BLOCK REPRESENTATION TYPE OF FROBENIUS KERNELS OF SMOOTH GROUPS

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Abstract. Let $\mathcal{G}$ be a smooth algebraic group, defined over an algebraically closed field k of characteristic $p \geq 3$. We determine the structure of the representation-finite and tame blocks of the algebras of distributions associated to the Frobenius kernels of $\mathcal{G}$.

0. Introduction

According to a fundamental theorem by Drozd [10], the category of finite-dimensional algebras over an algebraically closed field may be subdivided into the disjoint classes of representation-finite, tame, and wild algebras (cf. [12] for the formal definitions). For algebras belonging to the former two classes a classification of the indecomposable modules is often feasible.[1] By contrast, the module category of a wild algebra, being "at least as complicated" as that of any other algebra, does not afford such a classification.

In light of Drozd's result one is interested in criteria allowing to recognize what class a given algebra belongs to. Two well-understood cases are those concerning hereditary algebras and blocks of group algebras of finite groups. Regarding the latter, the determination of the blocks of finite representation type [24] is based on work by Dade [8], Janusz [32] and Kupisch [34], while the classification of tame blocks is due to Erdmann [12]. Recent work [20, 21, 17, 18] provides similar results for the principal blocks of the algebras of distributions associated to infinitesimal group schemes. The absence of a suitable analogue of a defect theory of blocks has been a major obstacle, which has so far prevented the establishment of general results on arbitrary blocks comparable to those pertaining to finite groups.

Experience shows that the representations of the Frobenius kernels of smooth groups tend to be more tractable than those of arbitrary groups. One manifestation of this phenomenon is the classical Lie-Kolchin Theorem, which implies that the distribution algebras of Frobenius kernels of solvable smooth groups are basic. In this paper we study the ramifications of smoothness for the block representation type of distribution algebras. Early results in this direction were obtained by Pollack [39], who showed that the restricted enveloping algebras of the classical simple Lie algebras have infinite representation type. The case of reduced enveloping algebras of reductive Lie algebras was recently settled by Gordon and Premet [25].

Our main results are summarized in the following:

Theorem. *Let $\mathcal{G}$ be a smooth algebraic k-group of characteristic $p \geq 3$ with unipotent radical $\mathcal{U}$ of dimension $n = \dim \mathcal{U}$, $\mathcal{B} \subset \mathrm{Dist}(\mathcal{G}_r)$ a block of the algebra of distributions of its r-th Frobenius kernel $\mathcal{G}_r$. Then the following statements hold:*

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[1] This concededly vague formulation is supposed to alert the reader to examples like the group algebra of the quaternion group of order eight. The classification of the indecomposables of this seemingly innocuous tame algebra has turned out to be a formidable task. As will be seen at the end of Section 4, the situation is somewhat better in our context. In fact, the tame blocks we are going to deal with are special biserial and of domestic representation type.

(1) *If $\mathcal{B}$ is representation-finite, then $\mathcal{B}$ is simple, or $r = 1$ and $\mathcal{B}$ is Morita equivalent to $k[X]/(X^{p^n})$ or to the bound quiver algebra $k[\tilde{A}_{p-1}]/\operatorname{Rad}(k[\tilde{A}_{p-1}])^{p^n}$.*

(2) *If $\mathcal{B}$ is tame, then $\mathcal{B}$ is Morita equivalent to the trivial extension of the path algebra of the Kronecker quiver by its dual module.*

Here $\tilde{A}_{p-1}$ denotes the (clockwise) oriented circle with p vertices. It is well-known [11, 22, 42] that the trivial extension of the path algebra of the Kronecker quiver occurs as the bound quiver algebra of the non-simple blocks of the restricted enveloping algebra of $\mathfrak{sl}(2)$ (for $p \geq 3$).

In Section 1 we introduce our major tool, Alperin's notion of complexity, and review a few of its basic properties. Our approach is based on a comparison with the case of reductive groups, which is well understood (cf. [14]). Theorem 2.1 determines those blocks, whose images in reductive factor groups are semisimple. This result plays a crucial rôle in the subsequent two sections, where refinements of the above Theorem (see (3.1),(4.6)) are established.

The final Section provides a few consequences concerning the structure of the stable Auslander-Reiten components of distribution algebras.

1. Preliminaries

Throughout, we will be working over an algebraically closed field k of characteristic $p > 0$. Unless mentioned otherwise, all algebras and modules are assumed to be finite dimensional. Given an associative k-algebra Λ and a (left) Λ-module M, we denote by $\operatorname{cx}_\Lambda(M) \in \mathbb{N}_0 \cup \{\infty\}$ the *complexity* of M. By definition, $\operatorname{cx}_\Lambda(M)$ is computable from a minimal projective resolution $(P_i)_{i \geq 0}$ of M via

$$\operatorname{cx}_\Lambda(M) := \min\{n \geq 0 \ ; \ \exists\, \lambda > 0 \text{ such that } \dim_k P_i \leq \lambda i^{n-1} \quad \forall\, i \geq 1\}.$$

The reader is referred to [1] for basic properties and other characterizations of this notion.

We shall be concerned with the case where $\Lambda = \operatorname{Dist}(\mathcal{H})$ is the *algebra of distributions* of an infinitesimal k-group $\mathcal{H}$. Basic properties of group schemes and their Hopf algebras can be found in [31, 44], which we will use as general references regarding these matters. As is well-known, the module category $\operatorname{mod}\operatorname{Dist}(\mathcal{H})$ is equivalent to the module category of $\mathcal{H}$. We will thus use the terms "$\mathcal{H}$-module" and "$\operatorname{Dist}(\mathcal{H})$-module" interchangeably, and write $\operatorname{cx}_{\mathcal{H}}(M) := \operatorname{cx}_{\operatorname{Dist}(\mathcal{H})}(M)$ for any $\mathcal{H}$-module M. If $\mathcal{H}' \subset \mathcal{H}$ is a closed subgroup scheme, then the restriction of an $\mathcal{H}$-module M to the subalgebra $\operatorname{Dist}(\mathcal{H}') \subset \operatorname{Dist}(\mathcal{H})$ will occasionally be denoted $M|_{\mathcal{H}'}$.

Thanks to [36, (2.6)] the algebra $\operatorname{Dist}(\mathcal{H})$ is a free $\operatorname{Dist}(\mathcal{H}')$-module. In particular, we have

$$\operatorname{cx}_{\mathcal{H}'}(M) \leq \operatorname{cx}_{\mathcal{H}}(M)$$

for every $\mathcal{H}$-module M.

The trivial module with underlying vector space k and action given by the co-unit $\varepsilon : \operatorname{Dist}(\mathcal{H}) \longrightarrow k$ will be denoted k. Since the distribution algebra of the group scheme $\alpha_{p^r} := \operatorname{Spec}(k[T]/(T^{p^r}))$ is the truncated polynomial ring $k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$, the Künneth formula readily yields

$$\operatorname{cx}_{\alpha_{p^r}}(k) = r.$$

The infinitesimal group scheme α_{p^r} is the r-th Frobenius kernel of the additive group $\alpha_k := \operatorname{Spec}_k(k[T])$. Given any algebraic k-group $\mathcal{G} := \operatorname{Spec}_k(k[\mathcal{G}])$ with coordinate algebra $k[\mathcal{G}]$, we let $k[\mathcal{G}]^{(r)}$ be the k-algebra, whose ring structure is that of $k[\mathcal{G}]$, and whose structure of a k-space is given by $\alpha \cdot x := \alpha^{p^{-r}} x$. We consider the algebraic group $\mathcal{G}^{(r)} := \operatorname{Spec}_k(k[\mathcal{G}]^{(r)})$, and denote by $\mathcal{G}_r$ the kernel of the iterated *Frobenius homomorphism* $F^r : \mathcal{G} \longrightarrow \mathcal{G}^{(r)}$, whose comorphism is the map $k[\mathcal{G}]^{(r)} \longrightarrow k[\mathcal{G}] \ ; \ x \mapsto x^{p^r}$. Since the Frobenius kernels of $\mathcal{G}$ coincide with those of its connected component $\mathcal{G}^0$, we shall assume $\mathcal{G}$ to be connected whenever this is convenient.

