REPRESENTATIONS OF FINITE GROUP SCHEMES AND MORPHISMS OF PROJECTIVE VARIETIES

ROLF FARNSTEINER

ABSTRACT. Given a finite group scheme $\mathcal{G}$ over an algebraically closed field k of characteristic $\mathrm{char}(k) = p > 0$, we introduce new invariants for a $\mathcal{G}$-module M by associating certain morphisms $\mathsf{im}^j_M : U_{M,j} \longrightarrow \mathrm{Gr}_{d_j}(M)$ $(1 \le j \le p-1)$ to M that take values in Grassmannians of M. These maps are defined for two classes of finite algebraic groups, infinitesimal group schemes and elementary abelian group schemes. They often give rise to degrees $\deg^j(M)$ ranging between 0 and $j\,\mathrm{rk}^j(M)$, where $\mathrm{rk}^j(M)$ is the generic j-rank of M. If M is a module of constant j-rank with dual module M^*, these data are linked by the formula

$$\deg^j(M) + \deg^j(M^*) = j\,\mathrm{rk}^j(M).$$

The extreme values for $\deg^j(M)$ are attained when the module M has the equal images property or the equal kernels property. For a self-dual module M of constant Jordan type our formula provides information concerning the indecomposable constituents of the pull-back $\alpha^*(M)$ of M along a p-point $\alpha : k[T]/(T^p) \longrightarrow k\mathcal{G}$.

INTRODUCTION

Let $\mathcal{G}$ be a finite group scheme over an algebraically closed field k of characteristic $p > 0$. Much of the recent work on the representations of $\mathcal{G}$ has focused on the investigation of invariants that are defined in terms of representation-theoretic support spaces, whose elements are equivalence classes of certain algebra homomorphisms $\alpha : k[T]/(T^p) \longrightarrow k\mathcal{G}$, the so-called p-points, cf. [9]. For each $\mathcal{G}$-module M, one can consider the linear operators $m \mapsto \alpha(T+(T^p))m$ of M along with their images, kernels and ranks. By specifying values of these data, one arrives at interesting full subcategories of the category $\mathrm{mod}\,\mathcal{G}$ of finite-dimensional $\mathcal{G}$-modules. In this article, we show how morphisms with values in Grassmannians can be employed to obtain new invariants for the objects of these categories.

One salient feature of the modular representation theory of finite groups is given by reduction to elementary abelian groups, with Quillen's Dimension Theorem being one notable instance. In our situation, basic algebro-geometric observations imply that our invariants are determined by their values on elementary abelian group schemes of rank 2. While these group schemes usually still have wild representation type, their modules enjoy properties that do not possess analogs in higher ranks.

Following a few preliminary observations concerning morphisms between projective varieties, we turn in Section 2 to the study of maps defined by modules. To a $\mathcal{G}$-module M, one associates its generic j-rank $\mathrm{rk}^j(M)$ $(1 \le j \le p-1)$, which is the maximal rank of the j-th powers of the aforementioned operators, cf. [10]. For elementary abelian group schemes and infinitesimal group schemes, one can define morphisms from quasi-projective varieties to Grassmannians $\mathrm{Gr}_{\mathrm{rk}^j(M)}(M)$ that turn out to encode structural properties of M. If $\mathcal{G}$ is an infinitesimal group scheme, the relevant varieties are open subsets $U_{M,j} \subseteq \mathrm{Proj}(V(\mathcal{G}))$ of the projectivized variety of infinitesimal one-parameter subgroups. For the so-called modules of constant j-rank, which are characterized by the condition $U_{M,j} = \mathrm{Proj}(V(\mathcal{G}))$, one particular case of interest arises when this variety coincides with a projective space $\mathbb{P}^n$. In the context of restricted Lie algebras (or infinitesimal groups of height 1), this happens when the nullcone $V(\mathfrak{g}) \subseteq \mathfrak{g}$ is a linear subspace, a condition that holds for nilpotent Lie algebras of nilpotent length $\le p$. In that case, the resulting morphisms

Date: October 14, 2016.
2010 *Mathematics Subject Classification.* Primary 14L15,16G10,17B50.
Supported by the D.F.G. priority program SPP1388 'Darstellungstheorie'.

$\mathsf{im}^j_M : \mathbb{P}(V(\mathfrak{g})) \longrightarrow \mathrm{Gr}_{\mathrm{rk}^j(M)}(M)$ are constant or have finite generic fibres. For $j = 1$, the map im^j_M is constant or injective, so that not all morphisms arise via this construction. Modules yielding constant morphisms enjoy the so-called equal images property. Their analogs for the group $\mathbb{Z}/(p)\times\mathbb{Z}/(p)$ were first investigated by Carlson-Friedlander-Suslin in [5].

Upon composing im^j_M with the appropriate Plücker embedding, we arrive at morphisms between projective spaces. The common degree of the homogeneous polynomials defining these maps is an interesting invariant, the j-degree $\deg^j(M)$ of M, which we study in Section 4. The degrees and ranks are linked via the following formula:

Theorem A. *Let $\mathfrak{g}$ be a restricted Lie algebra whose nullcone $V(\mathfrak{g})$ is a linear subspace of $\mathfrak{g}$. If M is a restricted $\mathfrak{g}$-module of constant j-rank, then*

$$\deg^j(M)+\deg^j(M^*) = j\,\mathrm{rk}^j(M).$$

Consequently, the categories of equal images modules and their duals, the equal kernels modules, which were defined in [5], comprise exactly those modules, whose j-degrees are 0 and $j\,\mathrm{rk}^j(M)$, respectively. Moreover, degrees may be used to distinguish modules having the same constant Jordan type. Being subadditive on exact sequences of modules of constant j-rank, and additive on sequences that are locally split, the functions $M \mapsto \deg^j(M)$ enjoy properties that are useful in the context of Auslander-Reiten theory. In the special case of elementary Lie algebras (or elementary abelian group schemes) of rank 2, the 1-degree of a module M of constant 1-rank coincides with the codimension of its "generic kernel" $\mathfrak{K}(M) \subseteq M$.

Our results concerning these Lie algebras can in principle be generalized in two directions. Since degrees of modules are determined by their values of restrictions to two-dimensional elementary Lie algebras, it seems reasonable to consider, for a $\mathfrak{g}$-module M of constant j-rank, the function $\mathbb{E}(2,\mathfrak{g}) \longrightarrow \mathbb{N}_0$; $\mathfrak{e} \mapsto \deg^j(M|_{\mathfrak{e}})$, that is defined on the projective variety of two-dimensional elementary subalgebras of $\mathfrak{g}$, see Section 4.3. In a different vein, the factorization property of p-points often allows the extension of results concerning abelian unipotent restricted Lie algebras to the general context of finite group schemes. A case in point is provided in Sections 5 and 6, where we exploit results by Tango [22, 23] on morphisms $\mathbb{P}^n \longrightarrow \mathrm{Gr}_d(V)$ to obtain information on modules of constant rank. As we show, a constant rank module over a finite group G has generic rank zero whenever its dimension is bounded by the p-rank $\mathrm{rk}_p(G)$ of G. Thus, non-trivial modules of constant rank for a p-elementary abelian group of rank r have dimension $\geq r{+}1$. In Section 6 we also introduce the aforementioned elementary abelian group schemes that generalize p-elementary abelian groups as well as elementary Lie algebras. Their sets of p-points carry the structure of a quasi-projective variety, so that modules over such groups afford j-degrees. The following result, which yields information on the Jordan types of self-dual modules, rests on the observation that Theorem A also holds for elementary abelian group schemes.

Theorem B. *Let $\mathcal{G}$ be a finite group scheme containing an elementary abelian subgroup scheme of rank ≥ 2. Suppose that M is a self-dual $\mathcal{G}$-module.*

(1) *If M has constant j-rank, then $\mathrm{rk}^j(M) \equiv 0 \bmod(2)$, whenever $j \equiv 1 \bmod(2)$.*
(2) *If M has constant Jordan type $\mathrm{Jt}(M) = \bigoplus_{i=1}^p a_i[i]$, then $a_i \equiv 0 \mod(2)$ whenever $i \equiv 0 \mod(2)$.*

The number a_i above is the multiplicity of the i-dimensional indecomposable $k[T]/(T^p)$-module $[i]$ as a direct summand of the $k[T]/(T^p)$-module $\alpha^*(M)$, obtained from M via pull-back along an arbitrary p-point $\alpha : k[T]/(T^p) \longrightarrow k\mathcal{G}$.

This paper mainly follows an algebraic approach which seems to be suitable for our purposes. Geometric aspects, related to alternative methods involving vector bundles, are only alluded to occasionally. I am grateful to Eric Friedlander and Julia Pevtsova for sharing their geometric insights with me.

1. MORPHISMS AND HOMOGENEOUS POLYNOMIALS

In this section we collect a few basic properties of certain morphisms that are relevant for our intended applications. Our main tool is the notion of a degree of a morphism $\varphi : X \longrightarrow Y$ between certain quasi-projective varieties.

1.1. **Morphisms between projective varieties.**

Throughout this section, k denotes an algebraically closed field. Recall that a polynomial $f \in k[X_0, \dots, X_n]$ is referred to as being *homogeneous of degree* d if f is a linear combination of monomials of degree d. We let $k[X_0, \dots, X_n]_d$ be the subspace of homogeneous polynomials of degree d and put $\deg(f) = d$ for every $f \in k[X_0, \dots, X_n]_d \smallsetminus \{0\}$.

Given $f, g \in k[X_0, \dots, X_n]$, we write $g|f$ to indicate that g divides f.

Lemma 1.1.1. *Let $f \neq 0$ be homogeneous of degree d. If $g|f$, then g is homogeneous of degree $\leq d$.*

Proof. We write $f = gh$ as well as $g = \sum_{i=\ell_1}^{\ell_2} g_i$, $h = \sum_{i=m_1}^{m_2} h_i$, where g_i, h_i are homogeneous of degree i and $g_{\ell_1}, g_{\ell_2}, h_{m_1}, h_{m_2}$ are not zero. Then we have

$$f = g_{\ell_2} h_{m_2} + \sum_{i<\ell_2+m_2} v_i,$$

where $v_i \in k[X_0, \dots, X_n]_i$. Since $g_{\ell_2} h_{m_2} \neq 0$, this readily yields $d = \ell_2 + m_2$. By the same token,

$$f = g_{\ell_1} h_{m_1} + \sum_{i>\ell_1+m_1} w_i \ ; \ \deg(w_i) = i,$$

so that $d = \ell_1 + m_1$. Hence $\ell_2 = d - m_2 \leq d - m_1 = \ell_1$, implying that $g = g_{\ell_1}$ is homogeneous of degree $\leq d$. □

For an n-dimensional k-vector space $V \neq (0)$, we let $\mathbb{P}(V)$ be the projective variety of one-dimensional subspaces of V. In particular, $\mathbb{P}^n = \mathbb{P}(k^{n+1})$ denotes the n-dimensional projective space, whose elements are of the form $(x_0 : x_1 : \cdots : x_n)$.

We are interested in morphisms $\varphi : X \longrightarrow Y$ between quasi-projective varieties that are given by homogeneous polynomials in the sense of the following:

Definition. Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be quasi-projective varieties, $\varphi : X \longrightarrow Y$ be a morphism. We say that φ *is homogeneous*, provided there exist homogeneous polynomials $f_0, \dots, f_m \in k[X_0, \dots, X_n]$ of the same degree such that

$$\varphi(x) = (f_0(x) : \cdots : f_m(x)) \quad \text{for every } x \in X.$$

In that case, $(f_0, \dots, f_m)$ is a *defining system for* φ and φ is said to be defined by $(f_0, \dots, f_m)$.

In this definition we tacitly assume that X does not intersect the zero locus $Z(f_0, \dots, f_m)$ of the polynomials $f_0, \dots, f_m$.

We begin with a few elementary observations concerning rational maps $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$. If $f \in k[X_0, \dots, X_n]$ is homogeneous, then $D(f) := \{x \in \mathbb{P}^n \ ; \ f(x) \neq 0\}$ is a well-defined open subset of $\mathbb{P}^n$.

Lemma 1.1.2. *Let $U \subseteq \mathbb{P}^n$ be a non-empty open subset, $\varphi : U \longrightarrow \mathbb{P}^m$ be a homogeneous morphism.*

(1) *Suppose there are polynomials $f_0, \dots, f_m \in k[X_0, \dots, X_n]_d$ and $g_0, \dots, g_m \in k[X_0, \dots, X_n]_{d'}$ such that*
   (a) $\gcd(f_0, \dots, f_m) = 1 = \gcd(g_0, \dots, g_m)$, *and*
   (b) *φ is defined by $(f_0, \dots, f_m)$ as well as $(g_0, \dots, g_m)$.*

Then there exists $\lambda \in k^{\times}$ *such that* $g_i = \lambda f_i$ *for* $i \in \{0, \dots, m\}$.

(2) *If* φ *is defined by* $(g_0, \dots, g_m)$, *then there exist homogeneous polynomials* h *and* $f_0, \dots, f_m \in k[X_0, \dots, X_n]$ *such that*
  (a) $U \subseteq D(h)$, $g_i = hf_i$ *for* $0 \leq i \leq m$, *and*
  (b) $\gcd(f_0, \dots, f_m) = 1$, *and*
  (c) φ *is defined by* $(f_0, \dots, f_m)$.

Proof. (1) By assumption (b), we have $f_i(u)g_j(u) = f_j(u)g_i(u)$ for all $u \in U$ and $i, j \in \{0, \dots, m\}$. As the open set U lies dense in the irreducible variety $\mathbb{P}^n$, this readily yields

$$(*) \qquad f_i g_j = f_j g_i \qquad \forall\, i, j \in \{0, \dots, m\}.$$

Let $i \in \{0, \dots, m\}$. If $f_i = 0$, then $(*)$ yields $f_j g_i = 0$ for all j. Since there is $j \in \{0, \dots, m\}$ such that $f_j \neq 0$, it follows that $g_i = 0$.

Suppose that $f_i \neq 0$, so that $g_i \neq 0$. For a prime polynomial $p \in k[X_0, \dots, X_n]$, we denote by $m_p(f_i)$ the multiplicity of p in f_i. Suppose that $m_p(f_i) > m_p(g_i)$. Then $(*)$ implies

$$m_p(f_j) = m_p(f_j g_i) - m_p(g_i) = m_p(f_i g_j) - m_p(g_i) = m_p(g_j) + m_p(f_i) - m_p(g_i) > m_p(g_j)$$

for every $j \in \{0, \dots, m\}$ with $f_j \neq 0$, so that p is a common divisor of the f_j, a contradiction. Thus, $m_p(f_i) \leq m_p(g_i)$, whence $m_p(f_i) = m_p(g_i)$ by symmetry. As a result, the polynomial g_i is a scalar multiple of f_i. The assertion now follows from $(*)$.

(2) Let h be a greatest common divisor of $g_0, \dots, g_m$, so that $g_i = hf_i$ for some $f_i \in k[X_0, \dots, X_n]$. By virtue of Lemma 1.1.1, the polynomials h and f_i are homogeneous. Hence the f_i are homogeneous of the same degree and have greatest common divisor 1.

Given $u = [x] \in U$, there is $i \in \{0, \dots, m\}$ such that

$$0 \neq g_i(x) = h(x) f_i(x),$$

so that $U \subseteq D(h)$. Since

$$\varphi(u) = (g_0(u) : \cdots : g_m(u)) = (f_0(u) : \cdots : f_m(u)) \quad \text{for all } u \in U,$$

the map φ is defined by $(f_0, \dots, f_m)$. □

Definition. Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be quasi-projective varieties, $\varphi : X \longrightarrow Y$ be a homogeneous morphism. An $(m+1)$-tuple $(f_0, \dots, f_m) \in k[X_0, \dots, X_n]_d^{m+1}$ is called a *reduced defining system for* φ, provided

(1) φ is defined by $(f_0, \dots, f_m)$, and
(2) $\gcd(f_0, \dots, f_m) = 1$.

Our next result shows that rational maps $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$ always afford reduced defining systems.

Lemma 1.1.3. *Let* $U \subseteq \mathbb{P}^n$ *be a non-empty open subset,* $\varphi : U \longrightarrow \mathbb{P}^m$ *be a morphism. Then* φ *is homogeneous.*

Proof. Since $U \subseteq \mathbb{P}^n$ is a noetherian topological space, an application of [13, (1.65)] provides non-empty open subsets $U_1, \dots, U_r$ of U, and homogeneous polynomials $f_{ij} \in k[X_0, \dots, X_n]$, where $0 \leq i \leq m$ and $1 \leq j \leq r$, such that

(a) $U = \bigcup_{j=1}^{r} U_j$,
(b) $U_j \subseteq \bigcup_{i=0}^{m} D(f_{ij})$,
(c) there exist $d_1, \dots, d_r \in \mathbb{N}_0$ such that $f_{ij} \in k[X_0, \dots, X_n]_{d_j}$ for $0 \leq i \leq m$ and $1 \leq j \leq r$,
(d) $\varphi(u) = (f_{0j}(u) : \cdots : f_{mj}(u))$ for all $u \in U_j$ and $j \in \{1, \dots, r\}$.

Since $U_j \subseteq \mathbb{P}^n$ is open, Lemma 1.1.2 shows that we may assume $\gcd(f_{0j}, \dots, f_{mj}) = 1$ for every $j \in \{1, \dots, r\}$. Given $j \in \{1, \dots, r\}$, we see that $U_j \cap U_1$ is a non-empty open subset of $\mathbb{P}^n$. By applying Lemma 1.1.2(1) to the morphism $\varphi|_{U_j \cap U_1}$ we find a scalar $\lambda_j \in k^\times$ such that $f_{ij} = \lambda_j f_{i1}$ for $0 \le i \le m$. Setting $f_i := f_{i1}$ for $0 \le i \le m$, we thus obtain for $u \in U_j$

$$\varphi(u) = (f_{0j}(u) : \cdots : f_{mj}(u)) = (f_0(u) : \cdots : f_m(u)).$$

Thanks to property (a), this identity holds for all $u \in U$. Consequently, the map φ is a homogeneous morphism. □

In view of the foregoing result, the following definition is meaningful:

Definition. Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map, $(f_0, \dots, f_m)$ be a reduced defining system for φ. Then

$$\deg(\varphi) = \deg(f_i) \ \ ; \ \ f_i \neq 0$$

is called the *degree* $\deg(\varphi)$ of φ.

Remarks. (1) The above definition should not be confused with the projective degree of a morphism, see [14, (19.4)]. If $\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ denotes the d-th Veronese embedding, then $\deg(\nu_d) = d$, while the degree of the image $\operatorname{im} \nu_d$ is d^n (cf. [14, (18.13)]), so that the map $\nu_d : \mathbb{P}^n \longrightarrow \operatorname{im} \nu_d$ has projective degree d^n.

(2) Let $A = (a_{ij}) \in \mathrm{GL}_{m+1}(k)$. If $(f_0, \dots, f_m) \in k[X_0, \dots, X_m]_d^{m+1}$, then $g_j := \sum_{i=0}^m a_{ij} f_i$ belongs to $k[X_0, \dots, X_n]_d$, and $\gcd(g_0, \dots, g_m) = \gcd(f_0, \dots, f_m)$. Accordingly, a linear change of coordinates does not affect the degree of a rational morphism $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$.

(3) A morphism $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^m$ defines a homomorphism $\varphi^* : \operatorname{Pic}(\mathbb{P}^m) \longrightarrow \operatorname{Pic}(\mathbb{P}^n)$ between the Picard groups of line bundles. Since these groups are isomorphic to $\mathbb{Z}$, the degree d of φ is given by $\varphi^*(\mathcal{O}_{\mathbb{P}^m}(1)) = \mathcal{O}_{\mathbb{P}^n}(d)$.

Corollary 1.1.4. *If $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^m$ is a morphism, then any defining system is a reduced defining system.*

Proof. Let $(g_0, \dots, g_m)$ be a defining system, $(f_0, \dots, f_m)$ be a reduced defining system for φ. Owing to Lemma 1.1.2(2), there exists a homogeneous polynomial h such that $\mathbb{P}^n = D(h)$ and $g_i = hf_i$ for $0 \le i \le m$. Since such a polynomial is necessarily constant, our assertion follows. □

Corollary 1.1.5. *Let $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^m$ and $\psi : \mathbb{P}^m \longrightarrow \mathbb{P}^s$ be morphisms.*

(1) *We have* $\deg(\psi \circ \varphi) = \deg(\psi) \deg(\varphi)$.

(2) *If $\psi \circ \varphi$ is constant, then ψ is constant or φ is constant.*

Proof. (1) If $f_0, \dots, f_m \in k[X_0, \dots, X_n]$ are homogeneous polynomials of degree d and $\nu \in \mathbb{N}_0^{m+1}$, then $f_0^{\nu_0} \cdots f_m^{\nu_m}$ is homogeneous of degree $d \cdot (\sum_{i=0}^m \nu_i)$. Consequently, $\psi \circ \varphi$ affords a defining system of homogeneous polynomials of degree $\deg(\psi) \deg(\varphi)$ and Corollary 1.1.4 yields the result.

(2) Since $\psi \circ \varphi$ is constant, there is $c = (c_0, \dots, c_s) \in k^{s+1} \smallsetminus \{0\}$ such that $[\psi \circ \varphi](x) = (c_0 : \cdots : c_s)$ for all $x \in \mathbb{P}^n$. This readily implies $0 = \deg(\psi \circ \varphi) = \deg(\psi) \deg(\varphi)$. As a result, one of the factors has degree 0 and the corresponding map is constant. □

Remarks. (1) The proof of (1) also shows that for morphisms $\varphi : U \longrightarrow V$ and $\psi : V \longrightarrow \mathbb{P}^s$, where $U \subseteq \mathbb{P}^n$ and $V \subseteq \mathbb{P}^m$ are non-empty open subset, we have $\deg(\psi \circ \varphi) \leq \deg(\psi)\deg(\varphi)$. However, this inequality may be strict.

(2) In view of the observations above, a morphism $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^m$ of degree 1 corresponds to an element $[A] \in \mathbb{P}(\mathrm{Mat}_{(m+1)\times(n+1)}(k))$ that is defined by a matrix A of rank $n+1$.

(3) According to Corollary 1.1.5, automorphisms of $\mathbb{P}^n$ have degree 1. Hence the canonical action of $\mathrm{GL}_{n+1}(k)$ on $\mathbb{P}^n$ induces an isomorphism $\mathrm{PGL}_{n+1}(k) \cong \mathrm{Aut}(\mathbb{P}^n)$.

Corollary 1.1.6. *Let $X \subseteq \mathbb{P}^n$ be a quasi-projective variety such that there exists a non-constant morphism $\omega_X : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ with $\omega_X(\mathbb{P}^1) \subseteq X$. If $\varphi : X \longrightarrow \mathbb{P}^m$ is a homogeneous morphism with defining system $(f_0, \dots, f_m) \in k[X_0, \dots, X_n]_d^{m+1}$, then*

$$d = \frac{\deg(\varphi\circ\omega_X)}{\deg(\omega_X)}.$$

Proof. Lemma 1.1.3 provides homogeneous polynomials $\omega_0, \dots, \omega_n \in k[X, Y]$ of degree $\ell > 0$ such that

$$\omega_X(x) = (\omega_0(x) : \cdots : \omega_n(x)) \qquad \forall\, x \in \mathbb{P}^1.$$

Note that the polynomials $g_i := f_i(\omega_0, \dots, \omega_n) \in k[X, Y]$ are homogeneous of degree $d\ell$, while

$$(\varphi \circ \omega_X)(x) = (g_0(x) : \cdots : g_m(x)) \quad \forall\, x \in \mathbb{P}^1.$$

According to Corollary 1.1.4, this implies that $\deg(\varphi \circ \omega_X) = d\ell$, as desired. □

Remarks. (1) In the situation above, the number

$$\deg(\varphi) := \frac{\deg(\varphi\circ\omega_X)}{\deg(\omega_X)} \in \mathbb{N}_0$$

does not depend on the choice of ω_X.

(2) If $X \subseteq \mathbb{P}^n$ is open, then Lemma 1.1.3 ensures that $\varphi : X \longrightarrow \mathbb{P}^m$ is homogeneous and the two notions of degree coincide. Hence Corollary 1.1.6 may be used to compute the degree of a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$.

1.2. **Tango's Theorem.** It is a well-known fact that the dimension of the image of a morphism $\varphi : \mathbb{P}^m \longrightarrow \mathbb{P}^n$ belongs to $\{0, m\}$, cf. [17, I.7,Prop.6]. In a series of articles, including [22, 23], H. Tango investigated morphisms $\mathbb{P}^m \longrightarrow \mathrm{Gr}(n, d)$ with values in the Grassmann variety $\mathrm{Gr}(n, d)$ of d-dimensional linear subspaces of $\mathbb{P}^n$. Letting $\mathrm{Gr}_d(V)$ be the Grassmann variety of d-dimensional subspaces of a finite-dimensional k-vector space V, we summarize some of his results as follows:

Theorem 1.2.1 (H. Tango)**.** *Let V be an n-dimensional vector space, $\varphi : \mathbb{P}^m \longrightarrow \mathrm{Gr}_d(V)$ be a morphism.*

(1) *If $n \leq m$, then φ is constant.*
(2) *Suppose that $n = m+1$.*
   (a) *If n is odd and $2 \leq d \leq n-2$, then φ is constant.*
   (b) *If d is odd while $3 \leq d \leq n-2$ and $(n, d) \neq (6, 3)$, then φ is constant.*

Proof. By definition, we have $\mathrm{Gr}_d(V) \cong \mathrm{Gr}(n-1, d-1)$. Now (1) and (2) are direct consequences of [22, (3.2)] and [23, Thm.], respectively. □

1.3. **Homogeneous morphisms of conical varieties.** Let $\mathfrak{M}$ be a finite-dimensional $k^\times$-module with weight space decomposition $\mathfrak{M} = \bigoplus_{i\geq 0} \mathfrak{M}_i$. A $k^\times$-stable Zariski closed subset $V \subseteq \mathfrak{M}$ such that $V \not\subseteq \mathfrak{M}_0$ is referred to as a *conical variety*. The commutative group $k^\times$ acts on the coordinate ring $k[\mathfrak{M}] = S(\mathfrak{M}^*)$ of polynomial functions on $\mathfrak{M}$ via

$$(\alpha.f)(m) := f(\alpha.m) \qquad \forall\, \alpha \in k^\times,\, m \in \mathfrak{M},\, f \in k[\mathfrak{M}]$$

such that each component $k[\mathfrak{M}]^{(d)} := S^d(\mathfrak{M}^*)$ is a $k^\times$-submodule. Since V is conical, its coordinate ring $k[V]$ inherits this action and there results a grading

$$k[V] = \bigoplus_{i\geq 0} k[V]_i,$$

where $k[V]_i = \{f \in k[V] \ ; \ f(\alpha.v) = \alpha^i f(v) \ \ \forall\, \alpha \in k^\times,\, v \in V\}$. The elements of $k[V]_i$ will be referred to as homogeneous polynomial functions of degree i.

