NILPOTENT OPERATORS, CATEGORIES OF MODULES, AND AUSLANDER-REITEN THEORY

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Introduction

These notes constitute an expanded version of a series of lectures given on the occasion of the 3rd Summer School and Workshop on Lie Theory and Representation Theory at East China Normal University, Shanghai in June of 2012. The purpose was to provide a survey of some recent developments of the representation theory of finite group schemes over fields of characteristic $p > 0$. Our focal point was representation-theoretic support spaces, which aim at understanding module categories of complicated algebras via the investigation of algebraic families of subalgebras, whose modules are completely understood. In this approach, the main technological device are varieties of nilpotent operators related to the fundamental notion of a p-point, set forth by Eric Friedlander and Julia Pevtsova in [17].

By way of motivation, we begin by reviewing some of the origins of the theory, starting with the modular representation theory of finite groups. Our emphasis is the detection of projectivity via restrictions to "small" subgroups, whose representations are more tractable. This leads to Carlson's rank varieties that are defined in terms of cyclic shifted subgroups. For restricted Lie algebras, we briefly discuss the analogous concept involving the nullcone. The cohomological side of the story is based on far-reaching results concerning the finite generation of cohomology rings. The origins again reside within the modular representation theory of finite groups and Quillen's seminal work concerning the spectrum of group cohomology. Since then much progress has been made to extend the applicability of these techniques to other classes of Hopf algebras. Although we shall be mainly interested in the ramifications of the Friedlander-Suslin Theorem concerning finite group schemes, we have formulated the main results for Hopf algebras with finitely generated cohomology. By work of Ginzburg-Kumar [20], small quantum groups at roots of unity also enjoy this property.

Section 2 begins with a short review of finite group schemes and presents two major results: the Friedlander-Suslin Theorem, which allows the definition of support varieties for finite group schemes, and infinitesimal one-parameter subgroups, which provide a representation-theoretic interpretation of supports for infinitesimal group schemes. The basic features of the theory of p-points, which unifies the various rank varieties discussed in earlier work, are delineated in Section 3. To ease on the technicalities we have dispensed with a discussion of the more general concept of a π-point. Having all our tools in hand, we turn in Section 4 to the main theme of these notes: For a finite group scheme $\mathcal{G}$, we introduce full subcategories that are defined via properties of nilpotent operators associated to p-points. Aside from listing the relevant results from the literature, we take a first look at these categories in the classical setting of Frobenius kernels of smooth reductive groups.

In Section 5, we return to the starting point of the theory, the modular representation theory of finite groups. The definition of a p-point is designed in such a way that essential features of arbitrary $\mathcal{G}$-modules may be detected via restriction to subgroup schemes that behave like elementary abelian groups. For this reason and the fact that a p-elementary abelian group E_r of rank r is wild whenever $r \neq 1$ and $(r, p) \neq (2, 2)$, these groups have been the focal point of much of the recent work.

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Questions in representation theory can sometimes be reduced to problems concerning modules of Loewy length 2. For E_r-modules, this naturally leads to the category $\mathrm{rep}(K_r)$ of representations of the r-Kronecker quiver K_r. Being a hereditary category, this category differs homologically from the Frobenius category $\mathrm{mod}\, E_r$. Section 5 begins with a quick tour of representations of quivers, and identifies the subcategories of $\mathrm{rep}(K_r)$ that correspond to the ones introduced in Section 4. Morphisms to Grassmannians for modules of constant rank over infinitesimal group schemes are a useful geometric tool. Using the flexibility of p-points, we provide an application of these morphisms concerning dimensions of modules of constant rank over elementary abelian groups and Frobenius kernels of reductive groups.

In Section 6 we introduce a fundamental tool from the representation theory of associative algebras, the Auslander-Reiten quiver. This directed graph organizes the indecomposable modules of a given algebra in such a way that neighbors are linked by morphisms that do not factor properly through other indecomposables. Our first example is the AR-quiver of the r-Kronecker quiver K_r. If $r = 2$ then the indecomposable modules belonging to our aforementioned subcategories comprise exactly two AR-components. For $r \geq 3$, a homological characterization in conjunction with results from the AR-theory of wild hereditary algebras show that the relevant modules occur in each AR-component. As Hopf algebras are self-injective, it is natural to study their stable Auslander-Reiten quivers. This quiver fits into the axiomatic framework of a stable representation quiver, so that its connected components may be described by undirected trees. In the context of finite group schemes, these trees turn out to be finite or infinite Dynkin diagrams or Euclidean diagrams. Modules belonging to a stable AR-component have the same support varieties and rank varieties, and the dimensions of the resulting geometric invariants of AR-components are related to the describing trees. In particular, components of tree type A_∞ usually occur "most often". The final section is concerned with new invariants of AR-components defined by p-points. In many cases, the Jordan types of the modules of a given component can be predicted from its tree class and the Jordan types of a few vertices. This leads to results on $\mathcal{G}$-modules and stable AR-components that lie beyond the reach of the techniques involving support varieties.

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1. Rank Varieties and Support Varieties

We fix the following conventions that will be in force throughout these notes:

- k denotes an algebraically closed field of characteristic $\mathrm{char}(k) = p > 0$.
- Unless stated otherwise, all k-vector spaces are assumed to be finite-dimensional.
- A k-algebra Λ is an associative algebra with identity over k. Modules are assumed to be left modules, with the identity operating as the identity operator. We denote by $\mathrm{mod}\,\Lambda$ the category of (finite-dimensional) Λ-modules.

1.1. **Detection of projectivity.** Important examples of associative algebras, that have often served as paradigms of the general theory, are the group algebras kG of finite groups. In the study of $\mathrm{mod}\, G := \mathrm{mod}\, kG$, attention first focused on the projective G-modules. If M is a projective G-module and $G' \subseteq G$ is a subgroup, then the restriction $M|_{G'}$ is also projective. This leads to the question, on which class of subgroups can projectivity effectively be detected. Using standard techniques from modular representation theory one can show that a module is projective, if its restriction $M|_P$ to a Sylow-p-subgroup enjoys this property. The following refinement of this result is much harder to prove:

Theorem 1.1.1 (Chouinard, [8]). *Let G be a finite group. A G-module M is projective if and only if its restriction $M|_E$ to every p-elementary abelian subgroup $E \subseteq G$ is projective.*

Recall that a finite group E is said to be *p-elementary abelian of rank r*, if $E \cong (\mathbb{Z}/(p))^r$. Let $g_1, \ldots, g_r \in E$ be generators. Setting $x_i := g_i - 1$, the map $X_i \mapsto x_i$ induces an isomorphism

$$k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p) \xrightarrow{\sim} kE$$

of k-algebras. For $\lambda = (\lambda_1, \ldots, \lambda_r) \in k^r$, we put $u_\lambda := \sum_{i=1}^r \lambda_i x_i$. The following result, which fails for non-algebraically closed fields, furnishes a criterion for the projectivity of an E-module:

Lemma 1.1.2 (Dade's Lemma, [9]). *Let E be p-elementary abelian of rank r. The E-module M is projective if and only if $M|_{k[u_\lambda]}$ is projective for every $\lambda \in k^r$.*

For future reference, let us make the following basic observations:

- The element u_λ usually does not belong to E.
- For $\lambda \neq 0$, the element u_λ is nilpotent of order p, that is, $k[u_\lambda] \cong k[X]/(X^p)$.
- kE is a free $k[u_\lambda]$-module.

The notion of the complexity of a G-module M (which can be defined for modules over arbitrary algebras Λ) can be viewed as a means to measure the degree to which a module departs from being projective.

Let $(a_i)_{i \geq 0}$ be a sequence of natural numbers. We call

$$\mathrm{gr}((a_i)_{i\geq 0}) := \min\{s \in \mathbb{N}_0 \cup \{\infty\} \; ; \; \exists \; \lambda > 0 \text{ such that } a_n \leq \lambda n^{s-1} \quad \forall \; n \geq 1\}$$

the *polynomial rate of growth* of the sequence $(a_i)_{i\geq 0}$. If $V_\bullet := (V_i)_{i\geq 0}$ is a sequence of finite-dimensional k-vector spaces, then we write $\mathrm{gr}(V_\bullet) := \mathrm{gr}((\dim_k V_i)_{i\geq 0})$. Let M be a G-module, $P^\bullet := (P^n)_{n\geq 0}$ be a minimal projective resolution of M, then

$$\mathrm{cx}_G(M) := \mathrm{gr}(P^\bullet)$$

is called the *complexity of M*. Note that a G-module M is projective if and only if $\mathrm{cx}_G(M) = 0$. Alperin and Evens proved the following generalization of Chouinard's theorem:

Theorem 1.1.3 ([1]). *Let G be a finite group, $\mathfrak{E}(G)$ be the set of p-elementary abelian subgroups of G. Then we have*

$$\mathrm{cx}_G(M) = \max_{E \in \mathfrak{E}(G)} \mathrm{cx}_E(M)$$

for every $M \in \mathrm{mod}\, G$.

Example. Let $G = (\mathbb{Z}/(p))^r$ be a p-elementary abelian group of rank r. Since

$$P^\bullet : \qquad \cdots \longrightarrow k\mathbb{Z}/(p) \longrightarrow k\mathbb{Z}/(p) \longrightarrow \cdots \longrightarrow k\mathbb{Z}/(p) \longrightarrow k \longrightarrow (0)$$

is a minimal projective resolution of the trivial $\mathbb{Z}/(p)$-module k, the trivial $\mathbb{Z}/(p)$-module has complexity $\mathrm{cx}_{\mathbb{Z}/(p)}(k) = 1$. The Künneth formula then shows that the r-fold tensor product $(P^\bullet)^{\otimes r}$ is a minimal projective resolution of the trivial G-module k. Consequently, $\mathrm{cx}_G(k) = r$.

1.2. **Cyclic shifted subgroups.** For p-elementary abelian groups, Carlson's rank varieties (cf. [4]), whose definition is motivated by Dade's Lemma, can be described as follows: Given a p-elementary abelian group E, one considers a subspace $V \subseteq kE$ of dimension $\dim_k V = \mathrm{rk}(E)$, whose nonzero elements $u \in V \smallsetminus \{0\}$ have the following property:

- $u^p = 0$, and $k[u]$ is a local algebra of dimension p,
- the module $kE|_{k[u]}$ is free.

For $V := \{u_\lambda \ ; \ \lambda \in k^r\}$, the second property follows from the definition.

Given $u \in V$, the element $1+u$ generates a subgroup $C_u \subseteq kE^\times$ of order p, a *cyclic shifted subgroup* of E. For an E-module M, we consider for $x \in kE$, the linear map

$$x_M : M \longrightarrow M \quad ; \quad m \mapsto x.m.$$

Thus, the subspace V gives rise to a variety of p-nilpotent operators on M. One defines the *rank variety* of M via

$$V(E)_M := \{u \in V \ ; \ M|_{k[u]} \text{ is not free}\} \cup \{0\}.$$

The name derives from the equivalence

$$u \in V(E)_M \ \Leftrightarrow \mathrm{rk}(u_M) < \frac{p-1}{p} \dim_k M \qquad (u \in V),$$

which also shows that $V(E)_M$ is a conical, Zariski closed subset of V. Dade's Lemma may now be rephrased by saying that M is projective if and only if $V(E)_M = \{0\}$. Note that $V(E)_k = V$ has dimension $\dim V(E)_k = \mathrm{rk}(E) = \mathrm{cx}_E(k)$.

1.3. **Nullcones of restricted Lie algebras.** In this section we consider restricted representations of restricted Lie algebras $(\mathfrak{g}, [p])$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ and recall that $U(\mathfrak{g})$ is a cocommutative Hopf algebra, whose comultiplication $\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes_k U(\mathfrak{g})$ and counit $\varepsilon : U(\mathfrak{g}) \longrightarrow k$ are determined by

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad ; \quad \varepsilon(x) = 0 \qquad \forall \ x \in \mathfrak{g},$$

respectively. The ideal $I := (\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\})$ satisfies

$$\Delta(I) \subseteq I \otimes_k U(\mathfrak{g}) + U(\mathfrak{g}) \otimes_k I \ \text{ and } \ \varepsilon(I) = (0).$$

This implies that the *restricted enveloping algebra*

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/I$$

of $(\mathfrak{g}, [p])$ inherits from $U(\mathfrak{g})$ the structure of a Hopf algebra. By the Theorem of Poincaré-Birkhoff-Witt, we have $\dim_k U_0(\mathfrak{g}) = p^{\dim_k \mathfrak{g}}$. We let $V(\mathfrak{g}) := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}$ be the *nullcone* of $\mathfrak{g}$. This is a conical, Zariski closed subset of $\mathfrak{g}$.

Let M be a $U_0(\mathfrak{g})$-module. In 1986, Friedlander and Parshall [16] introduced the rank variety

$$V(\mathfrak{g})_M := \{x \in V(\mathfrak{g}) \ ; \ M|_{k[x]} \text{ is not free}\} \cup \{0\}.$$

Given $x \in V(\mathfrak{g})$, the Poincaré-Birkhoff-Witt Theorem shows that $U_0(\mathfrak{g})$ is a free $k[x]$-module. Thus, the definition of $V(\mathfrak{g})_M$ parallels that of Carlson's rank varieties.

Examples. (1) Consider the restricted Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$ together the p-th power of matrices, so that $V(\mathfrak{sl}(2))$ is the set of nilpotent (2×2)-matrices. A matrix $x \in \mathrm{Mat}_2(k)$ belongs to $V(\mathfrak{sl}(2))$ if and only if $\mathrm{tr}(x) = 0 = \det(x)$. Thus, the variety $V(\mathfrak{sl}(2))$ is given by

$$V(\mathfrak{sl}(2)) = \{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathrm{Mat}_2(k) \ ; \ a^2 + bc = 0\},$$

so that $V(\mathfrak{sl}(2))$ is an irreducible variety of dimension $\dim V(\mathfrak{sl}(2)) = 2$.

There exists exactly one simple $U_0(\mathfrak{sl}(2))$-module $L(i)$ of dimension $i+1$ for $0 \leq i \leq p-1$. If $x \in V(\mathfrak{sl}(2)) \smallsetminus V(\mathfrak{sl}(2))_{L(i)}$, then $L(i)$ is a free module for the p-dimensional algebra $k[x]$. Thus, $p \mid \dim_k L(i)$ and $i = p-1$. Hence $L(i) = L(p-1)$ is the Steinberg module, which is projective. We therefore have

$$V(\mathfrak{sl}(2))_{L(i)} = \begin{cases} V(\mathfrak{sl}(2)) & i \neq p-1 \\ \{0\} & i = p-1. \end{cases}$$

(2) Let $\widetilde{\mathfrak{sl}(2)} := \mathfrak{sl}(2) \oplus k$ be a central extension of $\mathfrak{sl}(2)$, which splits as an extension of ordinary Lie algebras, and whose p-map is defined via the p-semilinear map

$$\psi : \mathfrak{sl}(2) \longrightarrow k \ \ ; \ \ \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \mapsto a^p.$$

This means that $(x, \alpha)^{[p]} = (x^{[p]}, \psi(x))$ for all $(x, \alpha) \in \widetilde{\mathfrak{sl}(2)}$. Then

$$V(\widetilde{\mathfrak{sl}(2)}) = (\ker \psi \cap V(\mathfrak{sl}(2)) \times k = (k \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \times k) \cup (k \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \times k)$$

is a two-dimensional, reducible variety.

1.4. **Support varieties for Hopf algebras.** In the sequel, we consider a finite-dimensional Hopf algebra Λ. By definition, Λ is an associative k-algebra together with k-linear maps $\Delta : \Lambda \longrightarrow \Lambda \otimes_k \Lambda$, $\varepsilon : \Lambda \longrightarrow k$, and $\eta : \Lambda \longrightarrow \Lambda$ such that

(1) Δ and ε are homomorphisms of k-algebras,
(2) $(\Delta \otimes \mathrm{id}_\Lambda) \circ \Delta = (\mathrm{id}_\Lambda \otimes \Delta) \circ \Delta$ (co-associativity),
(3) $\varepsilon \hat{\otimes} \mathrm{id}_\Lambda = \mathrm{id}_\Lambda = \mathrm{id}_\Lambda \hat{\otimes} \varepsilon$ (co-unit),
(4) $\eta \hat{\otimes} \mathrm{id}_\Lambda = \varepsilon 1 = \mathrm{id}_\Lambda \hat{\otimes} \eta$.

Here the symbol $f \hat{\otimes} g$ indicates (also in more general contexts) that the map $f \otimes g$ is followed by some multiplication.

The group algebra kG of a finite group G and the restricted enveloping algebra $U_0(\mathfrak{g})$ are examples of finite-dimensional Hopf algebras.

Let X, Y be Λ-modules. Given $n \geq 1$, the group $\mathrm{Ext}^n_\Lambda(X, Y)$ is usually computed via a projective resolution of M, or an injective resolution of N. There is another interpretation by equivalence classes of exact sequences

$$(0) \longrightarrow Y \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow X \longrightarrow (0).$$

Such sequences may be multiplied by splicing, i.e., there is a pairing

$$\mathrm{Ext}^m_\Lambda(Y, Z) \times \mathrm{Ext}^n_\Lambda(X, Y) \longrightarrow \mathrm{Ext}^{m+n}_\Lambda(X, Z)$$

which associates to a pair

$$(0) \longrightarrow Z \longrightarrow E'_m \longrightarrow \cdots \longrightarrow E'_1 \longrightarrow Y \longrightarrow (0) \ ; \ (0) \longrightarrow Y \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow X \longrightarrow (0)$$

of exact sequences its *Yoneda splice*

$$(0) \longrightarrow Z \longrightarrow E'_m \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E'_1 \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow X \longrightarrow (0).$$

This is the definition for $m, n \geq 1$, and there are also pairings in the remaining cases $m = 0$ or $n = 0$.

Given Λ-modules M, N, we consider the $\mathbb{Z}$-graded vector space

$$\mathrm{Ext}^\bullet_\Lambda(M, N) := \bigoplus_{n \geq 0} \mathrm{Ext}^n_\Lambda(M, N).$$

Using the Yoneda product, we see that $\mathrm{Ext}^\bullet_\Lambda(M, N)$ is a $\mathbb{Z}$-graded right module for the $\mathbb{Z}$-graded algebra $\mathrm{Ext}^\bullet_\Lambda(M, M)$ and $\mathbb{Z}$-graded left module for the $\mathbb{Z}$-graded algebra $\mathrm{Ext}^\bullet_\Lambda(N, N)$. Of course, these statements do not necessitate Λ to be a Hopf algebra.

The co-unit endows the space k with the structure of a Λ-module. We put $\mathrm{H}^\bullet(\Lambda,k) := \mathrm{Ext}^\bullet_\Lambda(k,k)$. This algebra is called the *cohomology ring* of Λ.

One special feature of Hopf algebras resides in their module categories affording tensor products: If M and N are Λ-modules, then $M\!\otimes_k\!N$ is a $\Lambda\!\otimes_k\!\Lambda$-module, whose structure may be pulled back to Λ:

$$a.v := \Delta(a).v \qquad \forall\ a \in \Lambda,\ v \in M\!\otimes_k\!N.$$

The co-unit property tells us that the canonical isomorphisms $M\!\otimes_k\!k \cong M \cong k\!\otimes_k\!M$ are in fact Λ-linear. An element of $\mathrm{H}^n(\Lambda,k)$ is given by a short exact sequence

$$(0) \longrightarrow k \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow k \longrightarrow (0).$$

Upon tensoring this sequence with M, we obtain an element

$$(0) \longrightarrow M \longrightarrow E_n\!\otimes_k\!M \longrightarrow \cdots \longrightarrow E_1\!\otimes_k\!M \longrightarrow M \longrightarrow (0)$$

of $\mathrm{Ext}^n_\Lambda(M,M)$. There results a homomorphism

$$\Phi_M : \mathrm{H}^\bullet(\Lambda,k) \longrightarrow \mathrm{Ext}^\bullet_\Lambda(M,M)$$

of $\mathbb{Z}$-graded k-algebras.

Why is it advantageous to look at $\mathrm{H}^\bullet(\Lambda,k)$? The following result provides a first idea:

Theorem 1.4.1 (cf. [28]). *Let Λ be a Hopf algebra. Then $\mathrm{H}^\bullet(\Lambda,k)$ is graded commutative, that is, we have*

$$yx = (-1)^{\deg(x)\deg(y)}xy$$

for any two homogeneous elements $x,y \in \mathrm{H}^\bullet(\Lambda,k)$.

In particular, the subring $\mathrm{H}^{\mathrm{ev}}(\Lambda,k) := \bigoplus_{n\ge 0}\mathrm{H}^{2n}(\Lambda,k)$ of even elements is commutative. Given a finite-dimensional Λ-module M, we shall view $\mathrm{Ext}^\bullet_\Lambda(M,M)$ as a left $\mathrm{H}^{\mathrm{ev}}(\Lambda,k)$-module via the map Φ_M. The application of geometric techniques requires certain finiteness conditions that we collect in the following:

Definition. We say that Λ is an *fg-Hopf algebra*, provided

(a) $\mathrm{H}^{\mathrm{ev}}(\Lambda,k)$ is a finitely generated k-algebra, and
(b) $\mathrm{Ext}^\bullet_\Lambda(M,M)$ is a finitely generated $\mathrm{H}^{\mathrm{ev}}(\Lambda,k)$-module for every $M \in \mathrm{mod}\,\Lambda$.