If $\mathcal{G}$ is an infinitesimal group, then there exists $r \geq 1$ such that $\mathcal{G} = \mathcal{G}_r$, and the minimal such number is called the *height* $\mathrm{ht}(\mathcal{G})$ of $\mathcal{G}$.

Recall that an algebraic group $\mathcal{G}$ is *smooth* if its coordinate algebra $k[\mathcal{G}]$ is reduced. As noted in [31, (I.2.8)] the functor

$$\mathcal{G} \longrightarrow \mathcal{G}(k)$$

is an equivalence between reduced algebraic group schemes and ordinary algebraic groups. Moreover, the $\mathcal{G}$-modules correspond to the rational $\mathcal{G}(k)$-modules. We will be mainly concerned with smooth groups, yet the reader should be aware of the subtleties arising from the necessity of working with reduced schemes.

Our first subsidiary result provides a lower bound for the complexity of simple modules.

Lemma 1.1. *Let $\mathcal{G}$ be a smooth algebraic k-group, $\mathcal{U} \subset \mathcal{G}$ a smooth, unipotent subgroup of dimension ≥ 1. If M is a $\mathcal{G}_r$-module such that k is a direct summand of $M|_{\mathcal{U}_r}$, then $r \leq \mathrm{cx}_{\mathcal{U}_r}(k) \leq \mathrm{cx}_{\mathcal{G}_r}(M)$.*

Proof. Owing to [9, (IV,§2,3.8)] the group $\mathcal{U}$ contains a subgroup of type α_k. Consequently, $\alpha_{p^r} \subset \mathcal{U}_r$, and we obtain

$$r = \mathrm{cx}_{\alpha_{p^r}}(k) \leq \mathrm{cx}_{\mathcal{U}_r}(k) \leq \mathrm{cx}_{\mathcal{U}_r}(M) \leq \mathrm{cx}_{\mathcal{G}_r}(M),$$

with the second inequality following from $\mathrm{cx}_{\mathcal{U}_r}(N \oplus N') = \max\{\mathrm{cx}_{\mathcal{U}_r}(N), \mathrm{cx}_{\mathcal{U}_r}(N')\}$. □

Let $\mathcal{G}$ be a finite algebraic k-group. Then $\mathrm{ord}(\mathcal{G}) := \dim_k k[\mathcal{G}]$ is called the *order* of $\mathcal{G}$.

Lemma 1.2. *Let $\mathcal{G}$ be a smooth algebraic k-group, $\mathcal{N} \lhd \mathcal{G}$ a smooth, normal subgroup. Then the canonical quotient map $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$ induces ismorphisms $\mathcal{G}_r/\mathcal{N}_r \cong (\mathcal{G}/\mathcal{N})_r$ for every $r \geq 1$.*

Proof. From the exact sequence $e_k \longrightarrow \mathcal{N} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N} \longrightarrow e_k$ we obtain an exact sequence

$$e_k \longrightarrow \mathcal{N}_r \longrightarrow \mathcal{G}_r \longrightarrow (\mathcal{G}/\mathcal{N})_r,$$

which induces a closed embedding $\mathcal{G}_r/\mathcal{N}_r \overset{\iota}{\hookrightarrow} (\mathcal{G}/\mathcal{N})_r$ (cf. [44, (15.3)]). Since $\mathcal{N}$, $\mathcal{G}$ and $\mathcal{G}/\mathcal{N}$ are smooth, a twofold application of [31, (I.9.6(2))] yields

$$\mathrm{ord}((\mathcal{G}/\mathcal{N})_r) = p^{r \dim \mathcal{G}/\mathcal{N}} = \frac{p^{r \dim \mathcal{G}}}{p^{r \dim \mathcal{N}}} = \frac{\mathrm{ord}(\mathcal{G}_r)}{\mathrm{ord}(\mathcal{N}_r)} = \mathrm{ord}(\mathcal{G}_r/\mathcal{N}_r).$$

As a result, the embedding ι is the desired isomorphism. □

Remark. Our restriction to groups of odd characteristic results primarily from the failure Lemma 1.2 in case $\mathcal{N}$ is not smooth. We shall repeatedly consider the extension

$$e_k \longrightarrow \mu_2 \longrightarrow \mathrm{SL}(2) \longrightarrow \mathrm{PSL}(2) \longrightarrow e_k,$$

whose kernel $\mu_2 := \mathrm{Spec}_k(k[T]/(T^2 - 1))$ is not smooth at even characteristic.

2. A Morita Equivalence

According to [14] the representation-finite and tame blocks of the Frobenius kernels of smooth reductive groups are well understood. In particular, the main result stated in the introduction holds in this context. In this section we study the case, where the smooth algebraic k-group $\mathcal{G}$ possesses an abelian unipotent radical $\mathcal{U}(\mathcal{G})$ on which $\mathcal{G}$ acts via characters. In this situation, certain blocks turn out to be Morita equivalent to distribution algebras of trigonalizable group schemes.

Given a smooth algebraic group $\mathcal{G}$, and a Borel subgroup $B \subset \mathcal{G}$, we set $\ell_r(\mathcal{G}) := r \dim \mathcal{G}/B$ for every $r \geq 1$. As all Borel subgroups of $\mathcal{G}$ are conjugate, the number $\ell_r(\mathcal{G})$ does not depend on the choice of B. Thanks to [4, (11.14)] we have

$$\ell_r(\mathcal{G}) = \ell_r(\mathcal{G}/\mathcal{U}(\mathcal{G}))$$

for every $r \geq 1$. For $q \geq 0$ we denote by μ_{p^q} the q-th Frobenius kernel of the multiplicative group $\mu_k := \mathrm{Spec}_k(k[T]_T)$. Thus, we have $\mu_{p^q}(R) = \{x \in R\ ;\ x^{p^q} = 1\}$ for every commutative k-algebra R. If $\mathcal{G}$ is an algebraic group, then

$$X(\mathcal{G}) := \mathrm{Hom}(\mathcal{G}, \mu_k)$$

denotes the *character group* of $\mathcal{G}$. As noted in [31, (I.2.4)], $X(\mathcal{G})$ is just the group of group-like elements of the Hopf algebra $k[\mathcal{G}]$. If $\mathcal{G}$ is infinitesimal, this group also identifies with the group $\mathrm{Alg}_k(\mathrm{Dist}(\mathcal{G}), k)$ of algebra homomorphisms $\mathrm{Dist}(\mathcal{G}) \longrightarrow k$, whose product is the convolution

$$(\lambda * \mu)(h) := \sum_{(h)} \lambda(h_{(1)})\mu(h_{(2)}),$$

where $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$ is the coproduct of the element $h \in \mathrm{Dist}(\mathcal{G})$. Since $X(\mathcal{G})$ is commutative, we shall write the group law additively.

If H is a cocommutative Hopf algebra with co-unit ε and antipode η, we denote by $H^\dagger := \ker \varepsilon$ the augmentation ideal and consider the (left) *adjoint representation*

$$h.x := \sum_{(h)} h_{(1)} x \eta(h_{(2)}) \qquad \forall\ h, x \in H.$$

In this fashion H obtains the structure of an *H-module algebra*, that is, we have

(i) $h.(ab) = \sum_{(h)} (h_{(1)}.a)(h_{(2)}.b)$ for all $h, a, b \in H$, and

(ii) $h.1 := \varepsilon(h)1 \quad \forall\ h \in H$.