In the above situation, we consider the projective varieties $\mathrm{Proj}(V) \subseteq \mathrm{Proj}(\mathfrak{M})$. The underlying sets are the $k^\times$-orbits of $V \smallsetminus \mathfrak{M}_0 \subseteq \mathfrak{M} \smallsetminus \mathfrak{M}_0$. The regular functions are locally given by fractions of homogeneous polynomial functions of the same degree. If the action of $k^\times$ on $\mathfrak{M}$ is just the restriction of the scalar multiplication, so that $\mathfrak{M} = \mathfrak{M}_1$, we retrieve the standard projective varieties $\mathbb{P}(V) \subseteq \mathbb{P}(\mathfrak{M})$. We write $V_0 := \{v \in V \ ; \ \alpha.v = v \ \ \forall\, \alpha \in k^\times\} = V \cap \mathfrak{M}_0$.

A morphism $\varphi : V \longrightarrow W$ between two conical varieties is called *homogeneous*, provided there exists $d \geq 0$ such that $\varphi(\alpha.v) = \alpha^d.\varphi(v)$ for all $\alpha \in k^\times$ and $v \in V$. This requirement is equivalent to the comorphism $\varphi^* : k[W] \longrightarrow k[V]$ satisfying $\varphi^*(k[W]_n) \subseteq k[V]_{nd}$ for all $n \geq 0$.

Lemma 1.3.1. *Let $\varphi : V \longrightarrow W$ be a homogeneous morphism of conical affine varieties. Then the following statements hold:*

(1) $\mathcal{O}_\varphi := \{[v] \in \mathrm{Proj}(V) \ ; \ \varphi(v) \notin W_0\}$ *is an open subset of* $\mathrm{Proj}(V)$.
(2) *The map* $\bar{\varphi} : \mathcal{O}_\varphi \longrightarrow \mathrm{Proj}(W) \ ; \ [v] \mapsto [\varphi(v)]$ *is a morphism of varieties.*

Proof. (1) Suppose that $\varphi : V \longrightarrow W$ has degree d. By definition, $W \subseteq \mathfrak{M}$ is a Zariski closed $k^\times$-stable subset of a $k^\times$-module $\mathfrak{M} = \bigoplus_{i\geq 0} \mathfrak{M}_i$. We choose a basis $\{v_1, \ldots, v_m\}$ of $\mathfrak{M}$ consisting of homogeneous vectors such that $\{v_1, \ldots, v_n\}$ is a basis of $\bigoplus_{i>0} \mathfrak{M}_i$. For each $j \in \{1, \ldots, n\}$, the coordinate function $\mathrm{pr}_j : \mathfrak{M} \longrightarrow k$ defines a homogeneous element of $k[W]$ of degree $\deg(v_j)$. Consequently, $\varphi_j := \mathrm{pr}_j \circ \varphi \in k[V]$ is homogeneous for $1 \leq j \leq n$ and $\mathcal{O}_\varphi = \mathrm{Proj}(V) \smallsetminus Z(\varphi_1, \ldots, \varphi_n)$, the complement of the zero locus of the φ_j, is an open subset of $\mathrm{Proj}(V)$.

(2) By assumption, we have $\varphi(\alpha.v) = \alpha^d.\varphi(v)$ for all $v \in V$, so that the map $\bar{\varphi}$ is well-defined and continuous. Let $U \subseteq \mathrm{Proj}(W)$ be an open set, $\rho : U \longrightarrow k$ be a regular function. Then $U' := \bar{\varphi}^{-1}(U)$ is open in $\mathcal{O}_\varphi$. Given $x \in U'$, there exist an open subset $U_1 \subseteq U$ containing $\bar{\varphi}(x)$ and $f, g \in k[W]$ homogeneous of the same degree with $U_1 \subseteq D(g)$ and such that $\rho(y) = \frac{f(y)}{g(y)}$ for all $y \in U_1$. Since φ is homogeneous, the functions $f \circ \varphi, g \circ \varphi \in k[V]$ are homogeneous of the same degree, and for $[u] \in \bar{\varphi}^{-1}(U_1)$, we have

$$\rho(\bar{\varphi}([u])) = \frac{f(\varphi(u))}{g(\varphi(u))},$$

so that $\rho \circ \bar{\varphi} : U' \longrightarrow k$ is regular at x. As a result, the map $\bar{\varphi} : \mathcal{O}_\varphi \longrightarrow \mathrm{Proj}(W)$ is a morphism. □

2. Modules for Infinitesimal Group Schemes and Maps to Grassmannians

Let $\mathcal{G}$ be a infinitesimal group scheme. In this section we associate to every $\mathcal{G}$-module M several morphisms im^j_M which take values in Grassmannians and are defined on open subsets of the projectivized rank variety $\mathrm{Proj}(V(\mathcal{G}))$ of infinitesimal one-parameter subgroups of $\mathcal{G}$.

2.1. **Preliminaries.** As before, k denotes an algebraically closed field. Let V be an n-dimensional k-vector space with basis $\{v_1, \dots, v_n\}$. Given $d \in \{1, \dots, n\}$, we denote by $\mathcal{S}(d)$ the set of d-element subsets of $\{1, \dots, n\}$. For $J \in \mathcal{S}(d)$ we put $V_J := \bigoplus_{j\in J} kv_j$ and define

$$v_J := v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_d} \in \bigwedge^d(V),$$

where $j_1 < j_2 < \cdots < j_d$ belong to J. For an endomorphism $f : V \longrightarrow V$, we denote by $\bigwedge^d(f)$ the unique endomorphism of $\bigwedge^d(V)$ such that $\bigwedge^d(f)(w_1 \wedge w_2 \wedge \cdots \wedge w_d) = f(w_1) \wedge f(w_2) \wedge \cdots \wedge f(w_d)$ for all $w_1, \dots, w_d \in V$. There results a homogeneous morphism $\mathrm{End}_k(V) \longrightarrow \mathrm{End}_k(\bigwedge^d(V))$; $f \mapsto \bigwedge^d(f)$ of degree d.

Lemma 2.1.1. *Given $d \in \{1, \dots, n\}$ and $J \in \mathcal{S}(d)$, the following statements hold:*

(1) *The set $\mathcal{O}_J := \{[f] \in \mathbb{P}(\mathrm{End}_k(V))\ ;\ \bigwedge^d(f)(v_J) \neq 0\}$ is an open subset of $\mathbb{P}(\mathrm{End}_k(V))$.*
(2) *The map $\overline{\mathrm{ev}}_J : \mathcal{O}_J \longrightarrow \mathbb{P}(\bigwedge^d(V))$; $[f] \mapsto [\bigwedge^d(f)(v_J)]$ is a morphism of varieties.*

Proof. Being the composite of homogeneous morphisms of degrees d and 1, the map

$$\mathrm{ev}_J : \mathrm{End}_k(V) \longrightarrow \bigwedge^d(V)\ ;\ f \mapsto \bigwedge^d(f)(v_J)$$

is a homogeneous morphism of conical varieties of degree d. Since $\mathcal{O}_J = \mathcal{O}_{\mathrm{ev}_J}$, our assertions follow from Lemma 1.3.1. □

Note that $\bigcup_{J\in\mathcal{S}(d)} \mathcal{O}_J = \{[f] \in \mathbb{P}(\mathrm{End}_k(V))\ ;\ \mathrm{rk}(f) \geq d\}$. Consequently, the subset

$$\mathbb{P}(\mathrm{End}_k(V))_d := \{[f] \in \mathbb{P}(\mathrm{End}_k(V))\ ;\ \mathrm{rk}(f) = d\}$$

of $\mathbb{P}(\mathrm{End}_k(V))$ is locally closed. Thus, $\mathbb{P}(\mathrm{End}_k(V))_d$ is a quasi-projective variety, which, being an orbit under the canonical action of $\mathrm{GL}(V)\times\mathrm{GL}(V)$, is irreducible. Setting $\mathcal{U}_J := \mathcal{O}_J \cap \mathbb{P}(\mathrm{End}_k(V))_d$, we obtain an open covering $\mathbb{P}(\mathrm{End}_k(V))_d = \bigcup_{J\in\mathcal{S}(d)} \mathcal{U}_J$.

We let

$$\mathsf{pl}_V : \mathrm{Gr}_d(V) \longrightarrow \mathbb{P}(\bigwedge^d(V))\ ;\ W \mapsto \bigwedge^d(W)$$

be the *Plücker embedding*, which identifies $\mathrm{Gr}_d(V)$ with the closed subset $\mathrm{im}\,\mathsf{pl}_V \subseteq \mathbb{P}(\bigwedge^d(V))$.

According to Lemma 2.1.1, the map

$$\overline{\mathrm{ev}}_J : \mathcal{U}_J \longrightarrow \mathbb{P}(\bigwedge^d(V))\ ;\ [f] \mapsto [\bigwedge^d(f)(v_J)]$$

is a morphism of varieties for every $J \in \mathcal{S}(d)$. These maps can be glued to a morphism $\mathbb{P}(\mathrm{End}_k(V))_d \longrightarrow \mathrm{Gr}_d(V)$.

Proposition 2.1.2. *Let $d \in \{1, \dots, n\}$. Then the following statements hold:*

(1) *The map*

$$\overline{\mathrm{ev}} : \mathbb{P}(\mathrm{End}_k(V))_d \longrightarrow \mathbb{P}(\bigwedge^d(V))\ ;\ [f] \mapsto \bigwedge^d(f(V))$$

is a morphism of varieties.

(2) *The map*

$$\overline{\mathrm{im}} : \mathbb{P}(\mathrm{End}_k(V))_d \longrightarrow \mathrm{Gr}_d(V)\ ;\ [f] \mapsto f(V)$$

is a morphism of varieties.

Proof. (1) Suppose that $f \in \mathcal{U}_J \cap \mathcal{U}_{J'}$ for two subsets $J, J' \in \mathcal{S}(d)$. Then $\bigwedge^d(f)(v_J), \bigwedge^d(f)(v_{J'})$ are nonzero elements of $\bigwedge^d(V)$ belonging to the subspace $\bigwedge^d(f(V)) \subseteq \bigwedge^d(V)$. Since $\mathrm{rk}(f) = d$, the space $\bigwedge^d(f(V))$ is one-dimensional, so that

$$\overline{\mathrm{ev}}_J([f]) = [\bigwedge^d(f)(v_J)] = [\bigwedge^d(f)(v_{J'})] = \overline{\mathrm{ev}}_{J'}([f]).$$

Owing to Lemma 2.1.1, the unique map $\overline{\mathrm{ev}} : \mathbb{P}(\mathrm{End}_k(V))_d \longrightarrow \mathbb{P}(\bigwedge^d(V))$, given by

$$\overline{\mathrm{ev}}|_{\mathcal{U}_J} = \overline{\mathrm{ev}}_J$$

for all $J \in \mathcal{S}(d)$ is a morphism of varieties.

(2) Let $J \in \mathcal{S}(d)$ be a d-element subset. Given $[f] \in \mathcal{U}_J$, we have $\bigwedge^d(f)(v_J) \neq 0$. Consequently, $\dim_k f(V_J) = d = \dim_k f(V)$, so that

$$(\mathsf{pl}_V \circ \overline{\mathrm{im}})([f]) = \bigwedge^d(f(V)) = \bigwedge^d(f(V_J)) = [\bigwedge^d(f)(v_J)] = \overline{\mathrm{ev}}([f]).$$

As a result, $\overline{\mathrm{ev}} = \mathsf{pl}_V \circ \overline{\mathrm{im}}$. Since $\overline{\mathrm{ev}}$ is a morphism, so is $\overline{\mathrm{im}}$. □

Let $f \in \mathrm{End}_k(V)$ and denote by $A(f) = (a_{ij}) \in \mathrm{Mat}_n(k)$ the $(n \times n)$-matrix representing f with respect to the basis $\{v_1, \dots, v_n\}$. We consider the associated map $g := \bigwedge^d(f) \in \mathrm{End}_k(\bigwedge^d(V))$. Given $(K', K) \in \mathcal{S}(d)^2$, we let $A(f)_{(K',K)}$ be the (K', K)-minor of A. Then we have

$$g(v_K) = \sum_{K' \in \mathcal{S}(d)} \det(A(f)_{(K',K)}) v_{K'} \qquad \forall\, K \in \mathcal{S}(d).$$

Thus, the coordinates of $\overline{\mathrm{ev}}([f])$ for $[f] \in \mathcal{U}_J$ are $[\det(A(f)_{(I,J)})]_{I \in \mathcal{S}(d)}$. As a result, the map

$$\overline{\mathrm{ev}} : \mathbb{P}(\mathrm{End}_k(V))_d \longrightarrow \mathbb{P}(\bigwedge^d(V)) \ ; \ [f] \mapsto \bigwedge^d(f(V))$$

is locally given by the homogeneous polynomials $(\det((X_{ij})_{(I,J)}))_{I \in \mathcal{S}(d)}$, where $\det((X_{ij})_{(I,J)}) \in k[X_{ij} \ ; \ 1 \leq i, j \leq n]_d$.

Example. Let V be a vector space of dimension $n \geq 2$. Then the morphism

$$\overline{\mathrm{ev}} : \mathbb{P}(\mathrm{End}_k(V))_1 \longrightarrow \mathbb{P}(V)$$

is not homogeneous.

Choosing a basis of V, we consider $(n \times n)$-matrices and observe that

$$\overline{\mathrm{ev}} : \mathbb{P}(\mathrm{Mat}_n(k))_1 \longrightarrow \mathbb{P}^{n-1}$$

associates to each element $[A] \in \mathbb{P}(\mathrm{Mat}_n(k))_1$ its one-dimensional column space.

For $j \in \{1, \dots, n\}$, there is a morphism

$$\psi_j : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}(\mathrm{Mat}_n(k))_1$$

sending $[x] \in \mathbb{P}^{n-1}$ to the class of the matrix, whose ℓ-th colum is of the form $\delta_{\ell,j} x^{\mathrm{tr}}$. We have $\overline{\mathrm{ev}} \circ \psi_j = \mathrm{id}_{\mathbb{P}^{n-1}}$.

Suppose that there are homogeneous polynomials $f_1, \dots, f_n \in k[X_{ij} \ ; \ 1 \leq i, j \leq n]$ of degree d that define the morphism $\overline{\mathrm{ev}}$. Then $\overline{\mathrm{ev}} \circ \psi_j$ is also defined by homogeneous polynomials of degree d, and an application of Corollary 1.1.4 conjunction with the above identity implies $d = \deg(\overline{\mathrm{ev}} \circ \psi_j) = \deg(\mathrm{id}_{\mathbb{P}^{n-1}}) = 1$. We thus write

$$f_\ell = \sum_{i,j=1}^{n} \alpha_{ij\ell} X_{ij}$$

for $\ell \in \{1,\ldots,n\}$. Since, for every $j \in \{1,\ldots,n\}$, we have

$$(x_1\!:\!x_2\!:\cdots:\!x_n) = [\sum_{i=1}^{n} \alpha_{ij\ell}x_i]_{1\le\ell\le n}$$

for all $(x_1 : x_2 : \cdots : x_n) \in \mathbb{P}^{n-1}$, it follows from Lemma 1.1.2 that there exist $\lambda_j \in k^\times$ with $\lambda_j X_\ell = \sum_{i=1}^{n} \alpha_{ij\ell}X_i$. Consequently, $\alpha_{ij\ell} = \delta_{\ell,i}\lambda_j$, so that

$$f_\ell = \sum_{j=1}^{n} \lambda_j X_{\ell j} \qquad 1\le\ell\le n.$$

Now let $x := (x_1, x_2, \ldots, x_n) \in k^n \smallsetminus \{0\}$ be such that $\sum_{j=1}^{n} \lambda_j x_j = 0$. If A_x denotes the $(n\times n)$-matrix all whose row vectors are x, then $[A_x] \in \mathbb{P}(\mathrm{Mat}_n(k))_1$, while

$$(f_1(A_x),\ldots,f_n(A_x)) = 0,$$

a contradiction.

Remark. With considerably more effort one can show that the morphism

$$\overline{\mathrm{ev}} : \mathbb{P}(\mathrm{End}_k(V))_d \longrightarrow \mathbb{P}(V)$$

is not homogeneous for an arbitrary $d \in \{1,\ldots,n-1\}$. Since the morphisms of interest for us will arise via composites of $\overline{\mathrm{ev}}$ with restrictions of representations $\varrho : A \longrightarrow \mathrm{End}_k(V)$, this appears to be the reason why our methods work best for those finite group schemes that are analogs of elementary abelian groups.

For future reference, we record a duality between the vector spaces $\bigwedge^d(V)$ and $\bigwedge^d(V^*)$. Let $\{\delta_1,\ldots,\delta_n\}$ be the basis of V^* that is dual to $\{v_1,\ldots,v_n\}$ and denote by $\delta_{I,J}$ the Kronecker symbol of $\mathcal{S}(d)^2$.

Lemma 2.1.3. *The following statements hold:*

(1) *For $I, J \in \mathcal{S}(d)$, we have*

$$\det((\delta_i(v_j))_{(I,J)}) = \delta_{I,J}.$$

(2) *The unique bilinear form $(\ ,\ ) : \bigwedge^d(V^*)\times\bigwedge^d(V) \longrightarrow k$, given by*

$$(f_1 \wedge f_2 \wedge \cdots \wedge f_d, w_1 \wedge w_2 \wedge \cdots \wedge w_d) \mapsto \det((f_i(w_j)))$$

is non-degenerate. □

2.2. **The morphisms im^j_M.** From now on k is assumed to be an algebraically closed field of characteristic $p > 0$. If $\mathcal{G}$ is a finite group scheme over k with coordinate ring $k[\mathcal{G}]$, then the dual Hopf algebra $k\mathcal{G} := k[\mathcal{G}]^*$ is the *algebra of measures on* $\mathcal{G}$. By general theory, finite-dimensional representations of $\mathcal{G}$ naturally correspond to finite-dimensional $k\mathcal{G}$-modules. We shall henceforth write $\mathrm{mod}\,\mathcal{G}$ for the category of finite-dimensional $\mathcal{G}$-modules and use both interpretations interchangeably. We refer to [24] for these matters.

If M is a finite-dimensional $\mathcal{G}$-module and $x \in k\mathcal{G}$, we let

$$x_M : M \longrightarrow M \ \ ; \ \ v \mapsto x.v$$

be the linear transformation effected by x.

Given $r \in \mathbb{N}$, we denote by $\mathbb{G}_{a(r)}$ the r-th Frobenius kernel of the additive group $\mathbb{G}_a = \mathrm{Spec}_k(k[T])$. We consider an infinitesimal group scheme $\mathcal{G}$ of height r along with the set $V(\mathcal{G})$ of its one-parameter subgroups. By definition, the elements of $V(\mathcal{G})$ are homomorphisms $\varphi : \mathbb{G}_{a(r)} \longrightarrow \mathcal{G}$ of group schemes. General theory then shows that

$$V(\mathcal{G}) = \mathrm{Hom}_{\mathrm{Hopf}}(k\mathbb{G}_{a(r)}, k\mathcal{G}) \subseteq \mathrm{Hom}_k(k\mathbb{G}_{a(r)}, k\mathcal{G}) := \mathfrak{M}$$

is an affine variety. In fact, $V(\mathcal{G})$ is the variety of k-rational points of the scheme of infinitesimal one-parameter subgroups introduced in [20].

Recall that the diagonalizable group $k^\times$ acts on $\mathbb{G}_{a(r)}$ via automorphisms. The action on $\mathbb{G}_{a(r)}$ corresponds to the operation of $k^\times$ on the coordinate ring $k[\mathbb{G}_{a(r)}] = k[T]/(T^{p^r})$ that is given by $\alpha.t^i := \alpha^{-i}t^i$, where $t := T+(T^{p^r})$. Consequently, $k^\times$ operates on $k\mathbb{G}_{a(r)} = k[\mathbb{G}_{a(r)}]^*$ via automorphisms of Hopf algebras and with set of weights $\{0,\dots,p^r-1\}$. We endow $\mathfrak{M}$ with the structure of a $k^\times$-module via

$$(\alpha.f)(u) := f(\alpha.u) \qquad \forall\, \alpha \in k^\times,\ f \in \mathfrak{M},\ u \in k\mathbb{G}_{a(r)},$$

so that $\mathfrak{M} = \bigoplus_{i=0}^{p^r-1} \mathfrak{M}_i$, with weight spaces

$$\mathfrak{M}_i = \{f \in \mathfrak{M}\ ;\ f((k\mathbb{G}_{a(r)})_j) = (0) \text{ for } j \neq i\} \cong \mathrm{Hom}_k((k\mathbb{G}_{a(r)})_i, k\mathcal{G}).$$

Note that $V(\mathcal{G})$ is $k^\times$-stable and that $V(\mathcal{G})_0 = \{\varepsilon\}$, where $\varepsilon : k\mathbb{G}_{a(r)} \longrightarrow k \subseteq k\mathcal{G}$ is the co-unit. Let $u_{r-1} \in k\mathbb{G}_{a(r)}$ be the linear map given by $u_{r-1}(t^j) = \delta_{j,p^{r-1}}$ for $0 \leq j \leq p^r-1$, so that $u_{r-1} \in k\mathbb{G}_{a(r)}$ is homogeneous of degree p^{r-1}.

Let $j \in \{1,\dots,p-1\}$. Given a $\mathcal{G}$-module M, the number

$$\mathrm{rk}^j(M) := \max\{\mathrm{rk}(\varphi(u_{r-1})^j_M)\ ;\ \varphi \in V(\mathcal{G})\}$$

is referred to as the *generic j-rank* of M. We call $\mathrm{rk}(M) := \mathrm{rk}^1(M)$ the *generic rank* of M.

For a vector space V, we let $\mathrm{Gr}_0(V)$ be the variety consisting of one point.

Theorem 2.2.1. *Let M be a $\mathcal{G}$-module such that $\mathrm{rk}^j(M) = d_j$. Then the following statements hold:*

(1) $U_{M,j} := \{[\varphi] \in \mathrm{Proj}(V(\mathcal{G}))\ ;\ \mathrm{rk}(\varphi(u_{r-1})^j_M) = d_j\}$ *is an open subset of* $\mathrm{Proj}(V(\mathcal{G}))$.

(2) *The map*

$$\mathsf{im}^j_M : U_{M,j} \longrightarrow \mathrm{Gr}_{d_j}(M)\ ;\ [\varphi] \mapsto \mathrm{im}\,\varphi(u_{r-1})^j_M$$

is a morphism of varieties.

Proof. (1) Recall that $k\mathcal{G}$ has the structure of a conical variety via the restriction of the scalar multiplication. We consider the evaluation map

$$u^*_{r-1} : V(\mathcal{G}) \longrightarrow k\mathcal{G}\ ;\ \varphi \mapsto \varphi(u_{r-1}).$$

Observing

$$u^*_{r-1}(\alpha.\varphi) = \varphi(\alpha.u_{r-1}) = \alpha^{p^{r-1}}\varphi(u_{r-1}) = \alpha^{p^{r-1}}u^*_{r-1}(\varphi)$$

for all $\alpha \in k^\times$, we conclude that u^*_{r-1} is a homogeneous morphism of degree p^{r-1}.

Let $\varrho : k\mathcal{G} \longrightarrow \mathrm{End}_k(M)$ be the representation afforded by M. By the above, the map

$$\omega^j : V(\mathcal{G}) \longrightarrow \mathrm{End}_k(M)\ ;\ \varphi \mapsto \varrho(\varphi(u_{r-1})^j)$$

is a homogeneous morphism of conical affine varieties of degree jp^{r-1}. As noted in Section 2.1,

$$O_{d_j} := \{[f] \in \mathbb{P}(\mathrm{End}_k(M))\ ;\ \mathrm{rk}(f) \geq d_j\}$$

is an open subset of $\mathbb{P}(\mathrm{End}_k(M))$. Since ω^j is homogeneous, the set

$$U_{M,j} = \{[\varphi] \in \mathrm{Proj}(V(\mathcal{G}))\ ;\ [\omega^j(\varphi)] \in O_{d_j}\}$$

is open in $\mathrm{Proj}(V(\mathcal{G}))$.

(2) If $d_j = 0$, then $\mathrm{Gr}_{d_j}(M)$ is a point. Hence we may assume that $d_j > 0$. Lemma 1.3.1(2) now implies that

$$\overline{\omega}^j : U_{M,j} \longrightarrow \mathbb{P}(\mathrm{End}_k(M))\ ;\ [\varphi] \mapsto [\varrho(\varphi(u_{r-1})^j)],$$

is a morphism, whose image is contained in $\mathbb{P}(\mathrm{End}_k(V))_{d_j}$. In view of

$$\mathsf{im}^j_M = \overline{\mathrm{im}} \circ \overline{\omega}^j,$$

our assertion follows from Proposition 2.1.2. □

By combining im^j_M with the Plücker embedding $\mathsf{pl}_M : \mathrm{Gr}_{d_j}(M) \longrightarrow \mathbb{P}(\bigwedge^{d_j}(M))$, we obtain a morphism

$$\mathsf{pl}_M \circ \mathsf{im}^j_M : U_{M,j} \longrightarrow \mathbb{P}(\bigwedge^{d_j}(M))$$

such that $\mathsf{pl}_M \circ \mathsf{im}^j_M = \overline{\mathrm{ev}} \circ \overline{\omega}^j$. The Example of Section 2.1 indicates that this morphism may not be homogeneous.

Given a $\mathcal{G}$-module M and a one-parameter subgroup $\varphi \in V(\mathcal{G})$, we denote by $\varphi^*(M)$ the $k\mathbb{G}_{a(r)}$-module with underlying k-space M and action given by

$$a\centerdot m := \varphi(a)m \qquad \forall\, a \in k\mathbb{G}_{a(r)},\ m \in M.$$

In [21] the authors define a scheme, whose variety of k-rational points is the *rank variety*

$$V(\mathcal{G})_M := \{\varphi \in V(\mathcal{G})\ ;\ \varphi^*(M)|_{k[u_{r-1}]} \text{ is not projective}\}$$

of M. Note that $V(\mathcal{G})_M$ is a conical subset of $V(\mathcal{G})$. If $V(\mathcal{G})_M \subsetneq V(\mathcal{G})$, then $\mathrm{rk}^j(M) = (p-j)\frac{\dim_k M}{p}$ and $U_{M,j} = \mathrm{Proj}(V(\mathcal{G}))\smallsetminus\mathrm{Proj}(V(\mathcal{G})_M)$.

Following [10], we say that a $\mathcal{G}$-module M has *constant j-rank*, provided $U_{M,j} = \mathrm{Proj}(V(\mathcal{G}))$. Modules of constant 1-rank are said to be of *constant rank*. We record the following direct consequence of Theorem 2.2.1.

Corollary 2.2.2. *Let M be a $\mathcal{G}$-module of constant j-rank $\mathrm{rk}^j(M) = d_j$. Then the map*

$$\mathsf{im}^j_M : \mathrm{Proj}(V(\mathcal{G})) \longrightarrow \mathrm{Gr}_{d_j}(M)\ \ ;\ \ [\varphi] \mapsto \mathrm{im}\,\varphi(u_{r-1})^j_M$$

is a morphism of varieties. □

Remark. One can equally well consider maps given by cokernels and kernels.

3. Morphisms for restricted Lie Algebras

In this section we consider representations of restricted Lie algebras. By general theory, these algebras correspond to infinitesimal groups of height 1, a class whose varieties of infinitesimal one-parameter subgroups are particularly tractable.