In this case, the maximal ideal spectrum $\mathcal{V}_\Lambda(k) := \mathrm{Maxspec}(\mathrm{H}^{\mathrm{ev}}(\Lambda,k))$ has the structure of a conical, affine variety. For every ideal $I \trianglelefteq \mathrm{H}^{\mathrm{ev}}(\Lambda,k)$, we denote by $Z(I) := \{\mathfrak{M} \in \mathrm{Maxspec}(\mathrm{H}^{\mathrm{ev}}(\Lambda,k))\ ;\ I \subseteq \mathfrak{M}\}$ the *zero locus* of I.

Definition. Let Λ be an fg-Hopf algebra, $M \in \mathrm{mod}\,\Lambda$. Then

$$\mathcal{V}_\Lambda(M) := Z(\ker\Phi_M \cap \mathrm{H}^{\mathrm{ev}}(\Lambda,k)) \subseteq \mathcal{V}_\Lambda(k)$$

is called the (*cohomological*) *support variety* of M.

By definition, $\mathcal{V}_\Lambda(M)$ is a conical, affine variety. In general, these varieties are hard to compute. One therefore needs an alternative description in terms of representation-theoretic support spaces. For the time being, we only collect the following important properties. A more detailed survey on fg-Hopf algebras and their support varieties can be found in [24].

Theorem 1.4.2. *Let Λ be an fg-Hopf algebra. Then the following statements hold:*

(1) *We have* $\dim \mathcal{V}_\Lambda(M) = \mathrm{cx}_\Lambda(M)$ *for every* $M \in \mathrm{mod}\,\Lambda$. *In particular,* M *is projective if and only if* $\mathcal{V}_\Lambda(M) = \{\bigoplus_{n\geq 1} \mathrm{H}^{2n}(\Lambda, k)\}$.
(2) *If* M *is in indecomposable, then* $\mathrm{Proj}(\mathcal{V}_\Lambda(M))$ *is connected.*
(3) *If* $(0) \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow (0)$ *is an exact sequence, then* $\mathcal{V}_\Lambda(M) \subseteq \mathcal{V}_\Lambda(M') \cup \mathcal{V}_\Lambda(M'')$. *If the sequence splits, then equality holds.*

2. Hopf Algebras, Group Schemes, and One-Parameter Subgroups

2.1. **Hopf algebras.** In the previous section, we have taken advantage of the presence of tensor products in the category $\mathrm{mod}\,\Lambda$ of a Hopf algebra Λ. These algebras also enjoy important homological properties which guarantee that support varieties are non-trivial objects. We begin by reviewing basic properties of Hopf algebras and group schemes. Standard references are [22, 32]. A summary that is tailored to our present needs can be found in [14].

Definition. A k-algebra Λ is called *self-injective*, provided the regular Λ-module Λ is injective.

Theorem 2.1.1. *Suppose that* Λ *is self-injective.*

(1) *Let* $M \in \mathrm{mod}\,\Lambda$. *Then* M *is projective if and only if* M *is injective.*
(2) *A module* $M \in \mathrm{mod}\,\Lambda$ *is projective (injective) if and only if* M *has finite projective (injective) dimension.*
(3) *The Nakayama functor* $\mathcal{N}_\Lambda : \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,\Lambda\ ;\ M \mapsto \mathrm{Hom}_\Lambda(M,\Lambda)^*$ *is exact. It permutes the simple* Λ*-modules.*
(4) *If* $P(S)$ *is the projective cover of the simple module* S*, then* $\mathcal{N}_\Lambda(\mathrm{Soc}(P(S))) \cong S$.
(5) *Let* $P^\bullet \stackrel{\varepsilon}{\longrightarrow} M$ *be a minimal projective resolution, where* $P^\bullet = (P^n, \partial^n)_{n\geq 0}$. *If* M *is non-projective indecomposable, then* $\Omega^n_\Lambda(M) := \ker \partial^{n-1} (n \geq 2)\ ;\ \Omega_\Lambda(M) := \ker \varepsilon$ *is indecomposable.*

Part (2) shows in particular, that all modules over a self-injective algebra Λ of finite global dimension are projective, so that Λ is semi-simple.

Every automorphism $\omega : \Lambda \longrightarrow \Lambda$ defines an auto-equivalence $M \mapsto M^{(\omega)}$ of $\mathrm{mod}\,\Lambda$. One defines $M^{(\omega)}$ to be the Λ-module with underlying k-space M and action given by

$$a\centerdot m := \omega^{-1}(a)m \quad \forall\ a \in \Lambda,\ m \in M.$$

For Hopf algebras parts of the foregoing result may be refined. This requires some preparations. We shall denote the comultiplication by $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$. Thus, if M and N are Λ-modules, then we have

$$a\centerdot(m \otimes n) = \sum_{(a)} a_{(1)}.m \otimes a_{(2)}.n \quad \forall\ a \in \Lambda,\ m \in M,\ n \in N.$$

The space $\mathrm{Hom}_k(M,N)$ obtains the structure of a Λ-module via

$$(a\centerdot\varphi)(m) := \sum_{(a)} a_{(1)}.\varphi(\eta(a_{(2)}).m) \quad \forall\ a \in \Lambda,\ m \in M,\ \varphi \in \mathrm{Hom}_k(M,N).$$

Theorem 2.1.2. *Let* Λ *be a Hopf algebra.*

(1) *If* $P \in \mathrm{mod}\,\Lambda$ *is projective (injective), then* $P \otimes_k M$ *and* $M \otimes_k P$ *are projective (injective) for all* $M \in \mathrm{mod}\,\Lambda$.
(2) Λ *is self-injective.*

(3) *The space* $\int_{\Lambda}^{\ell} := \{x \in \Lambda \ ; \ ax = \varepsilon(a)x \ \ \forall \ a \in \Lambda\}$ *is one-dimensional.*
(4) *There exists an algebra homomorphism* $\zeta_\ell : \Lambda \longrightarrow k$ *such that* $xa = \zeta_\ell(a)x$ *for all* $x \in \int_{\Lambda}^{\ell}, \ a \in \Lambda$.
(5) *We have* $\mathcal{N}_\Lambda(M) \cong M^{(\nu^{-1})}$, *where* $\nu : \Lambda \longrightarrow \Lambda \ ; \ a \mapsto \sum_{(a)} \eta^{-2}(a_{(1)})\zeta_\ell(a_{(2)})$ *is a Nakayama automorphism of* Λ.

Remarks. (a) Part (3) actually implies that Λ is a Frobenius algebra.

(b) Suppose that Λ is an fg-Hopf algebra. If $\mathrm{H}^{\mathrm{ev}}(\Lambda, k)$ is finite-dimensional, then the fg-property implies that $\mathrm{Ext}^{\bullet}_{\Lambda}(k, S)$ is finite-dimensional for every simple Λ-module S. Consequently, there exists $m_0 > 0$ such that $\mathrm{Ext}^m_\Lambda(k, -) = (0)$ for all $m \geq m_0$. As a result, the Λ-module k has finite projective dimension and is therefore projective. By (1), this implies that Λ is semi-simple.

This shows that the variety $\mathcal{V}_\Lambda(k)$ has dimension $\neq 0$, whenever the underlying Hopf algebra is not semi-simple.

(c) The theory of Hopf algebras implies that the automorphism ν has finite order. This is important for questions concerning Auslander-Reiten theory, which we will discuss later.

(d) If Λ is a *cocommutative* Hopf algebra, that is, if $f_\Lambda \circ \Delta = \Delta$, where $f_\Lambda : \Lambda \otimes_k \Lambda \longrightarrow \Lambda \otimes_k \Lambda \ ; \ a \otimes b \mapsto b \otimes a$ is the *flip*, then $\eta^2 = \mathrm{id}_\Lambda$.

Examples. (1) If G is a finite group, then $e_G := \sum_{g \in G} g \in \int_{kG}^{\ell}$ and we have $e_G g = e_G$ for all $g \in G$, whence $\zeta_\ell = \varepsilon$ and $\nu = \mathrm{id}$. Thus, the Nakayama functor of kG is the identity.

(2) Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra with restricted enveloping algebra $U_0(\mathfrak{g})$. Then one can show that $\zeta_\ell(x) = \mathrm{tr}(\mathrm{ad}\, x)$ for all $x \in \mathfrak{g}$. This yields $\nu(x) = x + \mathrm{tr}(\mathrm{ad}\, x)1$ for every $x \in \mathfrak{g}$.

Let R be a k-algebra. Then $\mathrm{Hom}_k(\Lambda, R)$ obtains the structure of an associative k-algebra via the *convolution*

$$(\varphi * \psi)(a) := \sum_{(a)} \varphi(a_{(1)})\psi(a_{(2)}) \qquad \forall \ a \in \Lambda, \ \varphi, \psi \in \mathrm{Hom}_k(\Lambda, R).$$

This applies in particular to the dual space Λ^* of Λ. Since Λ is finite-dimensional, Λ^* has a Hopf algebra structure:

$$\Delta^*(\varphi) = \sum_{(\varphi)} \varphi_{(1)} \otimes \varphi_{(2)} \Leftrightarrow \varphi(ab) = \sum_{(\varphi)} \varphi_{(1)}(a)\varphi_{(2)}(b) \qquad \forall \ a, b \in \Lambda.$$

We also have

$$\eta^*(\varphi) = \varphi \circ \eta \ \text{ as well as } \ \varepsilon^*(\varphi) = \varphi(1) \qquad \forall \ \varphi \in \Lambda^*.$$

Examples. Let us take a look at the algebra structures of Λ^* for our two standard examples.

(1) Let G be a finite group. Then every $\varphi \in kG^*$ corresponds to a map $\varphi : G \longrightarrow k$ and we have $(\varphi * \psi)(g) = \varphi(g)\psi(g)$ for $\varphi, \psi \in kG^*, \ g \in G$. This implies that $kG^* \cong k^{|G|}$, a product of copies of the ground field k.
(2) If $(\mathfrak{g}, [p])$ is a restricted Lie algebra of dimension n, then the PBW-Theorem implies that $U_0(\mathfrak{g})^* \cong k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$.

2.2. **Group schemes.** As before, k denotes an algebraically closed field of characteristic $p > 0$. Let Comm_k and Gr be the categories of (possibly infinite-dimensional) commutative k-algebras and groups, respectively. A functor $\mathcal{G} : \mathrm{Comm}_k \longrightarrow \mathrm{Gr}$ is called an *affine group scheme*, provided there exists $k[\mathcal{G}] \in \mathrm{Comm}_k$ such that

$$\mathcal{G}(R) = \mathrm{Comm}_k(k[\mathcal{G}], R) \qquad \forall\ R \in \mathrm{Comm}_k\,.$$

By Yoneda's Lemma, the *coordinate ring* $k[\mathcal{G}]$ of $\mathcal{G}$ is a uniquely determined commutative Hopf algebra. The product on $\mathcal{G}(R)$ is given by convolution, so that the inverse of $g : k[\mathcal{G}] \longrightarrow R$ is $g \circ \eta$.

If A is a commutative Hopf algebra, then the associated affine group scheme is customarily denoted $\mathrm{Spec}(A)$.

The affine group scheme $\mathcal{G}$ is called *algebraic* or *finite* or *infinitesimal* if $k[\mathcal{G}]$ is finitely generated, finite-dimensional, or finite-dimensional and local, respectively. If $\mathcal{G}$ is a finite group scheme, then $k\mathcal{G} := k[\mathcal{G}]^*$ is a finite-dimensional cocommutative Hopf algebra, the "group algebra" of $\mathcal{G}$. Given a finite group scheme $\mathcal{G}$, we write $\mathrm{mod}\,\mathcal{G} := \mathrm{mod}\,k\mathcal{G}$ and refer to the objects as "$\mathcal{G}$-modules".

Examples. (1) Let $\mathrm{GL}(n) : \mathrm{Comm}_k \longrightarrow \mathrm{Gr}$ be the functor such that $\mathrm{GL}(n)(R)$ is the group of invertible $(n \times n)$-matrices with coefficients in R. Then $k[\mathrm{GL}(n)] = k[X_{ij}]_{\det(X_{ij})}$, the localization of the polynomial ring in n^2 variables at the polynomial $\det(X_{ij})$. Hence $\mathrm{GL}(n)$ is an algebraic group scheme.

(2) Let $r \geq 1$. Then $\mathrm{GL}(n)_r : \mathrm{Comm}_k \longrightarrow \mathrm{Gr}$, given by

$$\mathrm{GL}(n)_r(R) := \{(a_{ij}) \in \mathrm{GL}(n)(R)\ ;\ a_{ij}^{p^r} = \delta_{ij}\} \quad (R \in \mathrm{Comm}_k)$$

is an infinitesimal subgroup scheme of $\mathrm{GL}(n)$.

(3) We put $\mathbb{G}_a(R) = (R, +)$ for every $R \in \mathrm{Comm}_k$. Note that $\mathbb{G}_a = \mathrm{Spec}(k[T])$, with $\Delta(T) = T\otimes 1 + 1 \otimes T$.

(4) Let $r \geq 1$. Then the infinitesimal group scheme $\mathbb{G}_{a(r)} : \mathrm{Comm}_k \longrightarrow \mathrm{Gr}$ is defined via

$$\mathbb{G}_{a(r)}(R) = \{a \in R\ ;\ a^{p^r} = 0\} \quad (R \in \mathrm{Comm}_k).$$

(5) Let G be a finite group with group algebra kG. We consider the functor

$$\mathcal{G}_G : \mathrm{Comm}_k \longrightarrow \mathrm{Gr} \ \ ; \ \ R \mapsto \mathrm{Comm}_k(kG^*, R).$$

Here kG^* has the Hopf algebra structure inherited from kG. Direct computation shows that $kG \cong k\mathcal{G}_G$. Note that $k[\mathcal{G}_G] = kG^*$ is a reduced commutative k-algebra (no non-zero nilpotent elements).

(6) Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Since the dual algebra $U_0(\mathfrak{g})^*$ is local, $\mathcal{G}_{\mathfrak{g}} := \mathrm{Spec}(U_0(\mathfrak{g})^*)$ is an infinitesimal group scheme such that $k\mathcal{G}_{\mathfrak{g}} \cong U_0(\mathfrak{g})$.

The above examples examples reveal a close connection between finite-dimensional cocommutative Hopf algebras and finite group schemes. In fact, the assigment

$$\Lambda \mapsto \mathrm{Spec}(\Lambda^*)$$

gives rise to an equivalence of categories. Example (2) is a special instance of the following general construction:

Definition. Let $\mathcal{G} = \mathrm{Spec}(A)$ be an algebraic group scheme with coordinate ring A, whose augmentation ideal is denoted by $I := \ker \varepsilon$. Given $r > 0$, the factor algebra

$$A_r := A/(\{x^{p^r}\ ;\ x \in I\})$$

is a commutative local Hopf algebra, and the infinitesimal group scheme $\mathcal{G}_r := \mathrm{Spec}(A_r)$ is called the *r-th Frobenius kernel* of $\mathcal{G}$.

Let $\mathcal{G}$ be an affine group scheme. A derivation $d : k[\mathcal{G}] \longrightarrow k$ is a k-linear map, such that $d(ab) = \varepsilon(a)b+\varepsilon(b)a$. The vector space $\mathrm{Der}_k(k[\mathcal{G}], k)$ of all derivations is closed under the Lie bracket $[d, d'] := d * d' - d' * d$. Thus, $\mathrm{Der}(k[\mathcal{G}], k)$ is a restricted Lie algebra, whose p-map is the ordinary p-th power operator. It is referred to as the *Lie algebra* $\mathrm{Lie}(\mathcal{G})$ of $\mathcal{G}$. If $\mathcal{G}$ is algebraic, general theory shows that the first Frobenius kernel $\mathcal{G}_1$ of $\mathcal{G}$ corresponds to $\mathrm{Lie}(\mathcal{G})$: We have an isomorphism $k\mathcal{G}_1 \cong U_0(\mathrm{Lie}(\mathcal{G}))$ of Hopf algebras. In particular, there is an equivalence between the categories of restricted Lie algebras and infinitesimal group schemes of height 1.

Let Λ be an associative k-algebra, G be a finite group that acts on Λ via automorphisms. Then $\Lambda * G := \bigoplus_{g\in G} \Lambda g$ is an algebra, whose multiplication is based on the rule

$$ga = (g.a).g \qquad \forall\, a \in \Lambda,\, g \in G.$$

This algebra is called the *skew group algebra*. The standard example is given by the semi-direct product $H = N \rtimes G$ of a subgroup $G \subseteq H$ and a normal subgroup $N \trianglelefteq H$ of some group H. Then G acts on kN via conjugation and $kH \cong kN{*}G$.

If Λ is a Hopf algebra and G acts on Λ via homomorphisms of Hopf algebras, then $\Lambda{*}G$ is also a Hopf algebra. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, $G \subseteq \mathrm{Aut}_p(\mathfrak{g})$ be a finite subgroup of the automorphism group of $(\mathfrak{g}, [p])$. Then G acts on $U_0(\mathfrak{g})$ via automorphisms of Hopf algebras, and $U_0(\mathfrak{g}){*}G$ is a finite-dimensional cocommutative Hopf algebra, whose dual algebra is usually neither reduced nor local.

Theorem 2.2.1 (Friedlander-Suslin, [19]). *Let $\mathcal{G}$ be a finite group scheme. Then $k\mathcal{G}$ is an fg-Hopf algebra.*

For every $\mathcal{G}$-module M, we thus have a support variety $\mathcal{V}_{\mathcal{G}}(M)$, which is a closed conical subset of the maximal ideal spectrum of $\mathrm{H}^{\mathrm{ev}}(\mathcal{G}, k) := \mathrm{H}^{\mathrm{ev}}(k\mathcal{G}, k)$.

2.3. **Infinitesimal one-parameter subgroups.** In 1997 Friedlander, Suslin and Bendel generalized rank varieties of restricted Lie algebras to infinitesimal group schemes. Their "higher nullcones" are the schemes $\mathcal{HOM}(\mathbb{G}_{a(r)}, \mathcal{G})$ of infinitesimal one-parameter subgroups. We shall only consider the corresponding varieties of k-rational points.

Recall that $k[\mathbb{G}_{a(r)}] = k[T]/(T^{p^r})$. Setting $t := T + (T^{p^r})$, we consider elements $u_0, \dots, u_{r-1} \in k\mathbb{G}_{a(r)}$, where $u_i(t^j) := \delta_{p^i,j}$. One can show that $X_i \mapsto u_i$ induces an isomorphism

$$k[X_0, \dots, X_{r-1}]/(X_0^p, \dots, X_{r-1}^p) \longrightarrow k\mathbb{G}_{a(r)}$$

of k-algebras. Thus, the groups $\mathbb{G}_{a(r)}$ can be thought of as analogues of p-elementary abelian groups of rank r.

Definition. Let $\mathcal{G}$ be an algebraic group scheme. For $r \geq 1$, we define

$$V_r(\mathcal{G}) := \{\alpha : \mathbb{G}_{a(r)} \longrightarrow \mathcal{G} \ ; \ \alpha \text{ homomorphism of group schemes}\}.$$

In view of the above categorical equivalence, every $\alpha \in V_r(\mathcal{G})$ may be thought of as a homomorphism $\alpha : k\mathbb{G}_{a(r)} \longrightarrow k\mathcal{G}_r$ of Hopf algebras. This tells us that $V_r(\mathcal{G})$ is an affine variety. Since the image of $\alpha \in V_r(\mathcal{G})$ is isomorphic to $\mathbb{G}_{a(s)}$ for some $s \leq r$, we are in fact considering algebraic families of "p-elementary abelian subgroups" of ranks $\leq r$. The standard action of $k^\times$ on $k\mathbb{G}_{a(r)}$ endows $V_r(\mathcal{G})$ with the structure of a conical variety, allowing us to consider the projective space $\mathrm{Proj}(V_r(\mathcal{G}))$.

Examples. (1) $V_r(\mathbb{G}_a) \cong k^r$

(2) $V_r(\mathrm{GL}(n)) \cong \{(v_0, \dots, v_{r-1}) \in \mathrm{Mat}_n(k)^r \ ; \ v_i^p = 0 \text{ and } [v_i, v_j] = 0 \text{ for } 0 \leq i, j \leq r-1\}$.

If M is a $\mathcal{G}$-module, then pull-back along α yields a $k\mathbb{G}_{a(r)}$-module $\alpha^*(M)$, whose action is given by

$$a.m := \alpha(a)m \qquad \forall\ a \in k\mathbb{G}_{a(r)},\ m \in M.$$

We now turn to the special case of infinitesimal group schemes. If $\mathcal{G}$ is infinitesimal, then there exists $r \in \mathbb{N}$ such that $\mathcal{G}_r = \mathcal{G}$, and the minimal such r is called the *height* of $\mathcal{G}$. We then define $V(\mathcal{G}) := V_r(\mathcal{G})$.