If $\mathcal{N} \lhd \mathcal{G}$ is a normal subgroup of the infinitesimal group $\mathcal{G}$, then $\mathrm{Dist}(\mathcal{N})$ and all powers of $\mathrm{Dist}(\mathcal{N})^\dagger$ are $\mathrm{Dist}(\mathcal{G})$-submodules of $\mathrm{Dist}(\mathcal{G})$ (cf. [31, (I.7.8)]). Moreover, we have [35, (3.4.2)]

$$\mathrm{Dist}(\mathcal{N})^\dagger \, \mathrm{Dist}(\mathcal{G}) = \mathrm{Dist}(\mathcal{G}) \, \mathrm{Dist}(\mathcal{N})^\dagger.$$

An algebraic group $\mathcal{M} = \mathrm{Spec}_k(k[\mathcal{M}])$ is called *diagonalizable* or *multiplicative*, if its coordinate ring is the group algebra $k[A]$ of some abelian group A. A smooth multiplicative group scheme $\mathcal{M}$ is referred to as a *torus*.

Theorem 2.1. *Let $\mathcal{G}$ be an infinitesimal group, $\mathcal{U} \lhd \mathcal{G}$ a unipotent normal subgroup such that*

(a) *the adjoint $\mathrm{Dist}(\mathcal{G})$-module $\mathrm{Dist}(\mathcal{U})_{\mathrm{ad}}$ is a direct sum of one-dimensional $\mathrm{Dist}(\mathcal{G})$-modules,*

(b) *the group $\mathcal{G}/\mathcal{U}$ is the r-th Frobenius kernel of a smooth reductive algebraic group $\mathcal{G}'$.*

Let $\mathcal{B} \subset \mathrm{Dist}(\mathcal{G})$ be a block such that the image $\mathcal{B}'$ of $\mathcal{B}$ under the canonical map $\mathrm{Dist}(\mathcal{G}) \stackrel{\pi}{\longrightarrow} \mathrm{Dist}(\mathcal{G}/\mathcal{U})$ contains a simple block of $\mathrm{Dist}(\mathcal{G}/\mathcal{U})$. Then there exists a multiplicative infinitesimal group $\mathcal{M}$ of height $\mathrm{ht}(\mathcal{M}) \leq r$ acting on $\mathcal{U}$ such that

$$\mathcal{B} \cong \mathrm{Mat}_{p^{\ell_r(\mathcal{G}')}}(\mathrm{Dist}(\mathcal{U} \rtimes \mathcal{M})).$$

Proof. We proceed in several steps.

(i) *The cohomology group $H^1(\mathcal{U},K)$ is a direct sum of one-dimensional* $\mathrm{Dist}(\mathcal{G})$*-modules.*
We let $\mathrm{Dist}(\mathcal{G})$ operate on $\mathrm{Dist}(\mathcal{U})$ via the adjoint representation. Thus, $\mathrm{Dist}(\mathcal{G})$ also acts on the standard resolution of the trivial $\mathrm{Dist}(\mathcal{G})$-module k of the supplemented algebra $(\mathrm{Dist}(\mathcal{G}),\varepsilon)$ (cf. [6, (IX.6,X.1,X.2)]). This action is readily seen to commute with the differentials, and thus gives rise to a $\mathrm{Dist}(\mathcal{G})$-structure on the cohomology groups $H^n(\mathcal{U},M|_{\mathcal{U}})$ for any $\mathcal{G}$-module M. Moreover, for every $h \in \mathrm{Dist}(\mathcal{G})$, the left multiplication by h defines a natural transformation of the universal δ-functor $(H^n(\mathcal{U},-))_{n\geq 0}$. By general theory, [6, (III.5.2)] or [46, §2.4], the action is therefore uniquely determined on $H^0(\mathcal{U},-) \cong -^{\mathcal{U}}$. Consequently, our operation coincides with the standard action on the derived functors of $\operatorname{mod}\mathcal{G} \longrightarrow \operatorname{mod}\mathcal{G}$; $M \mapsto M^{\mathcal{U}}$. In particular, we have an isomorphism

$$H^1(\mathcal{U},k) \cong \mathrm{Hom}_k(\mathrm{Dist}(\mathcal{U})^\dagger/(\mathrm{Dist}(\mathcal{U})^\dagger)^2,k),$$

of $\mathrm{Dist}(\mathcal{G})$-modules, where the right-hand side is the dual of a subquotient of the adjoint $\mathrm{Dist}(\mathcal{G})$-module $\mathrm{Dist}(\mathcal{U})_{\mathrm{ad}}$. Our asssertion now follows from condition (a). ◇

We let $R \subset X(\mathcal{G})$ be the set of characters occurring in the above decomposition of $H^1(\mathcal{U},k)$. The subgroup of $X(\mathcal{G})$ generated by R will be denoted $X_{\mathcal{U}}$. Given $\lambda \in X(\mathcal{G})$, we consider the map

$$\psi_\lambda : \mathrm{Dist}(\mathcal{G}) \longrightarrow \mathrm{Dist}(\mathcal{G}) \ \ ; \ \ h \mapsto \sum_{(h)} \lambda(h_{(1)})h_{(2)}.$$

Note that ψ_λ is an automorphism of the associative algebra $\mathrm{Dist}(\mathcal{G})$. Moreover, given any $\mathrm{Dist}(\mathcal{G})$-module M, the $\mathrm{Dist}(\mathcal{G})$-module $M \otimes_k k_\lambda$ is isomorphic to the $\mathrm{Dist}(\mathcal{G})$-module M with underlying k-space M and action defined via

$$h.m := \psi_\lambda(h)m$$

for every $h \in \mathrm{Dist}(\mathcal{G})$ and $m \in M$.

(ii) *Let S be a simple projective $\mathcal{B}'$-module. Then $\{S \otimes_k k_\lambda \ ; \ \lambda \in X_{\mathcal{U}}\}$ is the set of simple $\mathcal{B}$-modules. In particular, the block ideal $\mathcal{B}' \subset$* $\mathrm{Dist}(\mathcal{G}/\mathcal{U})$ *is semisimple.*
Let V and W be simple $\mathcal{G}$-modules. Since the unipotent normal subgroup $\mathcal{U} \lhd \mathcal{G}$ acts trivially on V and W, we have isomorphisms

$$H^n(\mathcal{U},W) \cong H^n(\mathcal{U},k) \otimes_k W$$

of $\mathcal{G}/\mathcal{U}$-modules. Accordingly, the spectral sequence given in [31, (I.6.6(2))] has the form

$$\mathrm{Ext}^m_{\mathcal{G}/\mathcal{U}}(V,H^n(\mathcal{U},k)\otimes_k W) \Rightarrow \mathrm{Ext}^{m+n}_{\mathcal{G}}(V,W).$$

If W is a projective $\mathrm{Dist}(\mathcal{G}/\mathcal{U})$-module, then $H^n(\mathcal{U},k)\otimes_k W$ is an injective $\mathrm{Dist}(\mathcal{G}/\mathcal{U})$-module. Thus, if V or W is a projective $\mathrm{Dist}(\mathcal{G}/\mathcal{U})$-module, then the spectral sequence collapses to isomorphisms

$$(*) \qquad \mathrm{Ext}^n_{\mathcal{G}}(V,W) \cong \mathrm{Hom}_{\mathcal{G}/\mathcal{U}}(V,H^n(\mathcal{U},k)\otimes_k W) \qquad \forall\, n \geq 0.$$

Thanks to (i) the $\mathcal{G}$-module $H^1(\mathcal{U},k)$ decomposes as

$$(**) \qquad H^1(\mathcal{U},k) \cong \bigoplus_{\lambda\in R} n_\lambda k_\lambda,$$

where $n_\lambda \in \mathbb{N}$ for every $\lambda \in R$.