Throughout, $(\mathfrak{g}, [p])$ denotes a finite-dimensional restricted Lie algebra. The finite-dimensional quotient

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]}\ ;\ x \in \mathfrak{g}\})$$

of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is referred to as the *restricted enveloping algebra* of $\mathfrak{g}$. The representations of the restricted Lie algebra $(\mathfrak{g}, [p])$ are the modules for the Hopf algebra $U_0(\mathfrak{g})$. We refer the reader to [19] concerning restricted Lie algebras and their representations.

3.1. **Preliminaries.** The conical Zariski closed subset

$$V(\mathfrak{g}) := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}$$

is referred to as the *nullcone* of $\mathfrak{g}$. We say that $(\mathfrak{g}, [p])$ is *p-trivial*, provided $V(\mathfrak{g}) = \mathfrak{g}$. By Engel's theorem such a Lie algebra is necessarily nilpotent.

General theory shows that the cone $V(\mathfrak{g})$ plays the role of $V(\mathcal{G})$. Thus, for $j \in \{1, \ldots, p-1\}$, the *generic j-rank* of a $U_0(\mathfrak{g})$-module M is given by

$$\mathrm{rk}^j(M) := \max\{\mathrm{rk}(x_M^j) \ ; \ x \in V(\mathfrak{g})\},$$

and we say that M has *constant j-rank*, provided $\mathrm{rk}(x_M^j) = \mathrm{rk}(y_M^j)$ for all $x, y \in V(\mathfrak{g}) \smallsetminus \{0\}$. In this setting, Theorem 2.2.1 reads as follows:

Corollary 3.1.1. *Let M be a $U_0(\mathfrak{g})$-module of generic j-rank d_j. Then the following statements hold:*

(1) $U_{M,j} := \{[x] \in \mathbb{P}(V(\mathfrak{g})) \ ; \ \mathrm{rk}(x_M^j) = d_j\}$ *is an open subset of* $\mathbb{P}(V(\mathfrak{g}))$.

(2) *The map*

$$\mathsf{im}_M^j : U_{M,j} \longrightarrow \mathrm{Gr}_{d_j}(M) \ ; \ [x] \mapsto \mathrm{im}\, x_M^j$$

is a morphism of varieties. □

Modules of constant 1-rank are referred to as being of constant rank. Given a $U_0(\mathfrak{g})$-module M, we write $\mathrm{rk}(M) := \mathrm{rk}^1(M)$ as well as $\mathsf{im}_M := \mathsf{im}_M^1$.

Lemma 3.1.2. *Let M be a $U_0(\mathfrak{g})$-module of constant rank d such that $\mathsf{im}_M : \mathbb{P}(V(\mathfrak{g})) \longrightarrow \mathrm{Gr}_d(M)$ is constant. If $\mathfrak{g}$ contains a non-abelian p-trivial subalgebra, then $d = 0$.*

Proof. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a non-abelian p-trivial subalgebra. Then $U_0(\mathfrak{h})$ is a local algebra such that $\mathfrak{h} \subseteq \mathrm{Rad}(U_0(\mathfrak{h}))$, and the restriction $N := M|_{\mathfrak{h}}$ of M is a $U_0(\mathfrak{h})$-module of constant rank d.

We shall show inductively that $\mathrm{im}\, x_N \subseteq \mathrm{Rad}^n(N)$ for every $n \geq 1$ and $x \in \mathfrak{h}$, the case $n = 1$ following from $\mathfrak{h} \subseteq \mathrm{Rad}(U_0(\mathfrak{h}))$. Let $n \geq 2$. By assumption, there exists $x_0 \in [\mathfrak{h}, \mathfrak{h}] \smallsetminus \{0\}$. Writing $x_0 = \sum_{i=1}^{\ell} [a_i, b_i]$ with $a_i, b_i \in \mathfrak{h}$, the inductive hypothesis yields

$$\mathrm{im}(x_0)_N \subseteq \sum_{i=1}^{\ell} (a_i.b_i.N + b_i.a_i.N) \subseteq \sum_{i=1}^{\ell} (a_i.\mathrm{Rad}^{n-1}(N) + b_i.\mathrm{Rad}^{n-1}(N)) \subseteq \mathrm{Rad}^n(N).$$

Since the map im_N is constant, the inclusion between the extreme terms holds for every $x \in \mathfrak{h}$.

As there is $n \in \mathbb{N}$ with $\mathrm{Rad}^n(N) = (0)$, we see that $\mathrm{im}\, x_N = (0)$ for all $x \in \mathfrak{h}$. As a result, $d = \mathrm{rk}(N) = 0$. □

3.2. **Lie algebras with smooth nullcones.** Since $V(\mathfrak{g})$ is a conical variety, it is smooth if and only if it is a subspace of $\mathfrak{g}$. For such Lie algebras, the results of Section 1 come to bear.

The *descending central series* $(\mathfrak{g}^n)_{n \geq 1}$ of $\mathfrak{g}$ is defined via $\mathfrak{g}^1 := \mathfrak{g}$ and $\mathfrak{g}^{n+1} := [\mathfrak{g}, \mathfrak{g}^n]$. If $\mathfrak{g}^n = (0)$, then $\mathfrak{g}$ is said to be nilpotent of nilpotent length $\leq n$.

Example. If $\mathfrak{g}$ is nilpotent of nilpotent length $\leq p$, then, given $x, y \in V(\mathfrak{g})$, Jacobson's formula [19, (II.1)] yields

$$(x+y)^{[p]} = \sum_{i=1}^{p-1} s_i(x, y) \in \mathfrak{g}^p = (0),$$

so that $V(\mathfrak{g})$ is a subspace of $\mathfrak{g}$.

We begin by recording a direct consequence of a general result on morphisms between projective spaces.

Lemma 3.2.1. *Suppose that $V(\mathfrak{g})$ is a subspace of $\mathfrak{g}$ and let M be a $U_0(\mathfrak{g})$-module of constant j-rank d_j. Then the image of the morphism*

$$\mathsf{im}^j_M : \mathbb{P}(V(\mathfrak{g})) \longrightarrow \mathrm{Gr}_{d_j}(M)$$

has dimension $\dim \mathsf{im}^j_M(\mathbb{P}(V(\mathfrak{g}))) \in \{0, \dim V(\mathfrak{g})-1\}$.

Proof. Recall that the Plücker embedding $\mathsf{pl}_M : \mathrm{Gr}_{d_j}(M) \longrightarrow \mathbb{P}(\bigwedge^{d_j}(M))$ is an injective morphism of projective varieties. Thanks to [17, I.7,Prop.6], the conclusion of the Lemma holds for the morphism $\mathsf{pl}_M \circ \mathsf{im}^j_M$. Hence it is also valid for im^j_M. □

We denote by $C(\mathfrak{g}) := \{x \in \mathfrak{g} \,;\, [x, \mathfrak{g}] = (0)\}$ the *center* of $\mathfrak{g}$. We say that a $U_0(\mathfrak{g})$-module M has the *equal images property*, provided

$$\operatorname{im} x^j_M = \operatorname{im} y^j_M \quad \text{for all } x, y \in V(\mathfrak{g})\smallsetminus\{0\} \text{ and } j \in \{1, \dots, p-1\}.$$

We shall see in Theorem 3.2.1 below that this condition is equivalent to $\operatorname{im} x_M = \operatorname{im} y_M$ for all $x, y \in V(\mathfrak{g})\smallsetminus\{0\}$ in case $\mathfrak{g}$ is abelian. For elementary abelian groups, modules with the latter property were first systematically investigated in [5]. We denote by $\mathrm{EIP}(\mathfrak{g})$ the full subcategory of $\operatorname{mod} U_0(\mathfrak{g})$, whose objects are the equal images modules.

In view of Lemma 3.2.1, the morphism im^j_M is constant or its generic fiber has dimension 0. The following result demonstrates that im_M has more restrictive properties.

Theorem 3.2.2. *Let M be a $U_0(\mathfrak{g})$-module of constant rank d. Then the following statements hold:*

(1) *Suppose that $V(\mathfrak{g})$ is a subspace. Then the morphism $\mathsf{im}_M : \mathbb{P}(V(\mathfrak{g})) \longrightarrow \mathrm{Gr}_d(M)$ is injective or constant.*
(2) *If $V(\mathfrak{g}) \cap C(\mathfrak{g}) \neq (0)$ and im_M is constant, then $M \in \mathrm{EIP}(\mathfrak{g})$.*

Proof. (1) Let $r := \dim_k V(\mathfrak{g})$. By assumption, $\mathbb{P}(V(\mathfrak{g})) \cong \mathbb{P}^{r-1}$, so that we have a morphism

$$\mathsf{im}_M : \mathbb{P}^{r-1} \longrightarrow \mathrm{Gr}_d(M).$$

Suppose that im_M is not injective. Then there exist linearly independent elements $x, y \in V(\mathfrak{g})$ such that $\operatorname{im} x_M = \operatorname{im} y_M$. We consider the subspace $\mathfrak{u} := kx \oplus ky$ of $V(\mathfrak{g})$. Given $(\alpha, \beta) \in k^2\smallsetminus\{0\}$, we have

$$\operatorname{im}(\alpha x+\beta y)_M \subseteq \operatorname{im} x_M + \operatorname{im} y_M = \operatorname{im} x_M.$$

Since M has constant rank, we actually have equality. As a result, the morphism im_M is constant on $\mathbb{P}(\mathfrak{u}) = \mathbb{P}^1 \subseteq \mathbb{P}^{r-1}$. Owing to Corollary 1.1.5(2), this implies that the map im_M is constant.

(2) We show inductively that im^j_M is constant for every $j \in \{1, \dots, p-1\}$. Let $j > 1$ and let $z \in V(\mathfrak{g})$ be a non-zero central element. Given $x \in V(\mathfrak{g})\smallsetminus\{0\}$, the inductive hypothesis implies

$$\operatorname{im} x^j_M = x_M(\operatorname{im} x^{j-1}_M) = x_M(\operatorname{im} z^{j-1}_M) = z^{j-1}_M(\operatorname{im} x_M) = \operatorname{im} z^j_M.$$

As a result, the map im^j_M is constant.

It follows that $M \in \mathrm{EIP}(\mathfrak{g})$. □

Remark. Suppose that $V(\mathfrak{g})$ is a subspace of dimension ≥ 2. If M is a $U_0(\mathfrak{g})$-module of constant j-rank such that im^j_M is not constant, then [18, (4.1.6)] shows that the number of elements of a generic fiber of im^j_M is given by the separability degree $[k(V(\mathfrak{g})) : k(\mathsf{im}^j_M(V(\mathfrak{g})))]_s$ of the fields of rational functions. By the above, this field extension is purely inseparable if $j = 1$.

Corollary 3.2.3. *Suppose that $\mathfrak{g}$ is p-trivial, and let M be a $U_0(\mathfrak{g})$-module of constant rank d. Then the following statements hold:*

1. *The morphism $\mathsf{im}_M : \mathbb{P}(\mathfrak{g}) \longrightarrow \mathrm{Gr}_d(M)$ is injective or constant.*
2. *If im_M is constant, then $M \in \mathrm{EIP}(\mathfrak{g})$.*
3. *If im_M is constant and $\mathfrak{g}$ is not abelian, then $M \cong k^{\dim_k M}$.*

Proof. (1) This is a direct consequence of Theorem 3.2.2(1).

(2) Suppose that $\mathfrak{g} \neq (0)$. Since the p-trivial Lie algebra $\mathfrak{g}$ is nilpotent, it follows that $C(\mathfrak{g}) \neq (0)$. Theorem 3.2.2(2) now yields the result.

(3) Lemma 3.1.2 implies that $d = 0$, whence $x_M = 0$ for all $x \in \mathfrak{g}$. Consequently, M is a trivial $U_0(\mathfrak{g})$-module. □

Following [4], we refer to an abelian Lie algebra with trivial p-map as being *elementary*. These algebras appeared in Hochschild's work [15] on restricted cohomology, who called them strongly abelian. We denote by $\mathfrak{e}_r$ the, up to isomorphism, unique elementary Lie algebra of dimension r and note that the restricted enveloping algebra $U_0(\mathfrak{e}_r)$ is isomorphic (as an associative algebra) to the group algebra kE_r of the p-elementary abelian group E_r of rank r. As observed in [4], the set $\mathbb{E}(2,\mathfrak{g}) := \{\mathfrak{e} \subseteq \mathfrak{g}\ ;\ \mathfrak{e} \text{ is an elementary } p\text{-subalgebra of dimension } 2\}$ is a closed subset of the Grassmannian $\mathrm{Gr}_2(\mathfrak{g})$.

Examples. Theorem 3.2.2(1) may fail for modules of constant j-rank. Suppose that $p \geq 3$.

1. We consider the Lie algebra $\mathfrak{e}_2 = kx \oplus ky$ as well as $M := U_0(\mathfrak{e}_2)/\operatorname{Rad}^3(U_0(\mathfrak{e}_2))$. Using the standard basis $\{\bar{x}^i\bar{y}^j\ ;\ 0 \leq i{+}j \leq 2\}$, we see that M has constant 2-rank $\mathrm{rk}^2(M) = 1$. In this case, the morphism
$$\mathsf{im}^2_M : \mathbb{P}^1 \longrightarrow \mathbb{P}^5\ ;\ (\alpha{:}\beta) \mapsto (0{:}0{:}0{:}\alpha^2{:}2\alpha\beta{:}\beta^2)$$
is injective.
2. Let $N := M/k\bar{x}\bar{y}$. Then N has constant 2-rank $\mathrm{rk}^2(N) = 1$, and the map
$$\mathsf{im}^2_N : \mathbb{P}^1 \longrightarrow \mathbb{P}^5\ ;\ (\alpha{:}\beta) \mapsto (0{:}0{:}0{:}\alpha^2{:}0{:}\beta^2)$$
is not injective.

The injectivity of the map im^j_M in the first example above is a consequence of the following result, which holds for stable modules. Recall that the general linear group $\mathrm{GL}_r(k)$ acts $\mathfrak{e}_r$ via automorphism of restricted Lie algebras such that $\mathfrak{e}_r \smallsetminus \{0\}$ is an orbit. This implies that $\mathrm{GL}_r(k)$ acts transitively on $\mathbb{P}(\mathfrak{e}_r)$. Moreover, $\mathrm{GL}_r(k)$ operates on $U_0(\mathfrak{e}_r)$ via automorphisms of Hopf algebras. If M is a $U_0(\mathfrak{e}_r)$-module and $g \in \mathrm{GL}_r(\mathfrak{e}_r)$, then $M^{(g)}$ is the $U(\mathfrak{e}_r)$-module with underlying k-space M and $U_0(\mathfrak{e}_r)$-structure
$$u.m := (g^{-1}.u).m \quad \forall\, u \in U_0(\mathfrak{e}_r),\ m \in M.$$
We say that M is $\mathrm{GL}_r(k)$-*stable* if $M^{(g)} \cong M$ for all $g \in \mathrm{GL}_r(k)$. Since $\mathrm{GL}_r(k)$ acts transitively on $\mathfrak{e}_r \smallsetminus \{0\}$, every $\mathrm{GL}_r(k)$-stable module has constant j-rank for every $j \in \{1,\ldots,p{-}1\}$.

Proposition 3.2.4. *Let M be a $\mathrm{GL}_r(k)$-stable $U_0(\mathfrak{e}_r)$-module. If $j \in \{1,\ldots,p{-}1\}$ is such that im^j_M is not constant, then im^j_M is injective.*

Proof. By assumption, there exists for every $g \in \mathrm{GL}_r(k)$ an isomorphism $\kappa_g : M^{(g)} \longrightarrow M$ of $U_0(\mathfrak{e}_r)$-modules. This implies in particular the identities
$$(*) \qquad \operatorname{im}(g.x^j)_M = \kappa_g(\operatorname{im}(g.x^j)_{M^{(g)}}) = \kappa_g(\operatorname{im} x^j_M) \quad \forall\, x \in \mathfrak{e}_r, g \in \mathrm{GL}_r(k).$$
Suppose that $\operatorname{im} y^j_M = \operatorname{im} x^j_M$. Then we have $\operatorname{im}(g.y^j)_M = \kappa_g(\operatorname{im} y^j_M) = \kappa_g(\operatorname{im} x^j_M) = \operatorname{im}(g.x^j)_M$, so that $g.(\mathsf{im}^j_M)^{-1}(\mathsf{im}^j_M([x])) \subseteq (\mathsf{im}^j_M)^{-1}(\mathsf{im}^j_M(g.[x]))$ for every $g \in \mathrm{GL}_r(k)$ and $[x] \in \mathbb{P}(\mathfrak{e}_r)$. Hence

$(\mathsf{im}^j_M)^{-1}(\mathsf{im}^j_M(g.[x])) = g.(g^{-1}.(\mathsf{im}^j_M)^{-1}(\mathsf{im}^j_M(g.[x]))) \subseteq g.(\mathsf{im}^j_M)^{-1}(\mathsf{im}^j_M([x]))$, and we have equality. As $\mathrm{GL}_r(k)$ acts transitively on $\mathbb{P}(\mathfrak{e}_r)$, all fibers of im^j_M have the same number of elements. Since im^j_M is not constant, Lemma 3.2.1 ensures that all fibers are finite.

Let $[x] \in \mathbb{P}(\mathfrak{e}_r)$ and denote by $G_{[x]}$ the stabilizer of $[x]$ in $\mathrm{GL}_r(k)$. Let $[y]$ be another element of the fiber $(\mathsf{im}^j_M)^{-1}(\mathsf{im}^j_M([x]))$. For $g \in G_{[x]}$, we have, observing $(*)$,

$$\mathsf{im}^j_M(g.[y]) = \mathrm{im}(g.y^j)_M = \kappa_g(\mathrm{im}\, y^j_M) = \kappa_g(\mathrm{im}\, x^j_M) = \mathrm{im}(g.x^j)_M = \mathrm{im}\, x^j_M = \mathsf{im}^j_M([x]).$$

Accordingly, the orbit $G_{[x]}.[y]$ is contained in the finite fiber $(\mathsf{im}^j_M)^{-1}(\mathsf{im}^j_M([x]))$.

As $\mathrm{GL}_r(k)$ acts transitively on $\mathfrak{e}_r \smallsetminus \{0\}$, the stabilizer $G_{[x]}$ is isomorphic to the group of those matrices, whose first column is of the form $\alpha(1,0,\ldots,0)^{\mathrm{tr}}$ for some $\alpha \in k^\times$. Consequently, $G_{[x]} \cong k^\times \times (k^{r-1} \rtimes \mathrm{GL}_{r-1}(k))$ is connected. As $G_{[x]}.[y]$ is finite, we obtain $G_{[x]}.[y] = \{[y]\}$, so that $G_{[x]} \subseteq G_{[y]}$. By symmetry, we thus have $G_{[x]} = G_{[y]}$.

Suppose that $[x] \neq [y]$. Then $\{x, y\}$ and $\{x, x+y\}$ are parts of two bases of $\mathfrak{e}_r$, and we can find $g \in \mathrm{GL}_r(k) = \mathrm{GL}(\mathfrak{e}_r)$ such that $g.x = x$ and $g.y = x+y$. Since $[y] \neq [x+y]$ we have $g \in G_{[x]} \smallsetminus G_{[y]}$, a contradiction. As a result, the map im^j_M is injective. □

Remark. If P is a projective $U_0(\mathfrak{e}_r)$-module. Then P is $\mathrm{GL}_r(k)$-stable and Proposition 3.2.4 implies that im^j_M is injective. Since $M_n := U_0(\mathfrak{e}_r)/\operatorname{Rad}^n(U_0(\mathfrak{e}_r))$ is $\mathrm{GL}_r(k)$-stable, it follows that $\mathsf{im}^j_{M_n}$ is injective whenever $j < n$.

4. The degree of a module

In this section, we introduce new invariants for modules of restricted Lie algebras, called j-degrees. We study the behavior of degrees on short exact sequences, and prove a formula that relates the j-degrees of a module and its dual module to its j-rank. This yields information concerning Jordan types of self-dual modules along with the computation of the degrees of a number of modules over elementary Lie algebras.

4.1. **General properties.** Throughout, $(\mathfrak{g}, [p])$ is assumed to be a restricted Lie algebra. In the sequel, we shall be concerned with modules for restricted Lie algebras $\mathfrak{g}$, whose nullcones $V(\mathfrak{g})$ are subspaces. As noted earlier, every $U_0(\mathfrak{g})$-module M gives rise to morphisms

$$\mathsf{pl}_M \circ \mathsf{im}^j_M : U_{M,j} \longrightarrow \mathbb{P}\Big(\bigwedge^{\mathrm{rk}^j(M)}(M)\Big),$$

where $U_{M,j} \subseteq \mathbb{P}(V(\mathfrak{g})) \cong \mathbb{P}^{r-1}$ is a non-empty open subset. The results of Section 1 motivate the following:

Definition. Suppose that $V(\mathfrak{g})$ is a subspace of $\mathfrak{g}$, $j \in \{1,\ldots,p-1\}$. Let M be a $U_0(\mathfrak{g})$-module of generic j-rank $\mathrm{rk}^j(M) > 0$. Then

$$\deg^j(M) := \deg(\mathsf{pl}_M \circ \mathsf{im}^j_M)$$

is called the *j-degree* of M. We write $\deg(M) := \deg^1(M)$ and refer to $\deg(M)$ as the *degree* of M.

If $\mathrm{rk}^j(M) = 0$, then we define $\deg^j(M) := 0$.

We say that a $U_0(\mathfrak{g})$-module M is of *constant Jordan type*, provided M has constant j-rank for all $j \in \{1,\ldots,p-1\}$. In view of cf. [3, (1.7)], our definition coincides with the one given in [3] in the wider context of finite group schemes, see also Sections 4.4 and 6.3 below.

Remarks. (1) Directly from the definition we see that a $U_0(\mathfrak{g})$-module M of constant Jordan type has the equal images property if and only if $\deg^j(M) = 0$ for every $j \in \{1,\dots,p-1\}$. If, in addition, $V(\mathfrak{g}) \cap C(\mathfrak{g}) \neq (0)$, then $\deg(M) = 0$ already implies $M \in \mathrm{EIP}(\mathfrak{g})$.

(2) Let $\mathfrak{e}_2 := kx \oplus ky$ be the two-dimensional elementary Lie algebra. Suppose that M is a $U_0(\mathfrak{e}_2)$-module such that $y_M = 0$. Since $\mathrm{im}(\alpha x+\beta y)^j_M = \mathrm{im}\,\alpha^j x^j_M$ for all $\alpha, \beta \in k$, the maps im^j_M are constant, so that $\deg^j(M) = 0$ for all $j \in \{1,\dots,p-1\}$.

Proposition 4.1.1. *Suppose that $V(\mathfrak{g}) \subseteq \mathfrak{g}$ is a subspace. Let M be a $U_0(\mathfrak{g})$-module. Then we have*

$$\deg^j(M) \in \{0,\dots,j\,\mathrm{rk}^j(M)\}$$

for every $j \in \{1,\dots,p-1\}$.

Proof. Let $j \in \{1,\dots,p-1\}$ and put $d := \mathrm{rk}^j(M)$. Without loss of generality, we may assume that $d > 0$. We fix a basis $\{v_1,\dots,v_n\}$ of M and adopt the conventions of Section 2.1. For $J \in \mathcal{S}(d)$, we consider the open subset

$$\mathcal{U}_J := \{[f] \in \mathbb{P}(\mathrm{End}_k(M))_d \ ; \ \bigwedge^d(f)(v_J) \neq 0\}$$

of $\mathbb{P}(\mathrm{End}_k(M))_d$. Thanks to Proposition 2.1.2(1), the map

$$\overline{\mathrm{ev}} : \mathbb{P}(\mathrm{End}_k(M))_d \longrightarrow \mathbb{P}(\bigwedge^d(M)) \ \ ; \ \ [f] \mapsto \bigwedge^d(f(M))$$

is a morphism such that

$$\overline{\mathrm{ev}}([f]) = [\bigwedge^d(f)(v_J)] \qquad \text{for all } \ [f] \in \mathcal{U}_J.$$

Let $f \in \mathrm{End}_k(M)_d$ and denote by $A(f) = (a_{ij}) \in \mathrm{Mat}_n(k)$ the $(n\times n)$-matrix representing f with respect to the basis $\{v_1,\dots,v_n\}$. As noted earlier, we have

$$\bigwedge^d(f)(v_K) = \sum_{K' \in \mathcal{S}(d)} \det(A(f)_{(K',K)})v_{K'} \qquad \forall\, K \in \mathcal{S}(d).$$

Thus, the Plücker coordinates of $\overline{\mathrm{ev}}([f])$ for $[f] \in \mathcal{U}_J$ are $[\det(A(f)_{(I,J)})]_{I\in\mathcal{S}(d)}$. As a result, the map $\overline{\mathrm{ev}}|_{\mathcal{U}_J}$ is defined by homogeneous polynomials of degree d.

The canonical map $\varrho_{M,j} : U_{M,j} \longrightarrow \mathbb{P}(\mathrm{End}_k(M))_d$; $[x] \mapsto [x^j_M]$ is given by homogeneous polynomials of degree j. Setting $O_J := \varrho^{-1}_{M,j}(\mathcal{U}_J)$, we obtain an open cover $U_{M,j} = \bigcup_{J\in\mathcal{S}(d_j)} O_J$ of $U_{M,j}$ such that $\mathsf{pl}_M \circ \mathsf{im}^j_M|_{O_J} = \overline{\mathrm{ev}} \circ \varrho_{M,j}|_{O_J}$ is defined by homogeneous polynomials $g_{0,J},\dots,g_{m,J} \in k[X_0,\dots,X_{r-1}]$ of degree jd, whenever $O_J \neq \emptyset$.

According to Lemma 1.1.3, the morphism $\mathsf{pl}_M \circ \mathsf{im}^j_M : U_{M,j} \longrightarrow \mathbb{P}(\bigwedge^d(M))$ is homogeneous, and Lemma 1.1.2 yields $\deg^j(M) \leq jd$. □

Remark. Let M be a module for an arbitrary restricted Lie algebra $\mathfrak{g}$. The above arguments also show that the morphism

$$\mathsf{im}^j_M : U_{M,j} \longrightarrow \mathrm{Gr}_{\mathrm{rk}^j(M)}(M)$$

is locally defined by homogeneous polynomials of degree $j\,\mathrm{rk}^j(M)$.

Theorem 4.1.2. *Suppose that $V(\mathfrak{g}) \subseteq \mathfrak{g}$ is a subspace, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a p-subalgebra such that $\dim V(\mathfrak{h}) \geq 2$. If M is a $U_0(\mathfrak{g})$-module of constant j-rank, then we have* $\deg^j(M) = \deg^j(M|_{\mathfrak{h}})$.