Our rank varieties are subsets of $V(\mathcal{G})$, whose elements give rise to certain p-nilpotent operators. This is implemented by restricting morphisms to the element u_{r-1}. Let me explain this perhaps somewhat mysterious choice. One essential feature of a rank variety is the detection of projectivity. By our earlier results, this means that the rank variety of a projective module should be the "origin", which in our case is the trivial element $\varepsilon : k\mathbb{G}_{a(r)} \longrightarrow k \subseteq k\mathcal{G}$. Thus, we are looking for a p-nilpotent element $x \in k\mathbb{G}_{a(r)}$ such that $\alpha^*(k\mathcal{G})|_{k[x]}$ is projective for every $\alpha \in V(\mathcal{G}) \smallsetminus \{\varepsilon\}$. Given such an element α, the kernel $\ker \alpha \subsetneq \ker \varepsilon$ is a Hopf ideal. We observe the following facts:

(a) There exists $s \in \{0, \ldots, r-1\}$ such that $\ker \alpha = (\{u_0, \ldots, u_{s-1}\})$.
(b) Since $k\mathcal{G}$ is a free module for the Hopf subalgebra $\alpha(k\mathbb{G}_{a(r)}) \subseteq k\mathcal{G}$, it follows that $\alpha^*(k\mathcal{G})$ is free over $k\mathbb{G}_{a(r)}/\ker \alpha \cong k[u_s, \ldots, u_{r-1}]$.
(c) Since $k\mathbb{G}_{a(r)}$ is a free $k[u_{r-1}]$-module, part (b) shows that $k\mathbb{G}_{a(r)}/\ker \alpha$ is free over $k[u_{r-1}]$. Hence $\alpha^*(k\mathcal{G})$ is free over $k[u_{r-1}]$.

For a $\mathcal{G}$-module M, we define its *rank variety* via

$$V(\mathcal{G})_M := \{\alpha \in V(\mathcal{G})\ ;\ \alpha^*(M)|_{k[u_{r-1}]} \text{ is not projective}\}.$$

It turns out that this naturally generalizes the varieties defined for restricted Lie algebras. Property (c) thus tells us that $V(\mathcal{G})_P = \{\varepsilon\}$ whenever $P \in \operatorname{mod} \mathcal{G}$ is projective.

If $\mathcal{G}$ is an infinitesimal group, the conical affine variety $V(\mathcal{G})_M$ reflects cohomological properties of M and $M \mapsto V(\mathcal{G})_M$ interacts nicely with categorical constructions. For the time being, let me just quote the following fundamental result:

Theorem 2.3.1 (Suslin-Friedlander-Bendel, [29]). *Suppose that $\mathcal{G}$ is an infinitesimal group scheme. Then there exists a homeomorphism $\Psi : \mathcal{V}_{\mathcal{G}}(k) \longrightarrow V(\mathcal{G})$ such that $\Psi(\mathcal{V}_{\mathcal{G}}(M)) = V(\mathcal{G})_M$ for every $M \in \operatorname{mod} \mathcal{G}$. In particular, we have*

$$\dim V(\mathcal{G})_M = \operatorname{cx}_{\mathcal{G}}(M)$$

for every $\mathcal{G}$-module M.

3. p-Points and essential subgroups

3.1. p-**Points: Definition and basic properties.**

In 2005, Eric Friedlander and Julia Pevtsova found a far-reaching generalization of rank varieties for p-elementary abelian groups and infinitesimal groups. Their new concept of a p-point has led to a number of geometric invariants that constitute refinements of those provided by rank varieties. In addition, p-points are defined for finite group schemes, so that the theory applies to all cocommutative Hopf algebras.

Given any homomorphism $\alpha : A \longrightarrow B$ of k-algebras, we denote by $\alpha^* : \operatorname{mod} B \longrightarrow \operatorname{mod} A$ the functor given by pull-back along α. Note that α^* is an exact functor. The leitmotiv underlying the theory of p-points is to understand a B-module M for a complicated algebra B in terms of its pull-backs $\alpha^*(M)$ for a well-understood algebra A and a suitable family of algebra homomorphisms. If $B = k\mathcal{G}$ is the "group algebra" of an infinitesimal group $\mathcal{G}$, we considered algebra homomorphisms $k[u_{r-1}] \longrightarrow k\mathcal{G}$ that are restrictions of homomorphisms $k\mathbb{G}_{a(r)} \longrightarrow k\mathcal{G}$ of Hopf algebras.

As before, k denotes an algebraically closed field of characteristic $\operatorname{char}(k) = p > 0$. For notational convenience we put $\mathfrak{A}_p := k[T]/(T^p)$ and write $t := T + (T^p)$. The algebra $\mathfrak{A}_p$ has a particular simple representation theory. It is a local Nakayama algebra and has up to isomorphism exactly p indecomposable modules. We denote by $[i] := \mathfrak{A}_p/(t^i)$ the indecomposable $\mathfrak{A}_p$-module of dimension i $(1 \le i \le p)$.

A finite group scheme $\mathcal{U}$ is said to be *unipotent*, provided $k\mathcal{U}$ is a local algebra. In particular, a finite group is unipotent if and only if it is a p-group. Similarly, the group scheme $\mathcal{G}_{\mathfrak{g}}$ of a restricted Lie algebra is unipotent if and only if $(\mathfrak{g}, [p])$ is p-unipotent in the sense that $\mathfrak{g}^{[p]^n} = (0)$ for some $n \in \mathbb{N}$.

Definition. Let $\mathcal{G}$ be a finite group scheme. An algebra homomorphism $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{G}$ is called a *p-point*, if

(P1) α is left flat, and
(P2) there exists an abelian unipotent subgroup scheme $\mathcal{U} \subseteq \mathcal{G}$ with $\operatorname{im}\alpha \subseteq k\mathcal{U}$.

Two p-points α, β are *equivalent* ($\alpha \sim \beta$), if for every $M \in \operatorname{mod}\mathcal{G}$ we have

$$\alpha^*(M) \text{ is projective} \Leftrightarrow \beta^*(M) \text{ is projective.}$$

The sets of p-points and equivalence classes of p-points will be denoted $\operatorname{Pt}(\mathcal{G})$ and $\operatorname{P}(\mathcal{G})$, respectively.

Since this definition may look somewhat contrived, let us spend a few moments on the relevant conditions. Property (P1) means that the pull-back functor $\alpha^* : \operatorname{mod} k\mathcal{G} \longrightarrow \operatorname{mod}\mathfrak{A}_p$ sends projectives to projectives. In particular, $\alpha^*(k\mathcal{G})$ is a free $\mathfrak{A}_p$-module, so that each $\alpha \in \operatorname{Pt}(\mathcal{G})$ is injective.

Condition (P2) is more subtle. It allows us to reduce many questions concerning p-points to the consideration of abelian unipotent groups, which are easier to understand. Recall that an abelian p-group U is of the form $U = \bigoplus_{i=1}^r \mathbb{Z}/(p^{n_i})$, whence

$$kU \cong k[X_1, \dots, X_r]/(X_1^{p^{n_1}}, \dots, X_r^{p^{n_r}}).$$

The latter isomorphism turns out to hold for any finite abelian unipotent group scheme. Let us take a look at the case where $n_i = 1$ for $1 \le i \le r$. Recall that p-elementary abelian groups are examples of such group schemes.

Proposition 3.1.1 (cf. [4]). *Let $\mathcal{U}$ be a unipotent group scheme such that $k\mathcal{U} \cong k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$.*

(1) *A homomorphism $\alpha : \mathfrak{A}_p \longrightarrow k\mathcal{U}$ is left flat if and only if $\alpha(t) \in \operatorname{Rad}(k\mathcal{U}) \smallsetminus \operatorname{Rad}^2(k\mathcal{U})$.*
(2) *Two p-points $\alpha, \beta \in \operatorname{Pt}(\mathcal{U})$ are equivalent if and only if there exists $\omega \in k^\times$ such that $\alpha(t) - \omega\beta(t) \in \operatorname{Rad}^2(k\mathcal{U})$.*

Examples. (1) Let E_r be the p-elementary abelian group of rank r. Given $\lambda \in k^r \smallsetminus \{0\}$, the map

$$\alpha_\lambda : \mathfrak{A}_p \longrightarrow kE_r \quad ; \quad t \mapsto u_\lambda$$

is a p-point and $\operatorname{P}(E_r) = \{[\alpha_\lambda] \ ; \ \lambda \in k^r \smallsetminus \{0\}\} \cong \mathbb{P}^{r-1}$.

(2) We illustrate the relevance of condition (P2) by looking at a restricted Lie algebra $(\mathfrak{g}, [p])$. Let $\alpha : \mathfrak{A}_p \longrightarrow U_0(\mathfrak{g})$ be a p-point. Then α factors through the restricted enveloping algebra $U_0(\mathfrak{u})$ of a suitable abelian, p-unipotent p-subalgebra $\mathfrak{u} \subseteq \mathfrak{g}$. In this case, the generators $x_1, \dots, x_r$ of the algebra $U_0(\mathfrak{u})$ can be chosen to be elements of $\mathfrak{u}$ that satisfy $x_i^{p^{n_i}} = 0 = x_i^{[p]^{n_i}}$. A generalization of Proposition 3.1.1 shows that the p-point α is equivalent to $\alpha_\zeta : \mathfrak{A}_p \longrightarrow U_0(\mathfrak{u})$, given by

$$\alpha_\zeta(t) = \sum_{i=1}^r \zeta_i x_i^{p^{n_i-1}} = \sum_{i=1}^r \zeta_i x_i^{[p]^{n_i-1}} \in \mathfrak{u},$$

where $\zeta := (\zeta_1, \dots, \zeta_r) \in k^r \smallsetminus \{0\}$.

For each $x \in V(\mathfrak{g})\smallsetminus\{0\}$, we consider the p-point $\alpha_x : \mathfrak{A}_p \longrightarrow U_0(\mathfrak{g})$; $t \mapsto x$. The foregoing observation shows that the map

$$\Psi_{\mathfrak{g}} : \mathrm{Proj}(V(\mathfrak{g})) \longrightarrow \mathrm{P}(\mathfrak{g}) \ \ ; \ \ [x] \mapsto [\alpha_x]$$

is surjective.

Let $x, y \in V(\mathfrak{g})\smallsetminus\{0\}$ such that $\alpha_x \sim \alpha_y$. We consider the $U_0(\mathfrak{g})$-module $M_x := U_0(\mathfrak{g})\otimes_{k[x]} k$. The augmentation $\varepsilon : U_0(\mathfrak{g}) \longrightarrow k$ induces a $U_0(\mathfrak{g})$-linear surjection $M_x \longrightarrow k$ for which $1 \mapsto 1 \otimes 1$ defines a $k[x]$-linear splitting. Hence $M_x|_{k[x]}$ is not projective. Thus, $M_x|_{k[y]}$ is not projective, and the PBW-Theorem yields $y \in kx$. As a result, the map $\Psi_{\mathfrak{g}}$ is bijective.

Note that $\Psi_{\mathfrak{g}}(V(\mathfrak{g})_M) = \{[\alpha] \in \mathrm{P}(\mathfrak{g}) \ ; \ \alpha^*(M) \text{ is not projective}\}$ for every $M \in \mathrm{mod}\, U_0(\mathfrak{g})$. This motivates the following

Definition. Let M be a $\mathcal{G}$-module. Then

$$\mathrm{P}(\mathcal{G})_M := \{[\alpha] \in \mathrm{P}(\mathcal{G}) \ ; \ \alpha^*(M) \text{ is not projective}\}$$

is called the *p-support* of M.

The following result shows that $\mathrm{P}(\mathcal{G})$ is a topological space, which generalizes and unifies the earlier definitions of rank varieties:

Theorem 3.1.2 (Friedlander-Pevtsova, [17]). *Let $\mathcal{G}$ be a finite group scheme.*

(1) *The sets $\mathrm{P}(\mathcal{G})_M$ form the closed sets of a noetherian topology on $\mathrm{P}(\mathcal{G})$.*
(2) *Suppose that $\mathcal{G}$ is infinitesimal. There exists a homeomorphism $\Psi_{\mathcal{G}} : \mathrm{Proj}(V(\mathcal{G})) \longrightarrow \mathrm{P}(\mathcal{G})$ such that*

$$\Psi_{\mathcal{G}}(\mathrm{Proj}(V(\mathcal{G})_M)) = \mathrm{P}(\mathcal{G})_M \qquad \forall\ M \in \mathrm{mod}\,\mathcal{G}.$$

The proof of this result actually requires information on $\mathrm{P}(\mathcal{G})_{M_1} \cap \mathrm{P}(\mathcal{G})_{M_2}$, which is provided in the next result that lists representation-theoretic properties of $M \mapsto P(\mathcal{G})_M$:

Theorem 3.1.3 ([17]). *Let $\mathcal{G}$ be a finite group scheme.*

(1) *We have $\dim \mathrm{P}(\mathcal{G})_M = \mathrm{cx}_{\mathcal{G}}(M) - 1$ for every $M \in \mathrm{mod}\,\mathcal{G}$. In particular, M is projective if and only if $\mathrm{P}(\mathcal{G})_M = \emptyset$.*
(2) *If $(0) \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow (0)$ is exact, then $\mathrm{P}(\mathcal{G})_{M_2} \subseteq \mathrm{P}(\mathcal{G})_{M_1} \cup \mathrm{P}(\mathcal{G})_{M_3}$, with equality in case the sequence splits.*
(3) *We have $\mathrm{P}(\mathcal{G})_{M_1 \otimes_k M_2} = \mathrm{P}(\mathcal{G})_{M_1} \cap \mathrm{P}(\mathcal{G})_{M_2}$ for all $M_1, M_2 \in \mathrm{mod}\,\mathcal{G}$.*
(4) *If M is indecomposable, then $\mathrm{P}(\mathcal{G})_M$ is connected.*

Part (3) is rather subtle. It is false if one does not require condition (P2). Part (4) is based on work by Carlson (cf. [5]), who proved a corresponding result for support varieties of finite groups.

3.2. **Essential subgroups.** Recall the Alperin-Evens Theorem, according to which the complexity of a G-module is the maximum of the complexities of its restrictions with respect to all p-elementary abelian subgroups. We would like to interpret this result in the context of finite group schemes by reducing the computation of $\mathrm{P}(\mathcal{G})$ to the case of infinitesimal groups and elementary abelian groups. This amounts to a weak version of what is commonly referred to as Quillen stratification.

Recall that a closed embedding $\iota : \mathcal{G}' \hookrightarrow \mathcal{G}$ of finite group schemes induces an injective homomorphism $\iota : k\mathcal{G}' \hookrightarrow k\mathcal{G}$ and hence a continuous map

$$\iota_{*,\mathcal{G}'} : \mathrm{P}(\mathcal{G}') \hookrightarrow \mathrm{P}(\mathcal{G}) \ \ ; \ \ [\alpha] \mapsto [\iota \circ \alpha],$$

such that $\iota_{*,\mathcal{G}}(\mathrm{P}(\mathcal{G}')) \subseteq \mathrm{P}(\mathcal{G})$ is closed. We apply this to special subgroups.

Let $\mathcal{G}$ be a finite group scheme. Then there exists an infinitesimal normal subgroup scheme $\mathcal{G}^0 \trianglelefteq \mathcal{G}$ and a reduced subgroup scheme $\mathcal{G}_{\mathrm{red}} \subseteq \mathcal{G}$ such that $\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}}$. At the level of Hopf algebras, this means that

$$k\mathcal{G} = k\mathcal{G}^0 * \mathcal{G}(k)$$

is a skew group algebra, where $\mathcal{G}(k) = \mathrm{Comm}_k(k[\mathcal{G}], k)$ is the finite group of k-rational points of $\mathcal{G}$ which acts on $\mathcal{G}^0$ via automorphisms.

For every p-elementary abelian subgroup $E \subseteq \mathcal{G}(k)$, we consider the group scheme $(\mathcal{G}^0)^E \times E \subseteq \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}}$. (Strictly speaking, we take closed subgroups of $\mathcal{G}_{\mathrm{red}}$, whose groups of k-rational points are p-elementary abelian.) For the corresponding Hopf algebras, this amounts to considering the skew group algebras $k((\mathcal{G}^0)^E) * E = k((\mathcal{G}^0)^E) \otimes_k kE \subseteq k\mathcal{G}$.

Theorem 3.2.1 ([17]). *We have*

$$\mathrm{P}(\mathcal{G}) = \bigcup_{E \in \mathfrak{E}(\mathcal{G}(k))} \iota_{*,(\mathcal{G}^0)^E \times E}(\mathrm{P}((\mathcal{G}^0)^E \times E)).$$

Example. Let $p \geq 3$. Consider the finite group scheme $\mathcal{L} : \mathrm{Comm}_k \longrightarrow \mathrm{Gr}$, given by

$$\mathcal{L}(R) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2)(R) \ ; \ a^p = 1 = d^p \ , \ b^{p^2} = b^p \ , \ c^p = 0 \}$$

for every $R \in \mathrm{Comm}_k$. Then we have

$$\mathcal{L}(k) = \{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}(2)(k) \ ; \ b \in \mathbb{F}_p \} \cong \mathbb{Z}/(p),$$

while $\mathcal{L}^0 = \mathrm{SL}(2)_1$, where

$$\mathrm{SL}(2)_1(R) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{L}(R) \ ; \ a^p = 1 = d^p \ , \ b^p = 0 = c^p \}$$

for every $R \in \mathrm{Comm}_k$. Consequently,

$$k\mathcal{L} \cong U_0(\mathfrak{sl}(2)) * \mathbb{Z}/(p)$$

is a skew group algebra, with $\mathbb{Z}/(p)$ acting on $\mathfrak{sl}(2)$ via conjugation. One computes $\mathfrak{sl}(2)^{\mathbb{Z}/(p)} = \mathfrak{u}$, the subalgebra of strictly upper triangular matrices. In view of Theorem 3.2.1, we thus have to find the p-points of the algebras $U_0(\mathfrak{u}) \otimes_k k\mathbb{Z}/(p) \cong U_0(\mathfrak{u} \times \mathfrak{u})$ and $U_0(\mathfrak{sl}(2))$, respectively. It follows that $\dim \mathrm{P}(\mathcal{L}) = 1$, so that $\mathrm{cx}_{\mathcal{L}}(k) = 2$.

Our goal is to get information on a $\mathcal{G}$-module M by studying the behavior of the nilpotent operators

$$\alpha(t)_M : M \longrightarrow M \ \ ; \ \ m \mapsto \alpha(t).m \qquad (\alpha \in \mathrm{Pt}(\mathcal{G})).$$

This ultimately necessitates some control over the "subgroup of $\mathcal{G}$ generated by all elementary abelian groups". By the latter, we understand subgroup schemes $\mathcal{G}' \subseteq \mathcal{G}$ such that $k\mathcal{G}' = k(\mathbb{Z}/(p)^r)$. One class of examples is given by p-elementary abelian groups, another one is formed by the Frobenius kernels $\mathbb{G}_{a(r)}$ of the additive group $\mathbb{G}_a$.

Definition. A group scheme $\mathcal{E}$ is called *quasi-elementary abelian*, provided $\mathcal{E} \cong \mathbb{G}_{a(r)} \times E$ for some $r \geq 1$ and some p-elementary abelian group E.

Example. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra with associated group scheme $\mathcal{G}_{\mathfrak{g}}$. If $\mathcal{E} \subseteq \mathcal{G}_{\mathfrak{g}}$ is quasi-elementary abelian, then there exists a surjective homomorphism $k[\mathcal{G}_{\mathfrak{g}}] \twoheadrightarrow k[\mathcal{E}]$ of k-algebras. Since $\mathcal{E} \cong \mathbb{G}_{a(r)} \times E$, we obtain $k[\mathcal{E}] \cong k[\mathbb{G}_{a(r)}] \otimes_k (kE)^* \cong k[X]/(X^{p^r})^{|E|}$. Now $k[\mathcal{G}_{\mathfrak{g}}] = U_0(\mathfrak{g})^* \cong k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$ forces $E = (0)$ and $r = 1$, so that $\mathcal{E} \cong \mathbb{G}_{a(1)}$. This shows that the quasi-elementary subgroups of $\mathcal{G}_{\mathfrak{g}}$ correspond to the one-dimensional p-subalgebras of $\mathfrak{g}$ that are defined by elements of $V(\mathfrak{g}) \smallsetminus \{0\}$.

Definition. A subgroup scheme $\mathcal{H} \subseteq \mathcal{G}$ of $\mathcal{G}$ is called *essential*, provided $\iota_{*,\mathcal{H}}(\mathrm{P}(\mathcal{H})) = \mathrm{P}(\mathcal{G})$.