Now consider the simple $\mathcal{B}$-module S. Let T be another simple $\mathcal{G}$-module. By $(*)$ and $(**)$ we have isomorphisms

$$(\dagger) \quad \mathrm{Ext}^1_{\mathcal{G}}(S,T) \cong \bigoplus_{\lambda\in R} n_\lambda \,\mathrm{Hom}_{\mathcal{G}/\mathcal{U}}(S,T\otimes_k k_\lambda) \ \text{ and } \ \mathrm{Ext}^1_{\mathcal{G}}(T,S) \cong \bigoplus_{\lambda\in R} n_\lambda \,\mathrm{Hom}_{\mathcal{G}/\mathcal{U}}(T,S\otimes_k k_\lambda).$$

This readily implies that every simple $\mathcal{B}$-module is of the form $S \otimes_k k_\lambda$ for some $\lambda \in X_{\mathcal{U}}$. In particular, the block ideal $\mathcal{B}'$ is semisimple.

Let $e_{\mathcal{B}} \in \mathrm{Dist}(\mathcal{G})$ be the block idempotent of $\mathcal{B}$, and consider $H := \{\lambda \in X(\mathcal{G}) \ ; \ e_{\mathcal{B}} \cdot (S \otimes_k k_\lambda) \neq (0)\}$. Since $S \otimes_k k_\lambda$ is the twist of S by the automorphism $\psi_\lambda : \mathrm{Dist}(\mathcal{G}) \longrightarrow \mathrm{Dist}(\mathcal{G})$, it readily follows that $H = \{\lambda \in X(\mathcal{G}) \ ; \ \psi_\lambda(e_{\mathcal{B}}) = e_{\mathcal{B}}\}$. Consequently, H is a subgroup of $X(\mathcal{G})$, which, owing to (†), contains R. This implies $X_{\mathcal{U}} \subset H$, so that $S \otimes_k k_\lambda$ belongs to $\mathcal{B}$ for every $\lambda \in X_{\mathcal{U}}$. $\diamond$

(iii) *The set* $\{S \otimes_k k_\lambda \ ; \ \lambda \in X_{\mathcal{U}}\}$ *is a complete set of representatives of the simple* $\mathcal{B}$*-modules. Each simple* $\mathcal{B}$*-module has dimension* $p^{\ell_r(\mathcal{G}')}$.

Let $T' \subset \mathcal{G}'$ be a maximal torus, $B' \supset T'$ a Borel subgroup of $\mathcal{G}'$. Following the notation of [31, (II.3)] we define

$$Z'_r(\gamma) := \mathrm{ind}_{B'_r}^{\mathcal{G}'_r} \gamma$$

for $\gamma \in X(T')$. By [31, (II.3.7(8))], we have $\dim_k Z'_r(\gamma) = p^{\ell_r(\mathcal{G}')}$. Owing to [31, (II.3.10)] the simple $\mathcal{G}'_r$-module S is of the form $L_r(\gamma) := \mathrm{Soc}_{\mathcal{G}_r}(Z'_r(\gamma))$ for some $\gamma \in X(T')$. Since $L_r(\gamma)$ is injective, and $Z'_r(\gamma)$ is indecomposable, we obtain $L_r(\gamma) = Z'_r(\gamma)$. In particular, S has the asserted dimension.

Now consider $\omega \in X(\mathcal{G}'_r) \cong X(\mathcal{G})$. The tensor identity [31, (I.3.6)] implies $S \otimes_k k_\omega \cong Z'_r(\gamma + \omega|_{T_r})$. Thus, if $S \cong S \otimes_k k_\omega$, then $L_r(\gamma) \cong L_r(\gamma + \omega|_{T_r})$, and [31, (II.3.10)] in conjunction with [31, (II.3.7)] yields $\omega|_{T_r} = 0$. It now follows from [31, (II.3.3)] that $\omega = 0$, as desired. $\diamond$

We now turn to the investigation of the principal indecomposable $\mathcal{B}$-modules. Given $i \in \mathbb{N}_0$, we put $m_i := \dim_k(\mathrm{Dist}(\mathcal{U})^\dagger)^i / (\mathrm{Dist}(\mathcal{U})^\dagger)^{i+1}$.

(iv) *Let* S *be a simple* $\mathcal{B}$*-module with projective cover* $P(S)$*, and let* J *denote the Jacobson radical of* $\mathrm{Dist}(\mathcal{G})$*. Then the semisimple module* $J^i P(S)/J^{i+1}P(S)$ *has* m_i *constituents.*

Let $\mathrm{Dist}(\mathcal{U})^\dagger$ be the augmentation ideal of the Hopf algebra $\mathrm{Dist}(\mathcal{U})$, so that $\mathrm{Dist}(\mathcal{G})\,\mathrm{Dist}(\mathcal{U})^\dagger = \ker \pi = \mathrm{Dist}(\mathcal{U})^\dagger\,\mathrm{Dist}(\mathcal{G})$ (cf. [35, (3.4.2)]). We denote by $\pi^* : \mathrm{mod}\,\mathcal{G}/\mathcal{U} \longrightarrow \mathrm{mod}\,\mathcal{G}$ the pull-pack functor induced by π. Since the functors

$$\mathrm{Hom}_{\mathcal{G}}(P(S), -) \circ \pi^* \quad \text{and} \quad \mathrm{Hom}_{\mathcal{G}/\mathcal{U}}(P(S)/\mathrm{Dist}(\mathcal{U})^\dagger P(S), -)$$

are equivalent on the module category $\mathrm{mod}\,\mathcal{G}/\mathcal{U}$, we see that $P(S)/\mathrm{Dist}(\mathcal{U})^\dagger P(S)$ is a principal indecomposable $\mathcal{B}'$-module with top S. As $\mathcal{B}'$ is semisimple, it follows that $\mathrm{Rad}(P(S)) = \mathrm{Dist}(\mathcal{U})^\dagger P(S)$. Consequently,

$$J^i P(S) = (\mathrm{Dist}(\mathcal{U})^\dagger)^i P(S) \qquad \forall\, i \geq 0.$$

Since the $\mathrm{Dist}(\mathcal{U})$-module $P(S)|_{\mathcal{U}}$ is projective and therefore free, there exists $\ell > 0$ such that

$$P(S)|_{\mathcal{U}} \cong \bigoplus_{i=1}^{\ell} \mathrm{Dist}(\mathcal{U}).$$

The vector space isomorphisms $S \cong P(S)/JP(S) \cong P(S)/\mathrm{Dist}(\mathcal{U})^\dagger P(S) \cong k^\ell$, then imply $\ell = \dim_k S$. Consequently,

$$J^i P(S)/J^{i+1}P(S) \cong (\mathrm{Dist}(\mathcal{U})^\dagger)^i P(S)/(\mathrm{Dist}(\mathcal{U})^\dagger)^{i+1} P(S) \cong \bigoplus_{i=1}^{\ell} (\mathrm{Dist}(\mathcal{U})^\dagger)^i/(\mathrm{Dist}(\mathcal{U})^\dagger)^{i+1}$$

has dimension $m_i \dim_k S$. Thanks to (ii) all simple $\mathcal{B}$-modules have dimension $\dim_k S$, so that our assertion follows. $\diamond$

Recall that $X_{\mathcal{U}} \subset X(\mathcal{G})$ is a subgroup of the character group of $\mathcal{G}$, which we identify with the group of group-like elements of the function algebra $k[\mathcal{G}]$. Accordingly, $X_{\mathcal{U}}$ is a commutative p-group, so that the affine group scheme $\mathcal{M} := \mathrm{Spec}_k(k[X_{\mathcal{U}}])$, defined by the group algebra $k[X_{\mathcal{U}}] \subset k[\mathcal{G}]$, is a multiplicative infinitesimal group (cf. [44, (2.2)]).