Proof. The canonical inclusion $\mathfrak{h} \subseteq \mathfrak{g}$ determines a morphism $\iota : \mathbb{P}(V(\mathfrak{h})) \hookrightarrow \mathbb{P}(V(\mathfrak{g}))$ of degree 1 such that $\mathsf{pl}_{M|_\mathfrak{h}} \circ \mathsf{im}^j_{M|_\mathfrak{h}} = \mathsf{pl}_M \circ \mathsf{im}^j_M \circ \iota$. Since $V(\mathfrak{h}) = V(\mathfrak{g}) \cap \mathfrak{h}$ is a subspace of dimension ≥ 2, Corollary 1.1.5 gives rise to

$$\begin{aligned}\deg^j(M|_\mathfrak{h}) &= \deg(\mathsf{pl}_{M|_\mathfrak{h}} \circ \mathsf{im}^j_{M|_\mathfrak{h}}) = \deg(\mathsf{pl}_M \circ \mathsf{im}^j_M \circ \iota) = \deg(\mathsf{pl}_M \circ \mathsf{im}^j_M) \deg(\iota) \\ &= \deg(\mathsf{pl}_M \circ \mathsf{im}^j_M) = \deg^j(M),\end{aligned}$$

as desired. □

Corollary 4.1.3. *Suppose that $V(\mathfrak{g}) \subseteq \mathfrak{g}$ is a subspace and that $\mathbb{E}(2, \mathfrak{g}) \neq \emptyset$. A $U_0(\mathfrak{g})$-module M of constant Jordan type has the equal images property if and only if $\deg(M) = 0$.*

Proof. Suppose that $\deg(M) = 0$, and let $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})$. Then $\mathsf{im}_{M|_\mathfrak{e}}$ is constant and Corollary 3.2.3 ensures that $M|_\mathfrak{e}$ has the equal images property. As a result, $\deg^j(M|_\mathfrak{e}) = 0$ for every $j \in \{1, \dots, p-1\}$. Thanks to Theorem 4.1.2, this also holds for the j-degrees of M, so that im^j_M is constant for $1 \leq j \leq p-1$. Consequently, M has the equal images property. □

Remark. In the situation above, the condition $\mathbb{E}(2, \mathfrak{g}) \neq \emptyset$ does not follow from $\dim V(\mathfrak{g}) \geq 2$. Suppose that $p \geq 3$. Then the Heisenberg Lie algebra $\mathfrak{g} := kx \oplus ky \oplus kz$ with p-map $x^{[p]} = 0 = y^{[p]}$ and $z^{[p]} = z$ has a linear nullcone of dimension 2, while $\mathbb{E}(2, \mathfrak{g}) = \emptyset$.

Lemma 4.1.4. *Suppose that $(0) \longrightarrow N \stackrel{\iota}{\longrightarrow} E \stackrel{\pi}{\longrightarrow} M \longrightarrow (0)$ is a short exact sequence of $U_0(\mathfrak{g})$-modules, where $(\mathfrak{g}, [p])$ is a restricted Lie algebra such that $V(\mathfrak{g}) \subseteq \mathfrak{g}$ is a subspace. For $j \in \{1, \dots, p-1\}$, the following statements hold:*

1. *If $\mathrm{rk}^j(E) = \mathrm{rk}^j(M) + \mathrm{rk}^j(N)$, then $\deg^j(E) \geq \deg^j(M) + \deg^j(N)$.*
2. *If the sequence splits, then $\deg^j(E) = \deg^j(M) + \deg^j(N)$.*
3. *If E has constant j-rank and $N \subseteq \bigcap_{x \in V(\mathfrak{g}) \smallsetminus \{0\}} \mathrm{im}\, x^j_E$, then M has constant j-rank satisfying $\mathrm{rk}^j(E) = \mathrm{rk}^j(M) + \dim_k N$, while $\deg^j(E) = \deg^j(M)$.*
4. *If E has constant j-rank and $\sum_{x \in V(\mathfrak{g}) \smallsetminus \{0\}} \ker x^j_E \subseteq N$, then N has constant j-rank satisfying $\mathrm{rk}^j(E) = \mathrm{rk}^j(N) + \dim_k M$, while $\deg^j(E) = \deg^j(N) + j \dim_k M$.*

Proof. Let $\{v_1, \dots, v_\ell\}$ be a basis of E such that $\{v_1, \dots, v_n\}$ is a basis of N. We shall compute defining sets of polynomials via this basis. Base change amounts to composing morphisms by an automorphism of a suitable $\mathbb{P}^r$, which does not affect the degree.

Let $d_X := \mathrm{rk}^j(X)$ for $X \in \{M, N, E\}$. As before, we define $\mathcal{S}(d_E)$ to be the set of d_E-element subsets of $[1, \ell] := \{1, \dots, \ell\}$ and let $S_N(d_N)$ and $S_M(d_M)$ denote the corresponding sets for $[1, n]$ and $[n+1, \ell]$, respectively. For a subset $J \subseteq [1, \ell]$, we set $J_N := J \cap [1, n]$ as well as $J_M := J \cap [n+1, \ell]$ and put

$$\mathcal{T}(d_E) := \{J \in \mathcal{S}(d_E) \,;\; J_X \in \mathcal{S}_X(d_X) \text{ for } X \in \{M, N\}\}.$$

Given elements $v, w \in V$ of a vector space V, we shall write $v \approx w$ to indicate that $kv = kw$.

(1) We assume that $d_M, d_N \neq 0$, leaving the requisite modifications of the ensuing arguments for the remaining cases to the reader.

By choice of the basis, the vectors $w_j := \pi(v_j)$ form a basis $\{w_{n+1}, \dots, w_\ell\}$ of M. In view of our current assumption, there exists $J \in \mathcal{T}(d_E)$ such that

$$O := \{[x] \in \mathbb{P}(V(\mathfrak{g})) \,;\, \bigwedge^{d_N}(x^j_N)(v_{J_N}) \neq 0 \text{ and } \bigwedge^{d_M}(x^j_M)(w_{J_M}) \neq 0\}$$

is a non-empty open subset of $U_{N,j} \cap U_{M,j} \subseteq U_{E,j}$.

Let $(f_I)_{I\in\mathcal{S}_N(d_N)}$, $(g_{I'})_{I'\in\mathcal{S}_M(d_M)}$ and $(h_{I''})_{I''\in\mathcal{S}(d_M)}$ be reduced defining systems for the morphisms $\mathsf{pl}_N\circ\mathsf{im}^j_N$, $\mathsf{pl}_M\circ\mathsf{im}^j_N$ and

$$\zeta_J : O \longrightarrow \mathbb{P}(\bigwedge^{d_M}(E)) \ ; \ [x]\mapsto [\bigwedge^{d_M}(x^j_E)(v_{J_M})],$$

respectively. In view of

$$\bigwedge^{d_M}(x^j_M)(w_{J_M}) = \bigwedge^{d_M}(\pi)(\bigwedge^{d_M}(x^j_E)(v_{J_M})) \approx \sum_{I'\in\mathcal{S}_M(d_M)} h_{I'}(x)w_{I'} \qquad \forall\, x\in O$$

there exists $I'_0\in\mathcal{S}_M(d_M)$ such that $h_{I'_0}\neq 0$. By the same token, Lemma 1.1.2(2) provides a homogeneous polynomial h such that $O\subseteq D(h)$ and

$$(*) \qquad h_{I'} = hg_{I'} \quad \text{for all } \ I'\in\mathcal{S}_M(d_M).$$

Let $\mathcal{P}(d_E) := \{(I,I'')\in\mathcal{S}_N(d_N)\times\mathcal{S}(d_M) \ ; \ I\cap I''=\emptyset\}$, so that $\mathcal{T}(d_E)\subseteq\{I\sqcup I'' \ ; \ (I,I'')\in\mathcal{P}(d_E)\}$. Given $[x]\in O\subseteq U_{E,j}$, we have

$$\begin{aligned}
\bigwedge^{d_E}(x^j_E)(v_J) &= \bigwedge^{d_N}(x^j_N)(v_{J_N})\wedge\bigwedge^{d_M}(x^j_E)(v_{J_M})\\
&\approx \sum_{I\in\mathcal{S}_N(d_N)} f_I(x)v_I\wedge\sum_{I''\in\mathcal{S}(d_M)} h_{I''}(x)v_{I''}\\
&= \sum_{(I,I'')\in\mathcal{P}(d_E)} \pm f_I(x)h_{I''}(x)v_{I\sqcup I''}.
\end{aligned}$$

Since there exists $I_0\in\mathcal{S}_N(d_N)$ such that $f_{I_0}\neq 0$, we conclude that the left-hand side does not vanish on the non-empty open set $O' := O\cap D(f_{I_0})\cap D(h_{I'_0})$. Consequently, the morphism

$$\mathsf{pl}_E\circ\mathsf{im}^j_E|_{O'} : O' \longrightarrow \mathbb{P}(\bigwedge^{d_E}(E)) \ ; \ [x]\mapsto\overline{\mathrm{ev}}([x^j_E])$$

is defined by the polynomials $(\pm f_I h_{I''})_{(I,I'')\in\mathcal{P}(d_E)}$ together with zero polynomials. Thus, if $(\gamma_Q)_{Q\in\mathcal{S}(d_E)}$ is a reduced defining system for $\mathsf{pl}_E\circ\mathsf{im}^j_E$, then Lemma 1.1.2(2) provides a homogeneous polynomial g such that

$$(**) \qquad g\gamma_{I\sqcup I''} = \pm f_I h_{I''} \qquad \text{for all } (I,I'')\in\mathcal{P}(d_E).$$

Let p be an irreducible factor of g. Then there exist $K_0\in\mathcal{S}_N(d_N)$ and $K'_0\in\mathcal{S}_M(d_M)$ such that $p\nmid f_{K_0}$ and $p\nmid g_{K'_0}$. Thus, $p\nmid f_{K_0}g_{K'_0}$, while $(*)$ and $(**)$ imply $g\gamma_{K_0\sqcup K'_0} = \pm hf_{K_0}g_{K'_0}$. As a result, $m_p(g)\leq m_p(h)$, so that $h=gg'$ for some homogeneous polynomial g'. Identities $(*)$ and $(**)$ now yield

$$\gamma_{K_0\sqcup K'_0} = \pm g' f_{K_0}g_{K'_0},$$

whence

$$\begin{aligned}
\deg^j(E) &= \deg(\gamma_{K_0\sqcup K'_0}) = \deg(g')+\deg(f_{K_0})+\deg(g_{K'_0}) = \deg(g')+\deg^j(N)+\deg^j(M)\\
&\geq \deg^j(N)+\deg^j(M),
\end{aligned}$$

as desired.

(2) Since the sequence splits, we have $\mathrm{rk}^j(E)=\mathrm{rk}^j(N)+\mathrm{rk}^j(M)$. We adopt the notation from (1) and let $(f_I)_{I\in\mathcal{S}_N(d_N)}$ and $(g_{I'})_{I'\in\mathcal{S}_M(d_M)}$ be reduced defining systems for $\mathsf{pl}_N\circ\mathsf{im}^j_N$ and $\mathsf{pl}_M\circ\mathsf{im}^j_M$, respectively.

Then we have for $x \in O$,

$$0 \neq \bigwedge^{d_E}(x_E^j)(v_J) = \bigwedge^{d_N}(x_N^j)(v_{J_N}) \wedge \bigwedge^{d_M}(x_M^j)(v_{J_M}) \approx \sum_{I \in \mathcal{S}_N(d_N)} f_I(x)v_I \wedge \sum_{I' \in \mathcal{S}_M(d_M)} h_{I'}(x)v_{I'} = \sum_{(I,I') \in \mathcal{S}_N(d_N)\times\mathcal{S}_M(d_M)} f_I(x)h_{I'}(x)v_{I\sqcup I'}.$$

Since the polynomials $(f_I g_{I'})_{(I,I')\in\mathcal{S}_N(d_N)\times\mathcal{S}_M(d_M)}$ have greatest common divisor 1, our assertion follows.

(3) Let $\sigma : M \longrightarrow E$ be a k-linear splitting of π. If $d_E = 0$, then $N = (0)$, and there is nothing to be shown. Alternatively, let $x \in V(\mathfrak{g})\smallsetminus\{0\}$. Then $\dim_k \operatorname{im} x_M^j = \dim_k \operatorname{im} x_E^j - \dim_k(\operatorname{im} x_E^j \cap N) = \dim_k \operatorname{im} x_E^j - \dim_k N$, so that M has constant j-rank $d_M = d_E - \dim_k N$. The splitting property implies that

$$(\dagger) \qquad V = \sigma(\pi(V)) \oplus N$$

for every subspace $V \subseteq E$ containing N. In particular, $E = \operatorname{im}\sigma \oplus N$, and the map σ gives rise to an injective k-linear map

$$\omega : \bigwedge^{d_M}(M) \longrightarrow \bigwedge^{d_E}(E) \ ; \ x \mapsto \bigwedge^{d_M}(\sigma)(x) \wedge u,$$

where $u \in \bigwedge^{\dim_k N}(N) \smallsetminus \{0\}$. Being linear, ω induces a morphism $\bar{\omega} : \mathbb{P}(\bigwedge^{d_M}(M)) \longrightarrow \mathbb{P}(\bigwedge^{d_E}(E))$ of degree 1. Let $\lambda : \mathrm{Gr}_{d_M}(M) \longrightarrow \mathrm{Gr}_{d_E}(E)$ be defined via $\lambda(W) := \sigma(W)\oplus N$. In view of ($\dagger$), direct computation reveals that $\bar{\omega}\circ \mathsf{pl}_M = \mathsf{pl}_E \circ\lambda$ as well as $\lambda\circ \mathsf{im}_M^j = \mathsf{im}_E^j$, so that

$$\bar{\omega}\circ \mathsf{pl}_M \circ \mathsf{im}_M^j = \mathsf{pl}_E \circ\lambda\circ \mathsf{im}_M^j = \mathsf{pl}_E \circ \mathsf{im}_E^j\,.$$

Owing to Corollary 1.1.5, we thus arrive at

$$\deg^j(E) = \deg(\mathsf{pl}_E\circ\mathsf{im}_E^j) = \deg(\bar{\omega}\circ\mathsf{pl}_M\circ\mathsf{im}_M^j) = \deg(\mathsf{pl}_M\circ\mathsf{im}_M^j) = \deg^j(M),$$

as asserted.

(4) By assumption, we have $\ker x_N^j = \ker x_E^j$ for all $x \in V(\mathfrak{g})\smallsetminus\{0\}$, so that N has constant j-rank

$$\mathrm{rk}^j(N) = \dim_k N - \dim_k \ker x_E^j = \dim_k E - \dim_k \ker x_E^j - \dim_k M = \mathrm{rk}^j(E) - \dim_k M$$

for all $x \in V(\mathfrak{g})\smallsetminus\{0\}$.

Let $V := \langle\{v_{n+1},\ldots,v_\ell\}\rangle$, so that $E = N\oplus V$. If $x \in V(\mathfrak{g})\smallsetminus\{0\}$, the condition $\ker x_E^j \subseteq N$ implies

$$(\dagger\dagger) \qquad \operatorname{im} x_E^j = \operatorname{im} x_N^j \oplus x_E^j(V) \ \text{ as well as } \ \dim_k x_E^j(V) = \ell - n = \dim_k M.$$

Let $J \in \mathcal{S}(d_E)$ be such that $O_J := \{[x]\in\mathbb{P}(V(\mathfrak{g}))\ ;\ \bigwedge^{d_E}(x_E^j)(v_J)\neq 0\}\neq\emptyset$. Given $[x]\in O_J$, we have

$$0 \neq \bigwedge^{d_E}(x_E^j)(v_J) = \bigwedge^{|J_N|}(x_N^j)(v_{J_N}) \wedge \bigwedge^{|J_M|}(x_E^j)(v_{J_M}).$$

Hence $|J_N| \le d_N$ and ($\dagger\dagger$) implies $|J_M| \le \dim_k M$. Since

$$|J| = d_E = d_N + \dim_k M \ \text{ while } \ J = J_N \sqcup J_M,$$

we obtain $|J_N| = d_N$ and $|J_M| = \ell - n$, so that $J_M = [n+1,\ell]$.

Since $x_E^j|_V$ is injective for every $x \in V(\mathfrak{g})\smallsetminus\{0\}$, we have $\bigwedge^{\ell-n}(x_E^j)(v_{[n+1,\ell]}) \neq 0$ for all $[x]\in\mathbb{P}(V(\mathfrak{g}))$. There results a morphism

$$\xi : \mathbb{P}(V(\mathfrak{g})) \longrightarrow \mathbb{P}(\bigwedge^{\ell-n}(E)) \ ; \ [x] \mapsto \bigwedge^{\ell-n}(x_E^j)(v_{[n+1,\ell]}),$$

whose coordinates are $[\det(A(x^j_E)_{(Q,[n+1,\ell])})]_{Q\in \mathcal{S}(n-\ell)}$. Thus, ξ is defined by homogeneous polynomials of degree $j(\ell-n)$, so that $\deg(\xi) = j(\ell-n)$.

Let $(g_K)_{K\in\mathcal{S}_N(d_N)}$ and $(h_Q)_{Q\in\mathcal{S}(\ell-n)}$ be reduced defining systems for $\mathsf{pl}_N \circ \mathsf{im}^j_N$ and ξ, respectively. As before, we put $\mathcal{P}(d_E) = \{(K,Q) \in \mathcal{S}_N(d_N)\times\mathcal{S}(\ell-n)\ ;\ K\cap Q = \emptyset\}$. Given $[x] \in \mathbb{P}(V(\mathfrak{g}))$, there exists $J \in \mathcal{S}(d_E)$ such that $[x] \in O_J$. The observations above imply

$$\begin{aligned}\bigwedge^{d_E}(x^j_E)(v_J) &\approx \sum_{K\in\mathcal{S}_N(d_N)}\ \sum_{Q\in\mathcal{S}(\ell-n)} g_K(x)h_Q(x)v_K\wedge v_Q\\ &= \sum_{(K,Q)\in\mathcal{P}(d_E)} \pm g_K(x)h_Q(x)v_{K\sqcup Q}.\end{aligned}$$

As a result, the polynomials $(\pm g_K h_Q)_{(K,Q)\in\mathcal{P}(d_E)}$ together with zero polynomials constitute a defining system for $\mathsf{pl}_E\circ\mathsf{im}^j_E : \mathbb{P}(V(\mathfrak{g})) \longrightarrow \mathbb{P}(\bigwedge^{d_E}(E))$. Corollary 1.1.4 implies that the system is reduced, so that

$$\deg^j(E) = \deg^j(N)+\deg(\xi) = \deg^j(N)+j\dim_k M,$$

as desired. □

Remarks. (1) The additivity of the generic j-ranks is usually not a consequence of the exactness of the sequence. Consider the exact sequence

$$(0)\longrightarrow k\longrightarrow U_0(\mathfrak{e}_r)\longrightarrow U_0(\mathfrak{e}_r)/k\longrightarrow(0)$$

of modules of constant rank. Since $\mathrm{rk}(k) = 0$ and $\mathrm{rk}(U_0(\mathfrak{e}_r)) = p^{r-1}(p-1)$ while $\mathrm{rk}(U_0(\mathfrak{e}_r)/k) = p^{r-1}(p-1)-1$, we have $\mathrm{rk}(U_0(\mathfrak{e}_r)) > \mathrm{rk}(U_0(\mathfrak{e}_r)/k)+\mathrm{rk}(k)$.

(2) The additivity of generic j-ranks holds whenever the sequence is *locally split*, i.e. when the restricted sequence

$$(0)\longrightarrow N|_{k[x]}\longrightarrow E|_{k[x]}\longrightarrow M|_{k[x]}\longrightarrow(0)$$

is split exact for every $x\in V(\mathfrak{g})$. Here $k[x] = U_0(kx)$ is the subalgebra of $U_0(\mathfrak{g})$ generated by x.

(3) Lemma 4.1.4(2) does not hold for arbitrary exact sequences of modules, whose generic j-ranks are additive. We consider the Lie algebra $\mathfrak{e}_2 := kx\oplus ky$ as well as the $U_0(\mathfrak{e}_2)$-module $E := U_0(\mathfrak{e}_2)/\operatorname{Rad}^2(U_0(\mathfrak{e}_2))$ with its canonical basis $\{\bar{1},\bar{x},\bar{y}\}$. Let $N := k\bar{y}$. Then we have $\mathrm{rk}(E) = 1 = \mathrm{rk}(E/N)$, while $\mathrm{rk}(N) = 0$. The module E has degree $\deg(E) = 1$, while $y_{E/N} = 0$ and $y_N = 0$ imply $\deg(N) = 0 = \deg(E/N)$.

(4) If M is a $U_0(\mathfrak{e}_r)$-module of constant j-rank, then $\tilde{\mathfrak{K}}^j(M) := \sum_{x\in\mathfrak{e}_r\smallsetminus\{0\}}\ker x^j_M$ is a submodule of M and Lemma 4.1.4(4) shows that $\tilde{\mathfrak{K}}^j(M)$ has constant j-rank while $\deg^j(M) = \deg^j(\tilde{\mathfrak{K}}^j(M)) + j\dim_k M/\tilde{\mathfrak{K}}^j(M)$. In particular, we have $M = \tilde{\mathfrak{K}}^j(M)$, whenever $\deg^j(M) < j$.

Remark. Let $(\mathfrak{g},[p])$ be a restricted Lie algebra such that $\zeta : \mathbb{P}^1 \longrightarrow \mathbb{P}(\mathfrak{g})$ is a non-constant morphism that factors through $\mathbb{P}(V(\mathfrak{g}))$. If M is a $U_0(\mathfrak{g})$-module of constant j-rank, one may consider $\deg^j_\zeta(M) := \deg(\mathsf{pl}_M\circ\mathsf{im}^j_M\circ\zeta)$. For $\mathfrak{g} = \mathfrak{sl}(2)$, the "Veronese embedding"

$$\zeta : \mathbb{P}^1 \longrightarrow \mathbb{P}(\mathfrak{sl}(2))\ ;\ (x\!:\!y)\mapsto [\begin{pmatrix} xy & x^2\\ -y^2 & -xy\end{pmatrix}]$$

is a morphism of degree $\deg(\zeta) = 2$ such that $\operatorname{im}\zeta = \mathbb{P}(V(\mathfrak{sl}(2)))$ and $\mathbb{P}^1 \longrightarrow \mathbb{P}(V(\mathfrak{sl}(2)))$ is an isomorphism.

4.2. **The rank-degree formula.** Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, M be a $U_0(\mathfrak{g})$-module of dimension n. For $d \leq n$, the assignment

$$(\, , \,) : \bigwedge^d(M^*) \times \bigwedge^d(M) \longrightarrow k \ ; \ \ (f_1 \wedge f_2 \wedge \cdots \wedge f_d, m_1 \wedge m_2 \wedge \cdots \wedge m_d) \mapsto \det((f_i(m_j)))$$

defines a non-degenerate bilinear form, see Lemma 2.1.3. The following subsidiary result shows that this form is compatible with the actions of $U_0(\mathfrak{g})$ on M and M^*.

Lemma 4.2.1. *Let M be an n-dimensional $U_0(\mathfrak{g})$-module, $d \leq n$. For $j \in \{1, \ldots, p-1\}$. We have*

$$(\bigwedge^d(x^j_{M^*})(a), m) = (-1)^{jd}(a, \bigwedge^d(x^j_M)(m))$$

for all $x \in \mathfrak{g}$, $a \in \bigwedge^d(M^*), m \in \bigwedge^d(M)$.

Proof. It suffices to verify the assertion for $a = f_1 \wedge f_2 \wedge \cdots \wedge f_d$, and $m = m_1 \wedge m_2 \wedge \cdots \wedge m_d$. For $x \in \mathfrak{g}$ and $\ell \in \{1, \ldots, p-1\}$ we have

$$\begin{aligned} (\bigwedge^d(x^\ell_{M^*})(f_1 \wedge \cdots \wedge f_d), m_1 \wedge \cdots \wedge m_d) &= (x^\ell . f_1 \wedge \cdots \wedge x^\ell . f_d, m_1 \wedge \cdots \wedge m_d) \\ &= \det(((x^\ell . f_i)(m_j))) \\ &= \det(((-1)^\ell f_i(x^\ell . m_j))) = (-1)^{\ell d} \det((f_i(x^\ell . m_j))) \\ &= (-1)^{\ell d}(f_1 \wedge \cdots \wedge f_d, \bigwedge^d(x^\ell_M)(m_1 \wedge \cdots \wedge m_d)), \end{aligned}$$

as desired. □

Theorem 4.2.2. *Suppose that $V(\mathfrak{g}) \subseteq \mathfrak{g}$ is a subspace. Let M be a $U_0(\mathfrak{g})$-module of constant j-rank. Then we have*

$$\deg^j(M) + \deg^j(M^*) = j \operatorname{rk}^j(M).$$

Proof. We put $d := \operatorname{rk}^j(M) = \operatorname{rk}^j(M^*)$. If $d = 0$, then $\deg^j(M) = 0 = \deg^j(M^*)$. We therefore assume $d > 0$. Adopting our previous notation, we let $\{v_1, \ldots, v_n\}$ be a basis of M, with dual basis $\{\delta_1, \ldots, \delta_n\} \subseteq M^*$. Let $A(x^j_M) \in \operatorname{Mat}_n(k)$ be the $(n \times n)$-matrix representing x^j_M relative to $\{v_1, \ldots, v_n\}$. Given $I, J \in \mathcal{S}(d)$, Lemma 4.2.1 implies

$$(*) \qquad (\bigwedge^d(x^j_{M^*})(\delta_I), v_J) = (-1)^{jd}(\delta_I, \bigwedge^d(x^j_M)(v_J)) = (-1)^{jd} \det(A(x^j_M)_{(I,J)})$$

for every $x \in \mathfrak{g}$.

Now let $(f_I)_{I \in \mathcal{S}(d)}$ and $(g_I)_{I \in \mathcal{S}(d)}$ be reduced defining systems for the morphisms $\mathsf{pl}_{M^*} \circ \mathsf{im}^j_{M^*}$ and $\mathsf{pl}_M \circ \mathsf{im}^j_M$, respectively. Given $I \in \mathcal{S}(d)$, we consider the open set $O_I := \{[x] \in \mathbb{P}(V(\mathfrak{g})) \, ; \, \bigwedge^d(x^j_{M^*})(\delta_I) \neq 0\}$. Then we have

$$\bigwedge^d(x^\ell_{M^*})(\delta_I) \approx \sum_{K \in \mathcal{S}(d)} f_K(x)\delta_K \qquad \forall \, [x] \in O_I.$$

Consequently, Lemma 2.1.3 in conjunction with $(*)$ implies

$$(**) \qquad [f_J(x)]_{J \in \mathcal{S}(d)} = [\det(A(x^j_M)_{(I,J)})]_{J \in \mathcal{S}(d)} \qquad \forall \, [x] \in O_I,$$

where the elements belong to $\mathbb{P}^{\binom{n}{d}-1}$. Thus, we have two defining systems for the morphism

$$\mathsf{pl}_{M^*} \circ \mathsf{im}^j_{M^*}|_{O_I} : O_I \longrightarrow \mathbb{P}^{\binom{n}{d}-1}.$$

Note that the morphism $V(\mathfrak{g}) \longrightarrow k$; $x \mapsto \det(A(x^j_M)_{(I,J)})$ is given by a homogeneous polynomial $\gamma_{(I,J)}$ of degree jd. Thus, if $O_I \neq \emptyset$, then Lemma 1.1.2 provides a homogeneous polynomial h_I with $O_I \subseteq D(h_I)$ such that

$$(***) \qquad \gamma_{(I,J)} = h_I f_J \qquad \forall\, J \in \mathcal{S}(d).$$

For $O_I = \emptyset$, identity $(*)$ implies $\gamma_{(I,J)} = 0$, so that the choice $h_I = 0$ gives the same identity.