Example. Let G be a finite group. Since $\iota_{*,gEg^{-1}}(\mathrm{P}(gEg^{-1})) = \iota_{*,E}(\mathrm{P}(E))$ for every $g \in G$, Theorem 3.2.1 can be used to show that a subgroup $H \subseteq G$ is essential if and only if for every $E \in \mathfrak{E}(G)$ there is $g \in G$ with $gEg^{-1} \subseteq H$.

Let $\mathcal{G}$ be a finite group scheme. Then there exists the smallest subgroup scheme $\mathcal{G}_{\mathrm{e}} \subseteq \mathcal{G}$ that contains every quasi-elementary abelian subgroup of $\mathcal{G}$. Note that $\mathcal{G}_{\mathrm{e}}$ is a *characteristic* subgroup scheme, that is, $\mathcal{G}_{\mathrm{e}}$ is invariant under all automorphisms of the group scheme $\mathcal{G}$.

Remarks. (1) If G is a finite group, then G_{e} is the subgroup generated by all p-elementary abelian groups. Being a characteristic subgroup, $G_{\mathrm{e}} \trianglelefteq G$ is normal in G.

(2) Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then $(\mathcal{G}_{\mathfrak{g}})_{\mathrm{e}}$ corresponds to the p-subalgebra $\mathfrak{g}_{\mathrm{e}} \subseteq \mathfrak{g}$, generated by the nullcone $V(\mathfrak{g})$. If $\mathfrak{g} = \mathrm{Lie}(G)$ is the Lie algebra of an algebraic group G, then $\mathfrak{g}_{\mathrm{e}}$ is invariant under $\mathrm{Ad}(G)$ and hence a p-ideal of $\mathfrak{g}$.

Let $\mathfrak{g} := kd \oplus kx \oplus ky \oplus kz$, with bracket and p-map given by

$$[z, \mathfrak{g}] = (0)\ ,\ [d, x] = y\ ,\ [d, y] = x\ ,\ [x, y] = z\ \ ;\ \ d^{[p]} = d\ ,\ x^{[p]} = z\ ,\ y^{[p]} = 0 = z^{[p]}.$$

Then $\mathfrak{g}_{\mathrm{e}} = ky \oplus kz$ is not a p-ideal.

A finite group scheme $\mathcal{G}$ such that $k\mathcal{G}$ is semi-simple is called *linearly reductive*. The second observation above is a special case of the following result:

Lemma 3.2.2. *Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a finite, normal subgroup scheme of the algebraic group scheme $\mathcal{G}$.*

1. *If $\mathcal{G}$ is finite with $\mathcal{G}/\mathcal{N}$ being linearly reductive, then $\mathcal{N}_{\mathrm{e}} = \mathcal{G}_{\mathrm{e}}$.*
2. *If $\mathcal{G}$ is reduced, then $\mathcal{N}_{\mathrm{e}} \subseteq \mathcal{N}$ is a normal subgroup scheme of $\mathcal{G}$.*

Recall that for any finite group scheme $\mathcal{G}$, the finite group $\mathcal{G}(k)$ acts on $\mathcal{G}$ by conjugation. Our interest in $\mathcal{G}_{\mathrm{e}}$ derives from the following result:

Theorem 3.2.3 ([13]). *Let $\mathcal{G}$ be a finite group scheme. Then $\mathcal{G}_{\mathrm{e}}$ is the unique minimal $\mathcal{G}(k)$-invariant essential subgroup of $\mathcal{G}$.*

Proof. In generalization of our example concerning restricted Lie algebras, Friedlander and Pevtsova [17] have shown that every point $x \in \mathrm{P}(\mathcal{G})$ possesses a representative $\alpha \in \mathrm{Pt}(\mathcal{G})$ that factors through a quasi-elementary abelian subgroup $\mathcal{E} \subseteq \mathcal{G}$. Consequently, $\mathcal{G}_{\mathrm{e}}$ is a $\mathcal{G}(k)$-invariant, essential subgroup of $\mathcal{G}$.

Let $\mathcal{H} \subseteq \mathcal{G}$ be a $\mathcal{G}(k)$-invariant essential subgroup of $\mathcal{G}$. Using Theorem 3.2.1 one shows that the connected component $\mathcal{H}^0$ is an essential subgroup of $\mathcal{G}^0$. If $\mathcal{E} \subseteq \mathcal{G}$ is a quasi-elementary subgroup, then

$\mathcal{E}^0 \cong \mathbb{G}_{a(r)}$ is a subgroup of $\mathcal{G}^0$ and the group $\mathcal{H}^0 \cap \mathcal{E}^0$ is essential in $\mathcal{E}^0$. Since $\mathcal{H}^0 \cap \mathcal{E}^0 \cong \mathbb{G}_{a(s)}$ for some $s \le r$, it follows from Theorem 3.1.3 that

$$s = \dim \mathrm{P}(\mathcal{H}^0 \cap \mathcal{E}^0)+1 \ge \dim \mathrm{P}(\mathcal{E}^0)+1 = r.$$

Consequently, $\mathcal{H}^0 \cap \mathcal{E}^0 = \mathcal{E}^0$, so that $\mathcal{E}^0 \subseteq \mathcal{H}^0$. One now proceeds to prove $\mathcal{E}_{\mathrm{red}} \subseteq \mathcal{H}_{\mathrm{red}}$. As a result, $\mathcal{E}$ is contained in $\mathcal{H}$. Consequently, $\mathcal{G}_{\mathrm{e}} \subseteq \mathcal{H}$. □

Remark. Let $M \in \operatorname{mod} \mathcal{G}$ be a $\mathcal{G}$-module. The above result tells us that conditions imposed on M via p-points (such as $\alpha^*(M) \cong \beta^*(M)$ for all $\alpha, \beta \in \mathrm{Pt}(\mathcal{G})$) provide information on the restriction $M|_{\mathcal{G}_{\mathrm{e}}}$.

We turn to the classical context of Frobenius kernels of reductive groups:

Theorem 3.2.4 ([13]). *Let G be a reduced, reductive algebraic group scheme.*

1. *$(G_r)_{\mathrm{e}}$ is a normal subgroup scheme of G_r such that $(G_r)_{\mathrm{e}}T_r = G_r$ for every maximal torus $T \subseteq G$.*
2. *We have $(G_r, G_r) \subseteq (G_r)_{\mathrm{e}} \subseteq (G,G)_r$. Moreover, the group $G_r/(G_r)_{\mathrm{e}}$ is diagonalizable.*

Proof. We only treat the case $r = 1$ and consider the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Let $T \subseteq G$ be a maximal torus with Lie algebra $\mathfrak{t} := \mathrm{Lie}(T)$. Then $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$ is the root space decomposition of $\mathfrak{g}$ relative to T. Given $x \in \mathfrak{g}_\alpha$, general theory implies $x^{[p]} = 0$, whence $\mathfrak{g} = \mathfrak{t} + \mathfrak{g}_{\mathrm{e}}$. By Lemma 3.2.2, $\mathfrak{g}_{\mathrm{e}}$ is an ideal of $\mathfrak{g}$. Consequently, $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_{\mathrm{e}}$ and the factor algebra $\mathfrak{g}/\mathfrak{g}_{\mathrm{e}} \cong \mathfrak{t}/(\mathfrak{t} \cap \mathfrak{g}_{\mathrm{e}})$ is a torus. □

4. Subcategories defined by p-points

Let $\mathcal{G}$ be a finite group scheme, $M \in \operatorname{mod} \mathcal{G}$ be a $\mathcal{G}$-module. Given a p-point $\alpha \in \mathrm{Pt}(\mathcal{G})$, we consider the nilpotent operator

$$\alpha(t)_M : M \longrightarrow M \;\; ; \;\; m \mapsto \alpha(t).m.$$

In this section, we shall introduce three full subcategories of $\operatorname{mod} \mathcal{G}$ that are defined via properties of the maps $\alpha(t)_M$.

For $i \in \{1, \ldots, p\}$, recall that $[i] := \mathfrak{A}_p/\mathfrak{A}_p t^i$ is the (up to isomorphism) unique indecomposable $\mathfrak{A}_p$-module of dimension i. Consequently,

$$\alpha^*(M) \cong \bigoplus_{i=1}^{p} \alpha_i(M)[i],$$

where $\alpha_i(M) \in \mathbb{N}_0$ denotes the multiplicity of $[i]$ in $\alpha^*(M)$. We call the isomorphism class of $\alpha^*(M)$ the *Jordan type* $\mathrm{Jt}(M, \alpha)$ of M relative to α and consider the finite set

$$\mathrm{Jt}(M) := \{\mathrm{Jt}(M, \alpha) \; ; \; \alpha \in \mathrm{Pt}(\mathcal{G})\}$$

of Jordan types of M.

Definition. Let M be a $\mathcal{G}$-module.

1. We say that M has the *equal images property*, provided for every $j \in \{1, \ldots, p-1\}$ there exists a subspace $V_j \subseteq M$ such that

$$\operatorname{im} \alpha(t)_M^j = V_j \qquad \text{for all } \alpha \in \mathrm{Pt}(\mathcal{G}).$$

2. We say that M has *constant Jordan type*, provided $|\mathrm{Jt}(M)| = 1$.

(3) Given $j \in \{1,\ldots,p{-}1\}$, we say that M has *constant j-rank* d_j, provided $\mathrm{rk}(\alpha(t)^j_M) = d_j$ for every $\alpha \in \mathrm{Pt}(\mathcal{G})$. Modules of constant 1-rank are said to be of *constant rank*.

We denote by $\mathrm{EIP}(\mathcal{G})$, $\mathrm{CJT}(\mathcal{G})$ and $\mathrm{CR}_j(\mathcal{G})$ the full subcategories of $\mathrm{mod}\,\mathcal{G}$, whose objects have the equal images property, are of constant Jordan type, or have constant j-rank, respectively. The categories were introduced in [7], [6], and [18].

Linear algebra tells us that M has constant Jordan type if and only if M has constant j-rank for every $j \in \{1,\ldots,p{-}1\}$. We therefore obtain

$$\mathrm{EIP}(\mathcal{G}) \subseteq \mathrm{CJT}(\mathcal{G}) = \bigcap_{j=1}^{p-1} \mathrm{CR}_j(\mathcal{G}).$$

We are now going to discuss these categories in turn.

4.1. **Modules of constant j-rank.** The following observation illustrates that modules belonging to these subcategories cannot be distinguished by their support spaces:

Lemma 4.1.1. *Let M be a $\mathcal{G}$-module of constant j-rank. Then* $\mathrm{P}(\mathcal{G})_M \in \{\mathrm{P}(\mathcal{G}), \emptyset\}$.

Proof. Let α be a p-point. Then we have

$$\begin{aligned}\mathrm{rk}(\alpha(t)^j_M) &= \sum_{i=j+1}^{p} \alpha_i(M)(i{-}j) \leq \sum_{i=j+1}^{p} \alpha_i(M)(i{-}j\frac{i}{p}) = (\sum_{i=j+1}^{p} \alpha_i(M)i)(1{-}\frac{j}{p}) \\ &\leq (\sum_{i=1}^{p} \alpha_i(M)i)(1{-}\frac{j}{p}) = \frac{\dim_k M}{p}(p{-}j).\end{aligned}$$

If $\alpha^*(M)$ is projective, then $\alpha^*(M) = n[p]$, with $n = \frac{\dim_k M}{p}$ and we have equality throughout. Alternatively, one of the two inequalities must be strict. Since M has constant j-rank, we conclude that $\mathrm{P}(\mathcal{G})_M \in \{\mathrm{P}(\mathcal{G}), \emptyset\}$. □

Examples. (1) Let $\mathfrak{a}_2$ be the two-dimensional restricted Lie algebra with trivial bracket and trivial p-map, so that $U_0(\mathfrak{a}_2) = k[X,Y]/(X^p,Y^p)$ with canonical generators $x, y \in \mathfrak{a}_2$. We define a $U_0(\mathfrak{a}_2)$-module M of dimension $2n$ by specifying two linear transformations $x_M, y_M \in \mathrm{End}_k(M)$. To that end, we write $M = V \oplus V$ as a sum of two n-dimensional subspaces and give the following matrix representations:

$$x_M := \left(\begin{smallmatrix} 0 & 0 \\ I_n & 0 \end{smallmatrix}\right) \ ; \ y_M := \left(\begin{smallmatrix} 0 & 0 \\ J_n & 0 \end{smallmatrix}\right),$$

where I_n and J_n are the identity matrix and the Jordan matrix of size n, respectively. It follows that $\mathrm{Rad}^2(M) = (0)$, so that $u_M^2 = 0$ for all $u \in \mathrm{Rad}(U_0(\mathfrak{a}_2))$. As a result, M has constant j-rank for every $j \in \{2,\ldots,p{-}1\}$. On the other hand, $\mathrm{rk}(x_M) = n$, while $\mathrm{rk}(y_M) = n{-}1$, so that M does not have constant rank.

(2) Let $\mathrm{SL}(2)_2$ be the second Frobenius kernel of $\mathrm{SL}(2)$, that is,

$$\mathrm{SL}(2)_2(R) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \ ; \ a^{p^2} = 1 = d^{p^2} \ , \ b^{p^2} = 0 = c^{p^2}\} \qquad (R \in \mathrm{Comm}_k).$$

There are precisely p^2 simple $\mathrm{SL}(2)_2$-modules $L(i)$ with $0 \leq i \leq p^2{-}1$. Then $L(i_1{+}pi_2)$ has constant j-rank if and only if $j > i_1{+}i_2$ or $i_1 = p{-}1 = i_2$.

(3) Let $\mathcal{G}$ be a finite group scheme, and suppose that $\zeta \in \mathrm{H}^{2n+1}(\mathcal{G},k)\smallsetminus\{0\}$. We consider the *Carlson module* L_ζ of ζ. By definition, $L_\zeta = \ker\hat{\zeta}$, where $\hat{\zeta} : \Omega^{2n+1}_{\mathcal{G}}(k) \longrightarrow k$ is the $\mathcal{G}$-linear map corresponding to ζ. Then there exists $n_\zeta \in \mathbb{N}_0$ such that

$$\mathrm{Jt}(L_\zeta,\alpha) = \begin{cases} 2[p-1] \oplus n_\zeta[p] & \text{if } \alpha^\bullet(\zeta) = 0 \\ [p-2] \oplus (n_\zeta+1)[p] & \text{if } \alpha^\bullet(\zeta) \neq 0. \end{cases}$$

Here $\alpha^\bullet : \mathrm{H}^\bullet(\mathcal{G},k) \longrightarrow \mathrm{H}^\bullet(\mathfrak{A}_p,k)$ denotes the map induced by $\alpha \in \mathrm{Pt}(\mathcal{G})$. Consequently, L_ζ usually has constant j-rank for $j \leq p-2$, but not for $j = p-1$.

The last example above also shows that the category $\mathrm{CR}_j(\mathcal{G})$ is not closed under taking kernels. The right-hand terms of the exact sequence

$$(0) \longrightarrow L_\zeta \longrightarrow \Omega^{2n+1}_{\mathcal{G}}(k) \longrightarrow k \longrightarrow (0)$$

are modules of constant Jordan type and thus in particular of constant $(p-1)$-rank, while L_ζ may not enjoy this property.

Let V be a k-vector space. Given $d \in \mathbb{N}$, the set

$$\mathrm{End}_k(V)_d := \{f \in \mathrm{End}_k(V) \ ; \ \mathrm{rk}(f) \leq d\}$$

is a Zariski closed subset of the affine variety $\mathrm{End}_k(V)$. We would like to have an analog for equivalence classes of p-points. Let $M \in \mathrm{mod}\,\mathcal{G}$ be a $\mathcal{G}$-module. Given $j \in \{1,\ldots,p-1\}$, we let $\mathrm{rk}^j(M) := \max\{\mathrm{rk}(\alpha(t)^j_M) \ ; \ \alpha \in \mathrm{Pt}(\mathcal{G})\}$ be the *generic j-rank* of M. The name derives from the following basic result:

Theorem 4.1.2 (Friedlander-Pevtsova, [18]). *Let $\mathcal{G}$ be a finite group scheme, M be a $\mathcal{G}$-module. Given $j \in \{1,\ldots,p-1\}$, the following statements hold:*

(1) *If $\mathrm{rk}(\alpha(t)^j_M) = \mathrm{rk}^j(M)$, then $\mathrm{rk}(\beta(t)^j_M) = \mathrm{rk}^j(M)$ for all $\beta \sim \alpha$.*
(2) *The set*

$$\Gamma^j(\mathcal{G})_M := \{x \in \mathrm{P}(\mathcal{G}) \ ; \ \mathrm{rk}(\alpha(t)^j_M) < \mathrm{rk}^j(M) \quad \forall\ \alpha \in x\}$$

is a closed subspace of $\mathrm{P}(\mathcal{G})$.

Corollary 4.1.3. *Let M be a module over a finite group scheme $\mathcal{G}$.*

(1) *M has constant j-rank if and only if $\Gamma^j(\mathcal{G})_M = \emptyset$.*
(2) *If $\mathcal{G}$ is infinitesimal, then M has constant j-rank if and only if $\mathrm{rk}(\alpha(u_{r-1})^j_M) = \mathrm{rk}^j(M)$ for all $\alpha \in V(\mathcal{G})\smallsetminus\{\varepsilon\}$.*

Proof. (1) Suppose that $\Gamma^j(\mathcal{G})_M = \emptyset$. Let $\alpha \in \mathrm{Pt}(\mathcal{G})$ be a p-point. By assumption, there exists $\beta \sim \alpha$ such that $\mathrm{rk}(\beta(t)^j_M) = \mathrm{rk}^j(M)$. Part (1) of Theorem 4.1.2 now implies $\mathrm{rk}(\alpha(t)^j_M) = \mathrm{rk}^j(M)$.

(2) Considering the case of a restricted Lie algebra $(\mathfrak{g},[p])$, we recall the bijective map

$$\Psi_{\mathfrak{g}} : \mathrm{Proj}(V(\mathfrak{g})) \longrightarrow \mathrm{P}(\mathfrak{g}) \ \ ; \ \ [x] \mapsto [\alpha_x].$$

Suppose that $\mathrm{rk}(x^j_M) = \mathrm{rk}^j(M)$ for all $x \in V(\mathfrak{g})\smallsetminus\{0\}$. Let α be a p-point. Then there exists $x \in V(\mathfrak{g})\smallsetminus\{0\}$ such that $\alpha \sim \alpha_x$. Theorem 4.1.2(1) now yields $\mathrm{rk}(\alpha(t)^j_M) = \mathrm{rk}(\alpha_x(t)^j_M) = \mathrm{rk}(x^j_M) = \mathrm{rk}^j(M)$, so that M has constant j-rank. □

Proposition 4.1.4. *Let $M \in \mathrm{mod}\,\mathcal{G}$.*

(1) *If $\mathrm{P}(\mathcal{G})_M \neq \mathrm{P}(\mathcal{G})$, then $\Gamma^j(\mathcal{G})_M = \mathrm{P}(\mathcal{G})_M$.*
(2) *We have $\Gamma^j(\mathcal{G})_M = \Gamma^j(\mathcal{G})_{\Omega^{2n}_{\mathcal{G}}(M)}$ for all $n \in \mathbb{Z}$.*

Remark. The second part implies that the even Heller shifts of a module of constant j-rank also have constant j-rank, a result which will be needed in our discussion of the Auslander-Reiten quiver. Note, however, that $\mathrm{CR}_j(\mathcal{G})$ is not necessarily closed under odd powers of the Heller operator.

4.2. **Modules of constant Jordan type.** The category $\mathrm{CJT}(\mathcal{G})$ of modules of constant Jordan type was introduced by Carlson-Friedlander-Pevtsova in 2008, [6]. Here are some fundamental properties:

Theorem 4.2.1 (Carlson-Friedlander-Pevtsova, [6]). *Let $M \in \mathrm{CJT}(\mathcal{G})$.*

(1) $\Omega^n_{\mathcal{G}}(M) \in \mathrm{CJT}(\mathcal{G})$ *for every* $n \in \mathbb{Z}$.
(2) *If $N|M$ is a direct summand of M, then $N \in \mathrm{CJT}(\mathcal{G})$.*
(3) *If $N \in \mathrm{CJT}(\mathcal{G})$, then $M \otimes_k N \in \mathrm{CJT}(\mathcal{G})$.*

Part (1) follows from the isomorphisms $\alpha^*(\Omega^n_{\mathcal{G}}(M)) \cong \Omega^n_{\mathfrak{A}_p}(\alpha^*(M)) \oplus (\mathrm{proj.})$ and $\Omega_{\mathfrak{A}_p}([i]) \cong [p-i]$. The other parts are much harder to verify.

Examples. (1) Every projective $\mathcal{G}$-module P has constant Jordan type $\mathrm{Jt}(P) = \frac{\dim_k P}{p}[p]$.

(2) For $p \geq 3$, we consider Frobenius kernels of the algebraic group $\mathrm{SL}(2)$.

