LIE ALGEBRAS WITH A COALGEBRA SPLITTING

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1. Introduction and Preliminaries

In their recent article [5], the authors endow every finite-dimensional simple complex Lie algebra $\mathfrak{g}$ with a coalgebra structure such that the composition $\mu\circ\delta$ of the two structure maps $\delta : \mathfrak{g} \longrightarrow \mathfrak{g}\otimes_{\mathbb{C}}\mathfrak{g}$ and $\mu : \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g} \longrightarrow \mathfrak{g}$ coincides with the identity. Moreover, the dual algebra $(\mathfrak{g}^*, \delta^*)$ associated to the Lie coalgebra is isomorphic to $(\mathfrak{g}, \mu)$. The coalgebra map δ is given explicitly for $\mathfrak{sl}(n)$, those for the other types are obtained via embeddings $\mathfrak{g} \hookrightarrow \mathfrak{sl}(n)$.

The purpose of the present short note is to elicit the conceptual sources of [5], starting from the observation that the coalgebra maps defined in [5] are in fact homomorphisms of $\mathfrak{g}$-modules. For Lie algebras affording non-degenerate symmetric associative forms, such coalgebra maps naturally arise by dualizing the multiplication. This immediately implies the abovementioned duality, and the formulae displayed in [5, §4] can also be subsumed under our general approach.

By demanding that δ be $\mathfrak{g}$-linear, we depart from the usual compatibility condition of a Lie bialgebra, which requires δ to be a derivation. In that case, $\mu \circ \delta$ is a derivation of $\mathfrak{g}$. For fields of characteristic zero, only nilpotent Lie algebras afford invertible derivations (cf. [7]), so that non-zero Lie bialgebras over such fields never satisfy $\mu \circ \delta = \mathrm{id}_{\mathfrak{g}}$.

Let k be a field. Given a finite-dimensional k-vector space V, we consider the k-linear maps

$$\tau_V : V \otimes_k V \longrightarrow V \otimes_k V \quad ; \quad u \otimes v \mapsto v \otimes u$$

and

$$\xi_V : V \otimes_k V \otimes_k V \longrightarrow V \otimes_k V \otimes_k V \quad ; \quad u \otimes v \otimes w \mapsto v \otimes w \otimes u.$$

Identifying $(V \otimes_k \cdots \otimes_k V)^*$ with $V^* \otimes_k \cdots \otimes_k V^*$, the transpose maps satisfy

$$\tau_V^* = \tau_{V^*}^{-1} \ \text{ as well as } \ \xi_V^* = \xi_{V^*}^{-1}.$$

A pair $(\mathfrak{g}, \delta)$ consisting of a k-vector space $\mathfrak{g}$ and a k-linear map $\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes_k \mathfrak{g}$ is called a *Lie coalgebra* if

(a) $(\mathrm{id}_{\mathfrak{g}\otimes_k\mathfrak{g}} + \tau_{\mathfrak{g}}) \circ \delta = 0$, and
(b) $(\mathrm{id}_{\mathfrak{g}\otimes_k\mathfrak{g}\otimes_k\mathfrak{g}} + \xi_{\mathfrak{g}} + \xi_{\mathfrak{g}}^2) \circ (\mathrm{id}_{\mathfrak{g}} \otimes \delta) \circ \delta = 0$.

In the following, we consider a finite-dimensional Lie algebra $\mathfrak{g}$, defined over a field k (of arbitrary characteristic), whose bracket will be interpreted as a $\mathfrak{g}$-linear map $\mu : \mathfrak{g} \otimes_k \mathfrak{g} \longrightarrow \mathfrak{g}$. As usual, the *adjoint representation* $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is defined via

$$(\mathrm{ad}\, x)(x) = [x, y] \qquad \forall\ x, y \in \mathfrak{g}.$$

Throughout, we assume that $\mathfrak{g}$ possesses a non-degenerate, symmetric, associative form $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, with corresponding isomorphism $\Theta : \mathfrak{g} \longrightarrow \mathfrak{g}^*\ ;\ x \mapsto (x, -)$. We let $\mu^* : \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \otimes_k \mathfrak{g}^*$ be the transpose μ and define a $\mathfrak{g}$-linear map $\delta : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes_k \mathfrak{g}$ via

$$\delta := (\Theta^{-1} \otimes \Theta^{-1}) \circ \mu^* \circ \Theta.$$

Date: April 27, 2007.
2000 *Mathematics Subject Classification.* Primary 17B20, Secondary 17B62.

