Dynkin Diagrams, Support Spaces and Representation Type

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Advances in Group Theory and Applications
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Lecture II: Support Varieties and Rank Varieties of Restricted Lie Algebras

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The latter part is equivalent to saying that $\mathcal{G}^0$ contains no subgroup of type $\mathbb{G}_{a(1)}$.

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Multiplication by splicing.

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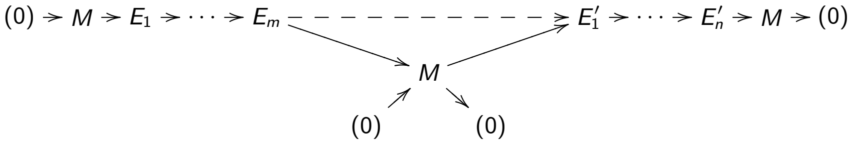

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2. *The homomorphism*

$$\Phi_M : \mathrm{H}^\bullet(\mathcal{G}, k) \longrightarrow \mathrm{Ext}^*_{\mathcal{G}}(M, M) \quad ; \quad [f] \mapsto [f \otimes \mathrm{id}_M]$$

is finite.

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- The $Z(I)$ are the closed subsets of the Zariski topology of $\mathrm{Maxspec}(R)$.
- If R is a finitely generated k-algebra, then $\mathrm{Maxspec}(R)$ is an affine variety.

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- How can support varieties be computed?

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(b) $x^p \in \mathrm{Lie}(\mathcal{G})$ for every $x \in \mathrm{Lie}(\mathcal{G})$.

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Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]}\ ;\ x \in \mathfrak{g}\})$$

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Let $\mathcal{G}$ be an infinitesimal group of height ≤ 1. Then there exists an isomorphism

$$k\mathcal{G} \cong U_0(\mathrm{Lie}(\mathcal{G}))$$

of Hopf algebras.

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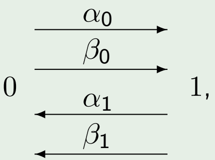

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This is a special biserial algebra.

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Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, M be a $U_0(\mathfrak{g})$-module. Then the following statements are equivalent:

1. *M is projective.*
2. $\mathcal{V}_{\mathfrak{g}}(M) = \{0\}$.

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- Periodic modules play an important role in Auslander-Reiten theory.