(a) Let $\mathcal{G} = \mathrm{SL}(2)_1$, so that we are looking at modules over $U_0(\mathfrak{sl}(2))$. The group $\mathrm{SL}(2)(k)$ acts on $V(\mathfrak{sl}(2)) \smallsetminus \{0\}$ via the adjoint representation with one orbit (the matrix $e := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ being a representative). Let $i \leq p-1$. Since $L(i)^{(\mathrm{Ad}(g))} \cong L(i)$ for all $g \in \mathrm{SL}(2)$, the simple module $L(i)$ has constant Jordan type $[i+1]$.

(b) Since $\mathrm{CJT}(\mathcal{G}) = \bigcap_{j=1}^{p-1} \mathrm{CR}_j(\mathcal{G})$, our earlier example shows that the trivial $\mathrm{SL}(2)_2$-module $L(0)$ and the Steinberg module $L(p^2-1)$ are the only simple $\mathrm{SL}(2)_2$- modules of constant Jordan type. This result actually holds for all Frobenius kernels $\mathrm{SL}(2)_r$ of height $r \geq 2$.

(3) Let $r \geq 2$ and consider the p-elementary abelian group E_r of rank r. Then $\mathrm{GL}(r)$ acts on kE_r via automorphisms such that $V(kE_r) \smallsetminus \{0\}$ consists of one orbit. Thus, every $M \in \operatorname{mod} kE_r$ satisfying $M^{(g)} \cong M$ for all $g \in \mathrm{GL}(r)$ has constant Jordan type. Let J be the radical of kE_r. Then J^m is a $\mathrm{GL}(r)$-submodule of kE_r and the action by $g \in \mathrm{GL}(r)$ induces an isomorphism $(J^m/J^n)^{(g)} \cong J^m/J^n$ for $m \leq n$. As a result, these kE_r-modules have constant Jordan type.

The above examples may be formalized by considering modules over algebras with a compatible action of some algebraic group. We are interested in the following situation: Let G be an algebraic group (a reduced algebraic group scheme) with Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$. Then G acts on $U_0(\mathfrak{g})$ via the adjoint representation $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(U_0(\mathfrak{g}))$. Given $M \in \operatorname{mod} U_0(\mathfrak{g})$ and $g \in G$, we put $M^{(g)} := M^{(\mathrm{Ad}(g))}$. We say that M is *G-stable*, provided $M^{(g)} \cong M$ for every $g \in G$. Every rational G-module gives rise to a G-stable $U_0(\mathfrak{g})$-module.

Example. Suppose that G is connected and let P be a principal indecomposable $U_0(\mathfrak{g})$-module. Then $\mathrm{Rad}^i(P)/\mathrm{Rad}^{i+j}(P)$ is G-stable for $i, j \geq 0$. In particular, every simple $U_0(\mathfrak{g})$-module is G-stable.

Theorem 4.2.2 ([13]). *Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of a reductive algebraic group G.*

(1) *We have $\mathrm{CR}_j(\mathfrak{g}) = \mathrm{CJT}(\mathfrak{g})$ for every $j \in \{1, \ldots, p-1\}$.*
(2) *Every $M \in \mathrm{CJT}(\mathfrak{g})$ is G-stable.*

Proof. As both parts are verified via reduction to the case $\mathfrak{g} = \mathfrak{sl}(2)$, let us concentrate on this case. Let M be a non-projective $U_0(\mathfrak{sl}(2))$-module of constant j-rank. By decomposing M into indecomposables, we arrive at a presentation

$$M = M_0 \oplus M_1 \oplus M_2,$$

where M_i is a direct sum of indecomposables, whose rank varieties have dimension i. By the classification of indecomposable $U_0(\mathfrak{sl}(2))$-modules, the modules M_0 and M_2 are $\mathrm{SL}(2)$-stable and of constant Jordan type. Consequently, M_1 has constant j-rank. In view of (4.1.1), this is only possible, if M_1 has no indecomposable constituents, whence $M_1 = (0)$. Thus, M is $\mathrm{SL}(2)$-stable and of constant Jordan type. □

4.3. **Equal images modules.** If $G = \mathbb{Z}/(p)$, then every indecomposable G-module has the equal images property, whence $\mathrm{EIP}(E_1) = \mathrm{mod}\, E_1$. It is therefore natural to look at the group $G = \mathbb{Z}/(p) \times \mathbb{Z}/(p)$. We shall do this in the next section. For the time being, we will discuss reductive Lie algebras. Our arguments rely on the well-known classification of the indecomposable $U_0(\mathfrak{sl}(2))$-modules.

Lemma 4.3.1. *Let $\mathcal{G}$ be a finite group scheme. Then $\mathrm{EIP}(\mathcal{G})$ is closed under direct sums and images.*

Note that this implies in particular that $\mathrm{EIP}(\mathcal{G})$ is closed under taking direct summands.

Example. Let $p \geq 3$ and consider the Lie algebra $\mathfrak{sl}(2)$. We claim that all equal images modules are trivial. Suppose that $M \in \mathrm{EIP}(\mathfrak{sl}(2))$ is a counter-example of minimal dimension. In particular, M is indecomposable. Suppose that $L(i) \subseteq M$. If $M = L(i)$, then $i = 0$. Alternatively, the equal images module $M/L(i) \neq (0)$ is trivial, so that we have a non-split exact sequence

$$(0) \longrightarrow L(i) \longrightarrow M \longrightarrow k^n \longrightarrow (0)$$

for some $n \geq 1$. Thus, $\mathrm{Ext}^1_{U_0(\mathfrak{sl}(2))}(k, L(i)) \neq (0)$, and standard $U_0(\mathfrak{sl}(2))$-theory forces $i = p-2$. Since $M \in \mathrm{EIP}(\mathfrak{sl}(2))$ is a rational $\mathrm{SL}(2)$-module, $\mathrm{im}\, x_M^j \subseteq L(i)$ is an $\mathrm{SL}(2)$-submodule for every $x \in V(\mathfrak{g})$. If $\mathrm{im}\, x_M \neq (0)$, then $\mathrm{im}\, x_M^2 \subsetneq \mathrm{im}\, x_M = L(p-2)$, whence $(0) = \mathrm{im}\, x_M^2 = \mathrm{im}\, x_{L(p-2)}$. As $\mathrm{Jt}(L(p-2)) = [p-2]$ this forces $p = 2$, a contradiction.

Theorem 4.3.2 ([13]). *Let $\mathcal{G}$ be a finite group scheme of characteristic $p \geq 3$. If*

(a) $\mathrm{SL}(2)_1 \subseteq \mathcal{G}$, *and*
(b) $\mathcal{G}_{\mathrm{e}} \trianglelefteq \mathcal{G}$ *is a normal subgroup scheme of $\mathcal{G}$,*

then $\mathrm{EIP}(\mathcal{G}) = \mathrm{mod}(\mathcal{G}/\mathcal{G}_{\mathrm{e}})$.

Proof. Let $M \in \mathrm{EIP}(\mathcal{G})$. Since M has constant Jordan type, and $M|_{\mathrm{SL}(2)_1} \in \mathrm{EIP}(\mathrm{SL}(2)_1)$, it follows that $\mathrm{Jt}(M) = (\dim_k M)[1]$. Now let $\mathcal{E} \subseteq \mathcal{G}$ be quasi-elementary. Since the standard generators of $k\mathcal{E}$ define p-points of $\mathcal{E}$, we conclude that these generators act trivially on M. Thus, $\mathcal{E}$ acts trivially on M and the same holds for $\mathcal{G}_{\mathrm{e}}$. The result now follows from condition (b). □

Corollary 4.3.3. *Let G be a reductive group of characteristic $p \geq 3$. Then every $M \in \mathrm{EIP}(G_r)$ is a direct sum of one-dimensional modules.*

Proof. If G is a torus, then every G_r-module is a sum of one-dimensional modules. Alternatively, G contains a copy of $\mathrm{SL}(2)$ or $\mathrm{PSL}(2)$, so that $\mathrm{SL}(2)_1 \subseteq \mathrm{SL}(2)_r \subseteq G_r$. According to Theorem 3.2.4, $(G_r)_{\mathrm{e}}$ is a normal subgroup of G_r such that the factor group $G_r/(G_r)_{\mathrm{e}}$ is diagonalizable. This implies that every $M \in \mathrm{mod}(G_r/(G_r)_{\mathrm{e}})$ is a direct sum of one-dimensional modules, and our assertion follows from Theorem 4.3.2. □

5. Modules for elementary abelian groups and Kronecker quivers

Recall that k denotes an algebraically closed field of characteristic $p > 0$. In Section 4 we introduced three of the following four full subcategories of modules over a finite group scheme $\mathcal{G}$. The additional class arises by dualizing equal images modules.

Definition. Let M be a $\mathcal{G}$-module.

(1) We say that M has the *equal images property*, provided there exist subspaces $V_j \subseteq M$ such that
$$\mathrm{im}\, \alpha(t)_M^j = V_j \qquad \text{for } j \in \{1, \ldots, p{-}1\} \text{ and } \alpha \in \mathrm{Pt}(\mathcal{G}).$$
The full subcategory of $\mathrm{mod}\, \mathcal{G}$ consisting of these modules is denoted $\mathrm{EIP}(\mathcal{G})$.

(2) We say that M has the *equal kernels property*, provided there exist subspaces $W_j \subseteq M$ such that
$$\ker \alpha(t)_M^j = W_j \qquad \text{for } j \in \{1, \ldots, p{-}1\} \text{ and } \alpha \in \mathrm{Pt}(\mathcal{G}).$$
The full subcategory of $\mathrm{mod}\, \mathcal{G}$ consisting of these modules is denoted $\mathrm{EKP}(\mathcal{G})$.

(3) We say that M has *constant Jordan type*, provided $|\,\mathrm{Jt}(M)| = 1$. The full subcategory of $\mathrm{mod}\, \mathcal{G}$ consisting of these modules is denoted $\mathrm{CJT}(\mathcal{G})$.

(4) Given $j \in \{1, \ldots, p{-}1\}$, we say that M has *constant j-rank d_j*, provided $\mathrm{rk}(\alpha(t)_M^j) = d_j$ for every $\alpha \in \mathrm{Pt}(\mathcal{G})$. The full subcategory of $\mathrm{mod}\, \mathcal{G}$ consisting of these modules is denoted $\mathrm{CR}_j(\mathcal{G})$. Modules of constant 1-rank are said to be of *constant rank*.

In this section we shall focus our attention on the case where $\mathcal{G} = E_r$ is a p-elementary abelian group of rank $r \geq 2$. Since $kE_r = k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)$, the associative algebra kE_r may be identified with $U_0(\mathfrak{a}_r)$, the restricted enveloping algebra of the r-dimensional abelian Lie algebra with trivial p-map. This means that we ignore some subtleties concerning tensor products.

5.1. **Representations of the r-Kronecker quiver.** Let $Q = (Q_v, Q_a)$ be a quiver (a directed graph) with finite set of vertices Q_v and finite set of arrows Q_a. A *representation* $V = ((V_i)_{i \in Q_v}, (V(\alpha)_{\alpha \in Q_a})$ over Q consists of a family $(V_i)_{i \in Q_v}$ of vector spaces. Moreover, for each arrow $\alpha : i \to j$, we have a k-linear map $V(\alpha) : V_i \longrightarrow V_j$. A *morphism* $\varphi : V \longrightarrow W$ is a family $(\varphi_i : V_i \longrightarrow W_i)_{i \in Q_v}$ of linear maps such that $\varphi_j \circ V(\alpha) = W(\alpha) \circ \varphi_i$ for every arrow $\alpha : i \to j$ of Q_a. We thus obtain the category $\mathrm{rep}(Q)$ of Q-representations.

We shall look at particular quivers, the r-Kronecker quivers K_r ($r \geq 2$). The quiver K_r has two vertices, 0 and 1, and r-arrows $\alpha_i : 0 \to 1$. Thus, $V \in \mathrm{rep}(K_r)$ is a pair of vector spaces (V_0, V_1), together with r linear maps $V_0 \longrightarrow V_1$. We shall employ $\mathrm{rep}(K_r)$ in order to study kE_r-modules of Loewy length ≤ 2.

There are the notions of simple and indecomposable objects of $\mathrm{rep}(K_r)$ that are defined in the canonical way. The two simple objects are $S_0 = (k, (0))$ and $S_1 := ((0), k)$. We record the following basic facts:

Lemma 5.1.1. *Let $V = (V_0, V_1) \in \mathrm{rep}(K_r)$ be indecomposable.*

(1) *If $V \not\cong S_1$, then $V_1 = \sum_{i=1}^r \mathrm{im}\, V(\alpha_i)$.*

(2) *If $V \not\cong S_0$, then $(0) = \bigcap_{i=1}^r \ker V(\alpha_i)$.*

Proof. (1) Put $W_1 := \sum_{i=1}^r \operatorname{im} V(\alpha_i) \subseteq V_1$, and pick $U_1 \subseteq V_1$ such that $V_1 = W_1 \oplus U_1$. Then $W := (V_0, W_1)$ together with the $V(\alpha_i)$, and $U := ((0), U_1)$ are subobjects of V such that $V = W \oplus U$. If $W = (0)$, then $V_0 = (0)$ and $V_1 = k$, by indecomposability. Thus, $V \cong S_1$, a contradiction. Consequently, $V = W$, so that $V_1 = W_1$.

(2) This is dual to (1). □

As before, we put $kE_r = k[x_1, \ldots, x_r]$. We define a functor $F : \operatorname{rep}(K_r) \longrightarrow \operatorname{mod} E_r$ as follows. Given $V = (V_0, V_1) \in \operatorname{rep}(K_r)$, we put

$$F(V) := V_0 \oplus V_1 \quad ; \quad x_i.(v_0{+}v_1) = V(\alpha_i)(v_0) \qquad \forall\ v_j \in V_j,\ i \in \{1, \ldots, r\},$$

as well as

$$F((\varphi_0, \varphi_1)) = \varphi_0 \oplus \varphi_1 \qquad \text{for all } \varphi_j : V_j \longrightarrow W_j.$$

Note that the E_r-module $F(V)$ has Loewy length $\ell\ell(F(V)) \leq 2$. Moreover, we have $F(S_0) = k = F(S_1)$.

Given an object $V \in \operatorname{rep}(K_r)$ and $\lambda = (\lambda_1, \ldots, \lambda_r) \in k^r$, we put $V(\lambda) := \sum_{i=1}^r \lambda_i V(\alpha_i)$.

Lemma 5.1.2. *The functor $F : \operatorname{rep}(K_r) \longrightarrow \operatorname{mod} E_r$ has the following properties:*

(1) *An object $V \in \operatorname{rep}(K_r)$ is indecomposable if and only if $F(V) \in \operatorname{mod} E_r$ is indecomposable.*
(2) *If $M \in \operatorname{mod} E_r$ is indecomposable of Loewy length ≤ 2, then there exists an indecomposable object $V \in \operatorname{rep}(K_r)$ such that $F(V) \cong M$.*
(3) *Suppose that $V \in \operatorname{rep}(K_r)$ is indecomposable and not simple.*
 (a) *$F(V) \in \operatorname{EIP}(E_r)$ if and only if $V(\lambda)$ is surjective for all $\lambda \in k^r \smallsetminus \{0\}$.*
 (b) *$F(V) \in \operatorname{EKP}(E_r)$ if and only if $V(\lambda)$ is injective for all $\lambda \in k^r \smallsetminus \{0\}$.*

Proof. (3a) Let $M := F(V)$. Since V is not simple, Lemma 5.1.1 implies $\operatorname{Rad}(M) = \langle \{x_i.V_0 \ ;\ 1 \leq i \leq r\} \rangle = V_1$. If $M \in \operatorname{EIP}(E_r)$, then all non-zero images $\operatorname{im}(\sum_{i=1}^r \lambda_i x_i)_M$ coincide, so that $\operatorname{im} V(\lambda) = V_1$ for all $\lambda \in k^r \smallsetminus \{0\}$.

For the reverse implication, let $\lambda \in k^r \smallsetminus \{0\}$. Then $\operatorname{im}(\sum_{i=1}^r \lambda_i x_i)_M = V_1 = \operatorname{Rad}(M)$. Since $\operatorname{Rad}^2(M) = (0)$, Proposition 3.1.1 now implies that $M \in \operatorname{EIP}(E_r)$. □

For $r = 2$, the indecomposable objects of $\operatorname{rep}(K_r)$ were classified by Kronecker. We summarize his result as follows: There are three classes of indecomposable objects $V = (V_0, V_1)$ of $\operatorname{rep}(\bullet \rightrightarrows \bullet)$, namely

- the *pre-projective objects* of dimension vectors $(\dim_k V_0, \dim_k V_1) = (n, n{+}1)$,
- the *pre-injective objects* of dimension vectors $(n{+}1, n)$, and
- the *regular objects* of dimension vectors (n, n).

Moreover, the objects belonging to the former two classes are uniquely determined by their dimension vectors.

It turns out that the indecomposable equal images modules (equal kernels modules) of Loewy length 2 correspond to the pre-injective objects (pre-projective objects) of $\operatorname{rep}(\bullet \rightrightarrows \bullet)$. Some E_2-modules associated to regular objects were considered in Section 4.1. Such modules belong to $\bigcap_{j=2}^{p-1} \operatorname{CR}_j(E_2) \smallsetminus \operatorname{CR}_1(E_2)$.

5.2. **Modules for $\mathbb{Z}/(p) \times \mathbb{Z}/(p)$.** Writing $kE_2 = k[x,y]$, we begin by introducing important examples of equal images modules over E_2.

Example. For $1 \le d \le p$ and $n \ge d$, we consider the E_2-module

$$W_{n,d} := (\bigoplus_{i=1}^{n} kE_2 v_i)/N_{n,d},$$

where $N_{n,d} := \langle \{x.v_1, y.v_n\} \cup \{x^d.v_i \ ; \ 2 \le i \le n\} \cup \{y.v_i - x.v_{i+1} \ ; \ 1 \le i \le n-1\}\rangle$. Then $W_{n,d}$ is an equal images module of constant Jordan type $\mathrm{Jt}(W_{n,d}) = \bigoplus_{i=1}^{d-1}[i] \oplus (n-d+1)[d]$. In particular, $W_{n,d}$ is indecomposable for $d \ge 2$, while $W_{n,1} \cong k^n$ is a trivial module.

The modules $W_{n,2}$ correspond to pre-injective objects. Thus (5.1.2) tells us that every equal images E_2-module of Loewy length ≤ 2 is of the form

$$M = \bigoplus_{i=1}^{m} W_{n_i,2} \oplus W_{n_{m+1},1}.$$

Building on this result, Carlson-Friedlander-Suslin established the following properties of equal images modules:

Theorem 5.2.1 (Carlson-Friedlander-Suslin, [7]). *Let $M \in \mathrm{EIP}(E_2)$.*

(1) *If $\dim_k \mathrm{Top}(M) = n$ and $\ell\ell(M) = d$, then there exists a surjection $W_{n,d} \twoheadrightarrow M$.*
(2) *If $\ell\ell(M) = d$, then there exist $a_1, \dots, a_d \neq 0$ such that $\mathrm{Jt}(M) = \bigoplus_{i=1}^{d} a_i[i]$.*

5.3. **Realizability.** One basic question is the realizability of Jordan types by indecomposable modules of constant Jordan type. The first interesting case concerns E_2, where Benson has investigated the question of modules affording only one non-projective block. Direct computation shows that

$$\mathrm{Jt}(\Omega^n_{E_2}(k)) = \begin{cases} [p-1] \oplus m_n[p] & n \text{ odd} \\ [1] \oplus m_n[p] & n \text{ even.} \end{cases}$$

Using restrictions to subgroups of type E_2 one can formulate Benson's result as follows:

Theorem 5.3.1 (Benson, [3]). *Let $\mathcal{G}$ be a finite group scheme that contains a quasi-elementary subgroup of rank $r \ge 2$. If $M \in \mathrm{CJT}(\mathcal{G})$ has Jordan type $\mathrm{Jt}(M) = [i] \oplus n[p]$ for some $i \in \{1, \dots, p-1\}$, then $i \in \{1, p-1\}$.*

If $\mathcal{G}$ contains a quasi-elementary abelian group $\mathcal{E}$ of rank ≥ 2, then $\mathrm{cx}_{\mathcal{G}}(k) \ge \mathrm{cx}_{\mathcal{E}}(k) \ge 2$. The example of the first Frobenius kernel $\mathrm{SL}(2)_1$ shows that the converse statement does not necessarily hold: We have $\mathrm{cx}_{\mathrm{SL}(2)_1}(k) = \dim V(\mathfrak{sl}(2)) = 2$, while every unipotent subalgebra $\mathfrak{u} \subseteq \mathfrak{sl}(2)$ has dimension ≤ 1. Moreover, the simple $\mathrm{SL}(2)_1$-module $L(i)$ has constant Jordan type $[i+1]$.

If $\mathcal{G}$ is reduced, so that $\mathcal{G}$ corresponds to the finite group $G := \mathcal{G}(k)$, then the Theorem of Alperin-Evens (1.1.3) implies that G possesses a p-elementary abelian subgroup $E \subseteq G$ of rank ≥ 2 if and only if $\mathrm{cx}_G(k) \ge 2$.