Using Sweedler notation, this amounts to

$$(*) \qquad \delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \ \Leftrightarrow \ (x,[y,z]) = \sum_{(x)} (x_{(1)},y)(x_{(2)},z) \qquad \forall\, x,y,z \in \mathfrak{g}.$$

Our first subsidiary result shows that Theorems 5.1 and 5.3 of [5] hold in this wider context.

Lemma 1.1. *The following statements hold:*

(1) $(\mathfrak{g},\delta)$ *is a Lie coalgebra.*

(2) *The Lie algebras* $(\mathfrak{g},\mu)$ *and* $(\mathfrak{g}^*,\delta^*)$ *are isomorphic.*

(3) *We have* $\operatorname{ad} = \eta\circ(\Theta\otimes \operatorname{id}_{\mathfrak{g}})\circ\delta$*, where* $\eta : \mathfrak{g}^*\otimes_k\mathfrak{g} \longrightarrow \operatorname{Hom}_k(\mathfrak{g},\mathfrak{g})$ *is given by* $\eta(f\otimes y)(x) = f(x)y$.

Proof. (1) In view of the above observations, we have

$$\begin{aligned}\tau_{\mathfrak{g}}\circ\delta &= \tau_{\mathfrak{g}}\circ(\Theta^{-1}\otimes\Theta^{-1})\circ\mu^*\circ\Theta = (\Theta^{-1}\otimes\Theta^{-1})\circ\tau_{\mathfrak{g}^*}\circ\mu^*\circ\Theta\\ &= (\Theta^{-1}\otimes\Theta^{-1})\circ(\mu\circ\tau_{\mathfrak{g}}^{-1})^*\circ\Theta = -\delta.\end{aligned}$$

Owing to

$$\begin{aligned}(\operatorname{id}_{\mathfrak{g}}\otimes\delta)\circ\delta &= (\Theta^{-1}\otimes\Theta^{-1}\otimes\Theta^{-1})\circ(\operatorname{id}_{\mathfrak{g}^*}\otimes\mu^*)\circ\mu^*\circ\Theta\\ &= (\Theta^{-1}\otimes\Theta^{-1}\otimes\Theta^{-1})\circ(\mu\circ(\operatorname{id}_{\mathfrak{g}}\otimes\mu))^*\circ\Theta,\end{aligned}$$

the Jacobi identity implies $(\operatorname{id}_{\mathfrak{g}\otimes_k\mathfrak{g}\otimes_k\mathfrak{g}} + \xi_{\mathfrak{g}} + \xi_{\mathfrak{g}}^2)\circ(\operatorname{id}_{\mathfrak{g}}\otimes\delta)\circ\delta = 0$.

(2) By dualizing the identity $(\Theta\otimes\Theta)\circ\delta = \mu^*\circ\Theta$ and identifying $\mathfrak{g}$ with $(\mathfrak{g}^*)^*$, we obtain

$$\delta^*\circ(\Theta^*\otimes\Theta^*) = \Theta^*\circ\mu,$$

so that $\Theta^* : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is an isomorphism between the Lie algebras $(\mathfrak{g},\mu)$ and $(\mathfrak{g}^*,\delta^*)$.

(3) Using $(*)$, we have $((\operatorname{ad} x)(y),z) = \sum_{(x)}(x_{(1)},y)(x_{(2)},z) = \sum_{(x)}(\Theta(x_{(1)})(y)x_{(2)},z)$ for every $z\in\mathfrak{g}$, whence $\operatorname{ad} x = \eta\circ(\Theta\otimes\operatorname{id}_{\mathfrak{g}})\circ\delta(x)$. □