For $J \in \mathcal{S}(d)$, we consider $U_J := \{[x] \in \mathbb{P}(V(\mathfrak{g}))\ ;\ \bigwedge^d(x^j_M)(v_J) \neq 0\}$. Let $J \in \mathcal{S}(d)$ be such that $U_J \neq \emptyset$. Given $[x] \in U_J$, we have

$$(\gamma_{(I,J)}(x))_{I \in \mathcal{S}(d)} \neq 0,$$

whence $f_J(x) \neq 0$. Moreover,

$$[\gamma_{(I,J)}(x)]_{I \in \mathcal{S}(d)} = [h_I(x) f_J(x)]_{I \in \mathcal{S}(d)} = [h_I(x)]_{I \in \mathcal{S}(d)} \qquad \forall\, [x] \in U_J,$$

implying that $(h_I)_{I \in \mathcal{S}(d)}$ and $(g_I)_{I \in \mathcal{S}(d)}$ are defining systems for the morphism $\mathsf{pl}_M \circ \mathsf{im}^j_M|_{U_J}$, with the latter being reduced. Consequently, Lemma 1.1.2 furnishes a homogeneous polynomial $\tilde{h}_J$ such that $U_J \subseteq D(\tilde{h}_J)$ and

$$h_I = \tilde{h}_J g_I \qquad \forall\, I \in \mathcal{S}(d).$$

Thus, if $U_J, U_{J'} \neq \emptyset$, then $\tilde{h} := \tilde{h}_J = \tilde{h}_{J'}$, so that $U_{J'} \subseteq D(\tilde{h})$. Since $\mathbb{P}(V(\mathfrak{g})) = \bigcup_{J \in \mathcal{S}(d)} U_J$, we conclude that $\mathbb{P}(V(\mathfrak{g})) \subseteq D(\tilde{h})$, implying that $\tilde{h}$ is constant.

Since the module M has constant j-rank $d > 0$, we obtain a morphism

$$\Phi : \mathbb{P}(V(\mathfrak{g})) \longrightarrow \mathbb{P}^{\binom{n}{d}^2 - 1}\ ;\ [x] \mapsto [\det(A(x^j_M)_{(I,J)})]_{(I,J) \in \mathcal{S}(d)^2}$$

of degree jd. It now follows from $(***)$ that $(g_I f_J)_{(I,J) \in \mathcal{S}(d)^2}$ is a reduced defining system for Φ, so that

$$j\,\mathrm{rk}^j(M) = \deg(\Phi) = \deg(g_I) + \deg(f_J) = \deg^j(M) + \deg^j(M^*)$$

for every pair $(I, J) \in \mathcal{S}(d)^2$ such that $g_I f_J \neq 0$. □

Let M be a $U_0(\mathfrak{g})$-module. Then M has the *equal j-images property* if im^j_M is constant. We say that M has the *equal j-kernels property*, provided there exists a subspace $W_j \subseteq M$ such that $\ker x^j_M = W_j$ for all $x \in V(\mathfrak{g}) \smallsetminus \{0\}$. If M has the equal j-kernels property for all $j \in \{1, \ldots, p-1\}$, then M has the *equal kernels property*. The full subcategory of equal kernels modules will be denoted $\mathrm{EKP}(\mathfrak{g})$.

Corollary 4.2.3. *Suppose that $V(\mathfrak{g}) \subseteq \mathfrak{g}$ is a subspace, and let M be a $U_0(\mathfrak{g})$-module of constant j-rank. Then M has the equal j-kernels property if and only if* $\deg^j(M) = j\,\mathrm{rk}^j(M)$.

Proof. A module M has the equal j-kernels property if and only if its dual module M^* has the equal j-images property. Since the latter property is equivalent to $\deg^j(M^*) = 0$, our assertion is a direct consequence of Theorem 4.2.2. □

Example. We consider the elementary Lie algebra $\mathfrak{e}_r$ and choose a basis $\{x_1, \ldots, x_r\} \subseteq \mathfrak{e}_r$. Using multi-index notation, we put $\tau := (p-1, \ldots, p-1) \in \mathbb{N}_0^r$ as well as $\epsilon_i := (\delta_{ij})_{1 \leq j \leq r}$ for $i \in \{1, \ldots, r\}$. Suppose that $p \geq 3$. Setting $v_r := \sum_{i=1}^r x^{\tau - 2\epsilon_i} \in U_0(\mathfrak{e}_r)$, we consider the submodule $M_{r+2} := kv_r \oplus \mathrm{Soc}_2(U_0(\mathfrak{e}_r))$ of $U_0(\mathfrak{e}_r)$. This module has Loewy length $\ell\ell(M_{r+2}) = 3$.

We write $M_{r+2} = kv_r \oplus \bigoplus_{i=1}^r kx_i.v_r \oplus kx^\tau$ and note that $x_i x_j v_r = \delta_{ij} x^\tau$. For $\lambda = (\lambda_1, \ldots, \lambda_r) \in k^r \smallsetminus \{0\}$, we put $x(\lambda) := \sum_{i=1}^r \lambda_i x_i$. Then we have

- $\mathrm{im}\, x(\lambda)_{M_{r+2}} = k(\sum_{i=1}^r \lambda_i x_i.v_r) \oplus kx^\tau$, as well as
- $\mathrm{im}\, x(\lambda)^2_{M_{r+2}} = k(\sum_{i=1}^r \lambda_i^2) x^\tau$.

Hence, for $r \geq 2$, the module M_{r+2} has constant rank $\mathrm{rk}(M_{r+2}) = 2$, but not constant 2-rank. As a result, M_{r+2} is neither an equal 1-images module nor an equal 1-kernels module, so that $\deg(M_{r+2}) = 1$.

4.3. **Degrees for modules of trigonalizable Lie algebras.** Our results concerning degrees of modules of constant j-rank depend on a technical assumption on the nullcone that in general is only known to hold for nilpotent Lie algebras of nilpotent length $\leq p$. By way of illustration, we indicate in this section a generalization concerning the behavior of degrees on varieties of elementary subalgebras. This is done in the context of trigonalizable Lie algebras; Lie algebras of algebraic groups will be addressed in future work.

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, M be a $U_0(\mathfrak{g})$-module. We consider the map

$$\mathsf{deg}^j_M : \mathbb{E}(2, \mathfrak{g}) \longrightarrow \mathbb{N}_0 \ ; \ \mathfrak{e} \mapsto \deg^j(M|_{\mathfrak{e}}).$$

According to Theorem 4.1.2, this map is constant whenever M is a module of constant j-rank and $V(\mathfrak{g})$ is a subspace of $\mathfrak{g}$ ($\mathbb{E}(2, \mathfrak{g}) \neq \emptyset$).

Given a p-subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, the canonical inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ induces a closed immersion $\mathbb{E}(2, \mathfrak{h}) \hookrightarrow \mathbb{E}(2, \mathfrak{g})$, whose image $\{\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g}) \ ; \ \mathfrak{e} \subseteq \mathfrak{h}\}$ will be identified with $\mathbb{E}(2, \mathfrak{h})$.

We denote by $(C_n(\mathfrak{g}))_{n\geq 1}$ the *ascending central series* of $\mathfrak{g}$, which is inductively defined via $C_1(\mathfrak{g}) := C(\mathfrak{g})$ and $C_{n+1}(\mathfrak{g}) := \{x \in \mathfrak{g} \ ; [x, \mathfrak{g}] \subseteq C_n(\mathfrak{g})\}$. Each member of this series is a p-ideal of $\mathfrak{g}$.

A restricted Lie algebra is called *trigonalizable*, provided every simple $U_0(\mathfrak{g})$-module is one-dimensional. In view of the Lie-Kolchin Theorem, the Lie algebra of a solvable algebraic group is trigonalizable.

Proposition 4.3.1. *Suppose that $p \geq 3$ and let $\mathfrak{g}$ be trigonalizable. If M is a $U_0(\mathfrak{g})$-module of constant j-rank, then deg^j_M is constant.*

Proof. Let $\mathfrak{a} \subseteq \mathfrak{g}$ be an abelian p-subalgebra. Then $\mathfrak{h}_{\mathfrak{a}} := \mathfrak{a}+C_3(\mathfrak{g})$ is a p-subalgebra such that $\mathfrak{h}_{\mathfrak{a}}^3 = (0)$, so that $\mathfrak{h}_{\mathfrak{a}}$ is a nilpotent subalgebra of nilpotent length ≤ 3. As $p \geq 3$, the nullcone $V(\mathfrak{h}_{\mathfrak{a}})$ is subspace of $\mathfrak{h}_{\mathfrak{a}}$.

We first assume $\mathfrak{g}$ to be unipotent. Suppose that $\dim V(C_3(\mathfrak{g})) = 1$. Then there exists a p-nilpotent element $x_0 \in C_3(\mathfrak{g})$ such that $C_3(\mathfrak{g}) = (kx_0)_p$ is a nil-cyclic Lie algebra (cf. [6, (4.3)]). Given $y \in \mathfrak{g}$, we have

$$(\mathrm{ad}\, x_0)^3(y) = 0.$$

As $p \geq 3$, this readily yields $x_0^{[p]} \in C(\mathfrak{g})$, so that $\dim_k C_3(\mathfrak{g})/C(\mathfrak{g}) \leq 1$. Thus, $C_2(\mathfrak{g}) = C(\mathfrak{g})$ or $C_2(\mathfrak{g}) = C_3(\mathfrak{g})$. As $\mathfrak{g}$ is nilpotent, we obtain $C_3(\mathfrak{g}) = \mathfrak{g}$ in either case. It follows that $\mathbb{E}(2, \mathfrak{g}) = \emptyset$, and there is nothing to be shown.

Alternatively, $\dim V(C_3(\mathfrak{g})) \geq 2$ and since $\mathfrak{g}$ is unipotent, there is $\mathfrak{e}_0 \in \mathbb{E}(2, C_3(\mathfrak{g}))$. Now let $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})$. By the above, $V(\mathfrak{h}_{\mathfrak{e}})$ is a subspace, so that Theorem 4.1.2 yields

$$\mathsf{deg}^j_M(\mathfrak{e}) = \mathsf{deg}^j_{M|_{\mathfrak{h}_{\mathfrak{e}}}}(\mathfrak{e}) = \deg^j(M|_{\mathfrak{h}_{\mathfrak{e}}}) = \mathsf{deg}^j_{M|_{\mathfrak{h}_{\mathfrak{e}}}}(\mathfrak{e}_0) = \mathsf{deg}^j_M(\mathfrak{e}_0),$$

as desired.

Let $\mathfrak{g}$ be trigonalizable. By general theory, $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{t}$ is a semi-direct sum, where $\mathfrak{u}$ is unipotent and $\mathfrak{t}$ is a torus. Since $\mathbb{E}(2, \mathfrak{g}) = \mathbb{E}(2, \mathfrak{u})$, our assertion follows directly from the first part of the proof. □

4.4. **Applications: Self-dual modules and exact sequences.** Given $x \in V(\mathfrak{g})\smallsetminus\{0\}$, the subalgebra $k[x] \subseteq U_0(\mathfrak{g})$ is isomorphic to the truncated polynomial ring $k[T]/(T^p)$. Setting $[i] := k[x]/(x^i)$ for $1 \le i \le p$, we obtain a full set $\{[1], \dots, [p]\}$ of representatives for the isomorphism classes of the indecomposable $k[x]$-modules. If M is a $U_0(\mathfrak{g})$-module, then

$$M|_{k[x]} \cong \bigoplus_{i=1}^{p} a_i(x)[i]$$

for some $a_i(x) \in \mathbb{N}_0$. This isomorphism class is the *Jordan type* $\mathrm{Jt}(M, x)$ *of* M *relative to* x. We denote by

$$\mathrm{Jt}(M) = \{\mathrm{Jt}(M, x)\ ;\ x \in V(\mathfrak{g})\smallsetminus\{0\}\}$$

the finite set of Jordan types of M. Thus, the $U_0(\mathfrak{g})$-module M has constant Jordan type if and only if $|\,\mathrm{Jt}(M)| = 1$. In that case, we write

$$\mathrm{Jt}(M) = \bigoplus_{i=1}^{p} a_i[i].$$

We refer the reader to [3] for more details.

Remark. Let M be an equal images module of constant rank $\mathrm{rk}(M) \neq 0$. Then M^* is an equal kernels module of constant rank $\mathrm{rk}(M^*) = \mathrm{rk}(M)$. By the above, we have $\deg(M) = 0 \neq \deg(M^*)$, while $\mathrm{Jt}(M) = \mathrm{Jt}(M^*)$. Thus, $\deg(M)$ may discern properties of modules that cannot be detected by means of their Jordan types.

In the sequel, we shall be concerned with self-dual modules, that is, $U_0(\mathfrak{g})$-modules M satisfying $M \cong M^*$. Such a module M can be characterized by the existence of a non-degenerate invariant bilinear form $(\ ,\ ) : M \times M \longrightarrow k$.

Let η denote the *antipode* of the Hopf algebra $U_0(\mathfrak{g})$, that is, the unique anti-automorphism of $U_0(\mathfrak{g})$ such that $\eta(x) = -x$ for all $x \in \mathfrak{g}$. A bilinear form $(\ ,\ ) : M \times M \longrightarrow k$ is referred to as *invariant*, if

$$(a.m, m') = (m, \eta(a).m') \qquad \forall\ a \in U_0(\mathfrak{g}),\ m, m' \in M.$$

We denote by ${}^{\perp}V$ and $V^{\perp}$ the left and right perpendicular spaces of a subspace $V \subseteq M$. If the form $(\ ,\ )$ is non-degenerate, then $\dim_k M = \dim_k V + \dim_k V^{\perp} = \dim_k V + \dim_k {}^{\perp}V$, so that ${}^{\perp}(V^{\perp}) = V = ({}^{\perp}V)^{\perp}$.

Example. By general theory, the Hopf algebra $U_0(\mathfrak{g})$ is a Frobenius algebra (cf. also [19, (V.4)]). Accordingly, $U_0(\mathfrak{g})$ possesses a non-degenerate bilinear form $\langle\ ,\ \rangle : U_0(\mathfrak{g}) \times U_0(\mathfrak{g}) \longrightarrow k$ such that

$$\langle ab, c\rangle = \langle a, bc\rangle \qquad \forall\ a, b, c \in U_0(\mathfrak{g}).$$

Hence the form $(\ ,\ ) : U_0(\mathfrak{g}) \times U_0(\mathfrak{g}) \longrightarrow k$, given by

$$(a, b) := \langle \eta(a), b\rangle$$

for all $a, b \in U_0(\mathfrak{g})$, is a non-degenerate invariant form of the regular module $U_0(\mathfrak{g})$.

Theorem 4.4.1. *Suppose that* $\mathbb{E}(2, \mathfrak{g}) \neq \emptyset$. *If* M *is a self-dual* $U_0(\mathfrak{g})$*-module, then the following statements hold:*

(1) *If* M *has constant* j*-rank, then* $\mathrm{rk}^j(M) \equiv 0 \bmod (2)$*, whenever* $j \equiv 1 \bmod (2)$.
(2) *If* M *has constant Jordan type* $\mathrm{Jt}(M) = \bigoplus_{i=1}^{p} a_i[i]$*, then* $a_i \equiv 0 \mod (2)$ *whenever* $i \equiv 0 \mod (2)$.

Proof. Let $\mathfrak{e} \in \mathbb{E}(2,\mathfrak{g})$. We consider the self-dual $U_0(\mathfrak{e})$-module $N := M|_{\mathfrak{e}}$.

(1) Since $N \cong N^*$, Theorem 4.2.2 implies $2\deg^j(N) = j\,\mathrm{rk}^j(N) = j\,\mathrm{rk}^j(M)$.

(2) Note that N has constant Jordan type $\mathrm{Jt}(N) = \mathrm{Jt}(M)$. Setting $\mathrm{rk}^0(N) = \dim_k N$ and $\mathrm{rk}^p(N) = 0 = \mathrm{rk}^{p+1}(N)$, we have $\mathrm{rk}^j(N) = \sum_{i=j+1}^p a_i(i-j)$, so that

$$\mathrm{rk}^{j-1}(N) - \mathrm{rk}^j(N) = \sum_{i\ge j} a_i.$$

As a result,

$$a_j = \mathrm{rk}^{j-1}(N) - 2\,\mathrm{rk}^j(N) + \mathrm{rk}^{j+1}(N).$$

Let $j \in \{1,\ldots,p\}$ be even. Then $j-1$ and $j+1$ are odd, and (1) shows that $\mathrm{rk}^{j-1}(N)$ and $\mathrm{rk}^{j+1}(N)$ are even. Consequently, a_j is even. □

Remarks. (1) Let $i \in \{0,\ldots,p-1\}$. Since the simple $U_0(\mathfrak{sl}(2))$-module $L(i)$ is self-dual and of constant Jordan type $\mathrm{Jt}(L(i)) = [i+1]$, Theorem 4.4.1 may fail if $\mathbb{E}(2,\mathfrak{g}) = \emptyset$, even though $\mathbb{P}(V(\mathfrak{g}))$ is isomorphic to a projective space.

(2) The proof of Theorem 4.4.1 shows that the conclusions are valid for those $U_0(\mathfrak{g})$-modules M, whose restriction $M|_{\mathfrak{e}}$ is self-dual for some $\mathfrak{e} \in \mathbb{E}(2,\mathfrak{g})$.

(3) Suppose that $p \ge 3$. The $U_0(\mathfrak{e}_2)$-module $H(\mathfrak{e}_2) := \mathrm{Rad}(U_0(\mathfrak{e}_2))/\mathrm{Soc}(U_0(\mathfrak{e}_2))$ is indecomposable, self-dual, and of constant Jordan type $\mathrm{Jt}(H(\mathfrak{e}_2)) = 2[p-1] \oplus (p-2)[p]$. Its second Heller shift $M := \Omega^2_{U_0(\mathfrak{e}_2)}(H(\mathfrak{e}_2))$ has constant Jordan type $\mathrm{Jt}(M) = 2[p-1] \oplus n[p]$ for some $n \in \mathbb{N}_0$ and thus also fulfills the conclusion of Theorem 4.4.1. The assumption $M \cong M^*$ implies, $\Omega^2_{U_0(\mathfrak{e}_2)}(H(\mathfrak{e}_2)) \cong \Omega^2_{U_0(\mathfrak{e}_2)}(H(\mathfrak{e}_2))^* \cong \Omega^{-2}_{U_0(\mathfrak{e}_2)}(H(\mathfrak{e}_2)^*) \cong \Omega^{-2}_{U_0(\mathfrak{e}_2)}(H(\mathfrak{e}_2))$, so that $\Omega^4_{U_0(\mathfrak{e}_2)}(H(\mathfrak{e}_2)) \cong H(\mathfrak{e}_2)$. Hence $H(\mathfrak{e}_2)$ is periodic and thus has a one-dimensional rank variety, which contradicts $H(\mathfrak{e}_2)$ being a module of constant Jordan type.

(4) Suppose that $V(\mathfrak{g})$ is a subspace of $\mathfrak{g}$. If M is self-dual and of constant j-rank, then Theorem 4.2.2 also implies $\deg^j(M) \equiv 0 \mod(j)$, whenever j is odd.

Examples. Suppose that $\mathfrak{g}$ is a Lie algebra of dimension r such that $V(\mathfrak{g})$ is a subspace of dimension ≥ 2. We fix $j \in \{1,\ldots,n\}$.

(1) Since $U_0(\mathfrak{g})$ is self-dual and of constant j-rank $p^{r-1}(p-j)$, Theorem 4.2.2 gives

$$\deg^j(U_0(\mathfrak{g})) = \frac{j(p-j)}{2}p^{r-1}.$$

(2) *Suppose that* $V(\mathfrak{g}) \subseteq \mathrm{Rad}(U_0(\mathfrak{g}))$*. We have*

$$\deg^j(M) = \deg^j(U_0(\mathfrak{g})) - j \ \textit{ for } \ M \in \{\mathrm{Rad}(U_0(\mathfrak{g})), \mathrm{Rad}(U_0(\mathfrak{g}))/\mathrm{Soc}(U_0(\mathfrak{g}))\}.$$

In view of $\mathrm{Rad}(U_0(\mathfrak{g})) \cong \mathrm{Rad}(U_0(\mathfrak{g})^*) \cong (U_0(\mathfrak{g})/\mathrm{Soc}(U_0(\mathfrak{g})))^*$, Theorem 4.2.2 and Lemma 4.1.4(3) yield

$$\begin{aligned}\deg^j(\mathrm{Rad}(U_0(\mathfrak{g}))) &= j\,\mathrm{rk}^j(U_0(\mathfrak{g})/\mathrm{Soc}(U_0(\mathfrak{g}))) - \deg^j(U_0(\mathfrak{g})/\mathrm{Soc}(U_0(\mathfrak{g})))\\ &= j\,\mathrm{rk}^j(U_0(\mathfrak{g})) - j - \deg^j(U_0(\mathfrak{g})) = \deg^j(U_0(\mathfrak{g})) - j.\end{aligned}$$

By the same token, we have $\deg^j(\mathrm{Rad}(U_0(\mathfrak{g}))/\mathrm{Soc}(U_0(\mathfrak{g}))) = \deg^j(\mathrm{Rad}(U_0(\mathfrak{g})))$ for $j \le p-2$.

(3) *Suppose that* $\mathbb{E}(2,\mathfrak{g}) \ne \emptyset$*. Then we have*

$$j\,\mathrm{rk}^j(P) = 2\deg^j(P)$$

for every projective $U_0(\mathfrak{g})$*-module* P*.*

Let $\mathfrak{e} \in \mathbb{E}(2, \mathfrak{g})$. For a projective $U_0(\mathfrak{g})$-module P, there exists $\ell \in \mathbb{N}_0$ such that $P|_{\mathfrak{e}} \cong \ell U_0(\mathfrak{e})$. Since P has constant j-rank, Lemma 4.1.4(2), Theorem 4.2.2 and Theorem 4.1.2 imply

$$j\,\mathrm{rk}^j(P) = j\ell\,\mathrm{rk}^j(U_0(\mathfrak{e})) = 2\ell\deg^j(U_0(\mathfrak{e})) = 2\deg^j(P).$$

(4) We consider the $U_0(\mathfrak{e}_2)$-modules $M_n := U_0(\mathfrak{e}_2)/\operatorname{Rad}^n(U_0(\mathfrak{e}_2))$ for $1 \le n \le 2p-2$. Since each module M_n is also a $\mathrm{GL}_2(k)$-module, and $\mathrm{GL}_2(k)$ acts on $\mathbb{P}(\mathfrak{e}_2)$ with one orbit, it has constant j-rank.

If $n \le p-1$, then M_n is an equal kernels module, so that Corollary 4.2.3 yields

$$\deg^j(M_n) = j\,\mathrm{rk}^j(M_n) = \begin{cases} 0 & n \le j \\ j\frac{(n-j)(n-j+1)}{2} & n \ge j+1. \end{cases}$$

Let $n \ge p$. Since $\operatorname{Rad}^{p-1}(U_0(\mathfrak{e}_2))$ has the equal images property, it follows that $\operatorname{Rad}^{p-1+j}(U_0(\mathfrak{e}_2)) = \operatorname{im} x^j_{\operatorname{Rad}^{p-1}(U_0(\mathfrak{e}_2))}$ for all $x \in \mathfrak{e}_2 \smallsetminus \{0\}$, so that $\operatorname{Rad}^n(U_0(\mathfrak{e}_2)) \subseteq \bigcap_{x \in \mathfrak{e}_2 \smallsetminus \{0\}} \operatorname{im} x^j_{U_0(\mathfrak{e}_2)}$ for $n \ge p-1+j$. Consequently, Lemma 4.1.4(3) implies

$$\deg^j(M_n) = \deg^j(U_0(\mathfrak{e}_2)) = \frac{pj(p-j)}{2}$$

for $n \ge p-1+j$.

The following results deal with the behavior of degrees on short exact sequences of modules of constant j-rank. They rest on technical conditions that are usually fulfilled for the almost split sequences of Auslander-Reiten theory.

Corollary 4.4.2. *Let $\mathfrak{g}$ be a restricted Lie algebra such that $V(\mathfrak{g})$ is a subspace of dimension ≥ 2. If $(0) \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow (0)$ is a short exact sequence of $U_0(\mathfrak{g})$-modules of constant j-rank with $\mathrm{rk}^j(E) = \mathrm{rk}^j(M) + \mathrm{rk}^j(N)$, then $\deg^j(E) = \deg^j(M) + \deg^j(N)$.*

Proof. Dualization provides a short exact sequence $(0) \longrightarrow M^* \longrightarrow E^* \longrightarrow N^* \longrightarrow (0)$ of $U_0(\mathfrak{g})$-modules of constant j-rank such that $\mathrm{rk}^j(E^*) = \mathrm{rk}^j(M^*) + \mathrm{rk}^j(N^*)$. Lemma 4.1.4(1) yields $\deg^j(E^*) \ge \deg^j(M^*) + \deg^j(N^*)$, while repeated application of Theorem 4.2.2 gives

$$\deg^j(E) = j\,\mathrm{rk}^j(E) - \deg^j(E^*) \le j\,\mathrm{rk}^j(M^*) + j\,\mathrm{rk}^j(N^*) - \deg^j(M^*) - \deg^j(N^*) = \deg^j(M) + \deg^j(N).$$

By applying Lemma 4.1.4(1) to the original sequence, we obtain the reverse inequality. □

Corollary 4.4.3. *Let $\mathfrak{g}$ be a restricted Lie algebra such that $V(\mathfrak{g})$ is a subspace of dimension ≥ 2. If $(0) \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow (0)$ is a locally split short exact sequence of $U_0(\mathfrak{g})$-modules such that $M, N \in \mathrm{EIP}(\mathfrak{g})$, then $E \in \mathrm{EIP}(\mathfrak{g})$.*

Proof. Let $j \in \{1, \ldots, p-1\}$. The equal images modules M and N have constant j-rank, and since the sequence is locally split, the $U_0(\mathfrak{g})$-module E has constant j-rank $\mathrm{rk}^j(E) = \mathrm{rk}^j(M) + \mathrm{rk}^j(N)$. Corollary 4.4.2 now implies

$$\deg^j(E) = \deg^j(M) + \deg^j(N) = 0.$$

As a result, the module E enjoys the equal images property. □

5. Low Dimensional Modules of Constant Rank

In the following, we will show how Tango's Theorem 1.2.1 can be applied in order to obtain information concerning low-dimensional modules of constant rank for finite group schemes. We begin by considering infinitesimal groups of height 1. Throughout, $(\mathfrak{g}, [p])$ denotes a restricted Lie algebra. A $U_0(\mathfrak{g})$-module M such that $\mathfrak{g}.M = (0)$ will be referred to as being *trivial*.