(v) *We have* $\mathcal{M} \cong \mathcal{G}/(\bigcap_{\lambda \in X_{\mathcal{U}}} \ker \lambda)$ *as well as* $\mathrm{ht}(\mathcal{M}) \leq r$.

The canonical embedding $\varphi^* : k[X_{\mathcal{U}}] \hookrightarrow k[\mathcal{G}]$ induces a quotient map $\varphi : \mathcal{G} \longrightarrow \mathcal{M}$, [44, (15.1)]. Given a commutative k-algebra R and an element $g \in \mathcal{G}(R) = \operatorname{Spec}_k(k[\mathcal{G}])(R)$, we have

$$g \in \ker \varphi \Leftrightarrow \varepsilon = \varphi(g) = g|_{k[X_U]} \Leftrightarrow g(\lambda) = 1 \ \ \forall \lambda \in X_{\mathcal{U}} \Leftrightarrow g \in \bigcap_{\lambda \in X_{\mathcal{U}}} \ker \lambda.$$

By general theory [44, (15.3,15.4)], the map φ therefore induces the asserted isomorphism.

Since $\mathcal{U}$ is unipotent, the canonical quotient map $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{U} \cong \mathcal{G}'_r$ induces an injection $X_{\mathcal{U}} \hookrightarrow \mathcal{G}'_r$. Consequently, $\mathcal{M}$ is also a factor group of $\mathcal{G}'_r$. The naturality of the Frobenius homomorphism now yields $\mathcal{M} = \mathcal{M}_r$, so that $\operatorname{ht}(\mathcal{M}) \leq r$. ◇

By assumption (a), we have a decomposition $\operatorname{Dist}(\mathcal{U}) = \bigoplus_{\lambda \in X(\mathcal{G})} \operatorname{Dist}(\mathcal{U})_\lambda$ of the $\operatorname{Dist}(\mathcal{G})$-module $\operatorname{Dist}(\mathcal{U})_{\mathrm{ad}}$ into $\operatorname{Dist}(\mathcal{G})$-eigenspaces.

(vi) *We have* $\operatorname{Dist}(\mathcal{U}) = \bigoplus_{\lambda \in X_{\mathcal{U}}} \operatorname{Dist}(\mathcal{U})_\lambda$.

Note that $\Lambda := \bigoplus_{\lambda \in X_{\mathcal{U}}} \operatorname{Dist}(\mathcal{U})_\lambda$ is a subalgebra of $\operatorname{Dist}(\mathcal{U})$, which, by definition of $X_{\mathcal{U}}$, satisfies $\Lambda + \operatorname{Rad}(\operatorname{Dist}(\mathcal{U}))^2 = \operatorname{Dist}(\mathcal{U})$. Consequently, [3, (1.2.8)] implies $\Lambda = \operatorname{Dist}(\mathcal{U})$, as desired. ◇

By virtue of (vi), the group $\bigcap_{\lambda \in X_{\mathcal{U}}} \ker \lambda$ acts trivially on $\operatorname{Dist}(\mathcal{U})$. Thus, it also acts trivially on $\mathcal{U}$, and the action of $\mathcal{G}$ on $\mathcal{U}$ factors through to $\mathcal{M}$. We may therefore consider the infinitesimal group

$$\mathcal{D} := \mathcal{U} \rtimes \mathcal{M}.$$

Let S be a simple $\mathcal{B}$-module with projective cover $P_{(0)}$. Then $P_\lambda := P_{(0)} \otimes_k k_{-\lambda}$ is the projective cover of the simple module $S_{(\lambda)} := S \otimes_k k_{-\lambda}$.

(vii) *Let* $P := \bigoplus_{\lambda \in X_{\mathcal{U}}} P_{(\lambda)}$. *Then* $\dim_k \operatorname{End}_{\mathcal{G}}(P) = \dim_k \operatorname{Dist}(\mathcal{D})$.

In view of (iii), $\dim_k \operatorname{Hom}_{\mathcal{G}}(P, M)$ counts the number of simple constituents of a semisimple $\mathcal{B}$-module M. By virtue of (iv) we therefore obtain

$$\dim_k \operatorname{Hom}_{\mathcal{G}}(P, J^iP/J^{i+1}P) = \sum_{\lambda \in X_{\mathcal{U}}} \dim_k \operatorname{Hom}_{\mathcal{G}}(P, J^iP_{(\lambda)}/J^{i+1}P_{(\lambda)}) = \operatorname{ord}(X_{\mathcal{U}})m_i$$

for every $i \geq 0$. Consequently,

$$\begin{aligned} \dim_k \operatorname{End}_{\mathcal{G}}(P) &= \sum_{i \geq 0} \dim_k \operatorname{Hom}_{\mathcal{G}}(P, J^iP/J^{i+1}P) = \operatorname{ord}(X_{\mathcal{U}})(\sum_{i \geq 0} m_i) \\ &= \operatorname{ord}(X_{\mathcal{U}}) \dim_k \operatorname{Dist}(\mathcal{U}) = \dim_k \operatorname{Dist}(\mathcal{D}), \end{aligned}$$

as asserted. ◇

(viii) *The canonical action of* $\operatorname{Dist}(\mathcal{U})$ *on* P *induces an isomorphism* $\operatorname{Dist}(\mathcal{D}) \cong \operatorname{End}_{\mathcal{G}}(P)$.

Let λ, γ be elements of $X_{\mathcal{U}}$. Owing to (vi) we have

$$hu = u\psi_\lambda(h)$$

for $h \in \operatorname{Dist}(\mathcal{G})$ and $u \in \operatorname{Dist}(\mathcal{U})_\lambda$. Consequently, the map

$$\omega_u^\gamma : P_{(\gamma)} \longrightarrow P_{(\gamma+\lambda)} \quad ; \quad m \mapsto u.m$$

is $\operatorname{Dist}(\mathcal{G})$-linear for every $u \in \operatorname{Dist}(\mathcal{U})_\lambda$. As a result, the left multiplication by $u \in \operatorname{Dist}(\mathcal{U})_\lambda$ induces an endomorphism $\omega_u : P \longrightarrow P$ of degree λ. Linear extension provides a homomorphism

$$\ell : \operatorname{Dist}(\mathcal{U}) \longrightarrow \operatorname{End}_{\mathcal{G}}(P) \quad ; \quad \sum_{\lambda \in X_{\mathcal{U}}} u_\lambda \mapsto \sum_{\lambda \in X_{\mathcal{U}}} \omega_{u_\lambda}$$

of $X_{\mathcal{U}}$-graded associative algebras.