We fix a basis $\{x_1,\ldots,x_n\}$ of $\mathfrak{g}$ along with its dual basis $\{y_1,\ldots,y_n\}$ relative to our form $(\ ,\ )$. Then

$$c_{\mathrm{ad}} : \mathfrak{g} \longrightarrow \mathfrak{g} \ \ ; \ \ x \mapsto \sum_{j=1}^n (\operatorname{ad} x_j)\circ(\operatorname{ad} y_j)(x)$$

is the *Casimir operator* of the adjoint representation $\operatorname{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ of $\mathfrak{g}$ (cf. [8, p.77f]). It is well-known that c_{ad} does not depend on the choice of the basis $\{x_1,\ldots,x_n\}$. Moreover, if κ is another non-degenerate, symmetric, associative form, then there exists a $\mathfrak{g}$-linear automorphism $f : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $\kappa(x,y) = (x,f(y))$ for every $x,y\in\mathfrak{g}$. Accordingly,

$$\tilde{c}_{\mathrm{ad}} := f^{-1}\circ c_{\mathrm{ad}}$$

is the Casimir operator associated to κ.

Lemma 1.2. *The following statements hold:*

(1) $\delta(x) = \sum_{j=1}^n [y_j,x]\otimes x_j$ *for all* $x\in\mathfrak{g}$.

(2) *If* $\mathfrak{n}\lhd\mathfrak{g}$ *is an ideal, then* $\delta(\mathfrak{n}) \subseteq \mathfrak{n}\otimes_k\mathfrak{n}$.

(3) $\mu\circ\delta = -c_{\mathrm{ad}}$.

Proof. (1) Let x be an element of $\mathfrak{g}$. Writing

$$(\operatorname{ad} x)(x_j) = \sum_{i=1}^{n} \alpha_{ij}(x)x_i \ \text{ as well as } \ (\operatorname{ad} x)(y_j) = \sum_{i=1}^{n} \beta_{ij}(x)y_i,$$

we obtain $\beta_{ij} = -\alpha_{ji}$ for $1 \le i,j \le n$ (cf. [6, p.27]). We put $\delta(x) = \sum_{r,s=1}^{n} \gamma_{rs}(x)y_r \otimes x_s$ and apply (1.1(3)) to see that

$$\begin{aligned}\sum_{i=1}^{n} \alpha_{ij}(x)x_i &= (\operatorname{ad} x)(x_j) = [\eta \circ (\Theta \otimes \operatorname{id}_{\mathfrak{g}}) \circ \delta(x)](x_j) = [\eta(\sum_{r,s=1}^{n} \gamma_{rs}(x)\Theta(y_r) \otimes x_s)](x_j) \\ &= \sum_{r,s=1}^{n} \gamma_{rs}(x)(y_r, x_j)x_s = \sum_{s=1}^{n} \gamma_{js}(x)x_s,\end{aligned}$$

whence $\gamma_{ij}(x) = \alpha_{ji}(x)$. Consequently,

$$\delta(x) = \sum_{i,j=1}^{n} \alpha_{ji}(x)y_i \otimes x_j = \sum_{i,j=1}^{n} -\beta_{ij}(x)y_i \otimes x_j = \sum_{j=1}^{n}[y_j, x] \otimes x_j,$$

as desired.

(2) Let $\mathfrak{n} \lhd \mathfrak{g}$ be an ideal. Owing to (1), we have $\delta(\mathfrak{n}) \subseteq \mathfrak{n} \otimes_k \mathfrak{g}$. Since $\tau \circ \delta = -\delta$, this implies

$$\delta(\mathfrak{n}) \subseteq (\mathfrak{n} \otimes_k \mathfrak{g}) \cap (\mathfrak{g} \otimes_k \mathfrak{n}) = \mathfrak{n} \otimes_k \mathfrak{n}.$$