Lemma 5.1. *Let* $\mathfrak{g}$ *be a* p*-trivial restricted Lie algebra of dimension* $\dim_k \mathfrak{g} \geq 2$*. If* $M \in \mathrm{EIP}(\mathfrak{g}) \cap \mathrm{EKP}(\mathfrak{g})$*, then* M *is trivial.*

Proof. Let $M \in \mathrm{EIP}(\mathfrak{g}) \cap \mathrm{EKP}(\mathfrak{g})$. Then M has constant Jordan type, so that Corollary 4.1.3 and Corollary 4.2.3 yield

$$\mathrm{rk}(M) = \deg(M) = 0.$$

As $\mathfrak{g}$ is p-trivial, this implies that M is a trivial $U_0(\mathfrak{g})$-module. □

To illustrate our geometric methods, we begin with the following elementary observation. Suppose that $\mathfrak{g}$ is an r-dimensional p-trivial restricted Lie algebra, and let M be a non-trivial $U_0(\mathfrak{g})$-module of constant rank. If $\varrho_M : U_0(\mathfrak{g}) \longrightarrow \mathrm{End}_k(M)$ is the representation afforded by M, then M being non-trivial implies $\mathrm{rk}(M) \neq 0$, so that $\varrho_M|_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathrm{End}_k(M)$ is injective. Engel's Theorem guarantees that $\mathrm{im}\, \varrho_M$ may be embedded into the space of strictly upper triangular matrices of size $\dim_k M$, so that $r \leq \binom{\dim_k M}{2}$. Hence every $U_0(\mathfrak{g})$-module of constant rank of dimension $\dim_k M \leq \sqrt{2r} + \frac{1}{2}$ is trivial. Tango's Theorem allows us to relax this condition.

Theorem 5.2. *Suppose that* $\mathfrak{g}$ *is* p*-trivial and let* M *be a* $U_0(\mathfrak{g})$*-module of constant rank.*

1. *If* $\dim_k \mathrm{Rad}(M) \leq \dim_k \mathfrak{g} - 1$*, then* M *has the equal images property.*
2. *If* $\dim_k M/\mathrm{Soc}(M) \leq \dim_k \mathfrak{g} - 1$*, then* M *has the equal kernels property.*
3. *If* $\max\{\dim_k \mathrm{Rad}(M), \dim_k M/\mathrm{Soc}(M)\} \leq \dim_k \mathfrak{g} - 1$*, then* M *is trivial.*
4. *If* $\dim_k M \leq \dim_k \mathfrak{g}$*, then* M *is trivial.*

Proof. We put $r := \dim_k \mathfrak{g}$. Since $\mathfrak{g}$ is p-trivial, the algebra $U_0(\mathfrak{g})$ is local and $\mathfrak{g} \subseteq \mathrm{Rad}(U_0(\mathfrak{g}))$. It follows that $\mathrm{im}\, x_M \subseteq \mathrm{Rad}(M)$ for every $x \in \mathfrak{g}$.

(1) Suppose that $\dim_k \mathrm{Rad}(M) \leq r-1$ and put $d := \mathrm{rk}(M)$. Corollary 3.1.1 ensures that

$$\mathsf{im}_M : \mathbb{P}^{r-1} \longrightarrow \mathrm{Gr}_d(M) \ ; \ [x] \mapsto \mathrm{im}\, x_M$$

is a morphism of projective varieties. Since $\mathsf{im}_M(x) \subseteq \mathrm{Rad}(M)$ for all $x \in \mathbb{P}^{r-1}$, the morphism im_M factors through $\mathrm{Gr}_d(\mathrm{Rad}(M)) \subseteq \mathrm{Gr}_d(M)$. As $\dim_k \mathrm{Rad}(M) \leq r-1$, we may invoke Theorem 1.2.1(1) to see that the map im_M is constant. Corollary 3.2.3 now shows that the $U_0(\mathfrak{g})$-module M has the equal images property.

(2) We consider the dual $U_0(\mathfrak{g})$-module M^*, so that $x_{M^*} = -(x_M)^{\mathrm{tr}}$ for all $x \in \mathfrak{g}$. Consequently,

$$(*) \qquad \ker x_{M^*} = \{\lambda \in M^* \ ; \ \lambda \circ x_M = 0\},$$

showing that M^* also has constant rank $\mathrm{rk}(M^*) = \mathrm{rk}(M)$.

Recall the canonical pairing $M^* \times M \longrightarrow k \ ; \ (f, m) \mapsto f(m)$. Since $(x.f)(\mathrm{Soc}(M)) = (0)$ for every $x \in \mathrm{Rad}(U_0(\mathfrak{g}))$, we have $\mathrm{Rad}(M^*) \subseteq {}^{\perp}\mathrm{Soc}(M)$. Let $f \in {}^{\perp}\mathrm{Soc}(M)$. If $U \subseteq M^*$ is a maximal submodule, then $U^{\perp} \subseteq M$ is simple, so that $f(U^{\perp}) = (0)$ and $f \in {}^{\perp}(U^{\perp}) = U$. This shows that $f \in \mathrm{Rad}(M^*)$. Consequently, $\mathrm{Rad}(M^*) = {}^{\perp}\mathrm{Soc}(M) \cong (M/\mathrm{Soc}(M))^*$.

Since $\dim_k \mathrm{Rad}(M^*) = \dim_k M/\mathrm{Soc}(M) \leq r-1$, part (1) implies that M^* has the equal images property. In view of $(*)$, the module M has the equal kernels property.

(3) Owing to (1) and (2), the $U_0(\mathfrak{g})$-module M belongs to $\mathrm{EIP}(\mathfrak{g}) \cap \mathrm{EKP}(\mathfrak{g})$. Now Lemma 5.1 forces M to be trivial.

(4) Since $\dim_k M \leq r$, the condition of (3) is fulfilled. □

Example. The bound of Theorem 5.2(4) cannot be improved: Let $V_{r+1} = \bigoplus_{i=1}^{r+1} kv_i$ be the $(r+1)$-dimensional vector space on which the elementary Lie algebra $\mathfrak{e}_r := \bigoplus_{i=1}^{r} kx_i$ acts via

$$x_i.v_j = \delta_{i,j}v_{r+1} \quad \text{for } 1 \leq i \leq r \text{ and } 1 \leq j \leq r+1.$$

This $U_0(\mathfrak{e}_r)$-module has constant rank 1. Note that V_{r+1} is isomorphic to $\mathrm{Soc}_2(U_0(\mathfrak{e}_r))$.

For non-abelian p-trivial Lie algebras we record the following sharpening:

Corollary 5.3. *Let $\mathfrak{g}$ be a non-abelian p-trivial Lie algebra. If M is a non-trivial $U_0(\mathfrak{g})$-module of constant rank, then* $\min\{\dim_k \mathrm{Rad}(M), \dim_k M/\mathrm{Soc}(M)\} \geq \dim_k \mathfrak{g}$.

Proof. If $\dim_k \mathrm{Rad}(M) \leq \dim_k \mathfrak{g}-1$, then Theorem 5.2(1) and Corollary 3.2.3 imply that M is trivial, a contradiction. The assumption $\dim_k M/\mathrm{Soc}(M) \leq \dim_k \mathfrak{g}-1$ yields the same conclusion for M^* and hence also for M. □

Corollary 5.4. *Let $\mathfrak{g}$ be p-trivial. If M is a $U_0(\mathfrak{g})$-module of constant rank and of dimension* $\dim_k \mathfrak{g}+1$, *then one of the following cases occurs:*

(a) $M \cong k^{\dim_k \mathfrak{g}+1}$*, or*
(b) $\mathfrak{g}$ *is abelian and* $M \cong \mathrm{Soc}_2(U_0(\mathfrak{g}))$, $U_0(\mathfrak{g})/\mathrm{Rad}^2(U_0(\mathfrak{g}))$.

Proof. Let $r := \dim_k \mathfrak{g}$, so that $\dim_k \mathrm{Rad}(M) \leq r$. If $\mathrm{rk}(M) = 0$, then $M \cong k^{r+1}$. We therefore assume that $\mathrm{rk}(M) \neq 0$.

Suppose that $\dim_k \mathrm{Rad}(M) \leq r-1$. By Corollary 5.3, $\mathfrak{g}$ is abelian and hence isomorphic to $\mathfrak{e}_r$. Moreover, Theorem 5.2 shows that $M \in \mathrm{EIP}(\mathfrak{e}_r)$. Thanks to [5, (1.9)], we have $\mathrm{Rad}(M) \in \mathrm{EIP}(\mathfrak{e}_r)$ and Theorem 5.2 implies $\mathrm{Rad}^2(M) = (0)$, whence $\ell\ell(M) = 2$. Since M is not trivial, Theorem 5.2 yields $\dim_k \mathrm{Soc}(M) = 1$, so that there exists an embedding $\iota : M \hookrightarrow U_0(\mathfrak{e}_r)$. As $\ell\ell(M) = 2$, this map factors through $\mathrm{Soc}_2(U_0(\mathfrak{e}_r))$. Thus, $\mathrm{im}\,\iota = \mathrm{Soc}_2(U(\mathfrak{e}_r))$ for dimension reasons.

Since M^* has constant rank $\mathrm{rk}(M^*) = \mathrm{rk}(M)$, the assumption $\dim_k M/\mathrm{Soc}(M) \leq r-1$ implies $M \cong \mathrm{Soc}_2(U_0(\mathfrak{e}_r))^* \cong U_0(\mathfrak{e}_r)/\mathrm{Rad}^2(U_0(\mathfrak{e}_r))$.

It thus remains to consider the case, where $\dim_k \mathrm{Rad}(M) = r$ and $\dim_k \mathrm{Soc}(M) = 1$. As M has constant rank $\mathrm{rk}(M) \neq 0$, this implies $\mathrm{Soc}(M) \subseteq \mathrm{im}\, x_M$ for all $x \in \mathfrak{g}\smallsetminus\{0\}$. Consequently, the factor module $M/\mathrm{Soc}(M)$ has constant rank. By Theorem 5.2, this module is trivial, whence $\mathrm{Rad}(M) \subseteq \mathrm{Soc}(M)$. As a result, $r = 1$, so that $\mathfrak{g} \cong \mathfrak{e}_1$ and $M \cong \mathrm{Soc}_2(U_0(\mathfrak{e}_1))$. □

Let $\mathfrak{g}$ be p-trivial. By the foregoing result, $U_0(\mathfrak{g})$-modules of constant rank of dimension $\dim_k \mathfrak{g}+1$ belong to $\mathrm{EIP}(\mathfrak{g}) \cup \mathrm{EKP}(\mathfrak{g})$. The example of Section 4.2 shows that for modules of dimension $\dim_k \mathfrak{g}+2$, this may not be the case.

Corollary 5.5. *Suppose that $\mathfrak{g}$ is p-trivial and let M be a $U_0(\mathfrak{g})$-module of constant rank such that $[i] \oplus n[1] \in \mathrm{Jt}(M)$ for some $n \geq 1$ and $2 \leq i \leq p$. Then we have $n \geq \dim_k \mathfrak{g}-i+1$.*

Proof. Since $i \geq 2$, the $U_0(\mathfrak{g})$-module M is not trivial. Consequently, Theorem 5.2 implies

$$\dim_k \mathfrak{g}+1 \leq \dim_k M = n+i,$$

as asserted. □

Definition. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then

$$\mathrm{rk}_{\mathrm{triv}}(\mathfrak{g}) := \max\{\dim_k \mathfrak{h} \; ; \; \mathfrak{h} \subseteq \mathfrak{g} \; p\text{-trivial subalgebra}\}$$

is called the *p-trivial rank of* $\mathfrak{g}$.

Since p-trivial Lie algebras of dimension > 0 have non-trivial centers, we see that $\mathbb{E}(2, \mathfrak{g}) \neq \emptyset$ if and only if $\mathrm{rk}_{\mathrm{triv}}(\mathfrak{g}) \geq 2$.

Let G be a reductive algebraic group. We denote by h_G the *Coxeter number* of G. The dimension of any maximal torus $T \subseteq G$ is called the *rank* $\mathrm{rk}(G)$ of G.

Proposition 5.6. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, M be a $U_0(\mathfrak{g})$-module of constant rank.*

(1) *If $\dim_k M \leq \mathrm{rk}_{\mathrm{triv}}(\mathfrak{g})$, then $\mathrm{rk}(M) = 0$.*
(2) *Suppose that $\mathfrak{g} = \mathrm{Lie}(G)$, where G is reductive such that $p \geq h_G$. If $\dim_k M \leq \frac{1}{2}(\dim G - \mathrm{rk}(G))$, then M is a direct sum of one-dimensional modules.*

Proof. (1) Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a p-trivial subalgebra of dimension $\mathrm{rk}_{\mathrm{triv}}(\mathfrak{g})$. Then $N := M|_{\mathfrak{h}}$ is a $U_0(\mathfrak{h})$-module of constant rank $\mathrm{rk}(N) = \mathrm{rk}(M)$. In view of Theorem 5.2(4), the module N is trivial so that $\mathrm{rk}(N) = 0$.

(2) Since $p \geq h_G$, the unipotent radical of a Borel subalgebra of $\mathfrak{g}$ is p-trivial, whence $\dim_k M \leq \mathrm{rk}_{\mathrm{triv}}(\mathfrak{g})$. Let $T \subseteq G$ be a maximal torus with root system Φ. In view of (1), the elements of $\bigcup_{\alpha \in \Phi} \mathfrak{g}_\alpha \subseteq V(\mathfrak{g})$ act trivially on M. Hence $\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ acts trivially on M, and our assertion follows from the decomposition $\mathfrak{g} = \mathrm{Lie}(T) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. □

6. Modules for finite group schemes

We now turn to the general case concerning modules over a finite group scheme $\mathcal{G}$. This requires the Friedlander-Pevtsova theory of p-points, set forth in a series of articles, beginning with [9]. Let $\mathfrak{A}_p := k[T]/(T^p)$ be the truncated polynomial ring with canonical generator $t := T + (T^p)$. For an algebra homomorphism $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ we denote by $\alpha^* : \mathrm{mod}\, k\mathcal{G} \longrightarrow \mathrm{mod}\, \mathfrak{A}_p$ the associated pull-back functor. We say that α is a *p-point*, provided

(P1) α is left flat, i.e. $\alpha^*(k\mathcal{G})$ is projective, and
(P2) there exists an abelian unipotent subgroup scheme $\mathcal{U} \subseteq \mathcal{G}$ such that $\mathrm{im}\, \alpha \subseteq k\mathcal{U}$.

The set of p-points of $\mathcal{G}$ will be denoted $\mathrm{Pt}(\mathcal{G})$. Two p-points α, β are said to be *equivalent* ($\alpha \sim \beta$) if we have

$$\alpha^*(M) \text{ is projective } \Leftrightarrow \beta^*(M) \text{ is projective}$$

for every $M \in \mathrm{mod}\, \mathcal{G}$. By results of [9], the space $\mathrm{P}(\mathcal{G}) := \mathrm{Pt}(\mathcal{G})/\!\sim$ of equivalence classes of p-points is a noetherian topological space, whose closed subsets are of the form

$$\mathrm{P}(\mathcal{G})_M := \{[\alpha] \; ; \; \alpha^*(M) \text{ is not free}\} \qquad (M \in \mathrm{mod}\, \mathcal{G}).$$

If $\mathcal{H} \subseteq \mathcal{G}$ is a subgroup of the finite algebraic group $\mathcal{G}$, then the canonical inclusion induces a continuous map

$$\iota_* : \mathrm{P}(\mathcal{H}) \longrightarrow \mathrm{P}(\mathcal{G})$$

which is usually not injective.

6.1. **Modules defined via p-points.** Using p-points one can extend the concepts of constant rank modules and equal images modules to $\mathcal{G}$-modules, cf. [10]. Let M be a $\mathcal{G}$-module. Given $j \in \{1, \ldots, p-1\}$, we let

$$\mathrm{rk}^j(M) := \max\{\mathrm{rk}(\alpha(t)^j_M) \ ; \ \alpha \in \mathrm{Pt}(\mathcal{G})\}$$

be the *generic j-rank* of M. We say that $M \in \mathrm{mod}\,\mathcal{G}$ has *constant j-rank* if $\mathrm{rk}(\alpha(t)^j_M) = \mathrm{rk}^j(M)$ for all $\alpha \in \mathrm{Pt}(\mathcal{G})$. Modules of constant 1-rank are referred to as being of *constant rank*. The $\mathcal{G}$-module M has the *equal images property*, if, for every $\ell \in \{1, \ldots, p-1\}$, there is a subspace $V_\ell \subseteq M$ such that $\mathrm{im}\,\alpha(t)^\ell_M = V_\ell$ for all $\alpha \in \mathrm{Pt}(\mathcal{G})$. When dealing with infinitesimal groups of height r we shall often identify $\mathfrak{A}_p$ with the subalgebra $k[u_{r-1}]$ of $k\mathbb{G}_{a(r)}$.

The following result shows that in the context of infinitesimal groups our new definitions are compatible with the previous ones.

Lemma 6.1.1. *Let $\mathcal{G}$ be an infinitesimal group scheme of height r, M be a $\mathcal{G}$-module.*

(1) *Let $j \in \{1, \ldots, p-1\}$. If $\mathrm{rk}(\alpha(u_{r-1})^j_M) = \mathrm{rk}(\beta(u_{r-1})^j_M)$ for all $\alpha, \beta \in V(\mathcal{G}) \smallsetminus \{\varepsilon\}$, then M has constant j-rank.*

(2) *Suppose there exist subspaces $V_1, \ldots, V_{p-1} \subseteq M$ such that $\mathrm{im}\,\alpha(u_{r-1})^\ell_M = V_\ell$ for all $\alpha \in V(\mathcal{G}) \smallsetminus \{\varepsilon\}$ and $\ell \in \{1, \ldots, p-1\}$. Then M has the equal images property.*

Proof. In view of [9, (3.8)], the map

$$(*) \qquad \Xi_{\mathcal{G}} : \mathrm{Proj}(V(\mathcal{G})) \longrightarrow \mathrm{P}(\mathcal{G}) \ ; \ [\alpha] \mapsto [\alpha|_{k[u_{r-1}]}]$$

is bijective.

(1) This is a direct consequence of [10, (3.8)] and $(*)$.

(2) We first assume that $\mathcal{G} = \mathcal{U}$ is an infinitesimal abelian unipotent group of height r. General theory [24, (14.4)] provides an isomorphism

$$k\mathcal{U} \cong k[X_1, \ldots, X_s]/(X_1^{p^{n_1}}, \ldots, X_s^{p^{n_s}}) \quad ; \quad n_i \in \mathbb{N}.$$

We write $v_i := X_i + (X_1^{p^{n_1}}, \ldots, X_s^{p^{n_s}})$ and put $kE := k[v_1^{p^{n_1-1}}, \ldots, v_s^{p^{n_s-1}}]$. Hence kE looks like the group algebra of a p-elementary abelian group of rank s. According to [7, (1.6)], there exists a kE-linear projection $\mathrm{pr}_E : k\mathcal{U} \longrightarrow kE$ with kernel $\ker \mathrm{pr}_E = \mathrm{Rad}(kU)kE$ that induces a bijection

$$\mathrm{pr}_{E,*} : \mathrm{P}(\mathcal{U}) \longrightarrow \mathrm{P}(E) \ ; \ [\alpha] \mapsto [\alpha_{(E)}],$$

where $\alpha_{(E)}$ is the unique p-point such that $\alpha_{(E)}(u_{r-1}) := \mathrm{pr}_E(\alpha(u_{r-1}))$. It now follows from $(*)$ and [12, (2.2)] in conjunction with [7, (1.4)] that there exist $\alpha_1, \ldots, \alpha_s \in V(\mathcal{U})$ and $\lambda_1, \ldots, \lambda_s \in k^\times$ such that

$$v_i^{p^{n_i-1}} \equiv \lambda_i \alpha_i(u_{r-1}) \quad \mathrm{mod}\,\mathrm{Rad}(kU)\,\mathrm{Rad}(kE).$$

In particular, $kU\,\mathrm{Rad}(kE) = (v_1^{p^{n_1-1}}, \ldots, v_s^{p^{n_s-1}}) = (\alpha_1(u_{r-1}), \ldots, \alpha_s(u_{r-1}))$, so that

$$\mathrm{im}(v_i)_M^{p^{n_i-1}} \subseteq V_1 \qquad \text{for all } i \in \{1, \ldots, s\}.$$

Now let $\alpha \in \mathrm{Pt}(\mathcal{U})$ be a p-point. Since $\alpha(u_{r-1})^p = 0$ there exist $f_i \in k\mathcal{U}$ such that

$$\alpha(u_{r-1}) = \sum_{i=1}^{s} v_i^{p^{n_i-1}} f_i.$$

As a result,

$$\mathrm{im}\,\alpha(u_{r-1})_M \subseteq \sum_{i=1}^{s} \mathrm{im}(v_i)_M^{p^{n_i-1}} \subseteq V_1.$$

By virtue of (1), the $\mathcal{U}$-module M has constant rank $\mathrm{rk}(M) = \dim_k V_1$, so that $\dim_k \alpha(u_{r-1})_M = V_1$. We conclude that $\mathrm{im}\,\alpha(u_{r-1})_M = V_1$.

Since $k\mathcal{U}$ is abelian, we readily obtain $\operatorname{im}\alpha(u_{r-1})^{\ell}_M = \operatorname{im}\beta(u_{r-1})^{\ell}_M$ for all $\alpha, \beta \in \operatorname{Pt}(\mathcal{U})$ and $\ell \in \{1, \ldots, p-1\}$. As a result, the module M has the equal images property.

In the general case, we let $\alpha \in \operatorname{Pt}(\mathcal{G})$ be a p-point. By definition, there exists an abelian unipotent subgroup scheme $\mathcal{U} \subseteq \mathcal{G}$ such that $\operatorname{im}\alpha \subseteq k\mathcal{U}$. Since $V(\mathcal{U}) \subseteq V(\mathcal{G})$ (cf. [20, (1.5)]), the first part of the proof implies that $\operatorname{im}\alpha(u_{r-1})^{\ell}_M = V_\ell$ for every $\ell \in \{1, \ldots, p-1\}$. This shows that the $\mathcal{G}$-module M has the equal images property. □

For an abelian unipotent group scheme $\mathcal{U}$, condition (P2) above is automatic, so that p-points $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{U}$ are flat algebra homomorphisms, a requirement that makes no reference to the coalgebra structure of $k\mathcal{U}$. As noted above, we have an isomorphism

$$k\mathcal{U} \cong k[X_1, \ldots, X_s]/(X_1^{p^{n_1}}, \ldots, X_s^{p^{n_s}}) \quad ; \quad n_i \in \mathbb{N}$$

of associative algebras. A truncated polynomial ring $k[X_1, \ldots, X_s]/(X_1^{p^{n_1}}, \ldots, X_s^{p^{n_s}})$ can be interpreted as the restricted enveloping algebra $U_0(\mathfrak{n})$ of the abelian restricted Lie algebra $\mathfrak{n} = \bigoplus_{i=1}^s \mathfrak{n}_{n_i}$, where $\mathfrak{n}_{n_i} := \bigoplus_{j=0}^{n_i-1} kx_i^{[p]^j} \;\; ; \;\; x_i^{[p]^{n_i-1}} \neq 0 = x_i^{[p]^{n_i}}$ is the n_i-dimensional nil-cyclic restricted Lie algebra and $s = \dim V(\mathfrak{n})$. It follows from [9, (3.8)] that $s = \dim \operatorname{P}(\mathcal{U})+1$. We shall exploit this observation to generalize some of our earlier results.

Theorem 6.1.2. *Let $\mathcal{U}$ be an abelian unipotent group scheme. Suppose that M is a $\mathcal{U}$-module of constant rank. Then the following statements hold:*

(1) *If $\dim_k \operatorname{Rad}(M) \leq \dim \operatorname{P}(\mathcal{U})$, then M has the equal images property.*
(2) *If $\max\{\dim_k \operatorname{Rad}(M), \dim_k M/\operatorname{Soc}(M)\} \leq \dim \operatorname{P}(\mathcal{U})$, then $\operatorname{rk}(M) = 0$.*
(3) *If $\dim_k M \leq \dim \operatorname{P}(\mathcal{U})+1$, then $\operatorname{rk}(M) = 0$.*

Proof. (1) Observing [24, (14.4)], we write $k\mathcal{U} \cong k[X_1, \ldots, X_s]/(X_1^{p^{n_1}}, \ldots, X_s^{p^{n_s}}) \cong U_0(\mathfrak{n})$, so that the spaces of flat points coincide. As $\mathfrak{n}$ is abelian, the nullcone $V(\mathfrak{n})$ is the elementary restricted Lie algebra $\mathfrak{e}_s$ of dimension $s = \dim V(\mathfrak{n}) = \dim \operatorname{P}(\mathcal{U})+1$. Moreover, since $V(\mathfrak{n}) = V(\mathfrak{e}_s) = \mathfrak{e}_s$, an application of [9, (3.8)] shows that the canonical inclusion $\iota : \mathfrak{e}_s \hookrightarrow \mathfrak{n}$ induces a homeomorphism

$$\iota_* : \operatorname{P}(\mathfrak{e}_s) \longrightarrow \operatorname{P}(\mathfrak{n}) \;\; ; \;\; [\alpha] \mapsto [\iota \circ \alpha].$$

If $\mathfrak{n} = \bigoplus_{i=1}^s \mathfrak{n}_{n_i}$, then $\mathfrak{e}_s = \bigoplus_{i=1}^s \mathfrak{n}_{n_i}^{[p]^{n_i-1}}$, whence $\operatorname{Rad}(M|_{\mathfrak{e}_s}) \subseteq \operatorname{Rad}(M)|_{\mathfrak{e}_s}$. By Theorem 5.2(1), the map $\mathsf{im}_M : \mathbb{P}(\mathfrak{e}_s) \longrightarrow \operatorname{Gr}_{\operatorname{rk}(M)}(M)$ is constant. Since $\mathfrak{e}_s$ is abelian, a consecutive application of Theorem 3.2.2 and Lemma 6.1.1 yields the assertion.

(2) Since $\operatorname{Soc}(M)|_{\mathfrak{e}_s} \subseteq \operatorname{Soc}(M|_{\mathfrak{e}_s})$, it follows from Theorem 5.2 that $M|_{\mathfrak{e}_s}$ is trivial. Hence Lemma 6.1.1 implies $\operatorname{rk}(M) = \max\{\operatorname{rk}(x_M) \; ; \; x \in \mathfrak{e}_s\} = 0$.

(3) This is a direct consequence of (2). □

In view of $\dim \operatorname{P}(\mathbb{G}_{a(r)})+1 = r$, the foregoing result shows in particular, that a $\mathbb{G}_{a(r)}$-module M of constant rank with $\dim_k M \leq r$ is trivial. This strengthens [11, (3.18)].

Given a finite group scheme $\mathcal{G}$, we let

$$\operatorname{rk}_{\mathrm{au}}(\mathcal{G}) := \max\{\dim \operatorname{P}(\mathcal{U})+1 \; ; \; \mathcal{U} \subseteq \mathcal{G} \text{ abelian unipotent subgroup}\}$$

be the *abelian unipotent rank* of $\mathcal{G}$. If G is a finite group, Quillen's Dimension Theorem [1, (5.3.8)] ensures that this number coincides with the *p-rank* $\operatorname{rk}_p(G)$ of G, that is, the maximum of all ranks of the p-elementary abelian subgroups of G. Thus, for finite groups G, we have $\operatorname{rk}_{\mathrm{au}}(G) = \dim \operatorname{P}(G)+1$. In general, there is an inequality $\operatorname{rk}_{\mathrm{au}}(\mathcal{G}) \leq \dim \operatorname{P}(\mathcal{G})+1$, with both numbers possibly being arbitrarily far apart: Assuming $p \geq 3$, we let $\mathfrak{h}_n$ be the $(2n+1)$-dimensional p-trivial Heisenberg algebra. Then $\operatorname{rk}_{\mathrm{au}}(\mathfrak{h}_n) = n+1$, while

$\dim \mathrm{P}(\mathfrak{h}_n)+1 = 2n+1$. In Section 6.2 below we will see that the abelian unipotent rank of a group scheme is computable via elementary abelian subgroup schemes.

