Dynkin Diagrams, Support Spaces and Representation Type

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Advances in Group Theory and Applications
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Lecture III: Binary Polyhedral Groups, McKay Quivers, and Tame Blocks

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- $\mathcal{B}_0(\mathcal{G}) \subseteq k\mathcal{G}$ is the principal block of $\mathcal{G}$.

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3. $k\mathcal{G}$ *has finite representation type.*

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- $k\mathcal{G}$ *is a Nakayama algebra.*

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Recall: If G is a finite group, then $p = 2$, and the Sylow-2-subgroups of G are dihedral, semidihedral, or generalized quaternion.

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(b) $\mathfrak{g}/C(\mathfrak{g}) \cong \mathfrak{sl}(2)$, *where $C(\mathfrak{g})$ denotes the center of $\mathfrak{g}$.*

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where $\psi : \mathfrak{sl}(2) \longrightarrow V$ is p-semilinear.

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- For finite groups, the Mackey decomposition theorem implies that subgroups of tame groups are always tame.

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- Fix a $\mathcal{G}$-module V. Then V defines an $(n \times n)$-matrix $(a_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$ via

$$V \otimes_k S_j \cong \bigoplus_{i=1}^{n} a_{ij} S_i \qquad 1 \leq j \leq n.$$

The integral $(n \times n)$-matrix (a_{ij}) describes the left multiplication by V in the Grothendieck ring $K_0(\mathcal{G})$ relative to the standard basis.

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- $G \subseteq \mathrm{SL}(2)(k)$ acts on $\mathfrak{sl}(2)$ via automorphisms, and
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- $\Rightarrow$ the McKay quiver $\Psi_{L(1)}(G)$ of G relative to the two-dimensional standard representation $L(1) = k^2$ is connected.

Binary Polyhedral Groups and McKay Graphs

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- $\rightsquigarrow$ connection between the Gabriel quiver of $k\mathcal{G}$ and the McKay quiver of $\mathcal{G}/\mathcal{N}$ relative to $\mathrm{Ext}^1_{\mathcal{N}}(L_i, L_j)$.

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- **Lemma:** The Gabriel quiver of Λ consists of the extended Dynkin diagrams that appear in the classification of the tame hereditary algebras.

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The algebra $T(\Lambda)$ is symmetric.

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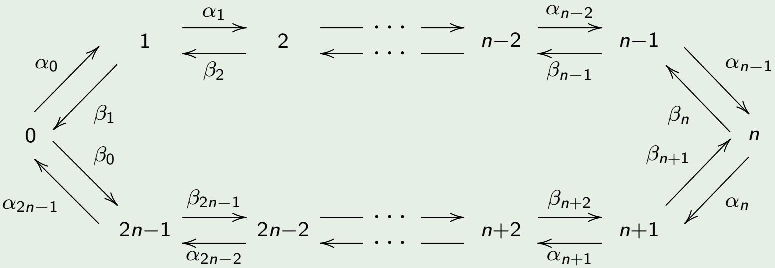

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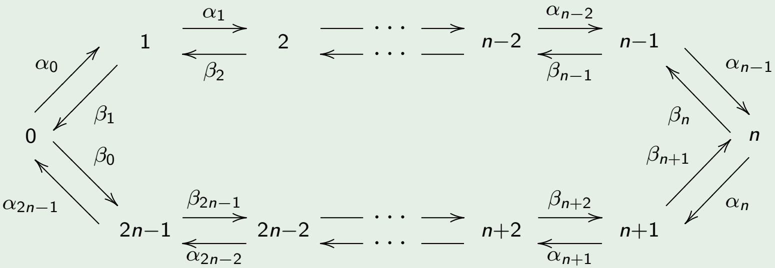

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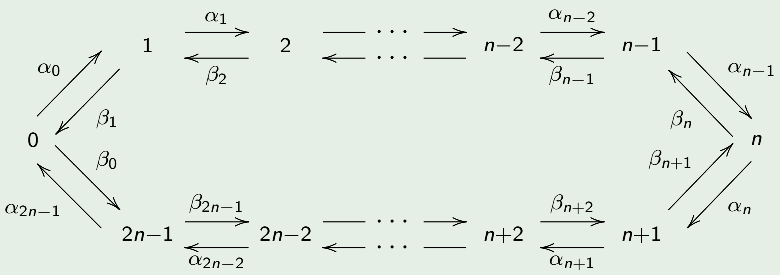

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$$\{\beta_{i+1}\alpha_i - \alpha_{i-1}\beta_i, \quad \alpha_{i+1}\alpha_i, \quad \beta_i\beta_{i+1} \quad ; \quad i \in \mathbb{Z}/(2n)\}.$$

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1. *There exists a linearly reductive group scheme $\tilde{\mathcal{G}} \subseteq \mathrm{SL}(2)$ such that the Gabriel quiver of $\mathcal{B}_0(\mathcal{G})$ is isomorphic to the McKay quiver $\Psi_{L(1)}(\tilde{\mathcal{G}})$.*

Theorem

Let $\mathcal{G}$ be a finite algebraic group of characteristic $p \geq 3$ such that $\mathcal{B}_0(\mathcal{G})$ tame.

1. *There exists a linearly reductive group scheme $\tilde{\mathcal{G}} \subseteq \mathrm{SL}(2)$ such that the Gabriel quiver of $\mathcal{B}_0(\mathcal{G})$ is isomorphic to the McKay quiver $\Psi_{L(1)}(\tilde{\mathcal{G}})$.*
2. *The block $\mathcal{B}_0(\mathcal{G})$ is Morita equivalent to a generalized trivial extension of a tame hereditary algebra.*

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- Hence $\mathcal{B}_0(\Lambda_n)$ is Morita equivalent to the trivial extension considered above.
- The other binary polyhedral groups give rise to the trivial extensions of the corresponding affine quivers.