(3) Application of μ to (1) yields

$$\mu(\delta(x)) = \sum_{j=1}^{n}[[y_j, x], x_j] = -\sum_{j=1}^{n}[x_j, [y_j, x]] = -c_{\text{ad}}(x),$$

as claimed. □

Example. For $n \ge 2$ we put $\mathbb{N}_n := \{1, \dots, n\}$ and consider the general linear Lie algebra $\mathfrak{g} := \mathfrak{gl}(n)$. Let $\{E_{r,s} \ ; \ (r,s) \in \mathbb{N}_n \times \mathbb{N}_n\}$ be the standard basis, given by the matrices $E_{r,s} := (\delta_{ir}\delta_{sj})_{1 \le i,j \le n}$. The trace form

$$(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \longrightarrow k \ \ ; \ \ (x,y) \mapsto \operatorname{tr}(xy)$$

is non-degenerate with $\{E_{s,r} \ ; \ (r,s) \in \mathbb{N}_n \times \mathbb{N}_n\}$ being the basis dual to the standard basis. Consequently, (1.2(1)) implies

$$\delta(x) = \sum_{r,s=1}^{n} [E_{s,r}, x] \otimes E_{r,s} \qquad \forall\, x \in \mathfrak{gl}(n).$$

In particular,

$$\delta(E_{i,j}) = \sum_{s=1}^{n} E_{s,j} \otimes E_{i,s} - E_{i,s} \otimes E_{s,j}.$$

Owing to (1.2(2)), the restriction of δ to the ideal $\mathfrak{n} := \mathfrak{sl}(n)$ defines a comultiplication, which is proportional to the one given in [5, §4] by the factor $-2n$. Note that the factor $2n$ just accounts for the passage from the trace form to the Killing form (cf. [6, p.31]). Letting $I_n \in \mathfrak{gl}(n)$ be the identity matrix, we have

$$(\mu \circ \delta)(E_{i,j}) = 2(\delta_{ij}I_n - nE_{i,j}),$$

so that $\mu \circ \delta|_{\mathfrak{sl}(n)} = -2n \operatorname{id}_{\mathfrak{sl}(n)}$.

The Lie algebra $\mathfrak{sl}(n)$ thus affords an $\mathfrak{sl}(n)$-linear comultiplication. However, if $2 < p := \operatorname{char}(k) \mid n$, then $\mathfrak{sl}(n)$ has center kI_n, which, by virtue of $\mathfrak{sl}(n) = [\mathfrak{sl}(n), \mathfrak{sl}(n)]$, is contained in the radical of any associative form on $\mathfrak{sl}(n)$. If $\{f_{i,j}\ ;\ (i,j) \in \mathbb{N}_n \times \mathbb{N}_n\} \subseteq \mathfrak{gl}(n)^*$ is the basis dual to the standard basis, then

$$\gamma : \mathfrak{gl}(n) \longrightarrow \mathfrak{gl}(n)^* \quad ; \quad E_{i,j} \mapsto f_{j,i}$$

defines an isomorphism $\mathfrak{gl}(n) \stackrel{\sim}{\longrightarrow} \mathfrak{gl}(n)^*$ that identifies $\mathfrak{sl}(n)$ with the ideal of those linear forms that annihilate I_n. (By (1.2(1)), the center $C(\mathfrak{g})$ of $\mathfrak{g}$ is contained in $\ker \delta$.)

2. Lie algebras with non-singular Casimir Operators

Retaining the notation of Section 1, we turn to those Lie algebras $\mathfrak{g} \neq (0)$ for which the map $\mu \circ \delta$ is invertible. According to (1.2(3)) this amounts to studying Lie algebras with non-singular Casimir operators. Our remarks concerning Casimir operators show that this property does not depend on the choice of the non-degenerate associative form.

Recall that a *restricted Lie algebra* $(\mathfrak{g}, [p])$ over a field of characteristic $\mathrm{char}(k) = p > 0$ is a Lie algebra $\mathfrak{g}$ together with a map $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}\ ;\ x \mapsto x^{[p]}$ that satisfies the formal properties of an associative p-th power operator. The reader is referred to [12, Chap.II] for basic properties of restricted Lie algebras.

As before, we are working over a field k of arbitrary characteristic. An early result of Dieudonné [3] states that every semisimple Lie algebra with a non-degenerate symmetric associative bilinear form is a direct sum of simple ideals. In our case, the semisimplicity of $\mathfrak{g}$ is a consequence of invertibility of c_{ad}.

