.1.2) there exists an element $g \in \mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k))$ such that

$$hxh^{-1} = gxg^{-1} \quad \forall\ x \in \mathcal{SC}_{(p^r)}.$$

Consequently, $gxg^{-1} = x \ \ \forall\ x \in \mathrm{SL}(2)_1$, whence $g = \pm I_2$. This shows that $h \in \mathrm{Cent}_{\mathcal{G}}(\mathcal{G}^0)(k) = \{1\}$. As a result, $\mathcal{H}$ is the trivial group for every $r \geq 1$, and there results an embedding

$$\mathcal{G} \hookrightarrow \mathcal{AUT}(\mathrm{SL}(2)_1) \cong \mathrm{PSL}(2).$$

Let $\hat{\mathcal{G}}$ be the inverse image of $\mathcal{G}$ under the canonical quotient map $\mathrm{SL}(2) \longrightarrow \mathrm{PSL}(2)$, so that $\mathcal{G} \cong P\hat{\mathcal{G}}$. By our above observations the principal block $\mathcal{B}_0(\hat{\mathcal{G}})$ is tame. Owing to (6.2.1) $\hat{\mathcal{G}}_{\mathrm{red}} \subset \mathrm{SL}(2)$ is a linearly reductive group.

If $r = 1$, then (3.3) provides an element $g \in \mathrm{SL}(2)(k)$ such that $g\hat{\mathcal{G}}_{\mathrm{red}}g^{-1}$ is one of the reduced binary polyhedral groups listed there. As $\hat{\mathcal{G}} = \mathrm{SL}(2)_1\hat{\mathcal{G}}_{\mathrm{red}}$, it follows that $g\hat{\mathcal{G}}g^{-1}$ belongs to the list of groups given by the examples of §5.2. Accordingly, $\mathcal{G} \cong P\hat{\mathcal{G}}$ is an amalgamated polyhedral group.

Alternatively, $r \geq 2$ and Lemma 6.1.2 implies the inclusion $\hat{\mathcal{G}}(k) \subset \mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k))$. This readily yields $\hat{\mathcal{G}}_{\mathrm{red}}(k) = T_{(n)}(k),\, N_{(n)}(k)$ for some n with $(n, p) = 1$, whence $\hat{\mathcal{G}}_{\mathrm{red}} = T_{(n)},\, N_{(n)}$. As a result, we obtain isomorphisms $\hat{\mathcal{G}} \cong \mathcal{SC}_{(m)}, \mathcal{SQ}_{(m)}$ for $m = np^r$, as desired. □

Let $\mathcal{G}$ be a finite algebraic k-group with multiplicative center $\mathcal{M}$. Then

$$\mathcal{N} := \mathcal{N}(\mathcal{G}) := \mathrm{Cent}_{\mathcal{G}}(\mathcal{G}^0/\mathcal{M})$$

is a normal subgroup of $\mathcal{G}$. Thanks to (1.2)

$$\mathcal{N} = \mathrm{Cent}_{\mathcal{G}^0}(\mathcal{G}^0/\mathcal{M}) \rtimes \mathcal{C}_{\mathcal{G}}$$

contains the maximal normal linearly reductive subgroup $\mathcal{G}_{\mathrm{lr}}$ of $\mathcal{G}$.

Proposition 7.1.3. *Let $\mathcal{G}$ be a finite algebraic group of characteristic $p \geq 3$. If $\mathcal{B}_0(\mathcal{G})$ is tame, then $\mathcal{G}/\mathcal{N}$ is an amalgamated polyhedral group and there exists $\ell \geq 0$ such that $\mathcal{N} \cong (\mathcal{M} \times (\mathcal{W}_\ell)_1) \rtimes \mathcal{C}_{\mathcal{G}}$.*

Proof. According to (7.1.1) the principal block $\mathcal{B}_0(\mathcal{G}^0)$ is tame. Since $\mathcal{N}^0 = \mathrm{Cent}_{\mathcal{G}^0}(\mathcal{G}^0/\mathcal{M})$ is a solvable, normal subgroup of $\mathcal{G}^0$ containing $\mathrm{Cent}(\mathcal{G}^0)$, [21, (3.5)] provides $\ell \geq 0$ with $\mathcal{N}^0 = \mathrm{Cent}(\mathcal{G}^0) \cong \mathcal{M} \times (\mathcal{W}_\ell)_1$. Consequently, $\mathcal{N} \cong (\mathcal{M} \times (\mathcal{W}_\ell)_1) \rtimes \mathcal{C}_{\mathcal{G}}$.

Let $\mathcal{G}' := \mathcal{G}/\mathcal{N}$ and denote by $\pi : \mathcal{G} \longrightarrow \mathcal{G}'$ the canonical quotient map. Directly from (1.1) we obtain

$$\mathcal{G}' \cong (\mathcal{G}^0/\mathrm{Cent}(\mathcal{G}^0)) \rtimes (\mathcal{G}_{\mathrm{red}}/\mathcal{C}_{\mathcal{G}}).$$

Thanks to [21, (3.5)] this implies $\mathcal{M}(\mathcal{G}') = e_k$, so that $\mathcal{N}(\mathcal{G}') = \mathrm{Cent}_{\mathcal{G}'}(\mathcal{G}'^0)$. The inverse image $\mathcal{H} \subset \mathcal{G}$ of $\mathcal{N}(\mathcal{G}')$ under π is a normal subgroup of $\mathcal{G}$. Thus, $\mathcal{H}^0$ is a normal subgroup of $\mathcal{G}^0$, whose derived group $(\mathcal{H}^0, \mathcal{H}^0)$ is contained in $\mathrm{Cent}(\mathcal{G}^0)$. As a result, the group $\mathcal{H}^0$ is solvable and, by [21, (3.5)], therefore contained in $\mathrm{Cent}(\mathcal{G}^0) = \mathcal{N}^0$.

Let h be an element of $\mathcal{H}(k)$. Then h operates on $\mathcal{G}^0/\mathcal{M}$ via an automorphism in such a way, that the induced action on $\mathcal{G}'^0 \cong \mathcal{SC}_{(p^r)}$ (cf. [21, (3.5)]) is trivial. Proposition 6.1.4 shows that h operates trivially on $\mathcal{G}^0/\mathcal{M}$. As a result, $\mathcal{H}_{\mathrm{red}}$ is a reduced subgroup of $\mathcal{G}$ such that $\mathcal{H}_{\mathrm{red}}(k) \subset \mathcal{C}_{\mathcal{G}}(k)$. Consequently, $\mathcal{H}_{\mathrm{red}} \subset \mathcal{C}_{\mathcal{G}} \subset \mathcal{N}$.

As an upshot of our discussion above, we have $\mathcal{H} \subset \mathcal{N}$, whence $\mathrm{Cent}_{\mathcal{G}'}(\mathcal{G}'^0) = \mathcal{N}(\mathcal{G}') = e_k$. We may now apply (7.1.2) to see that $\mathcal{G}'$ is an amalgamated polyhedral group. □

Remarks. Suppose that $\mathcal{B}_0(\mathcal{G})$ is tame.

(1) In view of (7.1.3) and [23, (3.1),(2.7)] the normal subgroup $\mathcal{N} \lhd \mathcal{G}$ has finite representation type. Moreover, (7.1.1) implies that $\mathcal{N} = \mathcal{G}_{\mathrm{lr}}$ is linearly reductive if $\mathcal{G}_{\mathrm{red}}/\mathcal{C}_{\mathcal{G}} \cong A_4, S_4$ or A_5.

(2) Thanks to (7.1.3) the exact sequence

$$e_k \longrightarrow \mathcal{N}/\mathcal{G}_{\mathrm{lr}} \longrightarrow \mathcal{G}/\mathcal{G}_{\mathrm{lr}} \longrightarrow \mathcal{G}/\mathcal{N} \longrightarrow e_k$$

defines an extension of an amalgamated polyhedral group by the first Frobenius kernel of the group $\mathcal{W}_\ell$ of Witt vectors of length ℓ.

Given $\ell \geq 1$, we let $\mathfrak{n}_\ell := \bigoplus_{i=0}^{\ell-1} kv_0^{[p]^i}$; $v_0^{[p]^{\ell-1}} \neq 0 = v_0^{[p]^\ell}$ be the ℓ-dimensional nil-cyclic restricted Lie algebra. Using the standard basis $\{e, h, f\}$ of $\mathfrak{sl}(2)$, we define a p-semilinear map $\psi_s^\ell : \mathfrak{sl}(2) \longrightarrow \mathfrak{n}_\ell$ via

$$\psi_s^\ell(e) = 0 = \psi_s^\ell(f) \quad ; \quad \psi_s^\ell(h) = v_0.$$

The adjoint action of SL(2) on $\mathfrak{sl}(2)$ gives rise to a character $\chi : \mathrm{Nor}_{\mathrm{SL}(2)}(T) \longrightarrow \mu_k$ such that

$$\mathrm{Ad}(g)(h \otimes 1) = \chi(g)(h \otimes 1)$$

for every $g \in \mathrm{Nor}_{\mathrm{SL}(2)}(T)(R)$ and every commutative k-algebra R. We let $\gamma : \mathrm{Nor}_{\mathrm{SL}(2)}(T) \longrightarrow \mathcal{AUT}(\mathfrak{n}_n)$ be the homomorphism given by

$$\gamma(g)(v_0 \otimes 1) = \chi(g)(v_0 \otimes 1) \quad \forall\ g \in \mathrm{Nor}_{\mathrm{SL}(2)}(T)(R),\ R \in M_k.$$

Since $T \subset \ker \chi$, the maps χ and γ factor through to homomorphisms

$$\hat{\chi} : \mathrm{Nor}_{\mathrm{SL}(2)}(T)/\mathcal{Z} \longrightarrow \mu_k \ \text{ and } \ \eta : \mathrm{Nor}_{\mathrm{SL}(2)}(T)/\mathcal{Z} \longrightarrow \mathcal{AUT}(\mathfrak{n}_\ell),$$

respectively. By definition, the map $\psi_s^\ell : \mathfrak{sl}(2) \longrightarrow \mathfrak{n}_\ell$ is $\mathrm{Nor}_{\mathrm{SL}(2)}(T)$-equivariant, and the group $\mathrm{Nor}_{\mathrm{SL}(2)}(T)$ acts on the restricted Lie algebra $\mathfrak{sl}(2)_s^\ell := \mathfrak{sl}(2)_{\psi_s}$ (cf. §6.1) via automorphisms:

$$g.(x, v) := (\mathrm{Ad}(g)(x), \gamma(g)(v)).$$

Following [21, §6] we denote by $\mathrm{SL}(2)_1^\ell$ the infinitesimal group of height 1 corresponding to the restricted Lie algebra $\mathfrak{sl}(2)_s^\ell$. In view of [13, (II,§7,4.3)] the above action induces an operation

$$\mathrm{Nor}_{\mathrm{SL}(2)}(T) \times \mathrm{SL}(2)_1^\ell \longrightarrow \mathrm{SL}(2)_1^\ell \quad ; \quad (g, x) \mapsto g.x$$

of $\mathrm{Nor}_{\mathrm{SL}(2)}(T)$ on $\mathrm{SL}(2)_1^\ell$ via automorphisms. For $m \in \mathbb{N}$ we consider the algebraic groups

$$\mathrm{SL}(2)_1^\ell \rtimes T_{(m)} \ \text{ and } \ \mathrm{SL}(2)_1^\ell \rtimes N_{(m)},$$

that are defined via this action. The factor groups of these groups by their largest normal linearly reductive subgroups will be denoted

$$PS\mathcal{C}_{[\ell,m]} \ \text{ and } \ PS\mathcal{Q}_{[\ell,m]},$$

respectively. Let $\mathcal{P} = PS\mathcal{C}_{(m)}, PS\mathcal{Q}_{(m)}$ be an amalgamated polyhedral group, $r := \mathrm{ht}(\mathcal{P}^0)$. Then the r-th Frobenius kernel T_r of the standard maximal torus $T \subset \mathrm{SL}(2)$ is contained in $\mathcal{P}^0 \cong S\mathcal{C}_{(p^r)}$, and the reduced group $\mathcal{P}_{\mathrm{red}}$ acts on $T_r = T_{(p^r)}$ via conjugation. Accordingly, $T_r \rtimes \mathcal{P}_{\mathrm{red}}$ is a closed subgroup of $\mathcal{P}$ such that $\mathcal{P} = \mathrm{SL}(2)_1 \cdot (T_r \rtimes \mathcal{P}_{\mathrm{red}}) = \mathcal{P}_1 \cdot (T_r \rtimes \mathcal{P}_{\mathrm{red}})$.

Our goal in this subsection is to show that "most" finite algebraic groups with tame principal blocks are either amalgamated polyhedral groups, or of types $PS\mathcal{C}_{[\ell,m]}$, $PS\mathcal{Q}_{[\ell,m]}$. To that end we require the following generalization of [21, (6.3)]. As before, we let $\mathrm{ht}(\mathcal{G}) := \mathrm{ht}(\mathcal{G}^0)$ be the height of a finite algebraic group $\mathcal{G}$.

