Dynkin Diagrams, Support Spaces and Representation Type

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Lecture II: Support Varieties and Rank Varieties of Restricted Lie Algebras
Recollection
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- $G$ is a finite group scheme over $k$. 

$G^0 \triangleleft G$ is a normal infinitesimal subgroup, $G^0(k) = \{1\}$.

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We are interested in the representation type of $kG$. 

First step: Semi-simplicity of $kG$.

Theorem (Nagata)

Let $G$ be a finite algebraic group. Then $kG$ is semi-simple if and only if

(a) $p \nmid \text{ord}(G(k))$,

(b) $G_0 \sim \prod G_{r_i}(r_i)$.

The latter part is equivalent to saying that $G_0$ contains no subgroup of type $G_a(1)$. 
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The latter part is equivalent to saying that $G_0$ contains no subgroup of type $G^1$. 
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**Theorem (Nagata)**

Let \( G \) be a finite algebraic group.

\( \text{\textit{ord}}(G) \) is the order of \( G \) in the group \( G(k) \).
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The latter part is equivalent to saying that \( G^0 \) contains no subgroup of type \( \mathbb{G}_a(1) \).
The Friedlander-Suslin Theorem

Definition

Let $M$ be a $G$-module. Then $\text{Ext}^*_{\mathbb{Z}}(M, M) := \bigoplus_{n \geq 0} \text{Ext}^n_{\mathbb{Z}}(M, M)$ is the Yoneda algebra of self-extensions of $M$.

Elements of $\text{Ext}^n_{\mathbb{Z}}(M, M)$:

$$(0) \rightarrow M \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_n \rightarrow M \rightarrow (0)$$

Multiplication by splicing.
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\[ \text{Ext}^m G(M, M) \times \text{Ext}^n G(M, M) \rightarrow \text{Ext}^{m+n} G(M, M) \]
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is the even cohomology ring of $G$. 

Theorem (Friedlander-Suslin, 1997)

1. The commutative $k$-algebra $H^\bullet(G, k)$ is finitely generated.

2. The homomorphism $\Phi_M : H^\bullet(G, k) \to \text{Ext}^\ast_G(M, M)$; $[f] \to [f \otimes \text{id}_M]$ is finite.
If $M = k$ is the trivial $G$-module, then

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**Theorem (Friedlander-Suslin, 1997)**

*Let $M$ be a $G$-module.*
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Support Varieties

Let $R$ be a commutative $k$-algebra. Then $\text{Maxspec}(R) := \{ M \subseteq R ; \text{maximal ideal} \}$ is called the maximal spectrum of $R$. Given $I \subseteq R$, we put $Z(I) := \{ M \in \text{Maxspec}(R) ; I \subseteq M \}$. The $Z(I)$ are the closed subsets of the Zariski topology of $\text{Maxspec}(R)$. If $R$ is a finitely generated $k$-algebra, then $\text{Maxspec}(R)$ is an affine variety.
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- The $Z(I)$ are the closed subsets of the Zariski topology of Maxspec($R$).
- If $R$ is a finitely generated $k$-algebra, then Maxspec($R$) is an affine variety.
Let $M$ be a $G$-module.
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**Definition**

The affine variety $V(M) := Z(\ker \Phi_M) \subseteq \text{Maxspec}(H^\bullet(G, k))$ is called the cohomological support variety of $M$.

Is this definition useful? How can support varieties be computed?
Let $M$ be a $G$-module.

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- Is this definition useful?
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Theorem

Let \( B \subseteq kG \) be a block, \( M \in \text{mod} B \).

1. If \( B \) is representation-finite, then \( \dim V(G) M \leq 1 \).
2. If \( B \) is tame, then \( \dim V(G) M \leq 2 \).

Example

Let \( kG = k \left( \mathbb{Z} / (p) \right)^r \).

Then \( H^\ast(G, k) := k\left[ X_1, \ldots, X_r \right] \otimes_k \Lambda(\{Y_1, \ldots, Y_r\}) \), with \( \deg(X_i) = 2 \), \( \deg(Y_i) = 1 \).

We thus obtain:

\[ V(G) k = \text{Maxspec}(H^\ast(G, k)) \sim A_r. \]

\( kG \) is representation-finite \( \Rightarrow r = 1. \)

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Theorem

Let $\mathcal{B} \subseteq kG$ be a block, $M \in \text{mod} \, \mathcal{B}$.

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$V(G) \sim \text{Maxspec}(H^\ast(G, k)) \sim A_r$.