Corollary 6.1.3. *Let M be a $\mathcal{G}$-module of constant rank. If $\dim_k M \leq \mathrm{rk}_{\mathrm{au}}(\mathcal{G})$, then $\mathrm{rk}(M) = 0$.*

Proof. Let $\mathcal{U} \subseteq \mathcal{G}$ be an abelian unipotent subgroup such that $\dim \mathrm{P}(\mathcal{U})+1 = \mathrm{rk}_{\mathrm{au}}(\mathcal{G})$. Since $M|_{\mathcal{U}}$ has constant rank, Theorem 6.1.2 implies that $\mathrm{rk}(M) = \mathrm{rk}(M|_{\mathcal{U}}) = 0$. □

Recall that a reduced finite group scheme $\mathcal{G}$ is completely determined by its finite group $\mathcal{G}(k)$ of k-rational points. Moreover, any finite group G gives rise to a reduced group scheme $\mathcal{G}_G = \mathrm{Spec}_k(kG^*)$, where $\mathcal{G}_G(k) = G$. We shall henceforth not distinguish between a finite group G and its associated reduced group scheme $\mathcal{G}_G$.

For a finite group $\mathcal{G}$ we denote by $\mathrm{cx}_{\mathcal{G}}(k)$ the *complexity* of the trivial $\mathcal{G}$-module, cf. [1, (§5.1)]. Thanks to [9, (5.6)], we have $\mathrm{cx}_{\mathcal{G}}(k) = \dim \mathrm{P}(\mathcal{G})+1$.

Since many of our results will require the assumption $\mathrm{rk}_{\mathrm{au}}(\mathcal{G}) \geq 2$, we indicate a structural ramification of this condition:

Lemma 6.1.4. *Let $\mathcal{G}$ be a finite group scheme such that $\mathrm{rk}_{\mathrm{au}}(\mathcal{G}) \geq 2$. Then there exists a subgroup scheme $\mathcal{E} \subseteq \mathcal{G}$ such that $\mathcal{E}$ is isomorphic to one of the following group schemes: $\mathbb{G}_{a(2)}, \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}, \mathbb{G}_{a(1)} \times E_1, E_2$.*

Proof. By assumption, there exists an abelian unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}$ such that $\dim \mathrm{P}(\mathcal{U}) \geq 1$. General theory provides a decomposition

$$\mathcal{U} = \mathcal{U}^0 \times \mathcal{U}_{\mathrm{red}},$$

where $\mathcal{U}^0$ is infinitesimal and $\mathcal{U}_{\mathrm{red}}$ is reduced. If both factors are non-trivial, then $\mathbb{G}_{a(1)} \subseteq \mathcal{U}^0$ and $E_1 \subseteq \mathcal{U}_{\mathrm{red}}$, so that $\mathcal{E} := \mathbb{G}_{a(1)} \times E_1$ is the desired group. If $\mathcal{U}^0 = e_k$, then Quillen's dimension theorem provides an elementary abelian subgroup $E_r \subseteq \mathcal{U}(k)$ such that $r = \dim \mathrm{P}(\mathcal{U})+1$. Thus, $\mathcal{E} := E_2 \subseteq \mathcal{U}_{\mathrm{red}}$ is a suitable subgroup. In the remaining case, $\mathcal{U}$ is an infinitesimal unipotent subgroup. Let $\mathfrak{u} := \mathrm{Lie}(\mathcal{U})$ be its Lie algebra, so that $V(\mathfrak{u})$ is an elementary subalgebra. If $\dim V(\mathfrak{u}) \geq 2$, then we have $\mathfrak{e}_2 \subseteq V(\mathfrak{u})$, which implies that $\mathcal{U}$ contains a subgroup $\mathcal{E}$ that is isomorphic to $\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$. Alternatively, $\dim V(\mathfrak{u}) = 1$, so that $\mathcal{U}$ contains exactly one subgroup of type $\mathbb{G}_{a(1)}$. If it contains no subgroups of type $\mathbb{G}_{a(2)}$, then [8, (5.2)] yields $1 = \mathrm{cx}_{\mathcal{U}}(k) = 1+\dim \mathrm{P}(\mathcal{U})$, a contradiction. □

6.2. **Modules for elementary abelian group schemes.** In this section, we introduce group schemes that are natural generalizations of p-elementary abelian groups and elementary restricted Lie algebras. Our motivation for considering this class rests on the observation that the space of p-points of an elementary abelian group scheme naturally carries the structure of a projective space, rendering our earlier results amenable to applications. Work by Friedlander-Pevtsova [9] implies, that the space of p-points of an abelian unipotent group scheme coincides with that of its unique maximal elementary abelian subgroup scheme, cf. Lemma 6.2.1 below.

Definition. An abelian group scheme $\mathcal{E}$ is called *elementary abelian*, provided there exist subgroups $\mathcal{E}_1, \ldots, \mathcal{E}_n \subseteq \mathcal{E}$ such that

(a) $\mathcal{E} = \mathcal{E}_1 \cdots \mathcal{E}_n$, and
(b) for each $i \in \{1, \ldots, n\}$, we have isomorphisms $\mathcal{E}_i \cong \mathbb{G}_{a(r_i)}$ or $\mathcal{E}_i \cong E_1$.

Since the group schemes E_1 and $\mathbb{G}_{a(1)}$ are simple, it follows that a reduced elementary abelian group scheme $\mathcal{E}$ is isomorphic to some E_r, while we have $\mathcal{E} \cong (\mathbb{G}_{a(1)})^r$ for every infinitesimal elementary abelian group scheme $\mathcal{E}$ of height 1. In the latter case, there are isomorphisms $\mathrm{Lie}(\mathcal{E}) \cong \mathfrak{e}_r$ and $k\mathcal{E} \cong U_0(\mathfrak{e}_r)$.

By general theory, an elementary abelian group scheme $\mathcal{E}$ is the direct product $\mathcal{E} = \mathcal{E}^0 \times \mathcal{E}_{\mathrm{red}}$ of its infinitesimal and reduced parts. By the above, we have $\mathcal{E}_{\mathrm{red}} \cong E_r$ for some $r \geq 0$.

The dimension $\dim_k k\mathcal{G}$ of a finite group scheme $\mathcal{G}$ is also referred to as the *order* of $\mathcal{G}$. Thus, an abelian unipotent subgroup $\mathcal{U} \subseteq \mathcal{G}$ is contained in an abelian unipotent subgroup $\mathcal{U}_0$, whose order is maximal subject to these properties. The group scheme $\mathcal{U}_0$ is maximal subject to being abelian and unipotent.

Our next result shows that Hopf algebras of elementary abelian group schemes are isomorphic (as associative algebras) to group algebras of p-elementary abelian groups.

Lemma 6.2.1. *Let $\mathcal{U}$ be a abelian unipotent finite group scheme. Then the following statements hold:*

(1) *There exists a unique elementary abelian subgroup $\mathcal{E}_{\mathcal{U}} \subseteq \mathcal{U}$ that contains any other elementary abelian subgroup of $\mathcal{U}$.*
(2) *The canonical map $\iota_* : \mathrm{P}(\mathcal{E}_{\mathcal{U}}) \longrightarrow \mathrm{P}(\mathcal{U})$ is a homeomorphism.*
(3) *If $\mathcal{U}$ is elementary abelian, then $k\mathcal{U} \cong k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$, where $n = \mathrm{rk}_{\mathrm{au}}(\mathcal{U})$.*

Proof. (1) We let $\mathcal{E}_{\mathcal{U}}$ be an elementary abelian subgroup of maximal order. If $\mathcal{E} \subseteq \mathcal{U}$ is elementary abelian, then $\mathcal{E}\mathcal{E}_{\mathcal{U}}$ is an elementary abelian subgroup of $\mathcal{U}$ containing $\mathcal{E}_{\mathcal{U}}$. Consequently, $\mathcal{E} \subseteq \mathcal{E}\mathcal{E}_{\mathcal{U}} = \mathcal{E}_{\mathcal{U}}$, as desired.

(2) We denote by $\iota : \mathcal{E}_{\mathcal{U}} \longrightarrow \mathcal{U}$ the canonical inclusion. Let $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{U}$ be a p-point. Thanks to [9, (4.2)], there exists a p-point $\beta \in \mathrm{Pt}(\mathcal{U})$ and an elementary abelian subgroup $\mathcal{E} \subseteq \mathcal{U}$ such that $\alpha \sim \beta$ and $\mathrm{im}\,\beta \subseteq k\mathcal{E}$. In view of (1), this implies that $[\alpha] \in \mathrm{im}\,\iota_*$.

Now suppose that $\alpha, \beta \in \mathrm{Pt}(\mathcal{E}_{\mathcal{U}})$ are p-points such that $\iota_*([\alpha]) = \iota_*([\beta])$. Let $M \in \mathrm{mod}\,\mathcal{E}_{\mathcal{U}}$. Since the abelian algebra $k\mathcal{U}$ is free over $k\mathcal{E}_{\mathcal{U}}$, we have $(k\mathcal{U} \otimes_{k\mathcal{E}_{\mathcal{U}}} M)|_{k\mathcal{E}_{\mathcal{U}}} \cong M^n$, where n is the rank of $k\mathcal{U}$ over $k\mathcal{E}_{\mathcal{U}}$. Thus, if $\alpha^*(M)$ is projective, then $(\iota \circ \alpha)^*(k\mathcal{U} \otimes_{k\mathcal{E}_{\mathcal{U}}} M) \cong \alpha^*(M)^n$ is projective. Since $\iota_*([\alpha]) = \iota_*([\beta])$, it follows that $\beta^*(M)^n \cong (\iota \circ \beta)^*(k\mathcal{U} \otimes_{k\mathcal{E}_{\mathcal{U}}} M)$ is also projective. Consequently, the module $\beta^*(M)$ is projective. As a result, $\alpha \sim \beta$, so that ι_* is injective. The same arguments show that $\iota_*(\mathrm{P}(\mathcal{E}_{\mathcal{U}})_M) = \mathrm{P}(\mathcal{U})_{k\mathcal{U} \otimes_{k\mathcal{E}_{\mathcal{U}}} M}$ for all $M \in \mathrm{mod}\,\mathcal{E}_{\mathcal{U}}$. Hence the continuous bijective map ι_* is closed and therefore a homeomorphism.

(3) By assumption, there exists a quotient map $\prod_{i=1}^n \mathcal{E}_i \longrightarrow \mathcal{U}$, where $\mathcal{E}_i = \mathbb{G}_{a(r_i)}, E_1$. There results a surjection

$$\gamma : \bigotimes_{i=1}^n k\mathcal{E}_i \twoheadrightarrow k\mathcal{U},$$

of Hopf algebras. As both algebras are local, we have $\gamma(\mathrm{Rad}(\bigotimes_{i=1}^n k\mathcal{E}_i)) = \mathrm{Rad}(k\mathcal{U})$. Since $x^p = 0$ for all $x \in \mathrm{Rad}(\bigotimes_{i=1}^n k\mathcal{E}_i)$, we conclude that $x^p = 0$ for all $x \in \mathrm{Rad}(k\mathcal{U})$.

As $\mathcal{U}$ is abelian an unipotent, general theory ([24, (14.4)]) provides an isomorphism

$$k\mathcal{U} \cong k[X_1, \ldots, X_n]/(X_1^{p^{a_1}}, \ldots, X_n^{p^{a_n}}),$$

where $a_i \in \mathbb{N}$ and $\mathrm{rk}_{\mathrm{au}}(\mathcal{U}) = \dim \mathrm{P}(\mathcal{U}) + 1 = \mathrm{cx}_{\mathcal{U}}(k) = n$ coincides with the complexity $\mathrm{cx}_{\mathcal{U}}(k)$ of the trivial $\mathcal{U}$-module (cf. [9, (5.6)]). By the above, we have $X_i^p \in (X_1^{p^{a_1}}, \ldots, X_n^{p^{a_n}})$. By applying the canonical map $\omega_i : k[X_1, \ldots, X_n] \longrightarrow k[X_i]$ sending X_j onto $\delta_{ij} X_i$, we see that $X_i^p \in (X_i^{p^{a_i}})$. This implies $a_i = 1$. □

Example. If $U = \mathcal{U}$ is reduced, then U is an abelian p-group and $\mathcal{E}_{\mathcal{U}}$ is the subgroup of elements of order $\leq p$.

Corollary 6.2.2. *Let $\mathcal{E}$ be a finite group scheme.*

(1) *If* $k\mathcal{E} \cong k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$*, then* $\mathcal{E}$ *is elementary abelian and* $n = \mathrm{rk_{au}}(\mathcal{E})$.
(2) *If* $\mathcal{E}$ *is elementary abelian and* $\mathcal{E}' \subseteq \mathcal{E}$ *is a subgroup, then* $\mathcal{E}'$ *is elementary abelian.*
(3) *If* $\mathcal{E}$ *is elementary abelian and* $\mathcal{E}' \subseteq \mathcal{E}$ *is a subgroup, then* $\mathcal{E}/\mathcal{E}'$ *is elementary abelian.*

Proof. (1) Since $k\mathcal{E}$ is abelian and local, the group scheme $\mathcal{E}$ is abelian unipotent and $n = \mathrm{rk_{au}}(\mathcal{E}) = \dim \mathrm{P}(\mathcal{E}){+}1$. We let $\mathcal{F} := \mathcal{E}_{\mathcal{E}}$ be the maximal elementary abelian subgroup of $\mathcal{E}$. In view of Lemma 6.2.1(2), the map $\iota_* : \mathrm{P}(\mathcal{F}) \longrightarrow \mathrm{P}(\mathcal{E})$ is a homeomorphism, so that $\dim \mathrm{P}(\mathcal{F}) = \dim \mathrm{P}(\mathcal{E})$. We thus conclude $\mathrm{rk_{au}}(\mathcal{F}) = \mathrm{rk_{au}}(\mathcal{F})$, so that Lemma 6.2.1(3) together with our current assumption implies $\dim_k k\mathcal{F} = p^n = \dim_k k\mathcal{E}$. As a result, the group $\mathcal{E} = \mathcal{F}$ is elementary abelian.

(2) Since $\mathcal{E}$ is elementary abelian, Lemma 6.2.1 implies that $x^p = 0$ for all $x \in \mathrm{Rad}(k\mathcal{E})$. As noted earlier, there is an isomorphism

$$k\mathcal{E}' \cong k[X_1, \ldots, X_r]/(X_1^{p^{n_1}}, \ldots, X_r^{p^{n_r}}),$$

where $n_i \geq 1$. As $k\mathcal{E}$ is abelian, we have $\mathrm{Rad}(k\mathcal{E}') \subseteq \mathrm{Rad}(k\mathcal{E})$, so that $n_i = 1$. Part (1) now shows that $\mathcal{E}'$ is elementary abelian.

(3) Put $\mathcal{F} := \mathcal{E}/\mathcal{E}'$, so that there exists a surjective homomorphism $\pi : k\mathcal{E} \longrightarrow k\mathcal{F}$ of Hopf algebras. Hence $k\mathcal{F}$ is local and $\mathrm{Rad}(k\mathcal{F}) = \pi(\mathrm{Rad}(k\mathcal{E}))$. As a result, $\mathcal{F}$ is unipotent and we have $y^p = 0$ for all $y \in \mathrm{Rad}(k\mathcal{F})$. The arguments of (2) now provide $\ell \in \mathbb{N}$ with that $k\mathcal{F} \cong k[X_1, \ldots, X_\ell]/(X_1^p, \ldots, X_\ell^p)$ and (1) implies that the group scheme $\mathcal{F}$ is elementary abelian. □

We turn to the definition of j-degrees for modules of constant j-rank over an elementary abelian group scheme $\mathcal{E}$ with $\mathrm{rk_{au}}(\mathcal{E}) = n$. In view of Lemma 6.2.1, the algebra of measures

$$k\mathcal{E} \cong k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$$

is a p-truncated polynomial ring. Given $x \in \mathrm{Rad}(k\mathcal{E})$, we denote by $\alpha_x : \mathfrak{A}_p \longrightarrow k\mathcal{E}$ the unique algebra homomorphism such that $\alpha_x(t) = x$. There results a bijection

$$\pi_{\mathcal{E}} : \mathrm{Rad}(k\mathcal{E}) \longrightarrow \mathrm{Alg}_k(\mathfrak{A}_p, k\mathcal{E}) \ ; \ x \mapsto \alpha_x$$

between $\mathrm{Rad}(k\mathcal{E})$ and the set of algebra homomorphisms from $\mathfrak{A}_p$ to $k\mathcal{E}$. Work of Carlson cf. [2, (6.2)] ensures that $\pi_{\mathcal{E}}$ restricts to a bijective map

$$\pi_{\mathcal{E}} : \mathrm{Rad}(k\mathcal{E}) \smallsetminus \mathrm{Rad}^2(k\mathcal{E}) \stackrel{\sim}{\longrightarrow} \mathrm{Pt}(\mathcal{E}).$$

Consequently, the open set

$$\mathbb{P}(\mathrm{Pt}(\mathcal{E})) := \{[x] \in \mathbb{P}(\mathrm{Rad}(k\mathcal{E})) \ ; \ x \notin \mathrm{Rad}^2(k\mathcal{E})\}$$

of the projective space $\mathbb{P}(\mathrm{Rad}(k\mathcal{E}))$ may be thought of as the projectivization of the set of p-points of $\mathcal{E}$. Let $V \subseteq \mathrm{Rad}(k\mathcal{E})$ be a subspace such that $\mathrm{Rad}(k\mathcal{E}) = V \oplus \mathrm{Rad}^2(k\mathcal{E})$. Thanks to [9, (2.12)], the map $\pi_{\mathcal{E}}$ induces a bijection

$$\bar{\pi}_{\mathcal{E}} : \mathbb{P}(V) \longrightarrow \mathrm{P}(\mathcal{E}) \ ; \ [x] \mapsto [\alpha_x].$$

In fact, it follows from [1, (5.8.2),(5.9.2)] that this map actually is a homeomorphism. Thus, each choice of a minimal set of generators of $k\mathcal{E}$ allows us to consider $\mathrm{P}(\mathcal{E})$ as a closed subset of $\mathbb{P}(\mathrm{Pt}(\mathcal{E}))$

Let $j \in \{1, \ldots, p{-}1\}$, M be an $\mathcal{E}$-module. Then

$$\mathbb{P}(\mathrm{Pt}(\mathcal{E}))_{M,j} := \{[x] \in \mathbb{P}(\mathrm{Pt}(\mathcal{E})) \ ; \ \mathrm{rk}(x_M^j) = \mathrm{rk}^j(M)\}$$

is an open subset of $\mathbb{P}(\mathrm{Pt}(\mathcal{E}))$ and hence of the projective space $\mathbb{P}(\mathrm{Rad}(k\mathcal{E}))$.

Theorem 6.2.3. *Let* M *be an* $\mathcal{E}$*-module,* $j \in \{1, \ldots, p{-}1\}$*. Then the following statements hold:*

(1) *The map*

$$\mathsf{Im}_M^j : \mathbb{P}(\mathrm{Pt}(\mathcal{E}))_{M,j} \longrightarrow \mathrm{Gr}_{\mathrm{rk}^j(M)}(M) \ ; \ [u] \mapsto \mathrm{im}(u_M^j)$$

is a morphism such that $\mathsf{pl}_M \circ \mathsf{Im}_M^j$ *is homogeneous.*

(2) *Suppose that M has constant j-rank. If* $\mathrm{rk_{au}}(\mathcal{E}) \geq 2$ *and* $V \subseteq \mathrm{Rad}(k\mathcal{E})$ *is a subspace such that* $\mathrm{Rad}(k\mathcal{E}) = V \oplus \mathrm{Rad}^2(k\mathcal{E})$*, then* $\deg(\mathsf{pl}_M \circ \mathsf{Im}^j_M) = \deg(\mathsf{pl}_M \circ \mathsf{Im}^j_M|_{\mathbb{P}(V)})$.

Proof. (1) Let $d := \mathrm{rk}^j(M)$ and denote by $\varrho : k\mathcal{E} \longrightarrow \mathrm{End}_k(M)$ the representation afforded by M. Then

$$\omega^j : \mathbb{P}(\mathrm{Pt}(\mathcal{E}))_{M,j} \longrightarrow \mathbb{P}(\mathrm{End}_k(M))_d \ ; \ [u] \mapsto [\varrho(u)^j]$$

is a morphism and Proposition 2.1.2 shows that Im^j_M also enjoys this property. Thanks to Lemma 1.1.3, the morphism $\mathsf{pl}_M \circ \mathsf{Im}^j_M$ is homogeneous.

(2) Let $\iota : \mathbb{P}(V) \longrightarrow \mathbb{P}(\mathrm{Rad}(k\mathcal{E}))$ be the morphism of degree 1 that is induced by the inclusion $V \subseteq \mathrm{Rad}(k\mathcal{E})$. As $\mathrm{im}\,\iota \subseteq \mathbb{P}(\mathrm{Pt}(\mathcal{E}))$, Corollary 1.1.6 and its succeeding remarks imply

$$\deg(\mathsf{pl}_M \circ \mathsf{Im}^j_M) = \deg(\mathsf{pl}_M \circ \mathsf{Im}^j_M \circ \iota),$$

as desired. □

Definition. Let $\mathcal{E}$ be an elementary abelian group scheme, M be an $\mathcal{E}$-module. Given $j \in \{1, \ldots, p-1\}$, the natural number

$$\deg^j(M) := \deg(\mathsf{pl}_M \circ \mathsf{Im}^j_M)$$

is called the *j-degree* of M.

Let M be a module for a finite group scheme $\mathcal{G}$. For an automorphism $\lambda \in \mathrm{Aut}(k\mathcal{G})$ of the associative k-algebra $k\mathcal{G}$, we consider the twisted module $M^{(\lambda)}$, which has underlying k-space M and action

$$a.m := \lambda^{-1}(a)m \qquad \forall\, a \in k\mathcal{G},\, m \in M.$$

Corollary 6.2.4. *Let M be an $\mathcal{E}$-module of constant j-rank, $\lambda \in \mathrm{Aut}(k\mathcal{E})$ be an automorphism. Then $M^{(\lambda)}$ has constant j-rank and*

$$\deg^j(M^{(\lambda)}) = \deg^j(M).$$

Proof. Since $\alpha(t)^j_{M^{(\lambda)}} = (\lambda^{-1} \circ \alpha)(t)^j_M$ and $\alpha \mapsto \lambda^{-1} \circ \alpha$ is a bijection of $\mathrm{Pt}(\mathcal{E})$, it readily follows that $M^{(\lambda)}$ has constant j-rank. For $\mathrm{rk_{au}}(\mathcal{E}) = 1$, there is nothing to be shown. Alternatively, we pick a subspace $V \subseteq \mathrm{Rad}(k\mathcal{E})$ of dimension $\mathrm{rk_{au}}(\mathcal{E}) \geq 2$ such that $\mathrm{Rad}(k\mathcal{E}) = V \oplus \mathrm{Rad}^2(k\mathcal{E})$. Since λ is an automorphism, the subspace $W := \lambda^{-1}(V)$ enjoys the same property. Moreover, λ^{-1} induces a morphism $\lambda^{-1} : \mathbb{P}(V) \longrightarrow \mathbb{P}(W)$ of degree 1. Now Theorem 6.2.3 in conjunction with Corollary 1.1.5 yields

$$\begin{aligned} \deg^j(M^{(\lambda)}) &= \deg(\mathsf{pl}_M \circ \mathsf{Im}^j_{M^{(\lambda)}}|_{\mathbb{P}(V)}) = \deg(\mathsf{pl}_M \circ \mathsf{Im}^j_M|_{\mathbb{P}(W)} \circ \lambda^{-1}) = \deg(\mathsf{pl}_M \circ \mathsf{Im}^j_M|_{\mathbb{P}(W)}) \\ &= \deg^j(M), \end{aligned}$$

as desired. □

Theorem 6.2.5. *Let $\mathcal{E}$ be an elementary abelian group scheme. If M is an $\mathcal{E}$-module of constant j-rank, then*

$$\deg^j(M) + \deg^j(M^*) = j\,\mathrm{rk}^j(M).$$

Proof. Setting $r := \mathrm{rk}_{\mathrm{au}}(\mathcal{E})$, we observe that Lemma 6.2.1 yields $k\mathcal{E} = U_0(\mathfrak{e}_r)$, where $\mathrm{Rad}(k\mathcal{E}) = \mathfrak{e}_r \oplus \mathrm{Rad}^2(k\mathcal{E})$. We denote by $\eta_{\mathcal{E}}$ and $\eta_{\mathfrak{e}_r}$ the antipodes of $k\mathcal{E}$ and $U_0(\mathfrak{e}_r)$, respectively. Direct computation shows that

$$M^* \cong (M^\sharp)^{(\lambda)},$$

where $M^\sharp$ is the dual of M as a $U_0(\mathfrak{e}_r)$-module and $\lambda := \eta_{\mathcal{E}} \circ \eta_{\mathfrak{e}_r}$.

By virtue of Theorem 6.2.3, we obtain

$$\deg^j(M) = \deg(\mathsf{pl}_M \circ \mathsf{Im}^j_M|_{\mathbb{P}(\mathfrak{e}_r)}) = \deg^j(\mathsf{pl}_M \circ \mathsf{im}^j_M),$$

implying that the j-degree of M as a $k\mathcal{E}$-module coincides with that of the $U_0(\mathfrak{e}_r)$-module M. A consecutive application of Theorem 4.2.2 and Corollary 6.2.4 thus yields

$$j\,\mathrm{rk}^j(M) = \deg^j(M) + \deg^j(M^\sharp) = \deg^j(M) + \deg^j((M^\sharp)^{(\lambda)}) = \deg^j(M) + \deg^j(M^*),$$

as desired. □

Our final results of this section show that degrees of modules may be computed on elementary abelian groups of rank 2.

Corollary 6.2.6. *Let $\mathcal{E}$ be an elementary abelian group scheme, $\mathcal{E}' \subseteq \mathcal{E}$ be a subgroup scheme such that $\mathrm{rk}_{\mathrm{au}}(\mathcal{E}') \geq 2$. If M is an $\mathcal{E}$-module of constant j-rank, then $\deg^j(M) = \deg^j(M|_{\mathcal{E}'})$.*

Proof. According to Corollary 6.2.2(2), the group $\mathcal{E}'$ is elementary abelian.