Lemma 7.1.4. *Let $e_k \longrightarrow (\mathcal{W}_\ell)_1 \longrightarrow \mathcal{G} \xrightarrow{\pi} \mathcal{P} \longrightarrow e_k$ be an exact sequence of finite algebraic groups with $\mathcal{P}$ being an amalgamated polyhedral group of type $PS\mathcal{C}_{(m)}$ or $PS\mathcal{Q}_{(m)}$. If $\mathcal{P} \not\cong \mathrm{SL}(2)_1$, then the following statements hold:*

(1) *Let $r := \mathrm{ht}(\mathcal{P})$. There exists a closed embedding $\sigma : T_r \rtimes \mathcal{P}_{\mathrm{red}} \hookrightarrow \mathcal{G}$ such $\pi \circ \sigma = \mathrm{id}_{T_r \rtimes \mathcal{P}_{\mathrm{red}}}$, and $\mathcal{G} = \mathcal{G}_1 \cdot \sigma(T_r \rtimes \mathcal{P}_{\mathrm{red}})$.*

(2) *Let σ be as in (1). The block $\mathcal{B}_0(\mathcal{G})$ is tame if and only if there exists an isomorphism $\Phi : \mathrm{SL}(2)_1^\ell \longrightarrow \mathcal{G}_1$ such that $\Phi(q.x) = \sigma(q)\Phi(x)\sigma(q)^{-1}$ for every $q \in T_r \rtimes \mathcal{P}_{\mathrm{red}}$ and $x \in \mathrm{SL}(2)_1^\ell$.*

Proof. (1) Since $(\mathcal{W}_\ell)_1$ is connected, we have $(\mathcal{W}_\ell)_1 \subset \mathcal{G}^0$ and (1.1) yields

$$\mathcal{P} \cong \mathcal{G}/(\mathcal{W}_\ell)_1 \cong (\mathcal{G}^0/(\mathcal{W}_\ell)_1) \rtimes \mathcal{G}_{\rm red}.$$

In particular, $\pi|_{\mathcal{G}_{\rm red}} : \mathcal{G}_{\rm red} \longrightarrow \mathcal{P}_{\rm red}$ is an isomorphism. Since $\mathrm{ht}(\mathcal{P}) = r$, we have $\mathcal{SC}_{(p^r)} = \mathcal{P}^0 \cong \mathcal{G}^0/(\mathcal{W}_\ell)_1$. As $\mathcal{SC}_{(p^r)}$ is semisimple and the automorphism scheme $\mathcal{AUT}((\mathcal{W}_\ell)_1)$ is solvable, our exact sequence defines a central extension

$$e_k \longrightarrow (\mathcal{W}_\ell)_1 \longrightarrow \mathcal{G}^0 \stackrel{\pi|_{\mathcal{G}^0}}{\longrightarrow} \mathcal{SC}_{(p^r)} \longrightarrow e_k.$$

Thus, setting $\mathcal{H} := \pi^{-1}(T_r \rtimes \mathcal{P}_{\rm red})$ we obtain an extension

$$(*) \qquad e_k \longrightarrow (\mathcal{W}_\ell)_1 \longrightarrow \mathcal{H} \stackrel{\pi|_{\mathcal{H}}}{\longrightarrow} T_r \rtimes \mathcal{P}_{\rm red} \longrightarrow e_k,$$

with $T_r = (T_r \rtimes \mathcal{P}_{\rm red})^0$ operating trivially on $(\mathcal{W}_\ell)_1$. Thanks to [13, (III,§6,6.5)] the canonical map

$$E\tilde{x}^1(\mathcal{P}_{\rm red}, (\mathcal{W}_\ell)_1) \longrightarrow E\tilde{x}^1(T_r \rtimes \mathcal{P}_{\rm red}, (\mathcal{W}_\ell)_1)$$

is bijective. In view of [13, (III,§6,4.5)] the left-hand side is trivial. As a result, the extension $(*)$ splits, and we obtain a closed embedding $\sigma : T_r \rtimes \mathcal{P}_{\rm red} \longrightarrow \mathcal{G}$ such that $\pi \circ \sigma = \mathrm{id}_{T_r \rtimes \mathcal{P}_{\rm red}}$. In particular, $\pi|_{\mathcal{G}^0} \circ \sigma|_{T_r} = \mathrm{id}_{T_r}$ and [21, (6.3)] yields $\mathcal{G}^0 = \mathcal{G}_1 \cdot \sigma(T_r)$. A comparison of orders now implies $\mathcal{G} = \mathcal{G}_1 \cdot \sigma(T_r \rtimes \mathcal{P}_{\rm red})$.

(2) Suppose that $\mathcal{B}_0(\mathcal{G})$ is tame. If $\ell = 0$, then the assertion follows from the isomorphism $\mathcal{G} \cong \mathcal{P}$. Alternatively, [21, (3.4)] implies that

$$(0) \longrightarrow \mathfrak{n}_\ell \longrightarrow \mathrm{Lie}(\mathcal{G}) \stackrel{d\pi}{\longrightarrow} \mathfrak{sl}(2) \longrightarrow (0)$$

is an exact sequence of restricted Lie algebras such that $d\pi(\mathrm{Ad}(g)(u)) = \mathrm{Ad}(\pi(g))(d\pi(u))$ for $g \in \mathcal{G}$ and $u \in \mathrm{Lie}(\mathcal{G})$. By general theory (cf. [20, §1]), there exists a p-semilinear form $\psi : \mathfrak{sl}(2) \longrightarrow \mathfrak{n}_\ell$ such that $\mathrm{Lie}(\mathcal{G}) = \mathfrak{sl}(2)^\ell_\psi$ (cf. §6.1 for the definition). The adjont action of $\mathcal{G}$ leaves $\mathfrak{sl}(2) = [\mathrm{Lie}(\mathcal{G}), \mathrm{Lie}(\mathcal{G})]$ and $\mathfrak{n}_\ell = \mathrm{Lie}((\mathcal{W}_\ell)_1)$ invariant. Since $(\mathcal{W}_\ell)_1$ acts trivially on $\mathrm{Lie}(\mathcal{G})$, the group $\mathcal{P}$ operates on $\mathrm{Lie}(\mathcal{G})$ such that

$$\pi(g).v = \mathrm{Ad}(g)(v) \qquad \forall\ g \in \mathcal{G},\ v \in \mathrm{Lie}(\mathcal{G}).$$

Let $\mathfrak{sl}(2)^{(1)}$ be the $\mathcal{G}$-module with underlying group $\mathfrak{sl}(2)$ and k-action given by $\alpha x = \alpha^{\frac{1}{p}} x$. According to [21, (3.3)], the linear map $\psi : \mathfrak{sl}(2)^{(1)} \longrightarrow \mathfrak{n}_\ell$ is $\mathcal{G}^0$-equivariant. The same arguments imply the $\mathcal{G}(k)$-equivariance of ψ, so that ψ is a homomorphism of $\mathcal{G}$-modules. A consecutive application of [21, (5.3)] and [20, (7.4)] shows that $\psi(h)$ generates $\mathfrak{n}_\ell$ as a restricted Lie algebra. Since $\mathcal{P}(k)$ operates on $k\psi(h)$ via the restriction of the character $\hat{\chi}_k : (\mathrm{Nor}_{\mathrm{SL}(2)}(T)/\mathcal{Z})(k) \longrightarrow k^\times$, we conclude that $T_r \rtimes \mathcal{P}_{\rm red}$ acts on $\mathfrak{n}_\ell$ via

$$q.v = \eta(q)(v) \quad \forall\ q \in T_r \rtimes \mathcal{P}_{\rm red},\ v \in \mathfrak{n}_\ell.$$

Accordingly, we have

$$\mathrm{Ad}(\sigma(q))(x, v) = (\mathrm{Ad}(q)(x), \eta(q)(v)))$$

for $q \in T_r \rtimes \mathcal{P}_{\rm red}$, $x \in \mathfrak{sl}(2)$, and $v \in \mathfrak{n}_\ell$.

Note that $ke \oplus kf$ and kh are $(T_r \rtimes \mathcal{P}_{\rm red})$-submodules of $\mathfrak{sl}(2)^{(1)}$. Let $\hat{T} := T/\mathcal{Z}$. Since $\mathcal{P} \neq \mathrm{SL}(2)_1$, the group $\mathcal{P} \cap \hat{T}$ acts non-trivially on ke and kf and trivially on $\mathfrak{n}_\ell$, so that $ke \oplus kf \subset \ker \psi$. As $\psi(h)$ generates the restricted Lie algebra $\mathfrak{n}_\ell$, there exist $\alpha \in k \setminus \{0\}$ and $c \in \mathfrak{n}_\ell^{[p]}$ such that

$$\psi(h) = \alpha v_0 + c.$$

Since $g.\psi(h) = \hat{\chi}_k(\pi(g))\psi(h)$ for every $g \in \mathcal{G}(k)$, the automorphism λ of $\mathfrak{n}_\ell$ that sends v_0 to $\alpha v_0 + c$ is $\mathcal{G}$-equivariant and satisfies

$$\lambda \circ \psi^\ell_s = \psi.$$

According to [21, (6.1)] we obtain an isomorphism

$$\varphi : \mathfrak{sl}(2)_s^\ell \longrightarrow \mathfrak{sl}(2)_\psi^\ell \quad ; \quad (x,v) \mapsto (x, \lambda(v))$$

of restricted Lie algebras such that

$$\begin{aligned} \varphi(q.(x,v)) &= \varphi(\mathrm{Ad}(q)(x), \eta(q)(v)) = (\mathrm{Ad}(q)(x), \lambda(\eta(q)(v))) = (\mathrm{Ad}(q)(x), \eta(q)(\lambda(v))) \\ &= \mathrm{Ad}(\sigma(q))(x, \lambda(v)) = \mathrm{Ad}(\sigma(q))(\varphi(x,v)) \end{aligned}$$

for every $q \in T_r \rtimes \mathcal{P}_{\mathrm{red}}$ and $v \in \mathfrak{n}_\ell$. Passage to the corresponding infinitesimal groups yields the desired isomorphism $\Phi : \mathrm{SL}(2)_1^\ell \xrightarrow{\sim} \mathcal{G}_1$.

Suppose conversely that we have an isomorphism $\Phi : \mathrm{SL}(2)_1^\ell \longrightarrow \mathcal{G}_1$ such that $\Phi(g.x) = \sigma(g)\Phi(x)\sigma(g)^{-1}$. Thanks to [20, (7.4)] the principal block $\mathcal{B}_0(\mathcal{G}_1) \cong \mathcal{B}_0(\mathrm{SL}(2)_1^\ell)$ is tame, and by (1) the factor group $\mathcal{G}^0/\mathcal{G}_1$ is multiplicative. According to [21, (5.3)] the group $\mathcal{G}^0$ also has a tame principal block. Since $\mathcal{G}_{\mathrm{red}} \cong \mathcal{P}_{\mathrm{red}}$ is a polyhedral group, Proposition 7.1.1 ensures that $\mathcal{B}_0(\mathcal{G})$ is tame. □

Theorem 7.1.5. *Let $\mathcal{G}$ be a finite algebraic k-group such that $\mathcal{G}_{\mathrm{lr}} = e_k$ and $\mathcal{G} \neq \mathcal{G}_1$. Then $\mathcal{B}_0(\mathcal{G})$ is tame if and only if $\mathcal{G}$ is either an amalgamated polyhedral group, or there exist $\ell, m \geq 1$ such that $\mathcal{G} \cong \mathcal{PSC}_{[\ell,m]}, \mathcal{PSQ}_{[\ell,m]}$.*

Proof. As noted earlier, every amalgamated polyhedral group has tame representation type. To prove the tameness of $\mathcal{B}_0(\mathcal{PSC}_{[\ell,m]})$ and $\mathcal{B}_0(\mathcal{PSQ}_{[\ell,m]})$, we consider the groups

$$\mathcal{G}_{[\ell,m]} := \mathrm{SL}(2)_1^\ell \rtimes T_{(m)},\ \mathrm{SL}(2)_1^\ell \rtimes N_{(m)} \text{ as well as } \mathcal{M}_{[\ell,m]} := \mathcal{M}(\mathcal{G}_{[\ell,m]}).$$

If $m = np^r$ with $(n,p) = 1$, then we have

$$\mathcal{G}^0_{[\ell,m]} = \mathrm{SL}(2)_1^\ell \rtimes T_r,$$

so that (1.2) gives

$$\bar{\mathcal{G}}_{[\ell,m]} := \mathcal{G}_{[\ell,m]}/(\mathcal{G}_{[\ell,m]})_{\mathrm{lr}} \cong ((\mathrm{SL}(2)_1^\ell \rtimes T_r)/\mathcal{M}_{[\ell,m]}) \rtimes ((\mathcal{G}_{[\ell,m]})_{\mathrm{red}}/\mathcal{C}_{\mathcal{G}_{[\ell,m]}}).$$

An application of [21, (6.4)] now ensures the tameness of $\mathcal{B}_0(\bar{\mathcal{G}}^0_{[\ell,m]})$. Thanks to (7.1.1) the block $\mathcal{B}_0(\bar{\mathcal{G}}_{[\ell,m]})$ is also tame, and the identification $\bar{\mathcal{G}}_{[\ell,m]} \cong \mathcal{PSC}_{[\ell,m]}$, $\mathcal{PSQ}_{[\ell,m]}$ yields our assertion.