Definition. Let $\mathcal{G}$ be a finite group scheme. A $\mathcal{G}$-module M is said to be *endotrivial*, provided

$$\mathrm{End}_k(M) \cong k \oplus (\mathrm{proj.}).$$

Theorem 5.3.2. *Let $\mathcal{G}$ be a finite group scheme, M be a $\mathcal{G}$-module.*

(1) *If $\mathcal{G} = E_r$ is elementary abelian, then M is endotrivial if and only if $M \cong \Omega^n_{E_r}(k) \oplus (\text{proj.})$ for some $n \in \mathbb{Z}$ (Dade, [9]).*

(2) *M is endotrivial if and only if $M \in \mathrm{CJT}(\mathcal{G})$ has constant Jordan type $\mathrm{Jt}(M) = [i] \oplus n[p]$ for some $i \in \{1, p-1\}$ (Carlson-Friedlander-Pevtsova, [6]).*

Proof. (2) If M is endo-trivial, then $M|_{\mathcal{E}}$ is endo-trivial for every quasi-elementary subgroup $\mathcal{E} \subseteq \mathcal{G}$. It now follows from (1) that $M|_{\mathcal{E}}$ has constant Jordan type $\mathrm{Jt}(M|_{\mathcal{E}}) = [i] \oplus n[p]$ for some $i \in \{1, p-1\}$. Since $\dim_k M \equiv i \mod(p)$, the Jordan type does not depend on the choice of $\mathcal{E}$. This implies that M has constant Jordan type.

If $\mathrm{Jt}(M) = [i] \oplus np$, then $\dim_k M \not\equiv 0 \mod(p)$, so that the $\mathcal{G}$-linear map $\mathrm{tr} : \mathrm{End}_k(M) \longrightarrow k \ ; \ f \mapsto \mathrm{tr}(f)$ is split surjective. One then shows $\mathrm{Jt}(\mathrm{End}_k(M)) = [1] \oplus m[p]$, so that $\mathrm{Jt}(\ker \mathrm{tr}) = m[p]$. Hence $\ker \mathrm{tr}$ is projective and M is endo-trivial. □

Examples. (1) Let $r \geq 2$. An E_r-module M has constant Jordan type $[i] \oplus m[p]$ with $i \in \{1, \ldots, p-1\}$ if and only if $M \cong \Omega^n_{E_r}(k) \oplus (\text{proj.})$ for some $n \in \mathbb{Z}$.

(2) An indecomposable $\mathrm{SL}(2)_1$-module M is endotrivial if and only if $M \cong \Omega^n_{\mathrm{SL}(2)_1}(L(i))$ for $n \in \mathbb{Z}$ and $i \in \{0, p-2\}$.

5.4. **Maps to Grassmannians.** Given a k-vector space V and $d \leq \dim_k V =: n$, we denote by $\mathrm{Gr}_d(V)$ the Grassmann variety of d-dimensional subspaces of V. Using the Plücker embedding, one shows that $\mathrm{Gr}_d(V)$ is a projective variety of dimension $d(n-d)$.

Let $\mathcal{G}$ be a finite group scheme. If M is a $\mathcal{G}$-module of constant j-rank d_j, then we have a map

$$\mathrm{Pt}(\mathcal{G}) \longrightarrow \mathrm{Gr}_{d_j}(M) \ ; \ \alpha \mapsto \mathrm{im}\, \alpha(t)^j_M.$$

This map between sets is well-behaved for infinitesimal group schemes:

Theorem 5.4.1 ([15]). *Let $\mathcal{G}$ be an infinitesimal group scheme. If M is a $\mathcal{G}$-module of constant j-rank d_j, then*

$$\mathrm{im}_{M,j} : \mathrm{Proj}(V(\mathcal{G})) \longrightarrow \mathrm{Gr}_{d_j}(M) \ ; \ [\alpha] \mapsto \mathrm{im}\, \alpha(u_{r-1})^j_M$$

is a morphism of projective varieties.

For a restricted Lie algebra $(\mathfrak{g}, [p])$, the foregoing result can be rephrased as follows: The nullcone $V(\mathfrak{g})$ of $\mathfrak{g}$ is a closed, conical subset of $\mathfrak{g}$, so that $\mathrm{Proj}(V(\mathfrak{g}))$ is a closed subset of $\mathrm{Proj}(\mathfrak{g}) = \mathbb{P}^{\dim_k \mathfrak{g} - 1}$. If M is a $U_0(\mathfrak{g})$-module of constant j-rank d_j, then

$$\mathrm{im}_{M,j} : \mathrm{Proj}(V(\mathfrak{g})) \longrightarrow \mathrm{Gr}_{d_j}(M) \ ; \ [x] \mapsto \mathrm{im}\, x^j_M.$$

Let us consider the case of an elementary abelian group, whose group algebra kE_r is isomorphic to the restricted enveloping algebra $U_0(\mathfrak{a}_r)$ of the strongly abelian r-dimensional restricted Lie algebra $\mathfrak{a}_r$. In that case, $\mathrm{Proj}(V(\mathfrak{a}_r)) = \mathrm{Proj}(\mathfrak{a}_r) = \mathbb{P}^{r-1}$, so that we are interested in morphisms

$$\mathbb{P}^{r-1} \longrightarrow \mathrm{Gr}_d(V).$$

Such maps were investigated by Hiroshi Tango in a series of articles. We only require the following:

Theorem 5.4.2 (Tango, [31]). *Let V be an n-dimensional vector space, $\varphi : \mathbb{P}^m \longrightarrow \mathrm{Gr}_d(V)$ be a morphism. If $n \leq m$, then φ is constant.*

This result provides some insight concerning the structure of modules of constant rank.

Theorem 5.4.3. *Let M be an E_r-module of constant rank.*

(1) *If $\dim_k \mathrm{Rad}(M) \le r-1$, then $M \in \mathrm{EIP}(E_r)$.*
(2) *If $\dim_k M \le r$, then M is trivial.*

Proof. (1) Suppose that $\dim_k \mathrm{Rad}(M) \le r-1$ and put $d := \mathrm{rk}^1(M)$. Theorem 5.4.1 provides a morphism

$$\mathrm{im}_{M,1} : \mathbb{P}^{r-1} \longrightarrow \mathrm{Gr}_d(M) \ ; \ [x] \mapsto \mathrm{im}\, x_M$$

of projective varieties. Since $\mathrm{im}\, x_M \subseteq \mathrm{Rad}(M)$ for all $x \in \mathbb{P}^{r-1}$, the morphism $\mathrm{im}_{M,1}$ factors through $\mathrm{Gr}_d(\mathrm{Rad}(M))$. As $\dim_k \mathrm{Rad}(M) \le r-1$, Theorem 5.4.2 shows that $\mathrm{im}_{M,1}$ is constant. Consequently, the E_r-module M has the equal images property.

(2) In view of (1), the module M has the equal images property. We consider the dual $U_0(\mathfrak{a}_r)$-module M^*, so that $x_{M^*} = -(x_M)^{\mathrm{tr}}$ for all $x \in \mathfrak{a}_r$. Consequently,

$$(*) \qquad \ker x_{M^*} = \{\lambda \in M^* \ ; \ \lambda \circ x_M = 0\},$$

so that M^* also has constant rank. Thus, M^* has the equal images property. In view of $(*)$, the module M has the equal kernels property. Consequently, $M \in \mathrm{EIP}(E_r) \cap \mathrm{EKP}(E_r)$. For $r \ge 2$, the latter subcategory can be shown to consist only of trivial modules. □

Example. Part (2) of Theorem 5.4.3 cannot be improved: Let $V_{r+1} := \bigoplus_{i=1}^{r+1} kv_i$ be an $(r{+}1)$-dimenional vector space on which the canonical generators act via

$$x_i.v_j = \delta_{i,j} v_{r+1} \quad \text{for } 1 \le i \le r \text{ and } 1 \le j \le r{+}1.$$

This module has constant Jordan type $\mathrm{Jt}(V_{r+1}) = (r-1)[1] \oplus [2]$.

Using Theorem 5.4.3 one can show that an $(r{+}1)$-dimensional E_r-module M of constant rank is either trivial or isomorphic to $\mathrm{Soc}_2(kE_r) \cong V_{r+1}$, or $kE_r/\mathrm{Rad}^2(kE_r)$.

One can generalize Theorem 5.4.3 to cover modules of constant ranks over arbitrary finite group schemes. This leads to:

Corollary 5.4.4. *Let G be a reductive group of semi-simple rank $\mathrm{rk}_{\mathrm{ss}}(G)$. If M is a G_r-module of constant rank such that $\dim_k M \le r\,\mathrm{rk}_{\mathrm{ss}}(G)$, then M is a direct sum of one-dimensional modules.*

6. Elements of Auslander-Reiten Theory

Throughout this section, Λ denotes an associative algebra over an algebraically closed field k. In general, one will not be able to classify indecomposable Λ-modules. Instead, one studies $\mathrm{mod}\,\Lambda$ via a directed graph, the so-called Auslander-Reiten quiver of Λ.

6.1. **Almost split sequences.** The vertices of the AR-quiver are the isoclasses of the indecomposable Λ-modules. The arrows arise via certain short exact sequences. Recall that an exact sequence

$$\mathfrak{E}_M : \qquad (0) \longrightarrow N \longrightarrow E_M \stackrel{\pi}{\longrightarrow} M \longrightarrow (0)$$

splits if and only if $\mathrm{Hom}_\Lambda(X,\pi) : \mathrm{Hom}_\Lambda(X,E_M) \longrightarrow \mathrm{Hom}_\Lambda(X,M)$ is surjective for every indecomposable Λ-module X.

Definition. A short exact sequence $\mathfrak{E}_M$ is *almost split*, provided

(a) M and N are indecomposable, and
(b) $\mathrm{Hom}_\Lambda(X,\pi)$ is surjective for every indecomposable Λ-module $X \not\cong M$, and
(c) $\mathrm{im}\,\mathrm{Hom}_\Lambda(M,\pi) = \mathrm{Rad}(\mathrm{End}_\Lambda(M))$.

Accordingly, a non-split short exact sequence $\mathfrak{E}_M$ with indecomposable extreme terms is almost split if and only if all non-isomorphisms $X \longrightarrow M$ of indecomposables factor through E_M. (As $\mathfrak{E}_M$ is non-split, isomorphisms $X \longrightarrow M$ cannot factor through E_M.)

Theorem 6.1.1 (Auslander-Reiten, [2]). *Let M be an indecomposable Λ-module.*

(1) *Suppose that M is non-projective.*
  (a) *There exists an almost split sequence $\mathfrak{E}_M$ terminating in M.*
  (b) *If $\mathfrak{E}'_M$ is another such sequence, then $\mathfrak{E}'_M \cong \mathfrak{E}_M$.*
  (c) *The initial term $\tau_\Lambda(M) := N$ of $\mathfrak{E}_M$ is uniquely determined (up to isomorphism).*
(2) *If M is non-injective, then there exists a unique almost split sequence with initial term M, whose terminal term is denoted $\tau_\Lambda^{-1}(M)$.*

Example. Consider the truncated polynomial ring $k[T]/(T^n)$. Then there are up to isomorphism exactly n indecomposable modules $[i] := k[T]/(T^i)$ $(1 \le i \le n)$. There are injections $[i] \hookrightarrow [i{+}1]$ and surjections $[i] \twoheadrightarrow [i{-}1]$. For $i \le n{-}1$ these are constituents of the almost split sequence

$$(0) \longrightarrow [i] \longrightarrow [i{-}1] \oplus [i{+}1] \longrightarrow [i] \longrightarrow (0)$$

terminating in $[i]$ $([0] := (0))$.

There is a recipe for the computation of the module $\tau_\Lambda(M)$: One picks a minimal projective presentation

$$(*) \qquad P^1 \stackrel{\alpha}{\longrightarrow} P^0 \longrightarrow M \longrightarrow (0).$$

Since the Nakayama functor $\mathcal{N}_\Lambda := \mathrm{Hom}_\Lambda(-,\Lambda)^*$ is right exact, we have an exact sequence

$$\mathcal{N}_\Lambda(P^1) \stackrel{\mathcal{N}_\Lambda(\alpha)}{\longrightarrow} \mathcal{N}_\Lambda(P^0) \longrightarrow \mathcal{N}_\Lambda(M) \longrightarrow (0),$$

and one defines

$$\tau_\Lambda(M) = \ker \mathcal{N}_\Lambda(\alpha).$$

Recall that an algebra Λ is called *hereditary*, provided every submodule of a projective module is projective. We compute τ_Λ for the two cases that are relevant for our purposes.

Lemma 6.1.2. *Let M be a non-projective, indecomposable Λ-module.*

(1) *If Λ is hereditary, then $\tau_\Lambda(M) \cong \mathrm{Ext}^1_\Lambda(M,\Lambda)^*$.*
(2) *If Λ is self-injective, then $\tau_\Lambda(M) \cong \mathcal{N}_\Lambda(\Omega^2_\Lambda(M)) \cong \Omega^2_\Lambda(\mathcal{N}_\Lambda(M))$.*

Proof. (1) Since Λ is hereditary, $(*)$ has in fact the form

$$(0) \longrightarrow P^1 \stackrel{\alpha}{\longrightarrow} P^0 \longrightarrow M \longrightarrow (0).$$

Application of $\mathrm{Hom}_\Lambda(-,\Lambda)$ yields an exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_\Lambda(P^0,\Lambda) \longrightarrow \mathrm{Hom}_\Lambda(P^1,\Lambda) \longrightarrow \mathrm{Ext}^1_\Lambda(M,\Lambda) \longrightarrow (0).$$

Dualizing this sequence, we arrive at:

$$(0) \longrightarrow \mathrm{Ext}^1_\Lambda(M,\Lambda)^* \longrightarrow \mathcal{N}_\Lambda(P^1) \stackrel{\mathcal{N}_\Lambda(\alpha)}{\longrightarrow} \mathcal{N}_\Lambda(P^0).$$

(2) We have an exact sequence

$$(0) \longrightarrow \Omega^2_\Lambda(M) \longrightarrow P^1 \stackrel{\alpha}{\longrightarrow} P^0 \longrightarrow M \longrightarrow (0).$$

As Λ is self-injective, $\mathcal{N}_\Lambda$ is exact, whence $\mathcal{N}_\Lambda(\Omega^2_\Lambda(M)) \cong \ker \mathcal{N}_\Lambda(\alpha)$. □

Remarks. (1) If $\Lambda = k[T]/(T^n)$ or $\Lambda = kG$ is the group algebra of a finite group G, then $\mathcal{N}_\Lambda \cong \mathrm{id}_{\mathrm{mod}\,\Lambda}$ and $\tau_\Lambda = \Omega^2_\Lambda$. (In these cases, Λ is a symmetric algebra.)

(2) Let Λ be a Hopf algebra. Then $\tau_\Lambda(M) \cong \Omega^2_\Lambda(M)^{(\nu)}$, where ν is the Nakayama automorphism of Λ.

Example. Let P be a non-simple projective-injective indecomposable Λ-module. Then $P/\mathrm{Soc}(P)$ is indecomposable, and

$$\mathfrak{E}_{P/\mathrm{Soc}(P)}: \qquad (0) \longrightarrow \mathrm{Rad}(P) \longrightarrow P \oplus (\mathrm{Rad}(P)/\mathrm{Soc}(P)) \longrightarrow P/\mathrm{Soc}(P) \longrightarrow (0)$$

is called the *standard almost split sequence involving* P.

6.2. **Auslander-Reiten components.** Using almost split sequences, we now define the following important invariant of the Morita equivalence class of a k-algebra Λ.

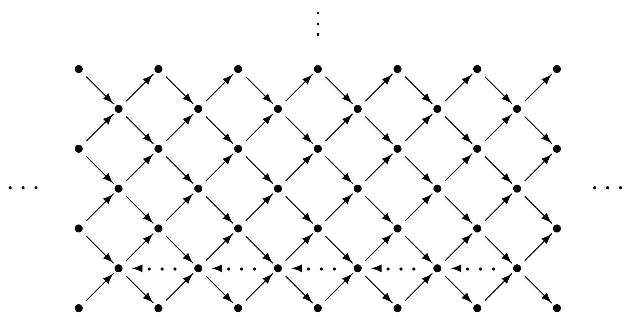
Definition. The Auslander-Reiten quiver $\Gamma(\Lambda)$ has the isoclasses of the indecomposable Λ modules as vertices. There is an arrow $X \to Y$, provided

- Y is not projective and X is a direct summand of the middle term E_Y of the almost split sequence

$$(0) \longrightarrow \tau_\Lambda(Y) \longrightarrow E_Y \longrightarrow Y \longrightarrow (0),$$

  terminating in Y, or
- Y is projective and X is a direct summand of $\mathrm{Rad}(Y)$.

This is actually only part of the story, because one should also incorporate the Auslander-Reiten translation as well as the multiplicity with which X occurs in E_Y. Leaving these technical issues aside, we consider components of the AR-quiver. These are connected in the sense that they cannot be decomposed into a disjoint union of subquivers that are "invariant" under τ_Λ. We shall see below that for algebras associated to Kronecker quivers and finite group schemes, components of type $\mathbb{Z}[A_\infty]$ occur most often. These components have the following shape, where the dotted arrow indicates the action of the AR-translation:



A module M belonging to the n-th row is said to have *quasi-length* $\mathrm{q}\ell(M) = n$. If $\mathrm{q}\ell(M) = 1$, then M is referred to as *quasi-simple*. This notion derives from the fact that the modules of such components afford a filtration whose factors are quasi-simple.

Definition. An indecomposable Λ-module M is said to be
- *pre-projective*, if there exists $n \geq 0$ such that $\tau_\Lambda^n(M)$ is projective,
- *pre-injective*, if there exists $n \geq 0$ such that $\tau_\Lambda^{-n}(M)$ is injective,
- *regular*, otherwise.

If Λ is self-injective, then every non-projective indecomposable Λ-module is regular.

Turning to $\mathrm{rep}(K_r)$, we consider the vector space kK_r with basis $\{e_0, e_1, \alpha_i \ ; \ 1 \leq i \leq r\}$ and multiplication

$$e_0^2 = e_0 \ \ ; \ \ e_1^2 = e_1 \ \ ; \ \ e_1\alpha_i = \alpha_i = \alpha_i e_0,$$

with all other products being zero. This makes kK_r into an associative k-algebra with identity element $1 := e_0+e_1$, the so-called *path algebra* of K_r. The algebra kK_r is hereditary, and the categories $\mathrm{mod}\, kK_r$ and $\mathrm{rep}(K_r)$ are equivalent.

Theorem 6.2.1. *The Auslander-Reiten quiver $\Gamma(K_r)$ of K_r ($r \geq 2$) has the following shape:*
1. *There exists one component containing all pre-projective objects.*
2. *There exists one component containg all pre-injective objects.*
3. *If $r = 2$, there are infinitely many components of type $\mathbb{Z}[A_\infty]/\langle\tau\rangle$, consisting of regular modules.*
4. *If $r \geq 3$, there are infinitely many components of type $\mathbb{Z}[A_\infty]$, consisting of regular modules (Ringel,* [27]*).*

6.3. K_r-**representations of constant rank.** Let $V = (V_0, V_1) \in \mathrm{rep}(K_r)$. We are going to define full subcategories of $\mathrm{CR}(K_r), \mathrm{EIP}(K_r)$ and $\mathrm{EKP}(K_r)$ by the following conditions:
- $V \in \mathrm{CR}(K_r) : \Leftrightarrow \exists\ d \in \mathbb{N}_0$ such that $\mathrm{rk}(V(\lambda)) = d$ for all $\lambda \in k^r \smallsetminus \{0\}$.
- $V \in \mathrm{EIP}(K_r) : \Leftrightarrow \mathrm{rk}(V(\lambda)) = \dim_k V_1$ for all $\lambda \in k^r \smallsetminus \{0\}$.
- $V \in \mathrm{EKP}(K_r) : \Leftrightarrow \mathrm{rk}(V(\lambda)) = \dim_k V_0$ for all $\lambda \in k^r \smallsetminus \{0\}$.

These are precisely those representations for which $F(V) \in \mathrm{mod}\, E_r$ belongs to the corresponding subcategory of $\mathrm{mod}\, E_r$. Note that $\mathrm{EIP}(K_r) \cup \mathrm{EKP}(K_r) \subseteq \mathrm{CR}(K_r)$.