Proposition 2.1. *Suppose that c_{ad} is non-singular. Then the following statements hold:*

(1) *The adjoint representation is a direct summand of $\mathfrak{g} \otimes_k \mathfrak{g}$.*

(2) *Every derivation of $\mathfrak{g}$ is inner.*

(3) *$\mathfrak{g} = \mathfrak{g}_1 \perp \mathfrak{g}_2 \perp \cdots \perp \mathfrak{g}_n$ is an orthogonal direct sum of simple Lie algebras, with each summand having a non-degenerate symmetric associative form with non-singular Casimir operator.*

(4) *There exist a $\mathfrak{g}$-linear comultiplication $\delta' : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes_k \mathfrak{g}$ such that $\mu \circ \delta' = \mathrm{id}_{\mathfrak{g}}$.*

(5) *If $\mathrm{char}(k) = p > 0$, then $\mathfrak{g}$ is a restricted Lie algebra with unique p-map and with each $\mathfrak{g}_i$ being a p-subalgebra.*

Proof. (1) Since μ is a split-surjective homomorphism of $\mathfrak{g}$-modules, our assertion follows.

(2) This is a direct consequence of Whitehead's Lemma, [8, p.77].

(3) Let $\mathfrak{n} \lhd \mathfrak{g}$ be an ideal. Directly from the definition we obtain $c_{\mathrm{ad}}(\mathfrak{n}) \subseteq \mathfrak{n}$, and our current assumption implies that this inclusion is in fact an equality. Now let $\mathfrak{n}$ be abelian. Thanks to (1.2(2),(3)), we have

$$\mathfrak{n} = c_{\mathrm{ad}}(\mathfrak{n}) = (\mu \circ \delta)(\mathfrak{n}) \subseteq \mu(\mathfrak{n} \otimes_k \mathfrak{n}) = (0).$$

On the other hand, the orthogonal complement $\mathfrak{n}^\perp$ of any ideal $\mathfrak{n} \lhd \mathfrak{g}$ is an ideal, and for $x, y \in \mathfrak{n} \cap \mathfrak{n}^\perp$ and $z \in \mathfrak{g}$, we have

$$([x, y], z) = (x, [y, z]) \in (\mathfrak{n}, \mathfrak{n}^\perp) = (0).$$

Thus, $\mathfrak{n} \cap \mathfrak{n}^\perp$ is abelian, so that $\mathfrak{n} \cap \mathfrak{n}^\perp = (0)$. As a result, any ideal $\mathfrak{n}$ of $\mathfrak{g}$ defines a direct sum decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{n}^\perp$$

of $\mathfrak{g}$-modules. This means that the adjoint representation endows $\mathfrak{g}$ with the structure of a semisimple $\mathfrak{g}$-module, whose simple constituents are simple Lie algebras. Consider two constituents $\mathfrak{g}_i \neq \mathfrak{g}_j$ of $\mathfrak{g}$, so that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_i \cap \mathfrak{g}_j = (0)$. Since $[\mathfrak{g}_j, \mathfrak{g}_j] = \mathfrak{g}_j$, we obtain

$$(\mathfrak{g}_i, \mathfrak{g}_j) = (\mathfrak{g}_i, [\mathfrak{g}_j, \mathfrak{g}_j]) \subseteq ([\mathfrak{g}_i, \mathfrak{g}_j], \mathfrak{g}_j) = (0).$$

Consequently, the above decomposition is an orthogonal decomposition, and the restriction $(\, , \,)|_{\mathfrak{g}_i \times \mathfrak{g}_i}$ is non-degenerate. By choosing dual bases $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ that are compatible with the orthogonal decomposition, we see that $c_{\mathrm{ad}}|_{\mathfrak{g}_i}$ is the Casimir operator of the form $(\, , \,)|_{\mathfrak{g}_i \times \mathfrak{g}_i}$. Thus, each $\mathfrak{g}_i$ also affords a non-singular Casimir operator.