Now let $\mathcal{G} \neq \mathcal{G}_1$ be an algebraic group with trivial linearly reductive radical and tame principal block. Owing to (7.1.3) there exists an amalgamated polyhedral group $\mathcal{P}$ and an exact sequence

$$e_k \longrightarrow (\mathcal{W}_\ell)_1 \longrightarrow \mathcal{G} \longrightarrow \mathcal{P} \longrightarrow e_k$$

such that $(\mathcal{W}_\ell)_1 \subset \mathrm{Cent}(\mathcal{G}^0)$. If $\ell = 0$, then our assertion follows. Alternatively, $\mathcal{G}^0 \ncong \mathrm{SL}(2)_1$, and (7.1.1) shows that $\mathcal{G}(k) \cong \mathcal{P}(k)$ is cyclic or dihedral. Hence there exist $m \in \mathbb{N}$ with $\mathcal{P} \cong \mathcal{PSC}_{(m)}, \mathcal{PSQ}_{(m)}$. The assumption $\mathcal{P} = \mathrm{SL}(2)_1$ implies that $\mathcal{G}$ is an infinitesimal group with tame principal block, which is a central extension of $\mathrm{SL}(2)_1$ by $(\mathcal{W}_\ell)_1$. A consecutive application of [21, (5.3)] and [21, (1.5)] then yields $\mathcal{G} = \mathcal{G}_1$, a contradiction. Accordingly, Lemma 7.1.4 provides a closed embedding $\sigma : T_r \rtimes \mathcal{P}_{\mathrm{red}} \hookrightarrow \mathcal{G}$ such that $\mathcal{G} = \mathcal{G}_1 \cdot \sigma(T_r \rtimes \mathcal{P}_{\mathrm{red}})$ as well as an isomorphism

$$\Phi : \mathrm{SL}(2)_1^\ell \longrightarrow \mathcal{G}_1$$

satisfying $\Phi(q.x) = \sigma(q)\Phi(x)\sigma(q)^{-1} \ \ \forall\, q \in T_r \rtimes \mathcal{P}_{\mathrm{red}},\ x \in \mathrm{SL}(2)_1^\ell$. Consequently,

$$\Psi : \mathrm{SL}(2)_1^\ell \rtimes (T_r \rtimes \mathcal{P}_{\mathrm{red}}) \longrightarrow \mathcal{G} \quad ; \quad (x,q) \mapsto \Phi(x)\sigma(q)$$

is a homomorphism of group schemes, which is readily seen to be a quotient map. Moreover, the canonical map $\mathrm{SL}(2)_1^\ell \rtimes (T_r \rtimes \mathcal{P}_{\mathrm{red}}) \longrightarrow T_r \rtimes \mathcal{P}_{\mathrm{red}}$ induces an isomorphism

$$\ker \Psi \xrightarrow{\sim} T_1,$$

so that $\ker \Psi \subset \mathcal{G}^0_{[\ell,m]}$ is a multiplicative normal subgroup of order p, which therefore coincides with $\mathcal{M}_{[\ell,m]}$. By definition, the reduced part $\mathcal{P}_{\rm red}$ of $\mathcal{P} = \mathcal{PSC}_{(m)}, \mathcal{PSQ}_{(m)}$ acts faithfully on $(\mathrm{SL}(2)_1^\ell \rtimes T_r)/\mathcal{M}_{[\ell,m]}$. Thus, $\mathcal{M}_{[\ell,m]} = (\mathrm{SL}(2)_1^\ell \rtimes (T_r \rtimes \mathcal{P}_{\rm red}))_{\rm lr}$, and Ψ induces an isomorphism

$$\bar{\mathcal{G}}_{[\ell,m]} \cong \mathcal{G}.$$

The desired result now follows from the isomorphism $\bar{\mathcal{G}}_{[\ell,m]} \cong \mathcal{PSC}_{[\ell,m]}$, $\mathcal{PSQ}_{[\ell,m]}$. □

Remark. The tame infinitesimal groups of height 1 with trivial multiplicative center were described in [20, (7.4)]. In contrast to the above result, there are usually infinitely many non-isomorphic such groups with unipotent center of length ℓ.

7.2. **The Gabriel Quiver.** Recall that the *separated quiver* Q_s of a quiver Q with vertex set $\{1, \dots, n\}$ has $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ and an arrow $\ell \to m'$ for every arrow $\ell \to m$ of Q (cf. [4, p.350]). The following well-known result, whose proof also provides first evidence for the importance of tame hereditary algebras, underscores the importance of separated quivers.

Theorem 7.2.1. *Let Λ be a finite dimensional k-algebra. If Λ is tame, then the separated quiver $\Gamma(\Lambda)_s$ of the Gabriel quiver $\Gamma(\Lambda)$ of Λ is a union of simply laced Dynkin diagrams or extended Dynkin diagrams of types $\tilde{A}$, $\tilde{D}$, $\tilde{E}$.*

Proof. Let J be the Jacobson radical of Λ. Then the algebra $\Lambda' := \Lambda/J^2$ is representation-finite or tame, has Jacobson radical $J' = J/J^2$ and Gabriel quiver $\Gamma(\Lambda') = \Gamma(\Lambda)$. According to [4, (III.2.5),(III.2.7)] the triangular matrix algebra

$$\Sigma := \begin{pmatrix} \Lambda'/J' & 0 \\ J' & \Lambda'/J' \end{pmatrix}$$

is hereditary with Gabriel quiver $\Gamma(\Sigma) \cong \Gamma(\Lambda')_s$. Thanks to [4, (III.2.2),(X.2.1)] the functor

$$F : \mathrm{mod}\ \Lambda' \longrightarrow \mathrm{mod}\ \Sigma \ \ ; \ \ M \mapsto \begin{pmatrix} M/J'M \\ J'M \end{pmatrix}$$

reflects isomorphisms and indecomposability. By the same token, F reaches all but finitely many indecomposable Σ-modules. Since F is right exact and commutes with direct sums, Watts' Theorem (cf. [54, (3.33)]) implies that F is equivalent to a functor of the form $X \otimes_{\Lambda'} -$ for the (Σ, Λ')-bimodule $X = F(\Lambda')$. As X is a direct summand of Σ, the algebra Σ is weakly tame or of finite representation type. Our assertion now follows from [46, (I.2.3)] and the classification of tame and representation-finite hereditary algebras over algebraically closed fields (cf. [6, (4.7.1)] and [4, (VIII.5.5)]). □

For each of the extended Dynkin diagrams $(\tilde{A}_n)_{n\geq 1}$, $(\tilde{D}_n)_{n\geq 4}$, and $(\tilde{E}_n)_{6\leq n\leq 8}$, we denote by the same letter the quiver in which each bond $\bullet$ — $\bullet$ is replaced by a pair of arrows $\bullet \overset{\longleftarrow}{\underset{\longrightarrow}{}} \bullet$. In particular, $\tilde{A}_1$ is the quiver of the principal block of $H(\mathrm{SL}(2)_1)$, which is also often denoted $\tilde{A}_{12}$. Observe that in each case the separated quiver is the union of at most two Diagrams of the same type, in which every vertex is either only a source or only a sink.

For future reference we record the following technical result.

Lemma 7.2.2. *Let $\mathcal{U} \lhd \mathcal{G}$ be a unipotent normal subgroup of a finite algebraic k-group $\mathcal{G}$, and denote by $\pi : H(\mathcal{G}) \longrightarrow H(\mathcal{G}/\mathcal{U})$ the canonical projection. Suppose that $\mathcal{B} \subset H(\mathcal{G})$ and $\mathcal{C} \subset H(\mathcal{G}/\mathcal{U})$ are blocks such that*

(a) *the Gabriel quiver $\Gamma(\mathcal{C})$ of $\mathcal{C}$ is of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$, and*
(b) *the block $\mathcal{B}$ is tame, and $\mathcal{C}$ is a direct summand of the the block ideal $\pi(\mathcal{B}) \subset H(\mathcal{G}/\mathcal{U})$.*

Then the natural map

$$\mathrm{Ext}^1_{\mathcal{G}/\mathcal{U}}(S,T) \longrightarrow \mathrm{Ext}^1_{\mathcal{G}}(S,T)$$

is bijective for any two simple $\mathcal{B}$-modules S, T. In particular, we have $\Gamma(\mathcal{B}) \cong \Gamma(\mathcal{C})$ for the Gabriel quivers of $\mathcal{B}$ and $\mathcal{C}$.

Proof. Since $\mathcal{U}$ is unipotent, $\mathcal{G}$ and its factor group $\mathcal{G}/\mathcal{U}$ have the same simple modules. Given two such modules S, T, the cohomology-five-term sequence of the spectral sequence $H^s(\mathcal{G}/\mathcal{U}, \mathrm{Ext}^t_{\mathcal{U}}(S,T)) \Rightarrow \mathrm{Ext}^{s+t}_{\mathcal{G}}(S,T)$ (cf. [35, (I.6.6)]) gives rise to an embedding

$$\mathrm{Ext}^1_{\mathcal{G}/\mathcal{U}}(S,T) \hookrightarrow \mathrm{Ext}^1_{\mathcal{G}}(S,T).$$

Thus, by (b), $\Gamma(\mathcal{C})$ is a subquiver of the connected quiver $\Gamma(\mathcal{B})$. By virtue of assumption (a), the former quiver is of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$. Consequently, adding an arrow to $\Gamma(\mathcal{C})$ yields a quiver, whose associated separated quiver is not a union of simply laced Dynkin diagrams or extended Dynkin diagrams of types $\tilde{A}$, $\tilde{D}, \tilde{E}$. As $\mathcal{B}$ is tame, this contradicts (7.2.1), and we obtain $\Gamma(\mathcal{B}) \cong \Gamma(\mathcal{G})$. □

We now turn to the proof of a more precise formulation of Theorem B. Recall that $\mathcal{M} \subset \mathcal{G}^0$ denotes the multiplicative center of the connected component of a finite algebraic k-group $\mathcal{G}$. Since $\mathcal{M} \lhd \mathcal{G}$ is normal in $\mathcal{G}$, the group $\mathcal{G}_{\mathrm{red}}$ acts on $\mathcal{G}^0/\mathcal{M}$ via conjugation. As before, we let $\mathcal{C}_{\mathcal{G}}$ denote the kernel of this action.

Recall the Frobenius endomorphism $F : \mathrm{SL}(2) \longrightarrow \mathrm{SL}(2)$, defined by raising each matrix entry to its p-th power. Given an SL(2)-module M, we denote by $M^{[1]}$ the Frobenius twist of M (cf. [35, (II.3.16)]). In our next result, we let D_n be the dihedral group of order $2n$.

Theorem 7.2.3. *Let $\mathcal{G}$ be a finite algebraic group of characteristic $p \geq 3$ with tame principal block. Then there exists a linearly reductive group $\tilde{\mathcal{G}} \subset \mathrm{SL}(2)$ such that the Gabriel quiver $\Gamma(\mathcal{G})$ of $\mathcal{B}_0(\mathcal{G})$ is isomorphic to the McKay quiver $\Upsilon_{L(1)}(\tilde{\mathcal{G}})$. Moreover, $\Gamma(\mathcal{G})$ belongs to the following list:*

$\mathcal{G}(k)/\mathcal{C}_{\mathcal{G}}(k)$	$\Gamma(\mathcal{G})$
$\mathbb{Z}/(n)$	$\tilde{A}_{2np^{r-1}-1}$
D_n	$\tilde{D}_{np^{r-1}+2}$
A_4	$\tilde{E}_6$
S_4	$\tilde{E}_7$
A_5	$\tilde{E}_8$

,

where $r := \mathrm{ht}(\mathcal{G}^0/\mathcal{M}) \geq 1$ and $(n,p) = 1$. Quivers of type $\tilde{E}$ occur only if $\mathcal{G}^0/\mathcal{M} \cong \mathrm{SL}(2)_1$.

Remark. As in Section 3, the groups in the left-hand column are assumed to be linearly reductive. Thus, we have $p \neq 2, 3$ for A_4, S_4, and $p \neq 2, 3, 5$ for A_5.

Proof. According to (6.2.1) the finite group has order prime to p, so that (1.2) yields $\mathcal{G}_{\mathrm{lr}} = \mathcal{M}\mathcal{C}_{\mathcal{G}}$. In view of (1.1) we thus have $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0((\mathcal{G}^0/\mathcal{M}) \rtimes (\mathcal{G}_{\mathrm{red}}/\mathcal{C}_{\mathcal{G}}))$ and we may assume that the groups $\mathcal{M}$ and $\mathcal{C}_{\mathcal{G}}$ are trivial. Thanks to (5.1.4), the principal block $\mathcal{B}_0(\mathcal{G}^0)$ is tame, and [21, (3.4)] provides $r \geq 1$ and an exact sequence

$$e_k \longrightarrow \mathrm{Cent}(\mathcal{G}^0) \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{SC}_{(p^r)} \longrightarrow e_k,$$

where the center $\mathrm{Cent}(\mathcal{G}^0)$ is isomorphic to $(\mathcal{W}_\ell)_1$ for a suitable $\ell \geq 0$.

We begin by considering the case $\ell = 0$, that is, $\mathcal{G}^0 \cong \mathcal{SC}_{(p^r)}$. Since $(\mathcal{SC}_{(p^r)})_1 = \mathrm{SL}(2)_1$ the conjugation action furnishes a homomorphism

$$\varrho : \mathcal{G} \longrightarrow \mathcal{AUT}(\mathrm{SL}(2)_1) \cong \mathrm{PSL}(2)$$

from $\mathcal{G}$ to the automorphism scheme $\mathcal{AUT}(\mathrm{SL}(2)_1)$ of $\mathrm{SL}(2)_1$. Setting $\mathcal{N} := \ker \varrho$, our current assumption implies $\mathcal{N} = e_k$ for $r = 1$. Alternatively, we consider an element $\lambda \in \mathcal{N}(k)$. Since $\mathcal{G}_{\mathrm{red}}$ acts faithfully on $\mathcal{G}^0$, we have $\mathcal{G}(k) \subset \mathrm{Aut}(\mathcal{SC}_{(p^r)})$, whence $\lambda|_{\mathrm{SL}(2)_1} = \mathrm{id}_{\mathrm{SL}(2)_1}$. By virtue of (6.1.2) there exists an element $g \in T(k)\langle\omega\rangle$ such that $\lambda = \iota_g$. We obtain $g \in \{I_2, -I_2\}$ so that $\lambda = 1$. Accordingly, $\mathcal{N}(k) = \{1\}$, and the group $\mathcal{N}$ is infinitesimal. As $p \geq 3$ the Frobenius kernel $\mathcal{N}_1 = \mathcal{N} \cap \mathcal{G}_1 = \mathrm{Cent}(\mathrm{SL}(2)_1)$ is trivial, whence $\mathcal{N} = e_k$. Consequently, ϱ is a closed embedding, and we may consider $\mathcal{G}$ a closed subgroup of $\mathrm{PSL}(2)$. Hence there exists a closed subgroup $\hat{\mathcal{G}} \subset \mathrm{SL}(2)$, and a quotient map $\pi : \hat{\mathcal{G}} \longrightarrow \mathcal{G}$, whose kernel $\mathcal{N}$ is a reduced group of order 2. In particular, π induces an isomorphism $\hat{\mathcal{G}}^0 \cong \mathcal{G}^0$. Consequently, (1.1(1)) yields

$$(*) \qquad \hat{\mathcal{G}}/\hat{\mathcal{G}}_1 \cong (\hat{\mathcal{G}}^0/\hat{\mathcal{G}}_1) \rtimes \hat{\mathcal{G}}_{\mathrm{red}} \cong (\mathcal{G}^0/\mathcal{G}_1) \rtimes \hat{\mathcal{G}}_{\mathrm{red}} \cong \mu_{p^{r-1}} \rtimes \hat{\mathcal{G}}_{\mathrm{red}},$$

so that $\hat{\mathcal{G}}/\hat{\mathcal{G}}_1$ is linearly reductive and operates faithfully on the Frobenius twist $L(1)^{[1]}$ of the standard $\mathrm{SL}(2)$-module. Since the group $\mathrm{SL}(2)$ is defined over $\mathbb{F}_p$, the Frobenius homomorphism induces an isomorphism $\mathrm{SL}(2)/\mathrm{SL}(2)_1 \cong \mathrm{SL}(2)$ (cf. [35, (I.9.5)]). Thus, we have an embedding

$$\hat{F} : \hat{\mathcal{G}}/\hat{\mathcal{G}}_1 \hookrightarrow \mathrm{SL}(2)/\mathrm{SL}(2)_1 \cong \mathrm{SL}(2),$$

which induces an isomorphism $\Upsilon_{L(1)}(\hat{F}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1)) \cong \Upsilon_{L(1)^{[1]}}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1)$. Now (3.3) in conjunction with $(*)$ and the isomorphism $\mathcal{G}(k) \cong \hat{\mathcal{G}}(k)/\mathcal{N}(k)$ (cf. [13, (III,§1,1.15)]) yields the asserted table, provided we show that $\Gamma(\mathcal{G}) \cong \Upsilon_{L(1)^{[1]}}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1)$.