Given $\gamma \in X(\mathcal{D}) \cong X(\mathcal{M}) = X_{\mathcal{U}}$, we denote by $e_\gamma \in \operatorname{End}_{\mathcal{G}}(P)$ the composite

$$P \longrightarrow P_{(\gamma)} \hookrightarrow P$$

of the canonical projection with the canonical injection. Since $\mathcal{M}$ is multiplicative, $\mathrm{Dist}(\mathcal{M})$ is a commutative, semisimple k-algebra (cf. [31, (I.2.11)]). We let $\mathrm{Dist}(\mathcal{M}) = \bigoplus_{\gamma \in X_{\mathcal{U}}} kv_\gamma$ be the block decomposition of $\mathrm{Dist}(\mathcal{M})$, with orthogonal primitive idempotents v_γ. As was observed in [21, (2.2)], we have

$$v_\gamma x = x v_{\gamma - \alpha}$$

for every element $x \in \mathrm{Dist}(\mathcal{U})_\alpha$. It follows that

$$\tilde{\ell} : \begin{cases} \mathrm{Dist}(\mathcal{D}) \longrightarrow \mathrm{End}_{\mathcal{G}}(P) \\ \sum_{\gamma \in X(\mathcal{M})} u_\gamma v_\gamma \mapsto \sum_{\gamma \in X(\mathcal{M})} \ell(u_\gamma) \circ e_\gamma \end{cases}$$

is a homomorphism of k-algebras. As $P_{(\lambda)}|_{\mathcal{U}} \cong \mathrm{Dist}(\mathcal{U}) \otimes_k S_{(\lambda)}$ is free, $\tilde{\ell}$ is injective and (vii) implies that $\tilde{\ell}$ is in fact an isomorphism. $\diamond$

According to (iii) we have an isomorphism

$$\mathcal{B} \cong p^{\ell_r(\mathcal{G}')} P$$

of $\mathrm{Dist}(\mathcal{G})$-modules. Observing [38, (3.4)] and (viii) we obtain isomorphisms

$$\mathcal{B} \cong \mathrm{End}_{\mathcal{G}}(\mathcal{B})^{\mathrm{op}} \cong \mathrm{Mat}_{p^{\ell_r(\mathcal{G}')}}(\mathrm{End}_{\mathcal{G}}(P))^{\mathrm{op}} \cong \mathrm{Mat}_{p^{\ell_r(\mathcal{G}')}}(\mathrm{End}_{\mathcal{G}}(P)^{\mathrm{op}}) \cong \mathrm{Mat}_{p^{\ell_r(\mathcal{G}')}}(\mathrm{Dist}(\mathcal{D})^{\mathrm{op}}).$$

Since the Hopf algebra $\mathrm{Dist}(\mathcal{D})$ is isomorphic to its opposite algebra, the asserted result follows. □

An algebraic k-group $\mathcal{G}$ is called *trigonalizable* if it is a closed subgroup of a group of invertible upper triangular matrices. In view of [9, (IV,§2,2.5,3.4)] this is equivalent to every simple $\mathcal{G}$-module being one-dimensional. In particular, an infinitesimal k-group $\mathcal{G}$ is trigonalizable if and only if its algebra $\mathrm{Dist}(\mathcal{G})$ of distributions is basic.

We let $V_{\mathcal{G}} : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}$ denote the *Verschiebung* of $\mathcal{G}$ (cf. [9, (IV,§3, n° 4; II, §7, n° 1)]). A commutative, unipotent infinitesimal k-group $\mathcal{U}$ is *V-uniserial* if the Verschiebung induces an exact sequence

$$\mathcal{U}^{(1)} \xrightarrow{V_{\mathcal{U}}} \mathcal{U} \longrightarrow \alpha_p \longrightarrow e_k.$$

The classification of the V-uniserial groups can be found in [16].

Corollary 2.2. *Let $\mathcal{G}$ be an infinitesimal k-group of characteristic $p \geq 3$, $\mathcal{U} \lhd \mathcal{G}$ a unipotent normal subgroup such that*

(a) *the adjoint* $\mathrm{Dist}(\mathcal{G})$*-module* $\mathrm{Dist}(\mathcal{U})_{\mathrm{ad}}$ *is a direct sum of one-dimensional* $\mathrm{Dist}(\mathcal{G})$*-modules,*

(b) *the group $\mathcal{G}/\mathcal{U}$ is the r-th Frobenius kernel of a smooth reductive algebraic group $\mathcal{G}'$.*

If $\mathcal{B} \subset \mathrm{Dist}(\mathcal{G})$ is a block such that the image $\mathcal{B}'$ of $\mathcal{B}$ under the canonical map $\mathrm{Dist}(\mathcal{G}) \xrightarrow{\pi} \mathrm{Dist}(\mathcal{G}/\mathcal{U})$ contains a simple block of $\mathrm{Dist}(\mathcal{G}/\mathcal{U})$, then $\mathcal{B}$ is either representation-finite or wild. Moreover, $\mathcal{B}$ is representation-finite if and only if $\mathcal{U}$ is V-uniserial.

Proof. According to (2.1) there exists an infinitesimal multiplicative group $\mathcal{M}$ such that the block $\mathcal{B}$ is Morita equivalent to $\mathrm{Dist}(\mathcal{U} \rtimes \mathcal{M})$. As $\mathcal{U} \rtimes \mathcal{M}$ is trigonalizable, we may apply [21, (2.6)] to see that the latter algebra is either representation-finite or wild. Moreover, an application of [20, (2.7)] shows that the former alternative occurs precisely when $\mathcal{U}$ is V-uniserial. □

We conclude this section by providing a criterion for the validity of the technical conditions of the foregoing results.

Proposition 2.3. *Let $\mathcal{G}$ be a smooth connected algebraic k-group with an abelian unipotent radical $\mathcal{U}$. If the automorphism group* $\mathrm{Aut}(\mathcal{U}_r)$ *is solvable, then the following statements hold:*

(a) *The adjoint* $\mathrm{Dist}(\mathcal{G}_r)$*-module* $\mathrm{Dist}(\mathcal{U}_r)_{\mathrm{ad}}$ *is a direct sum of one-dimensional* $\mathrm{Dist}(\mathcal{G}_r)$*-modules.*

(b) *The group $\mathcal{G}_r/\mathcal{U}_r$ is the r-th Frobenius kernel of a smooth reductive algebraic group $\mathcal{G}'$.*

Proof. We put $G := \mathcal{G}(k)$ and $U := \mathcal{U}(k)$, and let G act on $\mathcal{U}_r$ via conjugation. Thus, we have a homomorphism $\varrho : G \longrightarrow \mathrm{Aut}(\mathcal{U}_r)$ of algebraic groups. Since $\mathcal{U}$ is abelian, this homomorphism factors through to the reductive group $G' := G/U$. As $\mathrm{Aut}(\mathcal{U}_r)$ is solvable, it follows from [4, (14.2)] that the derived group (G', G') is contained in $\ker \varrho$. By the same token, there exists a torus T, a surjective homomorphism $\pi : G \longrightarrow T$, and a homomorphism $\omega : T \longrightarrow \mathrm{Aut}(\mathcal{U}_r)$ such that $\omega \circ \pi = \varrho$.

Recall that $\mathrm{Aut}(\mathcal{U}_r) \cong \mathrm{Aut}(k[\mathcal{U}_r])^{\mathrm{op}} \cong \mathrm{Aut}(\mathrm{Dist}(\mathcal{U}_r))$, where the latter group is the automorphism group of the Hopf algebra $\mathrm{Dist}(\mathcal{U}_r)$. The corresponding action of $\mathcal{G}$ on $\mathrm{Dist}(\mathcal{U}_r)$ is the adjoint representation of $\mathcal{G}$, which gives rise to the adjoint representation of the infinite-dimensional Hopf algebra $\mathrm{Dist}(\mathcal{G})$ on $\mathrm{Dist}(\mathcal{U}_r)$ (cf. [31, (I.7.7)]). By the above, the $\mathcal{G}$-module $\mathrm{Dist}(\mathcal{U}_r)$ is a direct sum of one-dimensional modules. Thus, the adjoint action $\mathrm{Ad} : \mathrm{Dist}(\mathcal{G}) \longrightarrow \mathrm{End}_k(\mathrm{Dist}(\mathcal{U}_r))$ enjoys the same property. Since $\mathrm{Dist}(\mathcal{G}_r) \subset \mathrm{Dist}(\mathcal{G})$ (cf. [31, (I.7.2(3))]), part (a) follows.

