Dynkin Diagrams, Support Spaces and Representation Type

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Advances in Group Theory and Applications
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Lecture I: Motivation and Basic Examples

Background

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- Let ind_Λ^d be the set of indecomposable modules of mod_Λ^d.

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Remarks. (1) If Λ is representation-finite, then there are only finitely many isoclasses of indecomposable Λ-modules (Brauer-Thrall II).
(2) If an algebra is wild, then its module category is at least as complicated as that of any other algebra (Drozd, 1977).

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   In either case, the indecomposable modules can be classified via the associated root system.

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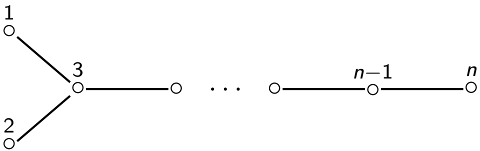

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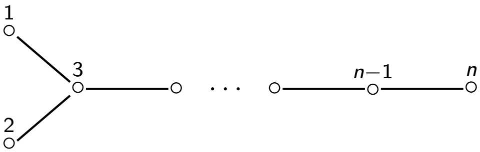

$\mathfrak{O}(2n, \mathbb{C})$

Dynkin diagrams

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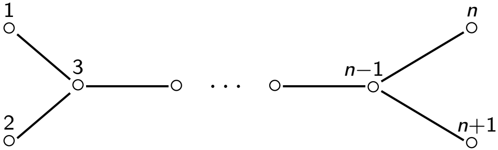

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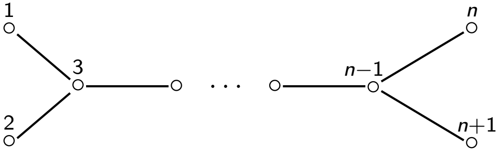

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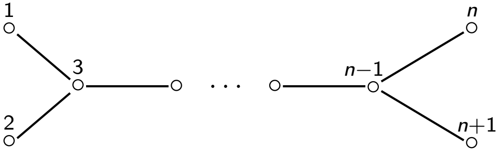

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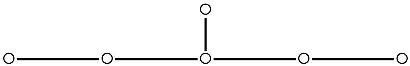
$\mathbb{E}_6$



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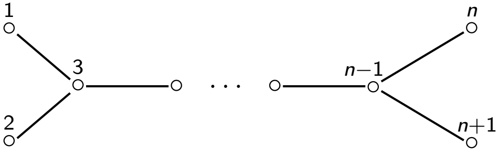

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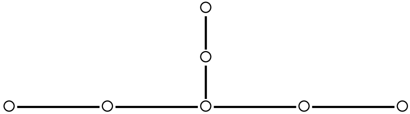

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  where the quiver $\tilde{A}_{p-1}$ is the clockwise oriented circle with p vertices.

Example

- Let $\mathrm{char}(k) = p > 0$, and consider the algebra given by
- $$\Lambda = k\langle t, x\rangle / (tx - xt - x, t^p - t, x^p).$$
- This is the natural presentation of the restricted enveloping algebra of the two-dimensional, non-abelian Lie algebra.
- The bound quiver presentation is
$$\Lambda \cong k\tilde{A}_{p-1} / \mathrm{rad}(k\tilde{A}_{p-1})^p,$$
where the quiver $\tilde{A}_{p-1}$ is the clockwise oriented circle with p vertices.
- Hence Λ is a Nakayama algebra.

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- About 20 years later, Karin Erdmann classified blocks of tame representation type via the stable Auslander-Reiten quiver.

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By definition, there exists a commutative k-algebra $k[\mathcal{G}]$ such that

$$\mathcal{G}(R) = \mathrm{Hom}_{M_k}(k[\mathcal{G}], R).$$

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- If $k[\mathcal{G}]$ is finitely generated, then $\mathcal{G}$ is an algebraic group.

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$$k[G] := kG^* \cong \bigoplus_{\lambda \in X(G)} k_\lambda$$

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We shall henceforth assume that $\mathrm{char}(k) = p > 0$.

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- The infinitesimal group $\mathcal{Q}_{(p^r)} = \mathrm{SL}(2)_1 T_r$ is the product of the first Frobenius kernel of SL(2) with the r-th Frobenius kernel of its standard maximal torus T.

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The latter part is equivalent to saying that $\mathcal{G}^0$ contains no subgroup of type $\mathbb{G}_{a(1)}$.