Let $\{M_1, \ldots, M_n\}$ be a complete set of representatives of the simple $(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1)$-modules. Directly from our table we see that the McKay graph $\bar{\Upsilon}_{L(1)^{[1]}}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1)$ has vertex chromatic number 2 (cf. [8, V,§1]). Hence there exists a 2-colouring

$$f : \{1, \ldots, n\} \longrightarrow \{0, p-2\}$$

such that $f(j) = 0$ if and only if M_j is linked to $M_1 \cong k$ by a path of even length.

Thanks to [35, (II.3.15)] and [36] the group $\hat{\mathcal{G}}$ and its normal subgroup $\hat{\mathcal{G}}_1 = \mathrm{SL}(2)_1$ satisfy the hypotheses of (2.1) and (2.3). The simple $\hat{\mathcal{G}}_1$-modules are given by the simple $\mathrm{SL}(2)$-modules $L(0), \ldots, L(p-1)$. Owing to [47, Theorem] we have

$$\mathrm{Ext}^1_{\hat{\mathcal{G}}_1}(L(i), L(j)) \cong \begin{cases} L(1)^{[1]} & i + j = p - 2 \\ (0) & \text{otherwise.} \end{cases}$$

Since $\hat{\mathcal{G}}/\hat{\mathcal{G}}_1$ is linearly reductive, (2.3(1)) now yields

$$\mathrm{Ext}^1_{\hat{\mathcal{G}}}(L(i) \otimes_k M_s, L(j) \otimes_k M_t) \cong \begin{cases} \mathrm{Hom}_{\hat{\mathcal{G}}/\hat{\mathcal{G}}_1}(M_s, L(1)^{[1]} \otimes_k M_t) & i + j = p - 2 \\ (0) & \text{otherwise.} \end{cases}$$

As $\hat{\mathcal{G}}/\hat{\mathcal{G}}_1$ acts faithfully on $L(1)^{[1]}$ we may apply (3.1) to see that the McKay quiver $\Upsilon_{L(1)^{[1]}}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1)$ is connected. This shows that the Gabriel quiver $\Gamma(\hat{\mathcal{G}})$ of the principal block $\mathcal{B}_0(\hat{\mathcal{G}})$ has vertex set $\{L(f(i)) \otimes_k M_i \ ; \ 1 \leq i \leq n\}$ and that

$$\Upsilon_{L(1)^{[1]}}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1) \longrightarrow \Gamma(\hat{\mathcal{G}}) \ \ ; \ \ M_i \mapsto L(f(i)) \otimes_k M_i$$

induces an isomorphism of quivers. Since $\mathcal{N}$ has order 2, (1.1(3)) provides an isomorphism $\mathcal{B}_0(\hat{\mathcal{G}}) \cong \mathcal{B}_0(\mathcal{G})$. As a result, the Gabriel quiver $\Gamma(\mathcal{G})$ is isomorphic to $\Upsilon_{L(1)^{[1]}}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1)$.

If $\ell \neq 0$, then the group $\mathcal{C} := \mathrm{Cent}(\mathcal{G}^0) \cong (\mathcal{W}_\ell)_1$ is unipotent. By the first part of our proof, the quiver $\Gamma(\mathcal{G}/\mathcal{C})$ is of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$. As $\mathcal{B}_0(\mathcal{G})$ is tame, Lemma 7.2.2 implies $\Gamma(\mathcal{G}/\mathcal{C}) = \Gamma(\mathcal{G})$.

We next show that $\mathrm{Cent}_{\mathcal{G}_{\mathrm{red}}}(\mathcal{G}^0)(k) = \mathrm{Cent}_{\mathcal{G}_{\mathrm{red}}}(\mathcal{G}^0/\mathcal{C})(k)$. One inclusion being trivial, we let g be an element of the right-hand group. This means that the inner automorphism $\iota_g \in \mathrm{Aut}(\mathcal{G}^0)$ effected by g induces the identity on $\mathcal{G}^0/\mathcal{C}$. Since the order of g is prime to p, the arguments employed in the proof of (6.1.3) imply $\iota_g = \mathrm{id}_{\mathcal{G}^0}$, so that $g \in \mathrm{Cent}_{\mathcal{G}_{\mathrm{red}}}(\mathcal{G}^0)(k)$.

Thanks to [21, (5.3),(1.5)] the infinitesimal groups $\mathcal{G}^0$ and $\mathcal{G}^0/\mathcal{C}$ have the same height. □

Remarks. (1) By combining the proof of (7.2.3) with [21, (3.5)] we see that the linearly reductive group $\tilde{\mathcal{G}}$ is isomorphic to $\hat{\mathcal{H}}/\hat{\mathcal{H}}_1$, where $\hat{\mathcal{H}}$ is a covering of the group $\mathcal{H} := \mathcal{G}/(\mathrm{Cent}(\mathcal{G}^0)\mathcal{G}_{\mathrm{lr}})$.

(2) The foregoing proof in conjunction with [28, Theorem2] also gives the dimensions and the positions of the simple $\mathcal{B}_0(\mathcal{G})$-modules within the quiver $\Gamma(\mathcal{G})$. For instance, if $\Gamma(\mathcal{G})$ is not of type $\tilde{A}$, then the trivial module $k = L(0) \otimes_k M_1$ is located at an end of $\Gamma(\mathcal{G})$. By the same token, the maximal possible dimension of a simple $\mathcal{B}_0(\mathcal{G})$-module is $6(p-1)$.

(3) McKay quivers also occur in the representation theory of quantized function algebras and half-quantum groups [10]. If G is an abelian group whose order is not divisible by p, and $V = \bigoplus_{i=1}^r n_i\, k_{\lambda_i}$ is a faithful G-module, then the characater group $X(G)$ is generated by $R := \{-\lambda_1, \ldots, -\lambda_r\}$ and the McKay quiver $\Psi_V(G)$ is a multiply-edged Cayley graph of $X(G)$ relative to R. Accordingly, [10, (6.7),(7.4)] can also be interpreted via McKay quivers.

Examples. (a) For a natural number $m \in \mathbb{N}$ we consider the finite algebraic groups

$$\mathcal{SC}_{(m)} = \mathrm{SL}(2)_1 T_{(m)} \quad \text{and} \quad \mathcal{SQ}_{(m)} = \mathrm{SL}(2)_1 N_{(m)}.$$

If $m = np^\ell$ with $(p,n) = 1$, then $(\mathcal{SC}_{(m)})^0 = \mathcal{SC}_{(p^r)} = (\mathcal{SQ}_{(m)})^0$ with $r := \max\{\ell, 1\}$, so that $\mathcal{M} = e_k$ in either case. Since $\mathcal{SC}_{(m)}(k) = T_{(n)}(k) \cong \mathbb{Z}/(n)$ and $\mathcal{SQ}_{(m)}(k) = T_{(n)}(k)\langle\omega\rangle \cong Q_n$, Theorem 7.2.3 implies

$$\Gamma(\mathcal{SQ}_{(m)}) = \tilde{D}_{np^{r-1}+2} \quad \text{and} \quad \Gamma(\mathcal{SC}_{(m)}) = \begin{cases} \tilde{A}_{2np^{r-1}-1} & n \text{ odd} \\ \tilde{A}_{np^{r-1}-1} & n \text{ even.} \end{cases}$$

For $n \neq 1$ the principal blocks $\mathcal{B}_0(\mathcal{SC}_{(m)})$ and $\mathcal{B}_0(\mathcal{SQ}_{(m)})$ are properly contained in the skew group algebras $\mathcal{B}_0(\mathcal{SC}_{(p^r)})[\mathbb{Z}/(n)]$ and $\mathcal{B}_0(\mathcal{SC}_{(p^r)})[Q_n]$, respectively (cf. (1.1(4)).

(b) By the same token, the principal blocks of the tame finite algebraic groups $\mathcal{S}\hat{T}$, $\mathcal{S}\hat{O}$ and $\mathcal{S}\hat{I}$ have Gabriel quivers $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$, respectively.

7.3. **Basic Algebras and Block Structure.** Turning to the determination of the basic algebras of the tame principal blocks, we let Q be one of the quivers $\tilde{A}_n (n \geq 2)$, $\tilde{D}_n (n \geq 4)$ or $\tilde{E}_n (6 \leq n \leq 8)$. Given a path q in the path algebra $k[Q]$, let $s(q)$ and $t(q)$ be its starting point and terminal point, respectively. We consider the set $Q(2)$ of paths of length 2 as well as $Q(2)_c \subset Q(2)$, the subset of oriented cycles. For an element $q \in Q(2)_c$, we let $m(q)$ be its midpoint. Set $Q(2)_{c,\ell} := \{(q_1 - q_2)^{p^\ell} \ ; \ q_i \in Q(2)_c \ , \ s(q_1) = s(q_2)\}$ for every $\ell \in \mathbb{N}_0$.

We let $\hat{\zeta} : \tilde{D}_n \longrightarrow \tilde{D}_n$ be an automorphism of order 2 fixing the branchpoints and not fixing any of the endpoints. (Unless $n = 4$, these properties uniquely determine $\hat{\zeta}$.) Set $\tilde{D}_n(2)_{\hat{\zeta}} := \{q \in \tilde{D}_n(2) \setminus \tilde{D}_n(2)_c \ ; \ t(q) = \hat{\zeta}(s(q))\}$, $\tilde{D}_n(2)^{\hat{\zeta}} := \{q_1 - q_2 \ ; \ q_i \in \tilde{D}_n(2)_c \ ; m(q_2) = \hat{\zeta}(m(q_1)) \neq m(q_1)\}$, as well as $q_{\hat{\zeta}} := \sum_{q \in \tilde{D}_n(2)_{\hat{\zeta}}} q$.

For $\ell \in \mathbb{N}_0$, we define the ideal $J(Q,\ell) \lhd k[Q]$ via

$$J(Q,\ell) := \begin{cases} \langle (Q(2) \setminus Q(2)_c) \cup Q(2)_{c,\ell} \rangle & \text{if } Q \not\cong \tilde{D}_n \text{ or } \ell = 0 \\ \langle (Q(2) \setminus (Q(2)_{\hat{\zeta}} \cup Q(2)_c) \cup Q(2)_{c,\ell} \cup Q(2)^{\hat{\zeta}} \cup \{q_{\hat{\zeta}}^{p^\ell}\} \rangle & \text{otherwise.} \end{cases}$$

We put $\mathcal{S}(Q,\ell) := k[Q]/J(Q,\ell)$, and (due to the occurrence of multiple arrows) define $\mathcal{S}(\tilde{A}_1,\ell)$ by "lenghtening" the commutativity relations of the trivial extension of the Kronecker algebra by p^ℓ (see [20, §7] for the precise definition). For type $\tilde{A}_n$ we require the number n to be odd, cf. Theorem 7.2.3. As before, we denote by $\Gamma(\mathcal{G})$ the Gabriel quiver of the principal block $\mathcal{B}_0(\mathcal{G})$.

Theorem 7.3.1. *Let $\mathcal{G}$ be a finite algebraic group of characteristic $p \geq 3$. If $\mathcal{B}_0(\mathcal{G})$ has tame representation type, then $\mathcal{B}_0(\mathcal{G})$ is Morita equivalent to $\mathcal{S}(\Gamma(\mathcal{G}),\ell)$, where ℓ is the length of the unipotent group* $\mathrm{Cent}(\mathcal{G}^0)/\mathcal{M}$.

Remark. In view of (7.2.3) we have $\mathrm{Cent}(\mathcal{G}^0)/\mathcal{M} = e_k$ for quivers of type $\tilde{E}_n$, so that $\ell = 0$ in that case (see also §7.4 below).

Proof. In virtue of (7.1.1) the algebras $\mathcal{B}_0(\mathcal{G})$ and $\mathcal{B}_0((\mathcal{G}^0/\mathcal{M}) \rtimes (\mathcal{G}_{\mathrm{red}}/\mathcal{C}_{\mathcal{G}}))$ are isomorphic, so we may assume that $\mathcal{M} = e_k = \mathcal{C}_{\mathcal{G}}$. A consecutive application of (5.1.4) and [21, (3.4)] provides $r \geq 1$ and $\ell \geq 0$ such that

$$\mathcal{G}^0/\mathrm{Cent}(\mathcal{G}^0) \cong \mathcal{SC}_{(p^r)} \quad \text{and} \quad \mathrm{Cent}(\mathcal{G}^0) \cong (\mathcal{W}_\ell)_1.$$

We write $\mathcal{C} := \mathrm{Cent}(\mathcal{G}^0)$ and observe that $H(\mathcal{C}) \cong u(\mathfrak{n}_\ell) \cong k[X]/(X^{p^\ell})$ is a truncated polynomial ring.