(4) Let $f : \mathfrak{g} \longrightarrow \mathfrak{g}$ be $\mathfrak{g}$-linear. Directly from (1.2(1)) we obtain $\delta \circ f = (f \otimes \mathrm{id}_{\mathfrak{g}}) \circ \delta$, whence $(\mathrm{id}_{\mathfrak{g}} \otimes f) \circ \delta = \tau_{\mathfrak{g}} \circ (f \otimes \mathrm{id}_{\mathfrak{g}}) \circ \delta = \tau_{\mathfrak{g}} \circ \delta \circ f = \delta \circ (-f)$. This implies

$$(\mathrm{id}_{\mathfrak{g}} \otimes (\delta \circ f)) \circ \delta \circ f = (\mathrm{id}_{\mathfrak{g}} \otimes \delta) \circ (\mathrm{id}_{\mathfrak{g}} \otimes f) \circ \delta \circ f = (\mathrm{id}_{\mathfrak{g}} \otimes \delta) \circ \delta \circ (-f^2).$$

As a result, $\delta \circ f$ is also a comultiplication.

By specializing $f = -c_{\mathrm{ad}}^{-1}$, we conclude from (1.2(3)) that $\delta' := \delta \circ f$ has the requisite property.

(5) By (2), the p-th power $(\mathrm{ad}\, x)^p$ of any inner derivation is inner. Thus, [12, (II.2.3)] provides a p-map on $\mathfrak{g}$. According to (3), the algebra $\mathfrak{g}$ has trivial center, so that [12, (II.2.2)] yields the unicity of the p-map. Clearly, each $\mathfrak{g}_i$ is a p-subalgebra. □

As a first consequence, we record the following generalization of [5, Thm.6.1]:

Theorem 2.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field k of characteristic* 0*. Then there exists a comultiplication $\delta' : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes_k \mathfrak{g}$ such that $\mu \circ \delta' = \mathrm{id}_{\mathfrak{g}}$.*

Proof. By Cartan's solvability criterion [12, (I.7.8)], the Lie algebra $\mathfrak{g}$ is a direct sum of simple Lie algebras, whose constituents possess non-degenerate Killing forms. Thus, the restriction $c_{\mathrm{ad}}|_{\mathfrak{g}_i}$ of the Casimir operator to each of these ideals is either invertible or zero. Since $\mathrm{tr}(c_{\mathrm{ad}}|_{\mathfrak{g}_i}) = \dim_k \mathfrak{g}_i {\cdot} 1 \in k \setminus \{0\}$, the former alternative applies, and our result follows from (2.1(4)). □

The foregoing result reduces us to the consideration of simple Lie algebras over fields of positive characteristic. If the base field k is algebraically closed, the Block-Wilson classification theorem [2] asserts that these are either classical or of Cartan type. The classical simple Lie algebras are described in [11, Chap.II], those of Cartan type can be found in [12, Chap.IV].

Thanks to [1, 9, 10], all classical simple Lie algebras of characteristic $p \geq 7$ possess a non-degenerate symmetric associative form. The following Lemma implies that certain simple Lie algebras do not afford invertible Casimir operators.

Lemma 2.3. *Let $\mathfrak{g}$ be a simple Lie algebra.*

(1) *If $\mathfrak{g} = \bigoplus_{i=-r}^{s} \mathfrak{g}_i$ $(r, s \geq 0)$ is $\mathbb{Z}$-graded and c_{ad} is non-singular, then $r = s$.*

(2) *If $\mathrm{char}(k) = p > 0$, then $\mathfrak{g} = \mathfrak{psl}(pn)$ does not possess a non-singular Casimir operator.*

Proof. (1) According to [4, (3.1)], we have

$$(\mathfrak{g}_i, \mathfrak{g}_j) = 0 \quad \text{for} \quad i + j \neq s - r.$$

Consequently, the linear map c_{ad} is homogeneous of degree $s - r$. Being non-singular, c_{ad} must have degree 0, whence $s = r$.

(2) Since $\mathfrak{psl}(pn)$ is the derived algebra of $\mathfrak{gl}(pn)/kI_{pn}$, not all derivations of $\mathfrak{psl}(pn)$ are inner. Thus, (2.1(2)) implies the assertion. □

We now consider the classical simple Lie algebras of types B_ℓ, C_ℓ and D_ℓ over an algebraically field k of arbitrary characteristic. By definition, these algebras are Lie algebras of linear transformations on an n-dimensional vector space V, with $n = 2\ell + 1$ for type B_ℓ and $n = 2\ell$, otherwise. If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is one of these algebras, then the trace form

$$(x, y) = \mathrm{tr}(xy) \qquad \forall \; x, y \in \mathfrak{g}$$

is non-degenerate.