Property (b) is a direct consequence of (1.2). □

3. Representation-Finite Blocks

Let $\mathcal{G}$ be a smooth, connected algebraic group of characteristic $p \geq 3$, and assume that the principal block $\mathcal{B}_0(\mathcal{G}_r) \subset \mathrm{Dist}(\mathcal{G}_r)$ is representation-finite. A consecutive application of [20, (2.2)] and [28, (12.1)] shows that the group $\mathcal{G}$ is solvable. It is thus a semidirect product $\mathcal{G} = \mathcal{U} \rtimes \mathcal{T}$ of a unipotent, smooth normal group $\mathcal{U}$ and a torus $\mathcal{T}$ (cf. [4, (10.6)]). If $\mathcal{B}_0(\mathcal{G}_r)$ is not simple, then $\mathcal{U} \neq e_k$ and (1.1) yields $r = 1$. Now [20, (2.7)] implies that $\mathcal{G}_1 \cong (\mathcal{W}_n)_1 \rtimes \mathcal{T}_1$ is the semidirect product of the first Frobenius kernel of the group $\mathcal{W}_n$ of *Witt vectors of length n* (cf. [9, (V,§1,1.6)]) with a multiplicative group of height ≤ 1. Thanks to [20, (2.4)] there exists $s \in \{0, 1\}$ with $\mathcal{B}_0(\mathcal{G}_1) \cong \mathrm{Dist}((\mathcal{W}_n)_1 \rtimes \mu_{p^s})$. Our next result shows that, up to Morita equivalence, all representation-finite blocks of Frobenius kernels of smooth groups are of this form.

In the sequel we denote by $\mathrm{Lie}(\mathcal{G})$ the Lie algebra of the algebraic group $\mathcal{G}$. It is well-known that $\mathrm{Lie}(\mathcal{G})$ is a restricted Lie algebra, whose p-map we write $x \mapsto x^{[p]}$. We refer the reader to [43] for the general theory of restricted Lie algebras, and note that $\mathrm{Lie}(\mathcal{W}_n)$ is the nil-cyclic Lie algebra of dimension n, i.e., $\mathrm{Lie}(\mathcal{W}_n) \cong \bigoplus_{i=0}^{n-1} kx^{[p]^i}$, where $x^{[p]^n} = 0 \neq x^{[p]^{n-1}}$. According to [9, II,§7,n°4] the category of restricted Lie algebras is equivalent to the category of infinitesimal groups of height ≤ 1. In particular, given such an infinitesimal group $\mathcal{G}$ with Lie algebra $\mathfrak{g}$, the canonical homomorphism

$$U_0(\mathfrak{g}) \longrightarrow \mathrm{Dist}(\mathcal{G})$$

of Hopf algebras is an isomorphism. Here $U_0(\mathfrak{g})$ denotes the *restricted enveloping algebra* of $\mathfrak{g}$.

Theorem 3.1. *Let $\mathcal{G}$ be a smooth group with unipotent radical $\mathcal{U}$, $\mathcal{B} \subset \mathrm{Dist}(\mathcal{G}_r)$ a block of finite representation type.*

(1) *If $r \geq 2$, then $\mathcal{G}$ is reductive, and $\mathcal{B}$ is simple.*

(2) *If $r = 1$, then there exists $s \in \{0, 1\}$ such that $\mathcal{B} \cong \mathrm{Mat}_{p^{\ell_1(\mathcal{G})}}(\mathrm{Dist}((\mathcal{W}_n)_1 \rtimes \mu_{p^s}))$, where $n = \dim \mathcal{U}$.*

Proof. (1) Let S be a simple $\mathcal{B}$-module. Since $\mathcal{B}$ has finite representation type, the minimal projective resolution of S is periodic (cf. [26]), and we have $\mathrm{cx}_{\mathcal{G}_r}(S) \leq 1$. As S is simple, it follows that $S^{\mathcal{U}_r} = S$, and $S|_{\mathcal{U}_r}$ is a direct sum of copies of k. If $\dim \mathcal{U} \geq 1$, then (1.1) yields $r \leq 1$, a contradiction. Hence $\mathcal{G}$ is reductive, and the simplicity of $\mathcal{B}$ now follows from [14, (7.1)], which also holds for $p = 3$.

(2) If $\mathcal{U} = e_k$, then $\mathcal{G}$ is reductive and $\mathcal{B}$ is simple. Thus, we assume $\mathcal{U} \neq e_k$ and set $\mathcal{G}' := \mathcal{G}/\mathcal{U}$. Lemma 1.2 provides an exact sequence

$$e_k \longrightarrow \mathcal{U}_1 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{G}'_1 \longrightarrow e_k.$$

The arguments employed in part (1) now yield $\mathrm{cx}_{\mathcal{U}_1}(k) = 1$, and the two results of [16, (5.3)] imply $\mathcal{U}_1 \cong (\mathcal{W}_n)_1$ for some $n \geq 1$. As $\mathcal{U}$ is smooth, an application of [9, (II,§5,2.1)] gives

$$n = \dim_k \mathrm{Lie}((\mathcal{W}_n)_1) = \dim_k \mathrm{Lie}(\mathcal{U}_1) = \dim_k \mathrm{Lie}(\mathcal{U}) = \dim \mathcal{U}.$$

Let $\mathfrak{u} := \mathrm{Lie}(\mathcal{U})$, $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ and $\mathfrak{g}' := \mathrm{Lie}(\mathcal{G}')$. Passage to Lie algebras yields an exact sequence

$$(0) \longrightarrow \mathfrak{u} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}' \longrightarrow (0)$$

of restricted Lie algebras. Since $\mathfrak{u}$ is abelian, the Lie algebra $\mathfrak{g}'$ acts on $\mathfrak{u}$ via p-derivations. By definition, such a derivation $D : \mathfrak{u} \longrightarrow \mathfrak{u}$ satisfies

$$D(x^{[p]}) = (\mathrm{ad}\, x)^{p-1}(D(x))$$

for every element $x \in \mathfrak{u}$. Since $\mathfrak{u} = \mathrm{Lie}(\mathcal{W}_n)$ is the n-dimensional nil-cyclic restricted Lie algebra, it follows that every p-derivation vanishes on the $(n-1)$-dimensional subspace $\mathfrak{u}^{[p]}$. Direct computation shows that the restricted Lie algebra $\mathrm{Der}_p(\mathfrak{u})$ of p-derivations is the semidirect product of a one-dimensional torus and an $(n-1)$-dimensional abelian restricted Lie algebra. In particular, $\mathrm{Der}_p(\mathfrak{u})$ is solvable.

Let $T' \subset \mathcal{G}'$ be a maximal torus. By general theory [29, (26.2)], and our assumption $p \geq 3$, the Lie algebra $\mathfrak{g}'$ affords a root space decomposition

$$\mathfrak{g}' = \mathrm{Lie}(T') \oplus \bigoplus_{\alpha \in R} \mathfrak{g}'_\alpha,$$

such that every root space $\mathfrak{g}'_\alpha$ is contained in a subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2)$ (cf. also [29, (32.3)]). In particular, $[\mathfrak{s}_\alpha, \mathfrak{s}_\alpha] = \mathfrak{s}_\alpha$, so that $\mathfrak{s}_\alpha$ acts trivially on $\mathfrak{u}$. Consequently, the subspace $\bigoplus_{\alpha \in R} \mathfrak{g}'_\alpha$ centralizes $\mathfrak{u}$, implying that $\mathfrak{u}$ is a semisimple $\mathfrak{g}'$-module with one-dimensional constituents. Thus, $\mathfrak{u}$ is also a semisimple $U_0(\mathfrak{g})$-module, with $\mathfrak{u}^{[p]}$ being a trivial submodule of codimension 1. Hence there exists a (possibly trivial) character $\lambda : U_0(\mathfrak{g}) \longrightarrow k$ such that

$$\mathfrak{u} = \mathfrak{u}_\lambda + \mathfrak{u}_0.$$

Accordingly, the adjoint representation of $U_0(\mathfrak{g})$ on the subalgebra $U_0(\mathfrak{u})$ affords a decomposition

$$U_0(\mathfrak{u}) = \bigoplus_{i=0}^{p-1} U_0(\mathfrak{u})_{i\lambda}.$$

In the language of group schemes this translates into a decomposition

$$\mathrm{Dist}(\mathcal{U}_1) = \bigoplus_{i=0}^{p-1} \mathrm{Dist}(\mathcal{U}_1)_{i\lambda},$$

of $\mathrm{Dist}(\mathcal{U}_1)$ relative to the adjoint action of $\mathrm{Dist}(\mathcal{G}_1)$ on $\mathrm{Dist}(\mathcal{U}_1)$. Thus, (2.1) applies to $\mathcal{G}_1$ and $\mathcal{U}_1$, and there exists an infinitesimal multiplicative group $\mathcal{M}$ of height ≤ 1 such that

$$\mathcal{B} \cong \mathrm{Mat}_{p^{\ell_1(\mathcal{G})}}(\mathrm{Dist}((\mathcal{W}_n)_1 \rtimes \mathcal{M})$$