The goup $\mathcal{G}(k)$ acts on $\mathcal{G}_1$ and $\mathcal{C} \subset \mathcal{G}_1$ by conjugation. According to [13, (II,§7,n°4)] the former action is equivalent to the adjoint action of $\mathcal{G}(k)$ on the Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ of $\mathcal{G}$. Since the principal block $\mathcal{B}_0(\mathfrak{g})$ of the restricted enveloping algebra $U_0(\mathfrak{g})$ is tame (cf. [21, (5.3)]) and $T(\mathfrak{g}) = (0)$, [20, (7.4)] provides p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow \mathfrak{n}_\ell$ such that $\mathfrak{g} \cong \mathfrak{sl}(2)^\ell_\psi$. In view of [21, (6.1)] the map ψ is a homomorphism of $\mathcal{G}(k)$-modules if we let $\alpha \in k$ act on $\mathfrak{sl}(2)$ via $\alpha^{\frac{1}{p}}$. As before, we denote by $\{e,h,f\}$ the standard basis of $\mathfrak{sl}(2)$.

If $\ell \neq 0$, then (6.1.1(2)) implies that $k\psi(h) \subset \mathfrak{n}_\ell$ is a one-dimensional $\mathcal{G}(k)$-submodule of $\mathfrak{n}_\ell$. We put $v_0 := \psi(h)$ and let $\zeta : k[\mathcal{G}(k)] \longrightarrow k$ be the character corresponding to the one-dimensional $\mathcal{G}(k)$-module kv_0. By (6.1.1(2)) and (7.2.3) the homomorphism ζ is trivial if $\Gamma(\mathcal{G}) \cong \tilde{A}_n$, and of order 2 if $\Gamma(\mathcal{G}) \cong \tilde{D}_n$. Moreover, ζ is trivial on the unique normal cyclic subgroup of index 2 in case $\mathcal{G}(k) \cong D_n$ for some $n \geq 3$. We also define $\zeta := \varepsilon$ for $\ell = 0$.

Owing to [20, (7.4)] the element v_0 generates the restricted Lie algebra $\mathfrak{n}_\ell$. Thus, the algebra $H(\mathcal{C}) \cong u(\mathfrak{n}_\ell)$ is also generated by v_0, and

$$H(\mathcal{G})H(\mathcal{C})^\dagger = H(\mathcal{G})v_0$$

is a principal ideal of $H(\mathcal{G})$.

Setting $\tilde{\mathcal{G}} := \mathcal{G}/\mathcal{C} \cong (\mathcal{G}^0/\mathcal{C}) \rtimes \mathcal{G}_{\mathrm{red}}$, we obtain

$$H(\tilde{\mathcal{G}}) = H(\mathcal{G}^0/\mathcal{C})[\mathcal{G}(k)],$$

so that ζ defines a character $\zeta : H(\tilde{\mathcal{G}}) \longrightarrow k$ via $\zeta|_{H(\mathcal{G}^0/\mathcal{C})} = \varepsilon$. When convenient, we shall view ζ as a character of $H(\mathcal{G})$ that vanishes on the augmentation ideal $H(\mathcal{C})^\dagger$ of $H(\mathcal{C})$.

Every character $\omega : H(\mathcal{G}) \longrightarrow k$ induces via the convolution product $\mathrm{id}_{H(\mathcal{G})} * \omega$ an automorphism of $H(\mathcal{G})$. By definition, we have

$$(\mathrm{id}_{H(\mathcal{G})} * \omega)(h) = \sum_{(h)} h_{(1)}\omega(h_{(2)})$$

for every $h \in H(\mathcal{G})$. Note that $\omega \mapsto \mathrm{id}_{H(\mathcal{G})} * \omega$ defines an injective homomorphism from the character group $X(\mathcal{G})$ into the automorphism group of the associative algebra $H(\mathcal{G})$.

(7.3.1.1) *Let P be a principal indecomposable $H(\mathcal{G})$-module. Then P/v_0P is a principal indecomposable $H(\tilde{\mathcal{G}})$-module, and left multiplication by v_0 induces isomorphisms*

$$(P/v_0P) \otimes_k k_{i\zeta} \xrightarrow{\sim} v_0^i P/v_0^{i+1}P \qquad 0 \le i \le p^\ell - 1.$$

Since the principal ideal $H(\mathcal{G})H(\mathcal{C})^\dagger$ is nilpotent, the algebras $H(\mathcal{G})$ and $H(\tilde{\mathcal{G}})$ afford the same simple modules. This readily implies that P/v_0P is a principal indecomposable $H(\tilde{\mathcal{G}})$-module. As v_0 lies in the center of $H(\mathcal{G}^0)$, the definition of the character ζ readily implies

$$hv_0^i = v_0^i(\mathrm{id}_{H(\mathcal{G})} * (i\zeta))(h) \qquad \forall\ h \in H(\mathcal{G}).$$

For $0 \le i \le p^\ell - 1$ we consider the surjective k-linear map

$$\gamma_i : (P/v_0P) \otimes_k k_{i\zeta} \longrightarrow v_0^i P/v_0^{i+1}P \ \ ; \ \ (x + v_0P) \otimes s \mapsto sv_0^i x + v_0^{i+1}P.$$

The above formula implies that γ_i is $H(\tilde{\mathcal{G}})$-linear. Moreover, since P is projective and $H(\mathcal{G})$ is a projective $H(\mathcal{C})$-module (cf. [35, (I.8.16(3))]), the module $P|_{H(\mathcal{C})}$ is a projective module over the local algebra $H(\mathcal{C})$. Hence $P|_{H(\mathcal{C})}$ is free, so that $P|_{H(\mathcal{C})} \cong k[v_0]^m$ for some $m \ge 1$. Consequently, $v_0^iP/v_0^{i+1}P \cong k^m$ for $0 \le i \le p^\ell - 1$, implying that γ_i is an isomorphism. $\diamond$

(7.3.1.2) *Let P and Q be principal indecomposable $H(\mathcal{G})$-modules such that*

$$\mathrm{Hom}_{\tilde{\mathcal{G}}}(P/v_0P, (Q/v_0Q) \otimes_k k_{i\zeta}) = (0)$$

for every $i \in \{0, \dots, p^\ell - 1\}$. Then we have $\mathrm{Hom}_{\mathcal{G}}(P, Q) = (0)$.

Let f be an element of $\mathrm{Hom}_{\mathcal{G}}(P, Q)$. If $f(P) \subset v_0^iQ$ for some $i \in \{0, \dots, p^\ell - 1\}$, then f gives rise to a unique homomorphism $\Psi(f) \in \mathrm{Hom}_{\tilde{\mathcal{G}}}(P/v_0P, v_0^iQ/v_0^{i+1}Q)$ such that

$$\Psi(f) \circ \pi_P = \pi_Q \circ f.$$

Here π_P and π_Q denote the natural maps $P \longrightarrow P/v_0P$ and $v_0^iQ \longrightarrow v_0^iQ/v_0^{i+1}Q$, respectively. In view of (7.3.1.1) and our current assumption, we have $\Psi(f) = 0$, whence $f(P) \subset \ker \pi_Q = v_0^{i+1}Q$. Since $v_0^{p^\ell} = 0$, we conclude that $f = 0$. $\diamond$

We next consider the group $\tilde{\mathcal{G}}$, which amounts to assuming $\ell = 0$. The arguments of (7.2.3) now show that we may assume that $\tilde{\mathcal{G}} \subset \mathrm{PSL}(2)$ is a closed subgroup, whose first Frobenius kernel coincides with $\mathrm{SL}(2)_1$. Its pre-image in $\mathrm{SL}(2)$ has the same principal block, so we consider the case where $\tilde{\mathcal{G}}$ is a closed subgroup of $\mathrm{SL}(2)$ with $\tilde{\mathcal{G}}_1 = \mathrm{SL}(2)_1$. By the same token, the factor group $\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1$ is linearly reductive, and the Gabriel quiver of $\mathcal{B}_0(\tilde{\mathcal{G}})$ is the McKay quiver of $\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1$ relative to the faithful $(\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1)$-module $L(1)^{[1]}$.

As noted earlier, the assumptions of (2.3) hold for the simple $\tilde{\mathcal{G}}_1$-modules $L(i)$ and their projective covers $P(i)$. Thus, letting $\{M_1, \dots, M_n\}$ be a complete set of representatives of the isoclasses of the simple $(\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1)$-modules, the modules $\tilde{P}_{i,j} := P(i) \otimes_k M_j \ \ 0 \le i < p,\ 1 \le j \le n$ form a complete set of representatives of the isoclasses of the principal indecomposable $H(\tilde{\mathcal{G}})$-modules.

(7.3.1.3) *Let $i \in \{0, \dots, p-1\}$. Then we have* $\mathrm{Hom}_{\tilde{\mathcal{G}}}(\tilde{P}_{i,j}, \tilde{P}_{i,t}) = (0)$ *for $j \neq t$.*

Let $i \in \{0, \dots, p-2\}$. Owing to [30, Thm.3] each module $P(i)$ has an $\mathrm{SL}(2)$-composition series with composition factors

$$(*) \qquad L(i),\ L(p-2-i) \otimes_k L(1)^{[1]},\ L(i).$$

Since $\tilde{\mathcal{G}}_1 = \mathrm{SL}(2)_1$, we have

$$\dim_k \mathrm{Hom}_{\tilde{\mathcal{G}}_1}(P(i), P(i)) = 2.$$

In view of $(*)$ we see that the same holds for the spaces of $\tilde{\mathcal{G}}$-homomorphisms. The resulting identity

$$\mathrm{Hom}_{\tilde{\mathcal{G}}}(P(i), P(i)) = \mathrm{Hom}_{\tilde{\mathcal{G}}_1}(P(i), P(i)),$$

implies in particular that $\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1$ acts trivially on the latter space. We therefore obtain

$$\begin{aligned} \mathrm{Hom}_{\tilde{\mathcal{G}}}(\tilde{P}_{i,j}, \tilde{P}_{i,t}) &\cong (P(i)^* \otimes_k M_j^* \otimes_k P(i) \otimes_k M_t)^{\tilde{\mathcal{G}}} \cong (\mathrm{Hom}_{\tilde{\mathcal{G}}_1}(P(i), P(i)) \otimes_k M_j^* \otimes_k M_t)^{\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1} \\ &\cong \mathrm{Hom}_{\tilde{\mathcal{G}}_1}(P(i), P(i)) \otimes_k (M_j^* \otimes_k M_t)^{\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1} \\ &\cong \mathrm{Hom}_{\tilde{\mathcal{G}}_1}(P(i), P(i)) \otimes_k \mathrm{Hom}_{\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1}(M_j, M_t). \end{aligned}$$

Thus, $\mathrm{Hom}_{\tilde{\mathcal{G}}}(\tilde{P}_{i,j}, \tilde{P}_{i,t}) = (0)$ for $j \neq t$. Since $P(p-1) = L(p-1)$ Lemma 2.1 ensures that the foregoing identity also holds for $i = p-1$. ◇

(7.3.1.4) *The algebra $H(\tilde{\mathcal{G}})$ is symmetric.*

Let $\hat{\mathcal{G}} \subset \mathrm{SL}(2)$ be the covering group of $\tilde{\mathcal{G}} \subset \mathrm{PSL}(2)$. By virtue of (2.3) the module $P(0)$ is the projective cover of the trivial $H(\hat{\mathcal{G}})$-module. Directly from $(*)$ we obtain $\mathrm{Soc}(P(0)) = L(0) = \mathrm{Top}(P(0))$. Since $\mathcal{B}_0(\hat{\mathcal{G}}) \cong \mathcal{B}_0(\tilde{\mathcal{G}})$, the top and socle of the projective cover of the trivial $\tilde{\mathcal{G}}$-module are also isomorphic. In view of [35, (I.8.13)] this implies that the modular function $\xi : H(\tilde{\mathcal{G}}) \longrightarrow k$ coincides with the co-unit. Thanks to [26, (1.5)] the convolution $\xi * \mathrm{id}_{H(\tilde{\mathcal{G}})}$ is a Nakayama automorphism of the Frobenius algebra $H(\tilde{\mathcal{G}})$, so that $H(\tilde{\mathcal{G}})$ is symmetric. ◇

According to (7.3.1.1) the principal indecomposable $H(\mathcal{G})$-modules $\{P_{i,j} \ ; \ 0 \leq i < p, \ 1 \leq j \leq n\}$ satisfy $\tilde{P}_{i,j} \cong P_{i,j}/v_0 P_{i,j}$. Moreover, by (7.2.2), we have $\Gamma(\mathcal{G}) = \Gamma(\tilde{\mathcal{G}})$ for the Gabriel quivers of $\mathcal{B}_0(\mathcal{G})$ and $\mathcal{B}_0(\tilde{\mathcal{G}})$. Thus, (7.2.3) implies that the graph $\Gamma(\mathcal{G})$ has vertex chromatic number 2, and there exists a function

$$f : \{1, \ldots, n\} \longrightarrow \{0, p-2\}$$

such that

(a) $f(j) = 0$ if and only if M_j is linked to $M_1 \cong k$ by a path of even length, and

(b) $\{P_{f(1),1}, \ldots, P_{f(n),n}\}$ and $\{\tilde{P}_{f(1),1}, \ldots, \tilde{P}_{f(n),n}\}$ are complete sets of representatives of the isomorphism classes of the principal indecomposable modules for $\mathcal{B}_0(\mathcal{G})$ and $\mathcal{B}_0(\tilde{\mathcal{G}})$, respectively.

Thanks to (b) the module $P_g := \bigoplus_{i=1}^n P_{f(i),i}$ is a progenerator for $\mathcal{B}_0(\mathcal{G})$, so that $\mathrm{End}_{\mathcal{G}}(P_g)^{\mathrm{op}}$ is the basic algebra of $\mathcal{B}_0(\mathcal{G})$ (cf. [4, (II.2.6)]). Since the antipode $\eta : H(\mathcal{G}) \longrightarrow H(\mathcal{G})$ and the co-unit $\varepsilon : H(\mathcal{G}) \longrightarrow k$ satisfy $\varepsilon \circ \eta = \varepsilon$, it follows that $\eta(\mathcal{B}_0(\mathcal{G})) = \mathcal{B}_0(\mathcal{G})$. Consequently, the duality $M \mapsto M^*$ induces an isomorphism $\mathrm{End}_{\mathcal{G}}(P_g) \cong \mathrm{End}_{\mathcal{G}}(P_g)^{\mathrm{op}}$. We may therefore complete the proof by showing that $\Lambda := \mathrm{End}_{\mathcal{G}}(P_g)$ is isomorphic to $\mathcal{S}(\Gamma(\mathcal{G}), \ell)$.