In the sequel, we let κ be the Killing form of $\mathfrak{g}$. Since k is algebraically closed, there exists an element $m_{\mathfrak{g}} \in k$ such that

$$\kappa = m_{\mathfrak{g}}\,(\ ,\ ).$$

Suppose that $m_{\mathfrak{g}} \neq 0$. Letting c_κ and c_{tr} be the Casimir operators of $\mathfrak{g}$ relative to these two forms, our remarks preceding (1.2) yield

$$c_{\mathrm{tr}} = m_{\mathfrak{g}}\, c_\kappa.$$

If $\mathrm{char}(k) = 0$, then $c_\kappa = \mathrm{id}_{\mathfrak{g}}$, so that we have

$$c_{\mathrm{tr}} = m_{\mathfrak{g}}\, \mathrm{id}_{\mathfrak{g}}$$

in that case.

Proposition 2.4. *Suppose that $p := \mathrm{char}(k) > 2$. Let $\mathfrak{g}$ be a classical simple Lie algebra of type B_ℓ, C_ℓ, or D_ℓ. Then $\mathfrak{g}$ affords an invertible Casimir operator if and only if:*

(1) *$\mathfrak{g}$ is of type B_ℓ, and $p \nmid 2\ell - 1$, or*
(2) *$\mathfrak{g}$ is of type C_ℓ, and $p \nmid \ell + 1$, or*
(3) *$\mathfrak{g}$ is of type D_ℓ, and $p \nmid \ell - 1$.*

Proof. Let $\mathcal{C}(\mathfrak{g}) = (c_{ij}(\mathfrak{g}))$ be the Cartan matrix of $\mathfrak{g}$. Then we have

$$\det(\mathcal{C}(\mathfrak{g})) = \begin{cases} 2 & \text{if } \mathfrak{g} \text{ has type } B_\ell \text{ or } C_\ell \\ 4 & \text{if } \mathfrak{g} \text{ has type } D_\ell, \end{cases}$$

see [6, p.63].

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ be a root space decomposition relative to some Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Given $h \in \mathfrak{h}$, we have

$$\kappa(h,h) = \mathrm{tr}((\mathrm{ad}\, h)^2) = 2\sum_{\alpha > 0} \alpha(h)^2.$$

Accordingly, the constant $m_{\mathfrak{g}}$ is determined by

$$2\sum_{\alpha > 0} \alpha(h)^2 = m_{\mathfrak{g}}(h,h).$$

Using the Chevalley bases given on pages 139–141 of [8], we obtain

$$m_{\mathfrak{g}} = \begin{cases} 2\ell - 1 & \text{if } \mathfrak{g} \text{ has type } B_\ell \\ 2(\ell+1) & \text{if } \mathfrak{g} \text{ has type } C_\ell \\ 2(\ell-1) & \text{if } \mathfrak{g} \text{ has type } D_\ell. \end{cases}$$

We denote the aforementioned Chevalley basis by $\{h_1, \ldots, h_\ell\} \cup \{e_\alpha \ ;\ \alpha \in R\}$. Direct computation shows $(e_\alpha, e_{-\alpha}) \in \{1, 2\}$. Let $\{h'_1, \ldots, h'_\ell\}$ be the basis of $\mathfrak{h}$ that is dual to $\{h_1, \ldots, h_\ell\}$ relative to the non-degenerate form $(\ ,\ )|_{\mathfrak{h}\times\mathfrak{h}}$. Writing $h'_j = \sum_{i=1}^{\ell} s_{ij} h_i$, we obtain

$$\delta_{ij} = (h_i, h'_j) = \sum_{n=1}^{\ell} s_{nj}(h_i, h_n) = \sum_{n=1}^{\ell} \frac{1}{2} c_{in}(\mathfrak{g})(h_i, h_i) s_{nj},$$

so that the base change is described by the inverse of the matrix $(\frac{1}{2} c_{ij}(\mathfrak{g})(h_i, h_i))_{i,j} = \mathcal{C}(\mathfrak{g})$.