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Theorem

Let $\mathcal{B} \subseteq kG$ be a block, $M \in \text{mod} \mathcal{B}$.

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Let $kG = k \left( \mathbb{Z}/(p) \right)^r$. Then $H^\ast(G, k) := k \left[ X_1, \ldots, X_r \right] \otimes k \Lambda(Y_1, \ldots, Y_r)$.

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$V(G)_k = \text{Maxspec}(H^\ast(G, k)) \cong A_r$. 

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Theorem

Let \( \mathcal{B} \subseteq kG \) be a block, \( M \in \text{mod} \mathcal{B} \).

1. If \( \mathcal{B} \) is representation-finite, then \( \dim V(G)_{\mathcal{G}} \leq 1 \).
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Theorem

Let $B \subseteq kG$ be a block, $M \in \text{mod } B$.

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Let $kG = k(\mathbb{Z}/(p))^r$. Then

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Theorem

Let $\mathcal{B} \subseteq k\mathcal{G}$ be a block, $M \in \text{mod } \mathcal{B}$.

1. If $\mathcal{B}$ is representation-finite, then $\dim V(\mathcal{G})_M \leq 1$.
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Lie Algebras

**Definition**

Let $\Delta : kG \to kG \otimes_k kG$ denote the comultiplication of $kG$. Then 

$$\text{Lie}(G) := \{ x \in kG ; \Delta(x) = x \otimes 1 + 1 \otimes x \}$$

is called the Lie algebra of $G$.

Writing $[x, y] = xy - yx$, we have

(a) $[x, y] \in \text{Lie}(G)$ for every $x, y \in \text{Lie}(G)$, and

(b) $x_p \in \text{Lie}(G)$ for every $x \in \text{Lie}(G)$. 

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(a) $[x, y] \in \text{Lie}(G)$ for every $x, y \in \text{Lie}(G)$, and

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U_0(g) := U(g)/\langle \{ x^p - x^{[p]} \ ; \ x \in g \} \rangle
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The algebra $U_0(g)$ inherits the Hopf algebra structure from $U(g)$. 

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Definition

Let $G$ be an infinitesimal group. The minimal $r \geq 0$ such that $x^p = 0$ for all $x \in k[G]$ is called the height of $G$.

Proposition

Let $G$ be an infinitesimal group of height $\leq 1$. Then there exists an isomorphism $k[G] \cong U_0(\text{Lie}(G))$ of Hopf algebras.
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Examples

Let $g := \text{sl}(2)$ be the Lie algebra of trace zero $(2 \times 2)$-matrices. $U_0(\text{sl}(2))$ has exactly $p$ simple modules $L_0, \ldots, L_{p-1}$ with $\dim kL_i = i + 1$. For $p \geq 3$, the algebra $U_0(\text{sl}(2))$ has blocks $B_0, \ldots, B_{p-3}$, each $B_i$ possessing two simple modules $L_i$ and $L_{p-2-i}$.

There is one additional simple block $B_{p-1}$ belonging to the Steinberg module $L_{p-1}$.

Quiver and relations of $U_0(\text{sl}(2))$: (Drozd, Rudakov, Fischer early 1980's).
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This is a special biserial algebra.
Examples

Let $V$ be a $k$-vector space, $t: V \rightarrow V$ be a non-zero linear transformation satisfying $t^p = t$. Then $g(t, V) := kt \oplus V$ is a restricted Lie algebra via $[(\alpha t, v), (\beta t, w)] := (0, \alpha t(w) - \beta t(v))$ and $(\alpha t, v)[p] := (\alpha p t, \alpha p - 1 t - 1(v))$.

Abstract representation theory shows: $U_0(g(t, V))$ is representation-finite $\iff \dim_k V \leq 1$. $U_0(g(t, V))$ is tame $\iff \dim_k V = 2$ and $p = 2$. 
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Suppose that $p \geq 3$, and let $G$ be a solvable infinitesimal group. Then $B^0(G)$ is either representation-finite or wild.
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Definition

Let \((g, \mathcal{P})\) be a restricted Lie algebra. The conical variety \(V^g := \{ x \in g; x[\mathcal{P}] = 0 \}\) is called the nullcone of \(g\).

Let \(M\) be a \(U_0(g)\)-module. Then \(V^g(M) := \{ x \in V^g; M|_k[\{x\}] \text{ is not free} \} \cup \{0\}\) is called the rank variety of \(M\).