Let J be the Jacobson radical of $H(\mathcal{G})$. Since $\Gamma(\mathcal{G})$ is the Gabriel quiver of Λ, [4, (III.1.10)] provides a surjective homomorphism

$$\Omega : k[\Gamma(\mathcal{G})] \longrightarrow \Lambda,$$

sending an idempotent e_i to $\mathrm{id}_{P_{f(i),i}} \in \mathrm{Hom}_{\mathcal{G}}(P_{f(i),i}, P_{f(i),i}) \subset \Lambda$ and an arrow $\alpha : i \mapsto j$ to a homomorphism $\Omega(\alpha) \in \mathrm{Hom}_{\mathcal{G}}(P_{f(i),i}, JP_{f(j),j}) \setminus \mathrm{Hom}_{\mathcal{G}}(P_{f(i),i}, J^2 P_{f(j),j})$. We propose to show that $J(\Gamma(\mathcal{G}), \ell) \subset \ker \Omega$.

Recall the character $\zeta : H(\mathcal{G}) \longrightarrow k$ which is trivial unless $\ell \neq 0$ and $\Gamma(\mathcal{G})$ is of type $\tilde{D}_n$. Observe that tensoring with k_ζ induces an automorphism $\hat{\zeta}$ of order 2 of the McKay quiver $\Gamma(\mathcal{G}) \cong \tilde{D}_n$. The automorphism $\hat{\zeta}$ of $\Gamma(\mathcal{G})$ does not fix any of the endpoints of $\Gamma(\mathcal{G})$, as these correspond to one-dimensional modules (cf. [28]). Moreover, since ζ is trivial on the unique normal cyclic subgroup

N of $\tilde{\mathcal{G}}(k) \cong D_n$ $(n \geq 3)$, and all two-dimensional simple D_n-modules are induced from one-dimensional N-modules, an application of [6, (3.3.3)] implies that the isomorphism $\hat{\zeta}$ fixes all two-dimensional simple $\mathcal{G}(k)$-modules. If $n = 2$, then D_n has exactly one two-dimensional simple module, so that we arrive at the same result.

By abuse of notation, we denote by $\hat{\zeta} : k[\Gamma(\mathcal{G})] \longrightarrow k[\Gamma(\mathcal{G})]$ the corresponding automorphism of the path algebra. The auto-equivalence $M \mapsto M \otimes_k k_\zeta$ also induces an automorphism $\psi_\zeta : \Lambda \longrightarrow \Lambda$ such that

$$\psi_\zeta \circ \Omega = \Omega \circ \hat{\zeta}.$$

(7.3.1.5) *If $\rho \in k[\Gamma(\mathcal{G})]$ is a path of length 2 which is not a cycle, then $\rho \in \ker \Omega$ unless $\ell \neq 0$, $\Gamma(\mathcal{G}) = \tilde{D}_n$ and $t(\rho) = \hat{\zeta}(s(\rho))$.*

Suppose that $\rho = \alpha_j \alpha_i$, where α_i and α_j are arrows from the vertex i to the vertex j and from j to s, respectively. Thus, $\Omega(\rho) = \Omega(\alpha_j) \circ \Omega(\alpha_i)$ is an element of $\mathrm{Hom}_{\mathcal{G}}(P_{m,i}, J^2 P_{m,s})$, where $f(i) = m = f(s)$. A consecutive application of (7.3.1.3) and (7.3.1.2) implies $\mathrm{Hom}_{\mathcal{G}}(P_{m,i}, P_{m,s}) = (0)$, unless there is an element $t \in \{0, \ldots, p^\ell - 1\}$ such that $M_s \cong M_i \otimes_k k_{t\zeta}$. By assumption, we have $s \neq i$ so that $\Omega(\rho) = 0$ holds whenever $t\zeta = \varepsilon$. Recall that $\zeta = \varepsilon$ if $\ell = 0$ or $\Gamma(\mathcal{G}) \cong \tilde{A}_n$, and that ζ has order 2 if $\Gamma(\mathcal{G}) \cong \tilde{D}_n$. Accordingly, we only need to consider the case where $\ell \neq 0$, $\Gamma(\mathcal{G})$ is an extended Dynkin diagram of type $\tilde{D}_n$, and $\zeta = t\zeta$. Since $s \neq i$, the assumption $M_s \cong M_i \otimes_k k_\zeta$ thus implies $t(\rho) = s = \hat{\zeta}(i) = \hat{\zeta}(s(\rho))$. ◇

(7.3.1.6) *We have $\tilde{D}_n(2)^{\hat{\zeta}} \subset \ker \Omega$.*

Let q_1, q_2 be oriented cycles of $\tilde{D}_n$ originating in a branchpoint i and passing through j and $\hat{\zeta}(j) \neq j$, respectively. Since the path q_2 is uniquely determined by the given properties, we have $\hat{\zeta}(q_1) = q_2$. For the corresponding elements of Λ we thus obtain $\Omega(q_2) = \psi_\zeta(\Omega(q_1))$. However, since $P_{f(i),i} \otimes_k k_\zeta \cong P_{f(i),i}$, it follows that $\psi_\zeta(\Omega(q_1)) = \Omega(q_1)$, so that $q_1 - q_2 \in \ker \Omega$. ◇

Given a projective $\mathcal{B}_0(\mathcal{G})$-module P, we consider the projective $\mathcal{B}_0(\tilde{\mathcal{G}})$-module $\tilde{P} := P/v_0P$. The canonical map

$$\Psi_P : \mathrm{Hom}_{\mathcal{G}}(P, P) \longrightarrow \mathrm{Hom}_{\tilde{\mathcal{G}}}(\tilde{P}, \tilde{P})$$

is surjective, and $\ker \Psi_P = \{f \in \mathrm{Hom}_{\mathcal{G}}(P, P) \; ; \; f(P) \subset v_0 P\}$. By considering the progenerator P_g we obtain a surjection $\Lambda \longrightarrow \tilde{\Lambda}$ between the basic algebras of $\mathcal{B}_0(\mathcal{G})$ and $\mathcal{B}_0(\tilde{\mathcal{G}})$.

(7.3.1.7) *We have $q_{\hat{\zeta}}^{p^\ell} \in \ker \Omega$.*

Given $q \in \tilde{D}_n(2)_{\hat{\zeta}}$, it follows from (7.3.1.3) that $\Psi_{P_g}(\Omega(q)) = 0$. Consequently, $\Omega(q_{\hat{\zeta}})(P_g) \subset v_0 P_g$, so that $\Omega(q_{\hat{\zeta}}^{p^\ell}) = \Omega(q_{\hat{\zeta}})^{p^\ell} = 0$. ◇

(7.3.1.8) *Let $\Gamma(\mathcal{G}) \neq \tilde{A}_1$. If q_1, $q_2 \in \Gamma(\mathcal{G})_c(2)$ are oriented cycles such that $s(q_1) = s(q_2)$, then we have $(q_1 - q_2)^{p^\ell} \in \ker \Omega$.*

We let J be the Jacobson radical of $\mathcal{B}_0(\tilde{\mathcal{G}})$ and observe that the maps $\Psi_{P_g}(\Omega(q_1))$, $\Psi_{P_g}(\Omega(q_2))$ belong to $\mathrm{Hom}_{\tilde{\mathcal{G}}}(\tilde{P}_{f(j),j}, J^2 \tilde{P}_{f(j),j})$ for some $j \in \{1, \ldots, n\}$. In view of $(*)$ (see (7.3.1.3)) the latter space coincides with

$$\mathrm{Hom}_{\tilde{\mathcal{G}}}(\tilde{P}_{f(j),j}, L(f(j)) \otimes_k M_j) \cong \mathrm{Hom}_{\tilde{\mathcal{G}}}(L(f(j)) \otimes_k M_j, L(f(j)) \otimes_k M_j).$$

Since $q_1, q_2 \notin \ker(\Psi_{P_g} \circ \Omega)$ (cf. (7.3.1.1)), there exists a scalar $\alpha \in k \setminus \{0\}$ such that

$$\Psi_{P_g}(\Omega(q_2)) = \alpha \, \Psi_{P_g}(\Omega(q_1)).$$

By re-scaling an arrow, we may assume $\alpha = 1$. If $\Gamma(\tilde{\mathcal{G}}) \ncong \tilde{A}_n$ is a tree, then this can be done successively in such a way that all scalars equal 1 while retaining (7.3.1.6) and (7.3.1.7). Alternatively, the quiver $\Gamma(\mathcal{G})$ has vertex set $\mathbb{Z}/(n)$ for some $n \geq 3$, and arrows $\alpha_i : i \mapsto i+1$ and $\beta_i : i \mapsto i-1$. Thus, we have for every $i \in \mathbb{Z}/(n) = \{0, \ldots, n-1\}$ cycles

$$q_i := \beta_{i+1}\alpha_i \quad \text{and} \quad \hat{q}_i := \alpha_{i-1}\beta_i.$$

After re-scaling, we can assume that the corresponding elements $x_i := \Psi_{P_g}(\Omega(q_i))$ and $\hat{x}_i := \Psi_{P_g}(\Omega(\hat{q}_i))$ of $\tilde{\Lambda}$ satisfy

$$x_i = \hat{x}_i \quad 0 \leq i \leq n-2 \quad ; \quad \hat{x}_{n-1} = a\, x_{n-1}$$

for some $a \in k \setminus \{0\}$. Thanks to (7.3.1.4) the algebra $\mathcal{B}_0(\tilde{\mathcal{G}})$ is symmetric, so that $\tilde{\Lambda}$ also enjoys this property (cf. [17, (I.3.3)]). Hence there exists a linear form $\gamma : \tilde{\Lambda} \longrightarrow k$ such that
(a) $\gamma(xy) = \gamma(yx)$ for $x, y \in \tilde{\Lambda}$, and
(b) $\ker \gamma$ contains no non-zero left ideals of $\tilde{\Lambda}$.
In view of $(\ast)$ we have $J^3 = (0)$, so that $I := kx_{n-1}$ is a nonzero left ideal of $\tilde{\Lambda}$. Thus, condition (b) yields $\gamma(x_{n-1}) \neq 0$. Moreover, writing $\mu_i := \Psi_{P_g}(\Omega(\alpha_i))$ and $\nu_i := \Psi_{P_g}(\Omega(\beta_i))$ we obtain from (a) the identities

$$\gamma(\hat{x}_i) = \gamma(\mu_{i-1}\nu_i) = \gamma(\nu_i\mu_{i-1}) = \gamma(x_{i-1}) \quad 0 \leq i \leq n-1.$$

Moving around the circle, we see that

$$a\gamma(x_{n-1}) = \gamma(\hat{x}_{n-1}) = \gamma(x_{n-2}) = \gamma(\hat{x}_{n-2}) = \cdots = \gamma(\hat{x}_0) = \gamma(x_{n-1}),$$

whence $a = 1$. Consequently, all scalars are equal to 1 in this case as well.

We conclude that $f := \Omega(q_1 - q_2)$ belongs to $\ker \Psi_{P_g}$. Thus $f(P_g) \subset v_0 P_g$ and $f^{p^\ell}(P_g) \subset v_0^{p^\ell} P_g = (0)$, as desired. $\diamond$

Let P and Q be principal indecomposable $\mathcal{B}_0(\mathcal{G})$-modules and put $\tilde{P} := P/v_0P$, $\tilde{Q} := Q/v_0Q$. In view of (7.3.1.1) we have

$$\begin{aligned}
\dim_k \operatorname{Hom}_{\mathcal{G}}(P,Q) &= \sum_{i=0}^{p^\ell-1} \dim_k \operatorname{Hom}_{\mathcal{G}}(P, v_0^iQ/v_0^{i+1}Q) = \sum_{i=0}^{p^\ell-1} \dim_k \operatorname{Hom}_{\mathcal{G}}(P, (Q/v_0Q)\otimes_k k_{i\zeta}) \\
&= \sum_{i=0}^{p^\ell-1} \dim_k \operatorname{Hom}_{\mathcal{G}}(P/v_0P, (Q/v_0Q)\otimes_k k_{i\zeta}) \\
&= \frac{p^\ell+1}{2} \dim_k \operatorname{Hom}_{\tilde{\mathcal{G}}}(\tilde{P}, \tilde{Q}) + \frac{p^\ell-1}{2} \dim_k \operatorname{Hom}_{\tilde{\mathcal{G}}}(\tilde{P}, \tilde{Q}\otimes_k k_\zeta).
\end{aligned}$$

In the following, we let $(a_{ij})_{1\leq i,j\leq n}$ be the incidence matrix of the McKay quiver $\Upsilon_{L(1)^{[1]}}(\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1)$ of the linearly reductive group $\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1$. Thanks to $(\ast)$ the principal indecomposable module $\tilde{P}_{f(j),j}$ has a filtration

$$L(f(j))\otimes_k M_j,\ L(p-2-f(j))\otimes_k (L(1)^{[1]}\otimes_k M_j),\ L(f(j))\otimes_k M_j.$$

The second factor of the middle term decomposes into

$$L(1)^{[1]}\otimes_k M_j \cong \bigoplus_{i=1}^n a_{ij}M_i.$$

As a result, the Cartan invariants of $\mathcal{B}_0(\mathcal{G})$ are given by

$$\begin{aligned}\dim_k \operatorname{Hom}_{\mathcal{G}}(P_{f(i),i}, P_{f(j),j}) &= \frac{p^\ell+1}{2}\dim_k \operatorname{Hom}_{\tilde{\mathcal{G}}}(\tilde{P}_{f(i),i}, \tilde{P}_{f(j),j}) \\ &+ \frac{p^\ell-1}{2}\dim_k \operatorname{Hom}_{\tilde{\mathcal{G}}}(\tilde{P}_{f(i),i}, \tilde{P}_{f(j),j}\otimes_k k_\zeta) \\ &= \frac{p^\ell+1}{2}(2\delta_{ij}+a_{ij}) + \frac{p^\ell-1}{2}(2\delta_{i\hat{\zeta}(j)}+a_{i\hat{\zeta}(j)}).\end{aligned}$$

Thus, the principal indecomposable Λ-module $\operatorname{Hom}_{\mathcal{G}}(P_{f(i),i}, P_g)$ corresponding to the vertex i has dimension

$$\dim_k \operatorname{Hom}_{\mathcal{G}}(P_{f(i),i}, P_g) = \frac{p^\ell+1}{2}(2+n(i)) + \frac{p^\ell-1}{2}(2+n(i)) = p^\ell(2+n(i)),$$

where $n(i) = \sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{ji}$ is the number of arrows originating in the vertex$(f(i), i)$ of the McKay quiver $\Upsilon_{L(1)^{[1]}}(\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_1)$.