Let $\mathfrak{g}_{\mathbb{C}}$ be the complex Lie algebra of the same type, $\mathfrak{g}_{\mathbb{Z}} \subseteq \mathfrak{g}_{\mathbb{C}}$ be the integral form given by the Chevalley basis. By the above, the operator

$$4c_{\mathrm{tr}} = \sum_{i=1}^{\ell} (\mathrm{ad}\, h_i) \circ (\mathrm{ad}(4h'_i)) + \sum_{\alpha \in R} (\mathrm{ad}\, e_\alpha) \circ (\mathrm{ad}(\frac{4}{(e_\alpha, e_{-\alpha})} e_{-\alpha}))$$

sends $\mathfrak{g}_\mathbb{Z}$ to $\mathfrak{g}_\mathbb{Z}$. Since $\mathfrak{g} = \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} k$, the Casimir operator $4c_{\mathrm{tr}}$ on $\mathfrak{g}$ can be computed from $4c_{\mathrm{tr},\mathbb{Z}}$ via

$$4c_{\mathrm{tr}} = 4c_{\mathrm{tr},\mathbb{Z}} \otimes \mathrm{id}_k \,.$$

In view of $4c_{\mathrm{tr},\mathbb{Z}} = 4m_\mathfrak{g}\,\mathrm{id}_{\mathfrak{g}_\mathbb{Z}}$, we obtain

$$c_{\mathrm{tr}} = m_\mathfrak{g}\,\mathrm{id}_\mathfrak{g},$$

so that c_{tr} is invertible if and only if $m_\mathfrak{g} \not\equiv 0 \mod(p)$. □

According to [10] the exceptional Lie algebras of types G_2, F_4, E_6, E_7, E_8 have non-degenerate Killing forms for $p \geq 7$. The dimension formulas for these algebras provide a rough sufficient condition for the invertibility of the corresponding Casimir operators.

Let $\mathfrak{g}$ be a classical Lie algebra. A prime number $p > 7$ is called *suitable* for $\mathfrak{g}$ provided

$$\begin{array}{ll} p \nmid \ell+1 & \text{if } \mathfrak{g} \text{ has type } A_\ell \text{ or } C_\ell \\ p \nmid 2\ell-1 & \text{if } \mathfrak{g} \text{ has type } B_\ell \\ p \nmid \ell-1 & \text{if } \mathfrak{g} \text{ has type } D_\ell \\ p \neq 13 & \text{if } \mathfrak{g} \text{ has type } F_4 \text{ or } E_6 \\ p \neq 19 & \text{if } \mathfrak{g} \text{ has type } E_7 \\ p \neq 31 & \text{if } \mathfrak{g} \text{ has type } E_8. \end{array}$$

Summing up, we obtain the following result:

Theorem 2.5. *Let k be an algebraically closed field of characteristic $p > 7$, $\mathfrak{g}$ be a finite-dimensional Lie algebra over k.*

(1) *If $\mathfrak{g}$ possesses a non-singular Casimir operator, then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n$ is a direct sum of classical simple Lie algebras. Moreover, for each constituent of type A, B, C or D, the prime p is suitable.*

(2) *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n$ is a direct sum of classical simple Lie algebras, with p being suitable for each constituent, then $\mathfrak{g}$ possesses a non-singular Casimir operator and there exists a comultiplication $\delta' : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes_k \mathfrak{g}$ with $\mu \circ \delta' = \mathrm{id}_\mathfrak{g}$.*

Proof. (1) By (2.1(3),(5)) we may decompose $\mathfrak{g}$ into a direct sum of restricted simple Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, with each summand having a non-singular Casimir operator. By the Block-Wilson theorem [2], each $\mathfrak{g}_i$ is classical or of Cartan type. If L is a restricted Lie algebra of Cartan type, then $L = \bigoplus_{i=-r}^{s} L_i$ is $\mathbb{Z}$-graded, with $r \neq s$ (cf. [12, (IV.2-IV.5)]). Thanks to (2.3(1)), these algebras do not afford non-singular Casimir operators. A consecutive application of (2.3(2)) and (2.4) implies that p is suitable for the simple constituents of the given types.

(2) This is a consequence of the above remarks concerning the exceptional types, the example following (1.2), (2.4), as well as (2.1(4)). □

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