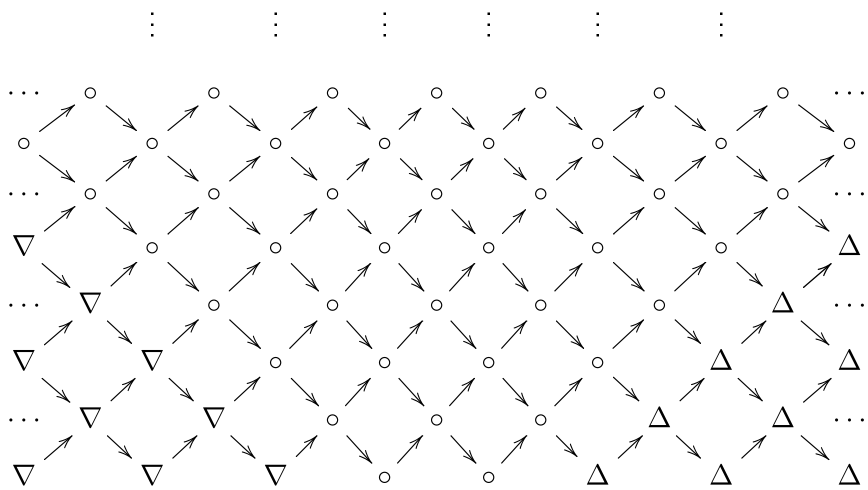
Lemma 6.3.1. *The category $\mathrm{CR}(K_r)$ is closed under direct summands.*

Proof. Given $V \in \operatorname{rep} K_r$, we consider $\operatorname{rk}(V) := \max\{\operatorname{rk}(V(\lambda))\ ;\ \lambda \in k^r \smallsetminus \{0\}\}$. Then $O_V := \{\lambda \in k^r \smallsetminus \{0\}\ ;\ \operatorname{rk}(V(\lambda)) = \operatorname{rk}(V)\}$ is a non-empty open subset of the irreducible variety $k^r \smallsetminus \{0\}$. If $V = V' \oplus V''$ belongs to $\operatorname{CR}(K_r)$, then $O_V = k^r \smallsetminus \{0\}$, while $O_{V'} \cap O_{V''} \neq \emptyset$ implies $\operatorname{rk}(V) = \operatorname{rk}(V') + \operatorname{rk}(V'')$. This readily gives $O_{V'} = k^r \smallsetminus \{0\} = O_{V''}$. □

In combination with our earlier observation, Lemma 6.3.1 implies $\operatorname{CR}(K_2) = \operatorname{EIP}(K_2) \cup \operatorname{EKP}(K_2)$. Moreover, the indecomposable objects in $\operatorname{EIP}(K_2)$ and $\operatorname{EKP}(K_2)$ make up the pre-injective and pre-projective components of the AR-quiver of K_2. Recent work by Julia Worch [34] shows that the situation is rather different for $r \geq 3$. Her results build on a homological characterization of the above subcategories that allow the application of other tools, such as the Auslander-Reiten formula.

Theorem 6.3.2 ([34]). *Let $r \geq 3$. If $\Theta \subseteq \Gamma(K_r)$ is a regular component, then $\Theta \cap \operatorname{EIP}(K_r)$ and $\Theta \cap \operatorname{EIP}(K_r)$ are disjoint cones.*

We obtain the following picture:



The left-hand cone consists of equal images modules while the right hand cone consists of modules having the equal kernels property. Given a regular component $\Theta \subseteq \Gamma(K_r)$, we are thus interested in understanding the central region. We let $w(\Theta)$ be the *width* of this region, that is, the minimal distance between two quasi-simple modules belonging to $\operatorname{EIP}(K_r)$ and $\operatorname{EKP}(K_r)$.

Proposition 6.3.3. *Let $r \geq 3$.*

(1) *For every $n \geq 0$, there exists a regular component $\Theta \subseteq \Gamma(K_r)$ such that $w(\Theta) \geq n$.*
(2) *If Θ is a regular component such that $w(\Theta) = 0$, then $\Theta \subseteq \operatorname{CR}(K_r)$.*

6.4. **The E_r-modules $M_{n,d}$ and $W_{n,d}$.** For $r = 2$, Carlson-Friedlander-Suslin [7] defined the equal images modules $W_{n,d}$ in an ad hoc fashion. In the general case, the following definition has proven to be useful. We consider the polynomial ring $R := k[X_1, \ldots, X_r]$ together with its augmentation ideal $I := (X_1, \ldots, X_r)$. Given $n \geq d$ and $d \leq p$, we put

$$M_{n,d} := I^{n-d}/I^n.$$

As $d \leq p$, the canonical R-action factors through $kE_r = k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$. Being a subquotient of the $\mathrm{GL}_r(k)$-module R, the module $M_{n,d}$ has constant Jordan type. The same holds for the dual E_r-module

$$W_{n,d} := M_{n,d}^*.$$

For $r = 2$, the latter modules are isomorphic to the modules considered in [7]. Note that

- $W_{n,d} \in \mathrm{EIP}(E_r)$ and $\mathrm{Jt}(W_{n,d}) = \bigoplus_{i=1}^d \binom{r+n-i-2+\delta_{i,d}}{n-i}[i]$,
- for $d \geq 2$ the modules $W_{n,d}$ are indecomposable with commutative endomorphism rings.

Since $W_{n,2}$ has Loewy length $\ell\ell(W_{n,2}) = 2$, it corresponds to a unique indecomposable object of $\mathrm{rep}(K_r)$, which we denote in the same fashion. The following result fails for $r = 2$:

Proposition 6.4.1. *Let $n, r \geq 3$. If $\Theta_n \subseteq \Gamma(K_r)$ is the AR-component containing $W_{n,2}$, then Θ_n is regular, $w(\Theta_n) = 0$, and $\Theta_n \subseteq \mathrm{CR}(K_r)$.*

7. The stable AR-quiver of a finite group scheme

In this section, we will see how support spaces can be used to obtain information on the AR-quiver of finite group schemes. Most of the methods actually apply in the context of fg-Hopf algebras, so that quantum groups at roots of unity can also be treated.

7.1. **Self-injective algebras.** The general set-up works in the context of self-injective algebras. Given such an algebra Λ, we recall that the Heller operator Ω_Λ defines a bijection on the set of non-projective indecomposable Λ-modules. The same holds for the Nakayama functor $\mathcal{N}_\Lambda$ and the AR-translation $\tau_\Lambda = \mathcal{N}_\Lambda \circ \Omega_\Lambda^2$.

Definition. The *stable Auslander-Reiten quiver* $\Gamma_s(\Lambda)$ of Λ is given by the following data:

- The vertices of $\Gamma_s(\Lambda)$ are the isomorphism classes of the non-projective indecomposable Λ-modules.
- There is an arrow $X \to Y$ if X is a non-projective direct summand of E_Y $(X|E_Y)$.
- The Auslander-Reiten translation $\tau_\Lambda : \Gamma_s(\Lambda) \longrightarrow \Gamma_s(\Lambda)$ is an automorphism.

If

$$\Lambda = \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots \oplus \mathcal{B}_r$$

is the block decomposition of Λ, then every block is a self-injective algebra. Moreover, $\mathrm{mod}\,\Lambda \cong \bigoplus_{i=1}^r \mathrm{mod}\,\mathcal{B}_i$, so that $\Gamma_s(\Lambda) = \bigsqcup_{i=1}^r \Gamma_s(\mathcal{B}_i)$.

Note that $\Gamma_s(\Lambda)$ is obtained from $\Gamma(\Lambda)$ by removing finitely many projective vertices along with all adjacent arrows. Since every projective indecomposable Λ-module is injective, the deletion only involves the standard almost split sequences

$$(0) \longrightarrow \mathrm{Rad}(P) \longrightarrow P \oplus (\mathrm{Rad}(P)/\mathrm{Soc}(P)) \longrightarrow P/\mathrm{Soc}(P) \longrightarrow (0).$$

Thus, removal of P has a major impact on a component Θ of $\Gamma(\Lambda)$ only if $\mathrm{Rad}(P)/\mathrm{Soc}(P) = (0)$, that is, if P is uniserial of length $\ell(P) = 2$. In that case, $\Theta = \Gamma(\mathcal{B})$ is the AR-quiver of a block $\mathcal{B} \subseteq \Lambda$ that is a Nakayama algebra of Loewy length 2.

The quiver $\Gamma_s(\Lambda)$ fulfills the axioms of a *stable representation quiver*. We require a few facts from the structure theory of such quivers. Let Δ be an arbitrary quiver. We define a new quiver $\mathbb{Z}[\Delta]$ with underlying set of vertices $\mathbb{Z} \times \Delta_v$. Each arrow $a \to b$ of Δ gives rise to arrows

$$(n, a) \to (n, b) \quad \text{and} \quad (n, b) \to (n+1, a).$$

There is an automorphism $\tau : \mathbb{Z}[\Delta] \longrightarrow \mathbb{Z}[\Delta]$, given by $\tau(n,a) := (n-1,a)$.

We have already seen an example for the tree A_∞: $\bullet \to \bullet \to \bullet \to \cdots$.

Theorem 7.1.1 (Riedtmann, [26]). *Let $\Theta \subseteq \Gamma_s(\Lambda)$ be a component.*

(1) *There exists a directed tree T_Θ and a subgroup $\Pi \subseteq \mathrm{Aut}(\mathbb{Z}[T_\Theta])$ such that $\Theta \cong \mathbb{Z}[T_\Theta]/\Pi$.*
(2) *The underlying graph $\bar{T}_\Theta$ is uniquely determined by Θ. It is called the tree class of Θ.*

One can show that $\mathrm{Aut}(\mathbb{Z}[A_\infty]) = \langle\tau\rangle$. The quivers $\mathbb{Z}[A_\infty]/\langle\tau^\ell\rangle$ $(\ell \geq 1)$ are tubes of rank ℓ. Modules belonging to components of this type are τ-periodic in the sense of the following

Definition. An indecomposable Λ-module M is called

(a) *periodic*, if there exists $n \in \mathbb{N}$ such that $\Omega^n_\Lambda(M) \cong M$, and
(b) *τ-periodic*, if there exists $n \in \mathbb{N}$ such that $\tau^n_\Lambda(M) \cong M$.

Remarks. (1) If M is periodic, then $\mathrm{cx}_\Lambda(M) = 1$. The converse does not hold in general.

(2) If Λ is *representation-finite*, that is, if Λ possesses only finitely many indecomposable modules, then every non-projective indecomposable module is periodic. We have already seen this for $\Lambda = k[T]/(T^n)$, where $\Omega^2_\Lambda([i]) \cong [i]$ for $1 \leq i \leq n-1$.

(3) If the Nakayama functor has finite order, then the two notions coincide. By general theory, this is the case for Hopf algebras.

Theorem 7.1.2 (Happel-Preiser-Ringel, [21]). *If a component $\Theta \subseteq \Gamma_s(\Lambda)$ contains a τ-periodic module, then $\Theta \cong \mathbb{Z}[A_\infty]/\langle\tau^\ell\rangle$ for some $\ell \geq 1$, or $\bar{T}_\Theta$ is a finite Dynkin diagram.*

In the latter case, the component Θ is finite and a theorem by Auslander implies that the elements of Θ are the non-projective indecomposable modules of a representation-finite block $\mathcal{B} \subseteq \Lambda$.

The structure of the "non-periodic" components was first investigated by Webb [33] for group algebras of finite groups. Using support varieties his results can be extended to finite group schemes [11]. All of this is done by exploiting the presence of certain functions $\Theta \longrightarrow \mathbb{N}$, which restrict the possible tree classes. For our purposes, it is most convenient to quote the following result:

Theorem 7.1.3 (Kerner-Zacharia, [23]). *Let $\Theta \subseteq \Gamma_s(\Lambda)$ be a component such that*

(a) *every $M \in \Theta$ has finite complexity, and*
(b) *Θ contains no τ-periodic module.*

Then $\Theta \cong \mathbb{Z}[\Delta]$, where $\Delta \in \mathcal{T} := \{A_\infty, D_\infty, A^\infty_\infty, \tilde{A}_{1,2}, \tilde{A}_n, \tilde{D}_n, \tilde{E}_r (6 \leq r \leq 8)\}$.

Remark. The foregoing result is not entirely accurate in that the isomorphism class of a component of type $\mathbb{Z}[\tilde{A}_n]$, whose tree class is A^∞_∞, depends on the number of clockwise oriented arrows. Since this will not play a role in the sequel, we will ignore this point.

7.2. **AR-components for fg-Hopf algebras.** Throughout, Λ is assumed to be an fg-Hopf algebra. In that case, we have support varieties at our disposal, which provide invariants for AR-components:

Lemma 7.2.1. *Let $\Theta \subseteq \Gamma_s(\Lambda)$ be a component. Then we have*
$$\mathcal{V}_\Lambda(M) = \mathcal{V}_\Lambda(N) \quad \forall\ M, N \in \Theta.$$

We can therefore speak of the *support variety $\mathcal{V}_\Lambda(\Theta)$ of Θ*. The following result tells us that usually components of type $\mathbb{Z}[A_\infty]$ occur most often.

Theorem 7.2.2. *Let $\Theta \subseteq \Gamma_s(\Lambda)$ be a component.*

(1) *If $\dim \mathcal{V}_\Lambda(\Theta) = 1$, then $\Theta \cong \mathbb{Z}[A_\infty]/\langle\tau^\ell\rangle$ or $\bar{T}_\Theta$ is a finite Dynkin diagram of type A, D, E.*
(2) *If $\dim \mathcal{V}_\Lambda(\Theta) = 2$, then $\Theta \cong \mathbb{Z}[\Delta]$, where $\Delta \in \mathcal{T}$.*
(3) *If $\dim \mathcal{V}_\Lambda(\Theta) \geq 3$, then $\Theta \cong \mathbb{Z}[A_\infty]$.*

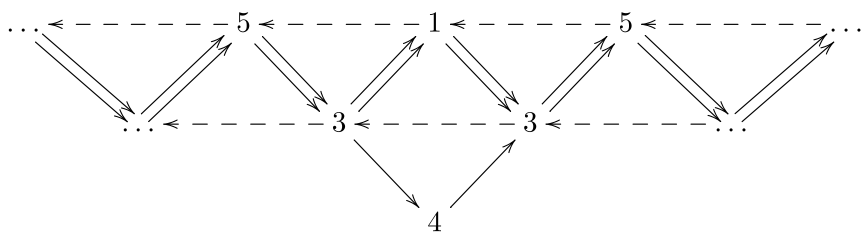
Remark. If $\Lambda = kG$ is the group algebra of a finite group, then a result by K. Erdmann [10] asserts that all AR-components belonging to a block of wild representation type have tree class A_∞. It is known that indecomposable modules of complexity ≥ 3 belong to wild blocks.

Since Theorem 7.2.2 only refers to the dimension of a variety, it is not clear what additional information varieties may provide. By way of illustration, let us consider the case of the restricted Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ of a connected reductive group G in case $\mathrm{char}(k) = p \geq 3$. By the above, a component $\Theta \subseteq \Gamma_s(U_0(\mathfrak{g}))$ gives rise to a rank variety $V(\mathfrak{g})_\Theta$. We begin with the example $\mathfrak{g} = \mathfrak{sl}(2)$.

Example. We consider $U_0(\mathfrak{sl}(2))$ for $p \geq 3$. Let $\{e, h, f\}$ be the standard basis of $\mathfrak{sl}(2)$, so that $\mathfrak{b} := kh \oplus ke$ is a Borel subalgebra. Given $\lambda \in \{0, \ldots, p-1\}$, we consider the *baby Verma module* $Z(\lambda) := U_0(\mathfrak{sl}(2)) \otimes_{U_0(\mathfrak{b})} k_\lambda$. Let $\Theta \subseteq \Gamma_s(U_0(\mathfrak{sl}(2))$ be a component.

(1) If $V(\mathfrak{sl}(2))_\Theta = kx$, then there exists $g \in \mathrm{SL}(2)$ and $\lambda \in \{0, \ldots, p-2\}$ such that $\mathrm{Ad}(g)(e) = x$ and $Z(\lambda)^{(g)} \in \Theta$. We have $\Theta \cong \mathbb{Z}[A_\infty]/\langle\tau\rangle$.
(2) If $V(\mathfrak{sl}(2))_\Theta = V(\mathfrak{sl}(2))$, then $\Theta \cong \mathbb{Z}[\tilde{A}_{1,2}]$ and there is $i \in \{0, \ldots, p-2\}$ with $L(i) \in \Theta$. Moreover, every $M \in \Theta$ is the restriction of a rational $\mathrm{SL}(2)$-module.

Note that rank varieties do not distinguish between the components containing a simple module. The tree $\tilde{A}_{1,2}$ is the Kronecker quiver $\bullet \rightrightarrows \bullet$. Below, we give the quiver $\mathbb{Z}[\tilde{A}_{1,2}]$ along with the lengths of its modules. The module of length 4 is projective and has to be deleted.



Corollary 7.2.3. *Let G be a connected reductive group of characteristic $p \geq 3$ with Lie algebra $\mathfrak{g}$. If $\Theta \subseteq \Gamma_s(U_0(\mathfrak{g}))$ is a component, then $\Theta \in \{\mathbb{Z}[A_\infty]/\langle\tau\rangle, \mathbb{Z}[A_\infty], \mathbb{Z}[D_\infty], \mathbb{Z}[A_\infty^\infty], \mathbb{Z}[\tilde{A}_{1,2}]\}$. Moreover, $\mathbb{Z}[\tilde{A}_{1,2}]$ occurs if and only if $\mathfrak{sl}(2)$ is a direct summand of $\mathfrak{g}$.*

Proof. Recall that G acts on $V(\mathfrak{g})$ via the adjoint representation.

Suppose that Θ is a component containing a simple $U_0(\mathfrak{g})$-module S. Since G is connected, we have $S^{(g)} \cong S$ for all $g \in G$. This implies that $V(\mathfrak{g})_\Theta = V(\mathfrak{g})_S$ is G-invariant.

(a) If $\dim V(\mathfrak{g})_\Theta = 1$, then $V(\mathfrak{g})_\Theta = kx$ is a p-unipotent ideal of $\mathfrak{g}$. By a result of Humphreys, $\mathfrak{g}$ does not afford such ideals. In particular, there are no finite components, and every representation-finite block $\mathcal{B} \subseteq U_0(\mathfrak{g})$ is simple (a full matrix ring over k).

(b) If $\dim \mathcal{V}(\mathfrak{g})_\Theta = 2$, then G acts on the one-dimensional projective variety $\mathrm{Proj}(V(\mathfrak{g})_\Theta)$. By Borel's Fixed Point Theorem, there is a point $x \in \mathrm{Proj}(V(\mathfrak{g}))$, whose centralizer is a parabolic subgroup P of codimension ≤ 1. By the above, the G-orbit is not a point, so that $\dim P = \dim G - 1$. This implies that $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathfrak{h}$, and that the component Θ is isomorphic to a component of $\Gamma_s(U_0(\mathfrak{sl}(2)))$ with two-dimensional support. Consequently, $\Theta \cong \mathbb{Z}[\tilde{A}_{1,2}]$.

If $\Theta \cong \mathbb{Z}[\Delta]$ is a component such that $\Delta \in \{\tilde{A}_n, \tilde{A}_{1,2}, \tilde{D}_n, \tilde{E}_r\}$, then $\dim V(\mathfrak{g})_\Theta = 2$, and general theory shows that we may assume without loss of generality that Θ contains a simple module. By (2), we have $\Theta \cong \mathbb{Z}[\tilde{A}_{1,2}]$, and the result thus follows from Theorem 7.2.2. □

Remarks. (1) By work of J. Külshammer [25], the foregoing result also holds for the small quantum group $u_\zeta(\mathfrak{g})$, associated to a complex semi-simple Lie algebra $\mathfrak{g}$ and a root of unity ζ of odd order.

(2) Using one-parameter subgroups one obtains an analogous result for the higher Frobenius kernels G_r of G.

The following result provides an example of how the stable AR-quiver can be used to obtain information on the structure of projective indecomposable modules.

Corollary 7.2.4. *Let G be a reductive group of characteristic $p \geq 3$ with Lie algebra $\mathfrak{g}$. Let S be a simple $U_0(\mathfrak{g})$-module, $\Theta_S \subseteq \Gamma_s(U_0(\mathfrak{g}))$ be the AR-component containing S.*

(1) *S is is the only simple module belonging to Θ_S.*

(2) *If $\Theta_S \cong \mathbb{Z}[A_\infty]$, then S is quasi-simple and $\mathrm{Rad}(P(S))/\mathrm{Soc}(P(S))$ is indecomposable.*

Proof. (2) Suppose we already know that S is quasi-simple. Then S has only one successor and since $\Omega_{U_0(\mathfrak{g})}$ induces an automorphism on $\Gamma_s(U_0(\mathfrak{g}))$ the same holds for $\Omega_{U_0(\mathfrak{g})}(S) \cong \mathrm{Rad}(P(S))$. The standard almost split sequence

$$(0) \longrightarrow \mathrm{Rad}(P(S)) \longrightarrow P(S) \oplus (\mathrm{Rad}(P(S))/\mathrm{Soc}(P(S))) \longrightarrow P(S)/\mathrm{Soc}(P(S)) \longrightarrow (0)$$

implies that these successors are the non-projective direct summands of $\mathrm{Rad}(P(S))/\mathrm{Soc}(P(S))$. Since this module has no non-zero projective direct summand, it is indecomposable. □

8. Invariants of AR-Components

Let $\mathcal{G}$ be a finite group scheme over k. So far, we have attached to each $\mathcal{G}$-module M its support variety $\mathcal{V}_\mathcal{G}(M)$ or its p-support $\mathrm{P}(\mathcal{G})_M$. The latter spaces are defined by those p-points $\alpha \in \mathrm{Pt}(\mathcal{G})$ for which $\alpha^*(M)$ is not projective. This is equivalent to saying that $\mathrm{Jt}(M, \alpha) \neq \frac{\dim_k M}{p}[p]$. We will now change our point of view and study the behavior of Jordan types within an AR-component. In this fashion, we obtain new invariants that often turn out to be finer than the ones given by supports.