Turning to the algebra $\mathcal{S}(\Gamma(\mathcal{G}), \ell)$, we denote by $\hat{P}_i$ the principal indecomposable $\mathcal{S}(\Gamma(\mathcal{G}), \ell)$-module belonging to the vertex $i \in \{1, \ldots, n\}$. Under the general hypothesis $\Gamma(\mathcal{G}) \not\cong \tilde{A}_1$, we first consider the case where $\ell = 0$ or $\Gamma(\mathcal{G}) \not\cong \tilde{D}_n$. Then the relations given in (7.3.1.5) and (7.3.1.8) imply

$$\dim_k \hat{P}_i = 2 + n(i)(2p^\ell - 1).$$

Thanks to Theorem 7.2.3 the quiver $\Gamma(\mathcal{G})$ is of type $\tilde{A}$ or $\tilde{D}$ whenever $\ell \neq 0$. In view of our current assumption we thus have $\ell = 0$ or $n(i) = 2$. In each of these instances we obtain $\dim_k \hat{P}_i = p^\ell(2 + n(i))$.

Next, we assume that $\ell > 0$ and $\Gamma(\mathcal{G}) \cong \tilde{D}_n$. If $n(i) = 2$, then i is neither a branchpoint nor an endpoint, and the above reasoning shows that $\dim_k \hat{P}_i = p^\ell(2 + n(i))$. Alternatively, detailed computations using (7.3.1.5-7.3.1.8) also yield this result.

As an upshot of our discussion, we obtain $\dim_k \Lambda = \dim_k \mathcal{S}(\Gamma(\mathcal{G}), \ell)$. Thanks to (7.3.1.5-7.3.1.8) the algebra Λ is a factor of $\mathcal{S}(\Gamma(\mathcal{G}), \ell)$ whenever $\Gamma(\mathcal{G}) \not\cong \tilde{A}_1$. Consequently, both algebras are isomorphic in all these cases.

If $\Gamma(\mathcal{G}) \cong \tilde{A}_1$, then (7.2.3) in conjunction with our assumption $\mathcal{M} = e_k = \mathcal{C}_{\mathcal{G}}$ implies that $\mathcal{G}$ is an infinitesimal group of height 1, and our assertion follows from [20, (7.1)]. □

Remarks. We collect a few data resulting from the proof of (7.3.1).

(1) The hearts (see Section 7.4 for the definition) of principal indecomposable $\mathcal{B}_0(\mathcal{G})$-modules may have up to three summands.

(2) Let $\mathcal{S}(\mathcal{G})$ be the set of isoclasses of simple $\mathcal{B}_0(\mathcal{G})$-modules. Then the length of the projective cover $P(S)$ of $S \in \mathcal{S}(\mathcal{G})$ is given by

$$l(P(S)) = p^\ell(2 + \sum_{T\in\mathcal{S}(\mathcal{G})} \dim_k \operatorname{Ext}^1_{\mathcal{G}}(S, T)).$$

(3) Consider the finite algebraic group $\mathcal{SC}_{(2)}$. Then we have $\Gamma(\mathcal{SC}_{(2)}) \cong \tilde{A}_3$ and $\mathcal{SC}^0_{(2)} \cong \mathrm{SL}(2)_1$. The proof of (7.3.1) implies

$$\dim_k \mathcal{B}_0(\mathcal{SC}_{(2)}) = 2\dim_k \mathcal{B}_0(\mathrm{SL}(2)_1).$$

Thus, the block $\mathcal{B}_0(\mathcal{SC}_{(2)})$ is properly contained in $\mathcal{B}_0(\mathrm{SL}(2)_1)[\mathcal{SC}_{(2)}(k)] \cong \mathcal{B}_0(\mathrm{SL}(2)_1)[\mathbb{Z}/(4)]$ (cf. (1.1(4)).

If the finite algebraic group $\mathcal{G}$ does not possess a non-trivial linearly reductive normal subgroup, then the foregoing arguments enable us to determine the block structure of the Hopf algebra $H(\mathcal{G})$. As we shall see, the representation-finite blocks of $H(\mathcal{G})$ are Nakayama algebras. Thanks to [37, Satz 8] an indecomposable self-injective Nakayama algebra is uniquely determined by its Loewy length and the number and dimensions of its simple modules. Accordingly, every such algebra is Morita equivalent to the unique indecomposable self-injective basic Nakayama algebra $\mathcal{N}(n, m)$ of Loewy length m and with n simple modules.

There are two approaches leading to our next result. Translation functors provide a conceptual background explaining the uniformity of the asserted block distribution. Rather than working out the details, we shall be content with the observation that the arguments employed in (7.3.1) do in fact apply to an entire class of blocks. We let $\lceil\,\rceil$ be the ceiling function.

Theorem 7.3.2. *Let $\mathcal{G}$ be a finite algebraic group of characteristic $p \geq 3$ with tame principal block. If $\mathcal{G}_{\rm lr} = e_k$, then the block structure of $H(\mathcal{G})$ is given as follows:*

(1) *The algebra $H(\mathcal{G})$ possesses exactly $\frac{p-1}{2}$ tame blocks $\mathcal{B}_0, \ldots, \mathcal{B}_{\frac{p-3}{2}}$, each of which is Morita equivalent to the principal block $\mathcal{B}_0 = \mathcal{B}_0(\mathcal{G})$.*

(2) *All other blocks of $H(\mathcal{G})$ are Nakayama algebras of Loewy length p^ℓ, where ℓ is the length of the unipotent group $\mathrm{Cent}(\mathcal{G}^0)$. If $f : \Gamma(\mathcal{G}) \longrightarrow \{0, 1\}$ is a 2-colouring of $\Gamma(\mathcal{G})$ which assigns 0 to the trivial module, then $H(\mathcal{G})$ possesses $|f^{-1}(0)|$ blocks of Morita type $\mathcal{N}(1, p^\ell)$ if $\ell = 0$ or $\Gamma(\mathcal{G})$ is not of type $\tilde{D}$. Alternatively, $\Gamma(\mathcal{G}) \cong \tilde{D}_s$ and there are $\lceil\frac{s-3}{2}\rceil$ blocks of Morita type $\mathcal{N}(1, p^\ell)$ having a simple module of dimension $2p$ and $\frac{3+(-1)^s}{2}$ blocks of Morita type $\mathcal{N}(2, p^\ell)$ with two simple modules of dimension p.*

In particular, the algebra $H(\mathcal{G})$ is tame.

Proof. We adopt the notation of (7.3.1). According to (7.1.1) the order of $\mathcal{G}(k)$ is not divisible by p, so that our assumption in conjunction with (1.2) yields $\mathcal{C}_{\mathcal{G}} = e_k$. By the same token, $\mathcal{B}_0(\mathcal{G}^0)$ is tame and [21, (3.4)] provides $\ell \geq 0$ and $r \geq 1$ such that

$$\mathcal{C} := \mathrm{Cent}(\mathcal{G}^0) \cong (\mathcal{W}_\ell)_1 \quad \text{and} \quad \mathcal{G}/\mathcal{C} \cong \mathcal{SC}_{(p^r)}.$$

We begin by assuming $\ell = 0$, and denote by $\hat{\mathcal{G}} \subset \mathrm{SL}(2)$ the covering group of the amalgamated polyhedral group $\mathcal{G} \subset \mathrm{PSL}(2)$ (cf. (7.1.2)). As before, we let $\{M_1, \ldots, M_n\}$ be a complete set of representatives of the isomorphism classes of the linearly reductive group $\hat{\mathcal{G}}/\hat{\mathcal{G}}_1$. Recall that the simple $\hat{\mathcal{G}}$-modules are of the form $L(i) \otimes_k M_s$ for $0 \leq i \leq p-1$ and $1 \leq s \leq n$. As noted in (7.2.3) the extensions between these modules are given by

$$\mathrm{Ext}^1_{\hat{\mathcal{G}}}(L(i) \otimes_k M_s, L(j) \otimes_k M_t) \cong \begin{cases} \mathrm{Hom}_{\hat{\mathcal{G}}/\hat{\mathcal{G}}_1}(M_s, L(1)^{[1]} \otimes_k M_t) & i + j = p-2 \\ (0) & \text{otherwise.} \end{cases}$$

For each $i \in \{0, \ldots, p-2\}$ we let $f_i : \bar{\Upsilon}_{L(1)^{[1]}}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1) \longrightarrow \{i, p-2-i\}$ be the 2-colouring that assigns i to the vertex $M_1 \cong k$. By the above formula, the connected component C_i of the Gabriel quiver of $H(\hat{\mathcal{G}})$ containing the simple module $L(i) \cong L(i) \otimes_k M_1$ has underlying vertex set $\{L(f_i(s)) \otimes_k M_s \ ; \ 1 \leq s \leq n\}$ and

$$\Upsilon_{L(1)^{[1]}}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1) \longrightarrow C_i \ \ ; \ \ M_s \mapsto L(f_i(s)) \otimes_k M_s$$

induces an isomorphism of quivers. In this fashion we obtain $p-1$ blocks $\hat{\mathcal{B}}_0, \ldots, \hat{\mathcal{B}}_{p-2}$ of $H(\hat{\mathcal{G}})$, each having Gabriel quiver $\Gamma(\hat{\mathcal{G}}) = \Gamma(\mathcal{G})$. Thanks to [30, Thm. 3] the arguments of (7.3.1) show that each $\hat{\mathcal{B}}_i$ is Morita equivalent to $\mathcal{S}(\Gamma(\mathcal{G}), 0)$. The remaining n blocks of $H(\hat{\mathcal{G}})$ are simple, each affording an irreducible module $L(p-1) \otimes_k M_s$ for some $s \in \{1, \ldots, n\}$.

Recall that $\hat{\mathcal{G}}/\mathcal{N} \cong \mathcal{G}$, where $\mathcal{N} = \mathrm{Cent}(\mathrm{SL}(2))$ is a reduced subgroup of order 2. The arguments of (1.1(3)) now show that the blocks of $H(\mathcal{G})$ are exactly those blocks of $H(\hat{\mathcal{G}})$ on which $\mathcal{N}$ acts trivially. Since the centralizer $\mathrm{Cent}_{\mathcal{N}}(M)$ of an $\mathcal{N}$-module M is a closed subgroup of $\mathcal{N}$ satisfying $\mathrm{Cent}_{\mathcal{N}}(M)(k) = \mathrm{Cent}_{\mathcal{N}(k)}(M)$ (cf. [35, (I.2.6(9),(12))]), the reducedness of $\mathcal{N}$ implies $\mathrm{Cent}_{\mathcal{N}}(M) = \mathcal{N}$ if and only if the element $c := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ operates trivially on M. As c acts on $L(i)$ via $(-1)^i$ exactly those blocks among $\hat{\mathcal{B}}_0, \ldots, \hat{\mathcal{B}}_{p-2}$ containing $L(2i)$ for some $0 \le i \le \frac{p-3}{2}$ occur as blocks of $H(\mathcal{G})$.

To determine the simple blocks of $H(\mathcal{G})$ we consider the colouring $f := f_0$ of $\Gamma(\hat{\mathcal{G}}) = \Gamma(\mathcal{G})$. Since $\mathcal{N}$ is reduced, the canonical quotient map $\hat{\mathcal{G}} \longrightarrow \hat{\mathcal{G}}/\hat{\mathcal{G}}_1$ induces an embedding $\mathcal{N} \hookrightarrow \hat{\mathcal{G}}/\hat{\mathcal{G}}_1$. By our above observations the element c acts on $L(1)^{[1]}$ via -1. Thus, if c acts on M_s via α, then it acts on $L(1)^{[1]} \otimes_k M_s$ via $-\alpha$, so that $-\alpha$ is the only eigenvalue of c on any neighbour of M_s within the McKay quiver $\Upsilon_{L(1)^{[1]}}(\hat{\mathcal{G}}/\hat{\mathcal{G}}_1)$. As a result, only those simple blocks corresponding to modules of the form $L(p-1) \otimes_k M_s$ with $f(s) = 0$ occur in $H(\mathcal{G})$. This concludes the proof for the case $\ell = 0$.

We now consider the case $\ell \neq 0$ and put $\tilde{\mathcal{G}} := \mathcal{G}/\mathcal{C}$, where $\mathcal{C} := \mathrm{Cent}(\mathcal{G}^0)$. Since $\mathcal{C}$ is unipotent, the groups $\mathcal{G}$ and $\tilde{\mathcal{G}}$ have the same simple modules. Thanks to the first part of our proof, the algebra $H(\tilde{\mathcal{G}})$ has tame blocks $\tilde{\mathcal{B}}_0, \ldots, \tilde{\mathcal{B}}_{\frac{p-3}{2}}$ with $\tilde{\mathcal{B}}_i$ containing the simple module $L(2i)$. For each $i \in \{0, \ldots, \frac{p-3}{2}\}$ we let $\mathcal{B}_i$ be the block of $H(\mathcal{G})$ having $L(2i)$ as a simple module. Since $\mathcal{B}_0(\mathcal{G}^0)$ is tame and $\mathcal{M} = e_k$ we obtain from [21, (7.1)] and (5.2.1) the tameness of $H(\mathcal{G})$. The first part of our proof ensures that the conditions of Lemma 7.2.2 apply to $\mathcal{B}_i$ and $\tilde{\mathcal{B}}_i$. Thus, the Gabriel quiver $\Gamma(\mathcal{B}_i)$ is canonically isomorphic to $\Gamma(\tilde{\mathcal{B}}_i)$. In particular, the blocks $\mathcal{B}_i$ are distinct. Owing to [35, (II.3.13)] each $L(i)$ is a self-dual SL(2)-module. It follows that $M \mapsto M^*$ restricts to a duality of $\mathrm{mod}\,\mathcal{B}_i$. Consequently, the arguments of (7.3.1) may be adopted mutatis mutandis to establish a Morita equivalence between $\mathcal{B}_i$ and $\mathcal{S}(\Gamma(\mathcal{G}), \ell)$.