Now let $V \subseteq \mathrm{Rad}(k\mathcal{E}')$ be a subspace such that $\mathrm{Rad}(k\mathcal{E}') = V \oplus \mathrm{Rad}^2(k\mathcal{E}')$. If $x \in V \setminus \{0\}$, then $\alpha_x : \mathfrak{A}_p \longrightarrow k\mathcal{E}'$ is a p-point (cf. [2, §6]). As $k\mathcal{E}$ is a free $k\mathcal{E}'$-module, the composite $\iota \circ \alpha_x$ of α_x and the natural inclusion $\iota : k\mathcal{E}' \longrightarrow k\mathcal{E}$ is a p-point. Thus, [2, §6] implies that $x \in \mathrm{Rad}(k\mathcal{E}) \setminus \mathrm{Rad}^2(k\mathcal{E})$. Hence $V \cap \mathrm{Rad}^2(k\mathcal{E}) = (0)$, and there exists a subspace $V \subseteq W \subseteq \mathrm{Rad}(k\mathcal{E})$ such that $\mathrm{Rad}(k\mathcal{E}) = W \oplus \mathrm{Rad}^2(k\mathcal{E})$. Since $\dim_k V = \mathrm{rk}_{\mathrm{au}}(\mathcal{E}') \geq 2$, we may apply Theorem 6.2.3 and Corollary 1.1.5 to arrive at

$$\deg^j(M) = \deg(\mathsf{pl}_M \circ \mathsf{Im}^j_M|_{\mathbb{P}(W)}) = \deg(\mathsf{pl}_M \circ \mathsf{Im}^j_M|_{\mathbb{P}(V)}) = \deg(\mathsf{pl}_M \circ \mathsf{Im}^j_{M|_{\mathcal{E}'}}|_{\mathbb{P}(V)}) = \deg^j(M|_{\mathcal{E}'}),$$

as desired. □

We denote by $Z(\mathcal{G})$ the *center* of the finite group scheme $\mathcal{G}$, cf. [16, (I.2.6)].

Corollary 6.2.7. *Let $\mathcal{G}$ be a finite group scheme, M be a $\mathcal{G}$-module of constant j-rank.*

(1) *If $\mathcal{E}, \mathcal{E}'$ are elementary abelian subgroups such that $\mathrm{rk}_{\mathrm{au}}(\mathcal{E} \cap \mathcal{E}') \geq 2$, then $\deg^j(M|_{\mathcal{E}}) = \deg^j(M|_{\mathcal{E}'})$.*

(2) *If $\mathrm{rk}_{\mathrm{au}}(Z(\mathcal{G})) \geq 2$, then there exists $d \in \mathbb{N}_0$ such that $\deg^j(M|_{\mathcal{E}}) = d$ for every elementary abelian subgroup $\mathcal{E} \subseteq \mathcal{G}$ such that $\mathrm{rk}_{\mathrm{au}}(\mathcal{E}) \geq 2$.*

Proof. (1) In view of Corollary 6.2.6 and our current assumption, we have $\deg^j(M|_{\mathcal{E}}) = \deg^j(M|_{\mathcal{E}\cap\mathcal{E}'}) = \deg^j(M|_{\mathcal{E}'})$.

(2) By assumption, there exists an elementary abelian subgroup $\mathcal{E}_0 \subseteq Z(\mathcal{G})$ such that $\mathrm{rk}_{\mathrm{au}}(\mathcal{E}_0) \geq 2$. Let $\mathcal{E} \subseteq \mathcal{G}$ be elementary abelian of rank $\mathrm{rk}_{\mathrm{au}}(\mathcal{E}) \geq 2$. Then $\mathcal{E}\mathcal{E}_0$ is elementary abelian, and a two-fold application of Corollary 6.2.6 gives

$$\deg^j(M|_{\mathcal{E}}) = \deg^j(M|_{\mathcal{E}\mathcal{E}_0}) = \deg^j(M|_{\mathcal{E}_0}),$$

so that the left-hand value does not depend on the choice of $\mathcal{E}$. □

Remark. Let $\mathcal{G}$ be a finite group scheme and let $\mathfrak{E}(2\!\uparrow, \mathcal{G})$ be the set of elementary abelian subgroups of $\mathcal{G}$ such that $\mathrm{rk_{au}}(\mathcal{E}) \geq 2$. We endow this set with the structure of a graph by postulating that two elements $\mathcal{E}, \mathcal{E}'$ are linked by a bond, whenever $\mathrm{rk_{au}}(\mathcal{E} \cap \mathcal{E}') \geq 2$. The above arguments show that the conclusion of Corollary 6.2.7(2) holds whenever $\mathfrak{E}(2\!\uparrow, \mathcal{G})$ is connected.

Since the full subcategory $\mathrm{EIP}(\mathcal{G}) \subseteq \operatorname{mod} \mathcal{G}$ of equal images modules is closed under images and direct sums, every $\mathcal{G}$-module M possesses a unique largest submodule $\mathfrak{K}(M)$ belonging to $\mathrm{EIP}(\mathcal{G})$, the so-called *generic kernel* of M.

Corollary 6.2.8. *Let $\mathcal{E}$ be an elementary abelian group scheme such that $\mathrm{rk_{au}}(\mathcal{E}) = 2$. If M is an $\mathcal{E}$-module of constant rank, then*

$$\deg(M) = \dim_k M/\mathfrak{K}(M).$$

Proof. We have $k\mathcal{E} \cong U_0(\mathfrak{e}_2)$, where $\mathrm{Rad}(k\mathcal{E}) = \mathfrak{e}_2 \oplus \mathrm{Rad}^2(k\mathcal{E})$. It thus follows from Theorem 6.2.3(2) that $\deg(M)$ coincides with the degree of M as a $U_0(\mathfrak{e}_2)$-module. We consider the exact sequence

$$(0) \longrightarrow \mathfrak{K}(M) \longrightarrow M \longrightarrow M/\mathfrak{K}(M) \longrightarrow (0).$$

As M has constant rank, [5, (7.6)] implies $\sum_{x \in \mathfrak{e}_2 \smallsetminus \{0\}} \ker x_M \subseteq \mathfrak{K}(M)$. The equal images module $\mathfrak{K}(M)$ has degree $\deg(\mathfrak{K}(M)) = 0$, and Lemma 4.1.4(4) yields $\deg(M) = \deg(\mathfrak{K}(M)) + \dim_k M/\mathfrak{K}(M) = \dim_k M/\mathfrak{K}(M)$. □

6.3. **Self-dual modules.** Let $\alpha \in \mathrm{Pt}(\mathcal{G})$ be a p-point. If M is a $\mathcal{G}$-module, then the isomorphism class

$$\mathrm{Jt}(M, \alpha) = [\alpha^*(M)]$$

of the $\mathfrak{A}_p$-module $\alpha^*(M)$ is the *Jordan type of M relative to α*. We denote by

$$\mathrm{Jt}(M) := \{\mathrm{Jt}(M, \alpha) \ ; \ \alpha \in \mathrm{Pt}(\mathcal{G})\}$$

the finite set of Jordan types of M and say that M has *constant Jordan type* if $|\mathrm{Jt}(M)| = 1$, cf. [3]. Since a $\mathcal{G}$-module has constant Jordan type if and only if it has constant j-rank for all $j \in \{1, \ldots, p-1\}$, Lemma 6.1.1 ensures that our present definition is compatible with the earlier one.

Theorem 6.3.1. *Let $\mathcal{G}$ be a finite group scheme such that $\mathrm{rk_{au}}(\mathcal{G}) \geq 2$. If M is a self-dual $\mathcal{G}$-module, then the following statements hold:*

1. *If M has constant j-rank, then $\mathrm{rk}^j(M) \equiv 0 \bmod (2)$, whenever $j \equiv 1 \bmod (2)$.*
2. *If M has constant Jordan type $\mathrm{Jt}(M) = \bigoplus_{i=1}^{p} a_i[i]$, then $a_i \equiv 0 \mod (2)$ whenever $i \equiv 0 \mod (2)$.*

Proof. Lemma 6.2.1 provides an elementary abelian subgroup $\mathcal{E}$ of abelian unipotent rank ≥ 2. We consider the self-dual $\mathcal{E}$-module $N := M|_{\mathcal{E}}$.

Let $j \in \{1, \ldots, p-1\}$ be odd. If M has constant j-rank, then N has constant j-rank $\mathrm{rk}^j(N) = \mathrm{rk}^j(M)$, and assertion (1) follows from Theorem 6.2.5. The second statement is a direct consequence of (1) and the formula $a_j = \mathrm{rk}^{j-1}(N) - 2\,\mathrm{rk}^j(N) + \mathrm{rk}^{j+1}(N)$. □

7. THE EQUAL IMAGES PROPERTY

We provide further applications of our morphisms im^j_M by establishing conditions for a module of constant rank to have the equal images property. Our starting point is the following observation concerning restricted Lie algebras:

Proposition 7.1. *Let $\mathfrak{g}$ be a p-trivial restricted Lie algebra, M be a $U_0(\mathfrak{g})$-module of constant rank. If there exist linearly independent elements $x, y \in \mathfrak{g}\smallsetminus\{0\}$ such that $\operatorname{im} x_M = \operatorname{im} y_M$, then M has the equal images property.*

Proof. By assumption, the map $\mathsf{im}_M : \mathbb{P}(\mathfrak{g}) \longrightarrow \operatorname{Gr}_{\operatorname{rk}(M)}(M)$ is not injective. The assertion thus follows from Corollary 3.2.3(2). □

Remark. Suppose that $r \geq 2$. If M is a $U_0(\mathfrak{e}_r)$-module of constant rank such that $M|_{\mathfrak{e}_2}$ has the equal images property, then Proposition 7.1 implies that M has the equal images property. In view of Corollary 6.2.6, an analogous result holds for elementary abelian group schemes.

Recall that $\mathbb{G}_{a(r)}$ denotes the r-th Frobenius kernel of the additive group $\mathbb{G}_a = \operatorname{Spec}(k[T])$. Thus, $k[\mathbb{G}_{a(r)}] = k[T]/(T^{p^r})$ is generated by the primitive element $t := T+(T^{p^r})$. If $u_0, \ldots, u_{r-1} \in k\mathbb{G}_{a(r)}$ are given by $u_i(t^j) := \delta_{p^i,j}$, then the map $X_i \mapsto u_i$ that is defined on the indeterminates $X_0, \ldots, X_{r-1}$ over k induces an isomorphism

$$k[X_0, \ldots, X_{r-1}]/(X_0^p, \ldots, X_{r-1}^p) \xrightarrow{\sim} k\mathbb{G}_{a(r)}.$$

Given a finite group scheme $\mathcal{G}$ and $r \in \mathbb{N}$, we denote by $\mathcal{G}_r$ the *r-th Frobenius kernel* of $\mathcal{G}$. By definition, $\mathcal{G}_r$ is the scheme-theoretic kernel of the r-th iterate $F^r : \mathcal{G} \longrightarrow \mathcal{G}^{(r)}$ of the Frobenius homomorphism $F : \mathcal{G} \longrightarrow \mathcal{G}^{(1)}$. For $\mathcal{G} = \mathbb{G}_{a(r)}$, F may be viewed as an endomorphism. Moreover, we have $\ker F^s = \mathbb{G}_{a(s)}$ and F^s induces an isomorphism $\mathbb{G}_{a(r)}/\mathbb{G}_{a(s)} \cong \mathbb{G}_{a(r-s)}$. Note that the induced homomorphism $F^s : k\mathbb{G}_{a(r)} \longrightarrow k\mathbb{G}_{a(r)}$ sends the canonical generator u_i onto u_{i-s}.

Theorem 7.2. *Let $\mathcal{G}$ be an infinitesimal group. If M is a $\mathcal{G}$-module of constant rank such that $M|_{\mathcal{G}_2} \in \operatorname{EIP}(\mathcal{G}_2)$, then M has the equal images property.*

Proof. Suppose that $\mathcal{G}$ has height r. For $r \leq 2$, there is nothing to be shown, so we assume that $r \geq 3$.

We first consider the case, where $\mathcal{G} = \mathbb{G}_{a(r)}$. The map $F^2 : \mathbb{G}_{a(r)} \longrightarrow \mathbb{G}_{a(r)}$ is given by $x \mapsto x^{p^2}$. This implies that $k[(\mathbb{G}_{a(r)})_2] = k[\mathbb{G}_{a(r)}]/(t^{p^2})$ as well as $k(\mathbb{G}_{a(r)})_2 = k[u_0, u_1]$. Setting $\mathfrak{e}_r := \bigoplus_{i=0}^{r-1} ku_i$, we have $k\mathbb{G}_{a(r)} = U_0(\mathfrak{e}_r)$ as well as $k(\mathbb{G}_{a(r)})_2 = U_0(\mathfrak{e}_2)$. Our assertion thus follows by applying Proposition 7.1 to the elements u_0 and u_1.

In the general case, we consider the inclusion $(\mathbb{G}_{a(r)})_2 = \mathbb{G}_{a(2)} \subseteq \mathbb{G}_{a(r)}$, which corresponds to the inclusion $k\mathbb{G}_{a(2)} = k[u_0, u_1] \subseteq k[u_0, \ldots, u_{r-1}] = k\mathbb{G}_{a(r)}$. By assumption, there exist subspaces $V_1, \ldots, V_{p-1} \subseteq M$ such that

$$(*) \qquad \operatorname{im} \psi(u_1)^j_M = V_j \qquad \forall\, \psi \in V(\mathcal{G}_2) \smallsetminus \{\varepsilon\},\ j \in \{1, \ldots, p-1\}.$$

We fix $j \in \{1, \ldots, p-1\}$ and let $\varphi \in V(\mathcal{G})\smallsetminus\{\varepsilon\}$. Then there exists $s \leq r-1$ such that $\ker \varphi = \mathbb{G}_{a(s)}$ and $\mathbb{G}_{a(r)}/\ker \varphi \cong \mathbb{G}_{a(r-s)}$. Our observations above now provide an injective homomorphism $\zeta : \mathbb{G}_{a(r-s)} \longrightarrow \mathcal{G}$ such that

$$\varphi = \zeta \circ F^s.$$

We consider the $\mathbb{G}_{a(r-s)}$-module $N := \zeta^*(M)$. It follows that N has constant rank $\operatorname{rk}(M)$.

If $s = r-1$, then $\zeta : \mathbb{G}_{a(1)} \longrightarrow \mathcal{G}$ satisfies

$$\varphi = \zeta \circ F^{r-1} = (\zeta \circ F) \circ F^{r-2}.$$

Accordingly, the map $\zeta \circ F : \mathbb{G}_{a(2)} \longrightarrow \mathcal{G}$ factors through $\mathcal{G}_2$ and $\zeta \circ F \in V(\mathcal{G}_2)\smallsetminus\{\varepsilon\}$. Consequently, $(*)$ yields

$$\operatorname{im}\varphi(u_{r-1})^j_M = \operatorname{im}(\zeta \circ F)(u_1)^j_M = V_j,$$

as desired.

Alternatively, $s \le r-2$, so that $\mathbb{G}_{a(2)} \subseteq \mathbb{G}_{a(r-s)}$. If $\lambda : \mathbb{G}_{a(2)} \longrightarrow \mathbb{G}_{a(2)}$ is a non-trivial homomorphism, then $\zeta \circ \lambda \in V(\mathcal{G}_2)\smallsetminus\{\varepsilon\}$, and

$$\operatorname{im}\lambda(u_1)^j_N = \operatorname{im}(\zeta \circ \lambda)(u_1)^j_M = V_j.$$

Consequently, $N|_{\mathbb{G}_{a(2)}} \in \operatorname{EIP}(\mathbb{G}_{a(2)})$, and the first part of the proof ensures that $N \in \operatorname{EIP}(\mathbb{G}_{a(r-s)})$. In particular, $\operatorname{im}\zeta(u_{r-s-1})^j_M = V_j$, whence $\operatorname{im}\varphi(u_{r-1})^j_M = \operatorname{im}(\zeta \circ F^s)(u_{r-1})^j_M = \operatorname{im}\zeta(u_{r-s-1})^j_M = V_j$. In view of Lemma 6.1.1, the module M thus has the equal images property. □

Remarks. (1) Since the restriction $M|_{\mathbb{G}_{a(1)}}$ of an arbitrary $\mathbb{G}_{a(r)}$-module M has the equal images property, the foregoing result may fail if $\mathcal{G}_2$ is replaced by $\mathcal{G}_1$.

(2) Since $\operatorname{rk}_{\mathrm{au}}(\mathbb{G}_{a(r)}) = r$, it follows from Corollary 6.2.6 that $\deg^j(M) = \deg^j(M|_{\mathbb{G}_{a(2)}})$ for every $\mathbb{G}_{a(r)}$-module of constant j-rank.

Lemma 7.3. *Let $\mathcal{U}$ be an abelian unipotent group scheme, M be a $\mathcal{U}$-module.*

(1) *If $M|_{\mathcal{E}_\mathcal{U}}$ has constant j-rank, then M has constant j-rank.*
(2) *If $M|_{\mathcal{E}_\mathcal{U}} \in \operatorname{EIP}(\mathcal{E}_\mathcal{U})$, then $M \in \operatorname{EIP}(\mathcal{U})$.*

Proof. Thanks to Lemma 6.2.1(2), the canonical inclusion $\iota : \mathcal{E}_\mathcal{U} \longrightarrow \mathcal{U}$ induces a bijection $\iota_* : \mathrm{P}(\mathcal{E}_\mathcal{U}) \longrightarrow \mathrm{P}(\mathcal{U})$.

(1) Given $j \in \{1, \dots, p-1\}$, we put

$$\mathrm{P}^j(\mathcal{U})_M := \{x \in \mathrm{P}(\mathcal{U})\ ;\ \text{there is } \alpha \in x \text{ such that } \operatorname{rk}(\alpha(t)^j_M) < \operatorname{rk}^j(M)\}.$$

Thanks to [10, (4.5)], $\mathrm{P}^j(\mathcal{U})_M$ is a closed subset of $\mathrm{P}(\mathcal{U})$ such that M has constant j-rank if and only if $\mathrm{P}^j(\mathcal{U})_M = \emptyset$. The observation above in conjunction with [10, (3.4)] yields

$$\mathrm{P}^j(\mathcal{U})_M = \mathrm{P}^j(\mathcal{E}_\mathcal{U})_{M|_{\mathcal{E}_\mathcal{U}}}.$$

By assumption, the latter set is empty, so that M has constant j-rank.

(2) By assumption, there exists a vector space $V \subseteq M$ such that

$$\operatorname{im}\alpha(t)_M = V \quad \text{for all } \alpha \in \operatorname{Pt}(\mathcal{E}_\mathcal{U}).$$

In particular, the module $M|_{\mathcal{E}_\mathcal{U}}$ has constant rank. By (1), this implies that M is a $\mathcal{U}$-module of constant rank.

Using the notation of the proof of Lemma 6.1.1(2), we observe that the identity $\mathrm{P}(\mathcal{U}) = \iota_*(\mathrm{P}(\mathcal{E}_\mathcal{U}))$, provides p-points $\alpha_1, \dots, \alpha_s \in \operatorname{Pt}(\mathcal{E}_\mathcal{U})$ such that

$$(v_1^{p^{n_1-1}}, \dots, v_s^{p^{n_s-1}}) = (\alpha_1(t), \dots, \alpha_s(t))$$

is the ideal of all elements of $u \in k\mathcal{U}$ with $u^p = 0$. It follows that

$$\operatorname{im}(v_j^{p^{n_j-1}})_M \subseteq V \quad \text{for all } j \in \{1, \dots, s\}.$$

Now let $\alpha \in \operatorname{Pt}(\mathcal{U})$ be a p-point. Then we have $\alpha(t) \in (v_1^{p^{n_1-1}}, \dots, v_s^{p^{n_s-1}})$, so that $\operatorname{im}\alpha(t)_M \subseteq V$. As M has constant rank, these spaces are in fact equal. Since $\mathcal{U}$ is abelian, this implies that M has the equal images property. □

Examples. Suppose that U is an abelian p-group, $U(p) := \{u \in U \ ; \ u^p = 1\}$.

(1) Let $N \in \mathrm{EIP}(U(p))$ and consider $M := kU \otimes_{kU(p)} N$. Since $M|_{U(p)} \cong N^{[U:U(p)]}$, it follows from Lemma 7.3 that M is an equal images module.
(2) Contrary to p-elementary abelian groups, modules of constant rank 0 may not be trivial. The U-module $M := kU \otimes_{kU(p)} k$ is a trivial $U(p)$-module, so that Lemma 6.2.1 implies that $M \in \mathrm{EIP}(U)$ has constant rank 0. However, if $U(p) \subsetneq U$, then M is not a trivial U-module.

Theorem 7.4. *Let $\mathcal{G}$ be a finite group scheme such that $\mathrm{rk}_{\mathrm{au}}(Z(\mathcal{G})) \geq 2$. If M is a $\mathcal{G}$-module of constant rank such that $M|_{Z(\mathcal{G})} \in \mathrm{EIP}(Z(\mathcal{G}))$, then $M \in \mathrm{EIP}(\mathcal{G})$.*

Proof. Since $\mathrm{rk}_{\mathrm{au}}(Z(\mathcal{G})) \geq 2$, Lemma 6.1.4 provides an elementary abelian subgroup $\mathcal{E}_0 \subseteq Z(\mathcal{G})$ such that $\mathrm{rk}_{\mathrm{au}}(\mathcal{E}_0) = 2$. We let $\alpha_{\mathcal{E}_0} \in \mathrm{Pt}(\mathcal{E}_0)$ be a p-point.

Let $\mathcal{U} \subseteq \mathcal{G}$ be a maximal abelian unipotent subgroup of $\mathcal{G}$. Being an image of the local algebra $k(\mathcal{E}_0 \times \mathcal{U}) \cong k\mathcal{E}_0 \otimes_k k\mathcal{U}$, the algebra $k(\mathcal{E}_0\mathcal{U})$ is local, so that the group $\mathcal{E}_0\mathcal{U}$ is unipotent. As $\mathcal{E}_0$ belongs to the center of $\mathcal{G}$, the group $\mathcal{E}_0\mathcal{U}$ is also abelian. We thus have $\mathcal{E}_0\mathcal{U} = \mathcal{U}$, whence $\mathcal{E}_0 \subseteq \mathcal{U}$ and $\mathcal{E}_0 \subseteq \mathcal{E}_{\mathcal{U}}$.

By our current assumption, the $\mathcal{E}_{\mathcal{U}}$-module $N := M|_{\mathcal{E}_{\mathcal{U}}}$ has constant rank, so that Corollary 6.2.6 yields $\deg(N) = \deg(N|_{\mathcal{E}_0})$. Since $N|_{\mathcal{E}_0} \in \mathrm{EIP}(\mathcal{E}_0)$, we obtain $\deg(N) = 0$. As $\mathcal{E}_{\mathcal{U}}$ is abelian, it follows that $N \in \mathrm{EIP}(\mathcal{E}_{\mathcal{U}})$, while Lemma 7.3 implies $M|_{\mathcal{U}} \in \mathrm{EIP}(\mathcal{U})$.

Now let $\alpha \in \mathrm{Pt}(\mathcal{G})$ be a p-point. Then there exists a maximal abelian unipotent group $\mathcal{U}$ containing $\mathrm{im}\,\alpha$. By the above, we have

$$\mathrm{im}\,\alpha(t)^j_M = \mathrm{im}\,\alpha_{\mathcal{E}_0}(t)^j_M \quad \text{for all } j \in \{1, \ldots, p-1\}.$$

As a result, the $\mathcal{G}$-module M has the equal images property. □

Example. Let U be a connected, unipotent algebraic group of dimension $\dim U \geq 1$. According to [18, (6.10)] there exists a subgroup $N \subseteq Z(U)$ such that $N \cong \mathbb{G}_a$. Consequently, $N_r \subseteq Z(U_r)$, so that $\mathrm{rk}_{\mathrm{au}}(Z(U_r)) \geq r$. Thus, for $r \geq 2$, a U_r-module M of constant rank is an equal images module whenever $M|_{Z(U_r)}$ enjoys this property.

Remark. Passage to dual modules provides analogous results for equal kernels modules. We leave the details to the interested reader.

References

[1] D. Benson, *Representations and Cohomology, II*. Cambridge Studies in Advanced Mathematics **31**. Cambridge University Press, 1991.
[2] J. Carlson, *The varieties and the cohomology ring of a module*. J. Algebra **85** (1983), 104–143.
[3] J. Carlson, E. Friedlander and J. Pevstova, *Modules of constant Jordan type*. J. reine angew. Math. **614** (2008), 191–234.
[4] ———, *Elementary subalgebras of Lie algebras*. J. Algebra **442** (2015), 155–189.
[5] J. Carlson, E. Friedlander and A. Suslin, *Modules for $\mathbb{Z}/p \times \mathbb{Z}/p$*. Commentarii Math. Helvetici **86** (2011), 609–657.
[6] R. Farnsteiner, *Periodicity and representation type of modular Lie algebras*. J. reine angew. Math. **464** (1995), 47–65.
[7] ———, *Support spaces and Auslander-Reiten components*. Contemp. Math. **442** (2007), 61–87.
[8] R. Farnsteiner, G. Röhrle and D. Voigt, *Infinitesimal unipotent group schemes of complexity* 1. Colloq. Math. **89** (2001), 179–192.
[9] E. Friedlander and J. Pevtsova, *Representation-theoretic support spaces for finite group schemes.* Amer. J. Math. **127** (2005), 379–420. Erratum: Amer. J. Math. **128** (2006), 1067–1068.
[10] ———, *Generalized support varieties for finite group schemes.* Documenta Math. **Extra Volume Suslin** (2010), 197–222.
[11] ———, *Constructions for infinitesimal group schemes*. Trans. Amer. Math. Soc. **363** (2011), 6007–6061.

[12] E. Friedlander, J. Pevtsova and A. Suslin, *Generic and maximal Jordan types*. Invent. math. **168** (2007), 485–222.
[13] U. Görtz and T. Wedhorn, *Algebraic Geometry I.* Advanced Lectures in Mathematics. Vieweg & Teubner, 2010.
[14] J. Harris, *Algebraic Geometry.* Graduate Texts in Mathematics **133**. Springer-Verlag, 1992.
[15] G. Hochschild, *Cohomology of restricted Lie algebras.* Amer. J. Math. **76** (1954), 555-580.
[16] J. Jantzen, *Representations of Algebraic Groups.* Mathematical Surveys and Monographs **107**. Amer. Math. Soc., 2003.
[17] D. Mumford, *The Red Book of Varieties and Schemes.* Lecture Notes in Mathematics **1358**. Springer-Verlag, 1999.
[18] T. Springer, *Linear Algebraic Groups*. Progress in Mathematics **9**. Birkhäuser-Verlag, 1981.
[19] H. Strade and R. Farnsteiner, *Modular Lie Algebras and their Representations.* Pure and Applied Mathematics **116**. Marcel Dekker, Inc. New York, 1988.
[20] A. Suslin, E. Friedlander and C. Bendel, *Infinitesimal 1-parameter subgroups and cohomology.* J. Amer. Math. Soc. **10** (1997), 693–728.
[21] ———, *Support varieties for infinitesimal group schemes.* J. Amer. Math. Soc. **10** (1997), 729–759.
[22] H. Tango, *On* $(n-1)$*-dimensional projective spaces contained in the Grassmann variety* $\mathrm{Gr}(n,1)$. J. Math. Kyoto Univ. **14** (1974), 415–460.
[23] ———, *On morphisms from projective space* $\mathbb{P}^n$ *to the Grassmann variety* $\mathrm{Gr}(n,d)$. J. Math. Kyoto Univ. **16** (1976), 201–207.
[24] W. Waterhouse, *Introduction to Affine Group Schemes.* Graduate Texts in Mathematics **66**. Springer-Verlag, 1979.

(R. Farnsteiner) MATHEMATISCHES SEMINAR, CHRISTIAN-ALBRECHTS-UNIVERSITÄT ZU KIEL, LUDEWIG-MEYN-STR. 4, 24098 KIEL, GERMANY
E-mail address: rolf@math.uni-kiel.de