Remark: Let \(x \in V^g\). Then \(x \in V^g(M) \iff \text{rk}(x^*M) < \dim_k k \mathcal{P}(\mathcal{P} - 1)\).
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Let \((g, [p])\) be a restricted Lie algebra. The conical variety

\[ V_g := \{ x \in g \mid x^{[p]} = 0 \} \]

is called the nullcone of \(g\). Let \(M\) be a \(U_0(g)\)-module. Then

\[ V_g(M) := \{ x \in V_g \mid M|_{k[x]} \text{ is not free} \} \cup \{0\} \]

is called the rank variety of \(M\).

Remark: Let \(x \in V_g\). Then \(x \in V_g(M) \iff \text{rk}(x_M) < \frac{\dim_k M}{p}(p-1)\).
Example

Let \( g = \text{sl}(2) \). Note that \( V_{\text{sl}(2)} \) is the set of nilpotent \((2 \times 2)\)-matrices, so that
\[
V_{\text{sl}(2)} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a^2 + bc = 0 \right\}.
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Thus, \( V_{\text{sl}(2)} \) is a two-dimensional, irreducible variety.

Recall that there are exactly \( p \)-simple \( U_0(\text{sl}(2)) \)-modules \( L(i) \) with \( 0 \leq i \leq p - 1 \) and \( \dim_k L(i) = i + 1 \).

Let \( x \in V_{\text{sl}(2)} \setminus V_{\text{sl}(2)}(L(i)) \).
\( \Rightarrow \) \( L(i) \) is a free module for the \( p \)-dimensional algebra \( k[x] \).
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\( L(i) = L(p - 1) \) is the Steinberg module, which is projective.

\( V_{\text{sl}(2)}(L(i)) = \left\{ V_{\text{sl}(2)} : i \neq p - 1 \right\} \setminus \{0\} \) for \( i = p - 1 \).
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Let $g = \mathfrak{sl}(2)$. 

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$L_i$ is a free module for the $p$-dimensional algebra $k[x]$. $p | \dim_k L_i$ and $i = p - 1$. $L_i = L_{p - 1}$ is the Steinberg module, which is projective.

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Notation:

Theorem (Jantzen, Friedlander-Parshall) Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then there exists a homeomorphism $\Psi: V(\mathfrak{g})_k \rightarrow V\mathfrak{g}$ such that $\Psi(V(\mathfrak{g})_M) = V\mathfrak{g}(M)$ for every $M \in \text{mod} U_0(\mathfrak{g})$.

Corollary Let $G$ be a finite algebraic group with Lie algebra $\mathfrak{g}$.

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Let $G$ be a finite algebraic group with Lie algebra $g$. 
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Let \((g, [p])\) be a restricted Lie algebra, \(M\) be a \(U_0(g)\)-module. Then the following statements are equivalent:

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Proof.

Suppose \( M \) is projective. Let \( x \in V(g(M)) \).

By the PBW-Theorem, \( U_0(g) \) is a free \( k[x] \)-module.

Hence \( M|_{k[x]} \) is projective, so that \( x = 0 \).

(2) \( \Rightarrow \) (1).

If \( V(g(M)) = \{0\} \), then \( \dim V(g(M)) = 0 \).

\( \Phi_M : H^\bullet(g, k) \to \text{Ext}^\bullet U_0(g)(M, M) \) is a finite morphism.

Since \( \dim H^\bullet(g, k) / \ker \Phi_M = 0 \), the algebra \( \text{Ext}^\bullet U_0(g)(M, M) \) is finite-dimensional.

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(1) \implies (2).

Suppose $M$ is projective. Let $x \in V_{g}(M)$. By the PBW-Theorem, $U_{0}(g)$ is a free $k[x]$-module. Hence $M|_{k[x]}$ is projective, so that $x = 0$.

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If $V_{g}(M) = \{0\}$, then $\dim V_{g}(M) = 0$. \(\Phi_{M}: H_{\bullet}(g, k) \rightarrow \text{Ext}^{\bullet}U_{0}(g)(M, M)\) is a finite morphism. Since $\dim H_{\bullet}(g, k)/\ker \Phi_{M} = 0$, the algebra $\text{Ext}^{\bullet}U_{0}(g)(M, M)$ is finite-dimensional. Hence there exists $n_{0} \in \mathbb{N}$ such that $\text{Ext}^{n}U_{0}(g)(M, -) = 0$ for all $n \geq n_{0}$.

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\begin{itemize}
  \item $\dim \mathcal{V}_g(M)$ has a representation-theoretic interpretation.
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  \item Periodic modules play an important role in Auslander-Reiten theory.
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