Let S, T be simple $\mathcal{G}$-modules belonging to the remaining blocks of $H(\mathcal{G})$, i.e., $S, T \in \{L(p-1) \otimes M_s \ ; \ 1 \le s \le n\}$. By the first part of our proof, S and T are projective $\tilde{\mathcal{G}}$-modules, and the spectral sequence $H^s(\tilde{\mathcal{G}}, \mathrm{Ext}^t_{\mathcal{C}}(S,T)) \Rightarrow \mathrm{Ext}^{s+t}_{\mathcal{G}}(S,T)$ yields isomorphisms

$$\mathrm{Ext}^1_{\mathcal{G}}(S,T) \cong \mathrm{Ext}^1_{\mathcal{C}}(S,T)^{\tilde{\mathcal{G}}} \cong (H^1(\mathcal{C},k) \otimes_k \mathrm{Hom}_k(S,T))^{\tilde{\mathcal{G}}}.$$

By general theory, the one-dimensional $\mathcal{G}$-module $H^1(\mathcal{C},k)$ is isomorphic to the dual of the adjoint module $H(\mathcal{C})^\dagger/(H(\mathcal{C})^\dagger)^2 \cong k_\zeta$, where $\zeta \in X(\mathcal{G})$ is the character defined in the proof of (7.3.1). Consequently, Schur's Lemma yields

$$\dim_k \mathrm{Ext}^1_{\mathcal{G}}(S,T) = \begin{cases} 1 & \text{if } T \cong S \otimes_k k_\zeta \\ 0 & \text{otherwise.} \end{cases}$$

If $\Gamma(\mathcal{G})$ is not of type $\tilde{D}$, then ζ is trivial, and each block corresponding to a module of type $L(p-1) \otimes_k M_s$ has one simple module. Alternatively, [28, Theorem 2] implies that all but four of the modules M_s have dimension 2. Since tensoring with k_ζ fixes the two-dimensional modules, our assertions concerning the number of simple modules in a block and the number of blocks follow. In view of [32, Thm. 9] the above formula for the extension groups identifies each of these blocks as a Nakayama algebra. The statement concerning the Loewy length is now a direct consequence of (7.3.1.1). □

Remark. Since $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{G}_{\mathrm{lr}})$, Theorem 7.3.2 gives the block structure of the algebra $H(\mathcal{G}/\mathcal{G}_{\mathrm{lr}})$, which, by the proof of (1.1(3)), is a block ideal of $H(\mathcal{G})$.

7.4. **Concluding Remarks.** The foregoing results have a number of consequences and motivate further investigations concerning the Auslander-Reiten theory of tame Hopf algebras. We shall only touch upon a few issues, which are related to earlier work on infinitesimal groups (cf. [21, 22]).

Proposition 7.4.1. *Let $\mathcal{G}$ be a finite algebraic group of characteristic $p \geq 3$. If $\mathcal{B}_0(\mathcal{G})$ is tame, then $H(\mathcal{G})$ is symmetric.*

Proof. Let $P(0)$ be the projective cover of the trivial $\mathcal{G}$-module. We propose to show that

$$\mathrm{Top}(P(0)) \cong \mathrm{Soc}(P(0)).$$

According to (1.1(3)) we may assume that $\mathcal{G}_{\mathrm{lr}} = e_k$. Owing to (7.1.1) and (1.2) the component $\mathcal{G}^0$ is an infinitesimal group with tame principal block and trivial multiplicative center. Thus, by [21, (3.4)] there exists $\ell \geq 0$ such that $\mathcal{C} := \mathrm{Cent}(\mathcal{G}^0) \cong (\mathcal{W}_\ell)_1$.

For $\ell = 0$ the assertion follows from (7.3.1.4). Alternatively, we write $H(\mathcal{C})^\dagger = H(\mathcal{G})v_0$ and consider the character $\zeta : H(\mathcal{G}) \longrightarrow k$ of order ≤ 2. Thanks to (7.3.1.1) there is an isomorphism

$$v_0^{p^\ell - 1} P(0) \cong (P(0)/v_0P(0)) \otimes_k k_{(p^\ell - 1)\zeta} \cong P(0)/v_0P(0)$$

of $\mathcal{G}$-modules. Since $P(0)/v_0P(0)$ is the projective cover of the trivial $\mathcal{G}/\mathcal{C}$-module (cf. (7.3.1.1)), we obtain

$$\mathrm{Soc}(P(0)) = \mathrm{Soc}(v_0^{p^\ell - 1}P(0)) \cong \mathrm{Soc}(P(0)/v_0P(0)) \cong \mathrm{Top}(P(0)/v_0P(0)) \cong \mathrm{Top}(P(0)),$$

as desired. As observed before, this implies that the co-unit ε is the modular function of $H(\mathcal{G})$ and [26, (1.5)] now yields the assertion. □

Let Λ be a k-algebra, M a Λ-module. Then $\mathrm{Ht}(M) := \mathrm{Rad}(M)/\mathrm{Soc}(M)$ is called the *heart* of M. Recall that a self-injective algebra Λ is *biserial* if the heart $\mathrm{Ht}(P)$ of every principal indecomposable Λ-module P is a direct sum of at most two uniserial modules. Following [55, p.174] we refer to Λ as *special biserial* if Λ is Morita equivalent to the bound quiver algebra $k[Q]/I$, where the bound quiver (Q, I) satisfies the following conditions:

(SB1) Each vertex of Q is the starting point and end point of at most two arrows.

(SB2) For any arrow α of Q there is at most one arrow β and one arrow γ such that $\alpha\beta,\ \gamma\alpha \notin I$.

According to [55, Lemma 1] special biserial algebras are biserial. In the modular representation theory of finite groups special biserial algebras occur in conjunction with 2-blocks with dihedral defect groups, see [17, VI].

Proposition 7.4.2. *Let $\mathcal{G}$ be a finite algebraic k-group of characteristic $p \geq 3$, $P(0)$ the projective cover of the trivial $\mathcal{G}$-module. Then the following statements are equivalent:*

(1) *The algebra $\mathcal{B}_0(\mathcal{G})$ is special biserial.*

(2) *The algebra $\mathcal{B}_0(\mathcal{G})$ is tame, and $\mathrm{Ht}(P(0))$ is decomposable.*

(3) *The algebra $\mathcal{B}_0(\mathcal{G}^0)$ is tame, $p \nmid \mathrm{ord}(\mathcal{G}(k))$, and $\mathcal{G}(k)/\mathcal{C}_{\mathcal{G}}(k)$ is cyclic.*

Proof. (1) ⇒ (2) Suppose that $\mathcal{B}_0(\mathcal{G})$ is special biserial. Thanks to [63, (2.4)] and [55, Lemma 1] the algebra $\mathcal{B}_0(\mathcal{G})$ is tame and biserial. Accordingly, $\mathrm{Ht}(P(0))$ is a direct sum of at most two uniserial modules. If $\mathrm{Ht}(P(0))$ is indecomposable, then there is exactly one arrow originating in the vertex k of the Gabriel quiver $\Gamma(\mathcal{G})$. Owing to (7.2.3) there then also exists a vertex which is the starting point of at least three arrows. Thus, $\mathcal{B}_0(\mathcal{G})$ is not special biserial, a contradiction.

(2) ⇒ (3) Since $\mathrm{Ht}(P(0))$ is decomposable, its top $\mathrm{Rad}(P(0))/\mathrm{Rad}^2(P(0))$ is not simple. Hence there are at least two arrows originating in the vertex of $\Gamma(\mathcal{G})$ corresponding to the trivial module. From (7.2.3) and [28, Theorem 2] it now follows that $\Gamma(\mathcal{G})$ is an extended Dynkin diagram of type

$\tilde{A}_n$. By the same token, the group $\mathcal{G}(k)/\mathcal{C}_{\mathcal{G}}(k)$ is cyclic, and the remaining assertions follow from (7.1.1).

(3) $\Rightarrow$ (1) According to (7.1.1) the algebra $\mathcal{B}_0(\mathcal{G})$ is tame. As $\mathcal{G}(k)/\mathcal{C}_{\mathcal{G}}(k)$ is cyclic, (7.2.3) shows that $\Gamma(\mathcal{G})$ is an extended Dynkin diagram of type $\tilde{A}_n$. The assertion now follows from (7.3.1). □

The class of tame algebras may be further subdivided according to the number of parametrizing families needed in each dimension. A representation-infinite k-algebra Λ is *domestic* if there exists a natural number $n \in \mathbb{N}$ with the property that for each $d > 0$ there exist at most n $(\Lambda, k[T])$-bimodules $X_1, \ldots, X_{\ell(d)}$ that are free $k[T]$-modules of rank d, such that all but finitely many isoclasses of indecomposable Λ-modules of dimension d are of the form $[X_i \otimes_{k[T]} k_\lambda]$ for some $i \in \{1, \ldots, \ell(d)\}$ and some algebra homomorphism $\lambda : k[T] \longrightarrow k$. In view of [11], this is equivalent to Ringel's original definition, cf. [52]. Examples of domestic algebras include the tame hereditary algebras ([12, 43]), 2-blocks of group algebras with defect a Klein four group (cf. [18]), as well as the tame blocks of the distribution algebras associated to Frobenius kernels of smooth groups (see [19]).

Given an algebra H, we denote by $T(H) := H \ltimes D(H)$ the *trivial extension of* H by its bimodule $D(H) := H^*$. Note that each basic algebra $\mathcal{S}(Q, 0)$ is isomorphic to the trivial extension of a radical square zero hereditary algebra of type Q. Trivial extensions are easily seen to be symmetric algebras, cf. [27, (V.1.1)].

A detailed analysis shows that there are analogs of (5.1.4) and (5.2.1) for domestic algebras. By combining the results of the foregoing subsections with those of [22], we thus arrive at the following characterization of domestic algebraic groups:

Proposition 7.4.3. *Let $\mathcal{G}$ be a finite algebraic group of characteristic $p \geq 3$. Then the following statements are equivalent:*

(1) *The principal block $\mathcal{B}_0(\mathcal{G})$ is domestic.*

(2) *The algebra $\mathcal{B}_0(\mathcal{G})$ is Morita equivalent to the trivial extension $T(H)$ of a radical square zero tame hereditary algebra H.*

(3) *The Hopf algebra $H(\mathcal{G})$ is domestic.*

Proof. (1) $\Rightarrow$ (2) The abovementioned result ensures that $\mathcal{B}_0(\mathcal{G}^0)$ is domestic or representation finite. Since [23, (3.1)] excludes the latter alternative, we may apply [22, (5.1)] to see that the group $\mathrm{Cent}(\mathcal{G}^0)/\mathcal{M}$ is trivial. Thus, (7.3.1) provides a Morita equivalence between $\mathcal{B}_0(\mathcal{G})$ and $\mathcal{S}(\Gamma(\mathcal{G}), 0)$, with the latter algebra being a trivial extension of a radical square zero tame hereditary algebra.

(2) $\Rightarrow$ (3) Thanks to [12, 43] every tame hereditary algebra is domestic. In view of [60] this also holds for the trivial extensions of these algebras. Consequently, $\mathcal{B}_0(\mathcal{G})$ is domestic. Thus, $\mathcal{B}_0(\mathcal{G}^0)$ also enjoys this property and [22, (5.1)] yields the domesticity of $H(\mathcal{G}^0)$. Thanks to (6.2.1) the characteristic of k does not divide the order of $\mathcal{G}(k)$, so that the analog of (5.2.1) implies that $H(\mathcal{G}) = H(\mathcal{G}^0)[\mathcal{G}(k)]$ is domestic.

(3) $\Rightarrow$ (1) By general theory, the block $\mathcal{B}_0(\mathcal{G})$ is domestic or representation-finite. In the latter case, [23, (3.1)] shows that $H(\mathcal{G})$ is of finite representation type, a contradiction. □

Remarks. (1) By the remark following (7.3.1), groups satisfying $\Gamma(\mathcal{G}) \cong \tilde{E}_n$ are domestic.

(2) The Auslander-Reiten theory of trivial extensions of hereditary algebras is well-understood (cf. [53, 60] and [27, (V.3.2)]). Let Δ be a Euclidean diagram, $k[\Delta]$ a corresponding radical square zero hereditary algebra. The stable Auslander-Reiten quiver of $T(k[\Delta])$ has two components of Euclidean type $\mathbb{Z}[\Delta]$, infinitely many components of type $\mathbb{Z}[A_\infty]/(\tau)$ (homogeneous tubes), and

two exceptional tubes of each type $\mathbb{Z}[A_\infty]/(\tau^{n_i})$. Here the n_i constitute the *tubular type* of $k[\Delta]$ (see [53, (3.6.5)]). These numbers also occur in the classification of the binary polyhedral groups (cf. [56, (4.4)]); the precise connection was given by Lusztig in [39]. According to [22, (4.1)] the tubular type of a domestic infinitesimal group $\mathcal{G}$ is (p^{r-1}, p^{r-1}), where $r := \mathrm{ht}(\mathcal{G}/\mathcal{M}(\mathcal{G}))$ is the height of the factor group $\mathcal{G}/\mathcal{M}(\mathcal{G})$. The number $2p^{r-1}$ is the maximal degree of a homogeneous system of parameters of the even cohomology ring $H^{\mathrm{ev}}(\mathcal{G}, k)$.

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