8.1. **Locally split sequences.** We denote by $\Gamma_s(\mathcal{G})$ stable AR-quiver of $k\mathcal{G}$. In preparation for further invariants, we consider the question when pull-backs of almost split sequences along p-points split.

Given a non-projective indecomposable $\mathcal{G}$-module M, we let

$$\mathfrak{E}_M : \qquad (0) \longrightarrow \tau_{\mathcal{G}}(M) \longrightarrow E_M \stackrel{\pi_M}{\longrightarrow} M \longrightarrow (0)$$

be the almost split sequence terminating in M. For a p-point $\alpha \in \mathrm{Pt}(\mathcal{G})$ we consider the exact sequence

$$\alpha^*(\mathfrak{E}_M) : \qquad (0) \longrightarrow \alpha^*(\tau_{\mathcal{G}}(M)) \longrightarrow \alpha^*(E_M) \longrightarrow \alpha^*(M) \longrightarrow (0).$$

The p-point α gives rise to induced modules $\mathsf{ind}(\alpha, i) = k\mathcal{G}\otimes_{\mathfrak{A}_p}[i]$ $(1 \le i \le p)$, where $k\mathcal{G}$ is viewed as a right $\mathfrak{A}_p$-module via α. Note that $\mathsf{ind}(\alpha, p)$ is projective, while $\mathrm{P}(\mathcal{G})_{\mathsf{ind}(\alpha,i)} = \{[\alpha]\}$ for $1 \le i \le p-1$.

Recall our earlier notation $\alpha^*(M) \cong \bigoplus_{i=1}^p \alpha_i(M)[i]$.

Lemma 8.1.1. *Let M be a non-projective indecomposable $\mathcal{G}$-module. Given $\alpha \in \mathrm{Pt}(\mathcal{G})$, the following statements are equivalent:*

1. *The sequence $\alpha^*(\mathfrak{E}_M)$ does not split.*
2. *There exists $i \in \{1, \dots, p-1\}$ such that M is a direct summand of* $\mathsf{ind}(\alpha, i)$.

Proof. (1) $\Rightarrow$ (2). We consider the canonical map $\mu : k\mathcal{G} \otimes_{\mathfrak{A}_p} \alpha^*(M) \longrightarrow M$. Under the adjoint isomorphism

$$\mathrm{Hom}_{k\mathcal{G}}(k\mathcal{G}\otimes_{\mathfrak{A}_p}\alpha^*(M), M) \cong \mathrm{End}_{\mathfrak{A}_p}(\alpha^*(M))$$

the map μ corresponds to $\mathrm{id}_{\alpha^*(M)}$.

If μ is not split surjective, then the almost split property provides a factorization $\mu = \pi_M \circ \omega$. Adjointness then yields $\mathrm{id}_{\alpha^*(M)} = \alpha^*(\pi_M) \circ \lambda$ for some $\lambda \in \mathrm{Hom}_{\mathfrak{A}_p}(\alpha^*(M), \alpha^*(E_M))$, so that $\alpha^*(\mathfrak{E}_M)$ splits, a contradiction. As a result, μ is split surjective, and the assertion now follows from M being a non-projective indecomposable module along with $k\mathcal{G}\otimes_{\mathfrak{A}_p}\alpha^*(M) \cong \bigoplus_{i=1}^p \alpha_i(M)\,\mathsf{ind}(\alpha, i)$. □

Definition. Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component. We say that Θ is *locally split*, provided the exact sequence $\alpha^*(\mathfrak{E}_M)$ splits for every $M \in \Theta$ and every $\alpha \in \mathrm{Pt}(\mathcal{G})$.

The following result essentially says that the Jordan types of modules belonging to an AR-component can be predicted from the knowledge of its tree class and a few vertices.

Theorem 8.1.2 ([12]). *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component.*

1. *If $\dim P(\mathcal{G})_\Theta \ge 1$, then Θ is locally split.*
2. *If Θ is locally split, then there exist functions $f_\Theta : \Theta \longrightarrow \mathbb{N}$ and $d^\Theta : \mathrm{Pt}(\mathcal{G}) \longrightarrow \mathbb{N}_0^{p-1}$ such that*

$$\alpha_i(M) = d_i^\Theta(\alpha) f_\Theta(M) \qquad 1 \le i \le p-1$$

for all $M \in \Theta$ and $\alpha \in \mathrm{Pt}(\mathcal{G})$.

Proof. (1) Suppose there is $M \in \Theta$ and $\alpha \in \mathrm{Pt}(\mathcal{G})$ such that $\alpha^*(\mathfrak{E}_M)$ does not split. Lemma 8.1.1 implies that $\mathrm{P}(\mathcal{G})_M \subseteq \mathrm{P}(\mathcal{G})_{\mathsf{ind}(\alpha,i)} \subseteq \{[\alpha]\}$, a contradiction. □

The functions f_Θ are well-known and depend only on the tree class of Θ. Here are some examples:

(a) If $\bar{T}_\Theta \cong A_\infty$, then $f_\Theta(M) = \mathrm{q}\ell(M)$ for all $M \in \Theta$. In particular, if $\alpha_i(M) = 1$ for some $\alpha \in \mathrm{Pt}(\mathcal{G})$ and $i \in \{1, \dots, p-1\}$, then $M \in \Theta$ is quasi-simple.

(b) If $\bar{T}_\Theta \cong \tilde{A}_{1,2}$, A_∞^∞, then $f_\Theta \equiv 1$.

8.2. **Jordan types and AR-components.** We introduce two invariants of modules: Let $M \in \operatorname{mod} \mathcal{G}$. Given $\alpha \in \operatorname{Pt}(\mathcal{G})$, we denote by

$$\operatorname{supp}_\alpha(M) := \{i \in \{1, \dots, p-1\} \ ; \ \alpha_i(M) \neq 0\}$$

the *α-support* of M. Let $j \in \{1, \dots, p-1\}$. Then

$$\mathfrak{rk}^j(M) := \{\operatorname{rk}(\alpha(t)_M^j) \ ; \ \alpha \in \operatorname{Pt}(\mathcal{G})\}$$

denotes the *set of j-ranks of M*.

Corollary 8.2.1. *Suppose that $\Theta \subseteq \Gamma_s(\mathcal{G})$ is locally split.*

(1) $|\mathfrak{rk}^j(M)| = |\mathfrak{rk}^j(N)|$ *for all $M, N \in \Theta$ and $j \in \{1, \dots, p-1\}$.*
(2) $|\operatorname{Jt}(M)| = |\operatorname{Jt}(N)|$ *for all $M, N \in \Theta$.*
(3) *If $M, N \in \Theta$, then $\operatorname{supp}_\alpha(M) = \operatorname{supp}_\alpha(N)$ for all $\alpha \in \operatorname{Pt}(\mathcal{G})$.*

Proof. (1) Let $M \in \Theta$ and $\alpha \in \operatorname{Pt}(\mathcal{G})$. Theorem 8.1.2 implies

$$\operatorname{rk}(\alpha(t)_M^j) = \sum_{i=j+1}^{p-1} \alpha_i(M)(i-j) + \alpha_p(M)(p-j) = f_\Theta(M) \sum_{i=j+1}^{p-1} d_i^\Theta(\alpha)(i-j) + \alpha_p(M)(p-j),$$

while

$$\alpha_p(M) = \frac{1}{p}(\dim_k M - \sum_{i=1}^{p-1} \alpha_i(M) i) = \frac{1}{p} \dim_k M - \frac{1}{p} f_\Theta(M) \sum_{i=1}^{p-1} d_i^\Theta(\alpha) i.$$

Hence there is a map $\zeta_j : \operatorname{Pt}(\mathcal{G}) \longrightarrow \mathbb{Q}$ such that

$$\operatorname{rk}(\alpha(t)_M^j) = f_\Theta(M)\zeta_j(\alpha) + \frac{p-j}{p} \dim_k M \qquad \text{for all } \alpha \in \operatorname{Pt}(\mathcal{G})).$$

Consequently, $|\mathfrak{rk}^j(M)| = |\operatorname{im} \zeta_j|$. □

Let us take another look at the example $\mathfrak{sl}(2)$ and see what these new invariants tell us about AR-components:

Example. Suppose that $p \geq 3$. For $i \in \{0, \dots, p-2\}$ we denote by Θ_i the component of $\Gamma_s(\mathfrak{sl}(2))$ containing the simple module $L(i)$. Let $\Theta \subseteq \Gamma_s(\mathfrak{sl}(2))$ be a component.

(1) If $\dim \operatorname{P}(\mathfrak{sl}(2))_\Theta = 0$, then $\Theta \cong \mathbb{Z}[A_\infty]/\langle \tau \rangle$. There exists $i_\Theta \in \{1, \dots, \frac{p-1}{2}\}$ such that

$$\operatorname{Jt}(M) = \{\operatorname{q\ell}(M)[p], [i_\Theta] \oplus [p-i_\Theta] \oplus (\operatorname{q\ell}(M)-1)[p]\}$$

for every $M \in \Theta$. This only tells us to which block of $U_0(\mathfrak{sl}(2))$ the component Θ belongs.
(2) If $\dim \operatorname{P}(\mathfrak{sl}(2))_\Theta = 1$, then Θ is locally split and $\Theta \cong \Theta_i$ for some $i \in \{0, \dots, p-2\}$. Since $L(i) \in \Theta$ has constant Jordan type $\operatorname{Jt}(L(i)) = [i+1]$, it follows from Theorem 8.1.2 that $M \in \Theta$ has constant Jordan type $\operatorname{Jt}(M) = [i+1] \oplus \alpha_p(M)[p]$. Since $\operatorname{supp}_\alpha(L(i)) = \{i+1\}$, we see that $\Theta_i \neq \Theta_j$ whenever $i \neq j$.

Corollary 8.2.2 (cf. [18, 6]). *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component.*

(1) *If $\Theta \cap \operatorname{CR}_j(\mathcal{G}) \neq \emptyset$, then $\Theta \subseteq \operatorname{CR}_j(\mathcal{G})$.*
(2) *If $\Theta \cap \operatorname{CJT}(\mathcal{G}) \neq \emptyset$, then $\Theta \subseteq \operatorname{CJT}(\mathcal{G})$.*

Proof. (1) Suppose that Θ is locally split. Given $M \in \Theta \cap \mathrm{CR}_j(\mathcal{G})$, we have $|\mathfrak{rk}^j(M)| = 1$, so that Corollary 8.2.1(1) implies $\Theta \subseteq \mathrm{CR}_j(\mathcal{G})$. Alternatively, Theorem 8.1.2 and Lemma 4.1.1 yield $\dim \mathrm{P}(\mathcal{G}) = \dim \mathrm{P}(\mathcal{G})_\Theta = 0$, so that Theorem 3.1.2 implies that $\mathrm{P}(\mathcal{G})$ is a singleton. The assertion now follows from Theorem 4.1.2.

(2) This is a direct consequence of (1). □

Regarding endotrivial modules, the above methods can be used to show that indecomposable endotrivial modules over group schemes of complexity $\mathrm{cx}_\mathcal{G}(k) \geq 3$ form τ-orbits of quasi-simple modules:

Theorem 8.2.3. *Let $\Theta \subseteq \Gamma_s(\mathcal{G})$ be a component containing an indecomposable endotrivial module M_0. If $\dim \mathrm{P}(\mathcal{G}) \geq 1$, then the following statements hold:*

(1) *If $\bar{T}_\Theta \cong \tilde{A}_{12}$, A_∞^∞, then every $\mathcal{G}$-module $M \in \Theta$ is endotrivial.*
(2) *If $\Theta \cong \mathbb{Z}[A_\infty]$, then $\mathrm{q}\ell(M_0) = 1$ and $M \in \Theta$ is endotrivial if and only if $M \cong \tau_\mathcal{G}^n(M_0)$ for some $n \in \mathbb{Z}$.*

Proof. Thanks to Theorem 5.3.2, a $\mathcal{G}$-module M is endotrivial if and only if M has constant Jordan type with $\alpha_i(M) = 1$ or $\alpha_{p-1}(M) = 1$ for all $\alpha \in \mathrm{Pt}(\mathcal{G})$. Theorem 8.1.2 now implies that this happens for $M \in \Theta$ if and only if $f_\Theta(M) = 1$. Since $f_\Theta \equiv 1$ and $f_\Theta = \mathrm{q}\ell$ for (1) and (2), respectively, the assertions follow immediately. □

Example. Let $\widetilde{\mathfrak{sl}(2)} := \mathfrak{sl}(2) \oplus kv_0$, be the 4-dimensional restricted Lie algebra considered in Section 1. The $U_0(\widetilde{\mathfrak{sl}(2)})$-modules k and $\mathrm{Rad}(P(k))$ belong to components Θ_k and $\Theta_{\mathrm{Rad}(P(k))}$ of type $\mathbb{Z}[A_\infty^\infty]$. As k and $\mathrm{Rad}(P(k))$ are endotrivial, Theorem 8.2.3 shows that every $M \in \Theta_k \cup \Theta_{\mathrm{Rad}(P(k))}$ is endotrivial.

8.3. **AR-components containing equal images modules.** With regard to Auslander-Reiten theory, equal images modules turn out to behave rather differently. As we shall see, they usually occupy only the boundaries of their AR-components.

Many results on $\mathrm{EIP}(\mathcal{G})$ ultimately hinge on the presence of sufficiently large abelian unipotent subgroups.

Definition. Let $\mathcal{G}$ be a finite group scheme. Then

$$\mathrm{rk_{au}}(\mathcal{G}) := \max\{\mathrm{cx}_\mathcal{U}(k) \ ; \ \mathcal{U} \subseteq \mathcal{G} \text{ abelian unipotent subgroup scheme}\}$$

is called the *abelian unipotent rank* of $\mathcal{G}$.

We always have $\mathrm{rk_{au}}(\mathcal{G}) \leq \mathrm{cx}_\mathcal{G}(k)$, and the Alperin-Evens Theorem 1.1.3 implies that equality holds for finite groups. By contrast, the $(2n{+}1)$-dimensional Heisenberg algebra $\mathfrak{h}_n = \bigoplus_{i=1}^n kx_i \oplus \bigoplus_{i=1}^n ky_i \oplus kz$, with non-zero brackets $[x_i, y_j] = \delta_{ij} z$ and trivial p-map ($p \geq 3$) has complexity $\mathrm{cx}_{\mathfrak{h}_n}(k) = 2n{+}1$, while $\mathrm{rk_{au}}(\mathfrak{h}_n) = n{+}1$.

For groups satisfying $\mathrm{rk_{au}}(\mathcal{G}) \geq 2$, properties of equal images modules may be checked by looking at $\mathbb{Z}/(p) \times \mathbb{Z}/(p)$.

Proposition 8.3.1 (Carlson-Friedlander-Suslin, [7]). *Let $\mathcal{G}$ be a finite group scheme with $\mathrm{rk_{au}}(\mathcal{G}) \geq 2$. If $M \in \mathrm{EIP}(\mathcal{G})$, then there exist $d \in \{1, \dots, p\}$ and $a_1, \dots, a_d \geq 1$ such that $\mathrm{Jt}(M) = \bigoplus_{i=1}^d a_i[i]$.*

Given a finite group scheme $\mathcal{G}$ and $d \in \{1, \ldots, p\}$, we denote by $\mathrm{EIP}(\mathcal{G})_d$ the full subcategory of $\mathrm{EIP}(\mathcal{G})$, such that $\mathrm{Jt}(M) = \bigoplus_{i=1}^{d} a_i[i]$ for every $M \in \mathrm{EIP}(\mathcal{G})_d$. If $\mathcal{E}$ is a quasi-elementary abelian group, $\mathrm{EIP}(\mathcal{E})_d$ consists of the equal images modules of Loewy length $\leq d$.

Recall that Corollary 4.1.4 shows that even Heller shifts of modules of constant j-rank also have constant j-rank. This is usually not the case for equal images modules:

Proposition 8.3.2. *Let $\mathcal{G}$ be a finite group scheme with $\mathrm{rk}_{\mathrm{au}}(\mathcal{G}) \geq 2$, $M \in \mathrm{EIP}(\mathcal{G}) \smallsetminus \{(0)\}$ be an equal images module.*

1. *If $\mathrm{Jt}(\Omega^2_{\mathcal{G}}(M)) = \mathrm{Jt}(M)$, then $\Omega^2_{\mathcal{G}}(M) \notin \mathrm{EIP}(\mathcal{G})$.*
2. *If $M \in \mathrm{EIP}(\mathcal{G})_{p-2}$, then $\Omega^j_{\mathcal{G}}(M) \notin \mathrm{EIP}(\mathcal{G})$ for $j \in \{-2, 2\}$.*

Proof. (2) Suppose that $\bigoplus_{i=1}^{p-2} a_i[i]$ is the Jordan type of M. Since

$$\alpha^*(\Omega^2_{\mathcal{G}}(M)) \cong \Omega^2_{\mathfrak{A}_p}(\alpha^*(M)) \oplus (\mathrm{proj.}) \cong \alpha^*(M) \oplus (\mathrm{proj.}),$$

for every $\alpha \in \mathrm{Pt}(\mathcal{G})$, there exists $a_p \in \mathbb{N}_0$ with $\mathrm{Jt}(\Omega^2_{\mathcal{G}}(M)) = \bigoplus_{i=1}^{p-2} a_i[i] \oplus a_p[p]$. If $a_p \neq 0$, then Proposition 8.3.1 yields $\Omega^2_{\mathcal{G}}(M) \notin \mathrm{EIP}(\mathcal{G})$. Alternatively, $\mathrm{Jt}(\Omega^2_{\mathcal{G}}(M)) = \mathrm{Jt}(M)$, and we may apply (1). □

A component $\Theta \subseteq \Gamma_s(\mathcal{G})$ is *regular*, provided it is a component of $\Gamma(\mathcal{G})$. In particular, there are only finitely many non-regular AR-components.

Theorem 8.3.3 ([13]). *Let $\mathcal{G}$ be a finite group scheme of abelian unipotent rank $\mathrm{rk}_{\mathrm{au}}(\mathcal{G}) \geq 2$, $\Theta \cong \mathbb{Z}[A_\infty]$ be a regular component. Then the following statements hold:*

1. *Every $M \in \Theta \cap \mathrm{EIP}(\mathcal{G})_{p-1}$ is quasi-simple, and $\Theta \cap \mathrm{EIP}(\mathcal{G})_{p-1}$ is finite.*
2. *If $\Theta \cap \mathrm{EIP}(\mathcal{G})_{p-2} \neq \emptyset$, then $\Theta \cap \mathrm{EIP}(\mathcal{G})$ is finite and every $M \in \Theta \cap \mathrm{EIP}(\mathcal{G})$ is quasi-simple.*

Remark. If $\mathcal{U}$ is a unipotent group scheme with $\mathrm{rk}_{\mathrm{au}}(\mathcal{U}) \geq 2$, then every component $\Theta \subseteq \Gamma_s(\mathcal{U})$ with $\Theta \cap \mathrm{EIP}(\mathcal{U}) \neq \emptyset$ is regular and a theorem of Erdmann [10] ensures that $\Theta \cong \mathbb{Z}[A_\infty]$.

The example $\mathbb{Z}/(p) \times \mathbb{Z}/(p)$ shows that statements (1) and (2) do not hold for $M \in \mathrm{EIP}(\mathcal{G})_p$ and $\mathrm{EIP}(\mathcal{G})_{p-1}$, respectively. Given $M \in \Gamma_s(\mathcal{G})$, we denote by $(\rightarrow M)$ the set of all predecessors of M.

Proposition 8.3.4. *Let $G = \mathbb{Z}/(p) \times \mathbb{Z}/(p)$ for $p \geq 3$. Let $\Theta \subseteq \Gamma_s(G)$ be a component and suppose that $n \geq p$ is minimal subject to $W_{n,p} \in \Theta$. Then $n \leq 2p-1$ and*

1. *if $n \neq 2p-1$, then $\Theta \cap \mathrm{EIP}(G) = (\rightarrow W_{n,p})$ and $\{W_{n+mp,p}\ ; m \geq 0\} = \{M \in \mathrm{EIP}(G) \cap \Theta\ ;\ \mathrm{q}\ell(M) = 1\}$,*
2. *if $n = 2p-1$, then $\Theta \cap \mathrm{EIP}(G) = (\rightarrow W_{p-1,p-1})$ and $\{W_{(m+2)p-1,p}\ ;\ m \geq 0\} \cup \{W_{p-1,p-1}\} = \{M \in \mathrm{EIP}(G) \cap \Theta\ ;\ \mathrm{q}\ell(M) = 1\}$.*

This result formally parallels our findings concerning higher Kronecker quivers in that cones again enter the stage.

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