In particular, $\mathrm{Dist}((\mathcal{W}_n)_1 \rtimes \mathcal{M})$ is connected, and [20, (2.1),(2.4)] show that $\mathcal{M} \cong \mu_{p^s}$ for $s \in \{0, 1\}$. □

Remarks. (1) The bound quiver presentations of the basic algebras $\mathrm{Dist}((\mathcal{W}_n)_1 \rtimes \mu_{p^s})$ are well-known (cf. for instance [19, §4]). Thus, (3.1) shows that every representation-finite block of $\mathrm{Dist}(\mathcal{G}_r)$ is Morita equivalent to

$$k[X]/(X^{p^n}) \quad \text{or} \quad k[\tilde{A}_{p-1}]/\operatorname{Rad}(k[\tilde{A}_{p-1}])^{p^n}.$$

In particular, the representation-finite blocks of $\mathrm{Dist}(\mathcal{G}_r)$ are Nakayama algebras. According to [13, (3.2)] the representation-finite blocks of infinitesimal groups of height 1 are Nakayama algebras with one or p simple modules, yet the dimensions of the simples and the Loewy lengths of the blocks are in general not known.

(2) In view of [19, (4.2)] there is no general upper bound for the number of simple modules of a representation-finite block of the algebra of distributions associated to an arbitrary infinitesimal group scheme.

(3) The blocks described in Theorem 3.1 all occur. Given $n \in \mathbb{N}$, we consider the smooth groups $\mathcal{W}_n$ and $\mathcal{W}_n \rtimes \mu_k$, where in the latter case the elements of μ_k act via their Teichmüller representatives (cf. [9, (V,§1,1.5)]). If $m = \ell_1(\mathcal{H})$ for some reductive group $\mathcal{H}$, we consider the smooth groups $\mathcal{G}(0) := \mathcal{H} \times \mathcal{W}_n$ and $\mathcal{G}(1) := \mathcal{H} \times (\mathcal{W}_n \rtimes \mu_k)$. If $\mathcal{B}$ is the block of $\mathcal{H}_1$ belonging to the first Steinberg module, then, by the connectedness of $\mathrm{Dist}((\mathcal{W}_n)_1 \rtimes \mu_p)$ (cf. [20, (2.4)]), $\mathcal{B} \otimes_k \mathrm{Dist}((\mathcal{W}_n)_1)$ and $\mathcal{B} \otimes_k \mathrm{Dist}((\mathcal{W}_n)_1 \rtimes \mu_p)$ are blocks of $\mathrm{Dist}(\mathcal{G}(s))$ that are isomorphic to $\mathrm{Mat}_{p^m}(\mathrm{Dist}((\mathcal{W}_n)_1 \rtimes \mu_{p^s}))$.

4. Blocks of Tame Representation Type

In this section we investigate occurrence of tame blocks in Frobenius kernels of smooth algebraic groups of characteristic $p \geq 3$. Throughout, $\mathcal{G}$ denotes a smooth connected algebraic k-group with unipotent radical $\mathcal{U} = \mathcal{U}(\mathcal{G})$. We begin with the following subsidiary result:

Lemma 4.1. *Let $\mathcal{N} \subsetneq \mathcal{U}$ be a normal subgroup of $\mathcal{G}$ and put $\tilde{\mathcal{G}} := \mathcal{G}/\mathcal{N}$. If $r \geq 2$ and $\mathcal{B} \subset \mathrm{Dist}(\mathcal{G}_r)$ is a tame block, then $\mathrm{Dist}(\tilde{\mathcal{G}}_r)$ also possesses a tame block.*

Proof. Thanks to (1.2) we have a surjection $\pi : \mathrm{Dist}(\mathcal{G}_r) \longrightarrow \mathrm{Dist}(\tilde{\mathcal{G}}_r)$. We note that $\tilde{\mathcal{B}} := \pi(\mathcal{B})$ is a block ideal, which is either tame or representation-finite (cf. [12, (I.4.7)]).

If $\tilde{\mathcal{B}}$ possesses a representation-finite block, then (3.1(1)) yields the reductivity of $\tilde{\mathcal{G}}$. Thus, $\mathcal{U}/\mathcal{N} = e_k$, a contradiction. It follows that every block of $\tilde{\mathcal{B}}$ is tame, as desired. □

Lemma 4.2. *Suppose that* $\mathrm{Dist}(\mathcal{G}_r)$ *affords a tame block. If $\mathcal{U} \neq e_k$, then the following statements hold:*

(1) $r \leq 2$.

(2) *If $r = 2$, then there exists a factor group $\hat{\mathcal{G}}$ of $\mathcal{G}$ by a connected unipotent normal subgroup such that $\mathcal{U}(\hat{\mathcal{G}}) \cong \alpha_k$ and $\mathrm{Dist}(\hat{\mathcal{G}}_r)$ has a tame block.*

(3) *If $r = 1$, then there exists a factor group $\hat{\mathcal{G}}$ of $\mathcal{G}$ by a connected unipotent normal subgroup such that $\mathcal{U}(\hat{\mathcal{G}}) \cong \alpha_k,\ \alpha_k \times \alpha_k$.*

Proof. Let $\mathcal{N}_0 \subset \mathcal{U}$ be a maximal, proper, smooth, connected, normal subgroup of $\mathcal{U}$, and consider $\hat{\mathcal{G}} := \mathcal{G}/\mathcal{N}_0$. A consecutive application of [9, (IV,§4,3.4)] and [9, (IV,§4,3.13)] provides a normal subgroup $\mathcal{N} \cong \alpha_k^n$ of $\mathcal{U}(\hat{\mathcal{G}})$. By maximality of $\mathcal{N}_0$, we have $\mathcal{N} = \mathcal{U}(\hat{\mathcal{G}})$.

Let $\mathcal{B} \subset \mathrm{Dist}(\mathcal{G}_r)$ be a tame block, and denote by $\hat{\mathcal{B}}$ the image of $\mathcal{B}$ under the canonical surjection $\mathrm{Dist}(\mathcal{G}_r) \longrightarrow \mathrm{Dist}(\hat{\mathcal{G}}_r)$, see (1.2). Then $\hat{\mathcal{B}}$ is representation-finite or tame, so that a combination of [26] and [41, Thm.2] ensures the existence of simple $\mathrm{Dist}(\hat{\mathcal{G}}_r)$-module S of complexity ≤ 2. Owing to (1.1), we obtain

$$2 \geq \mathrm{cx}_{\hat{\mathcal{G}}_r}(S) \geq \mathrm{cx}_{(\alpha_{p^r})^n}(k) = nr,$$

proving that $r \leq 2$ and $n \leq 2$, as well as $n = 1$ for $r = 2$. In view of (4.1) the algebra $\mathrm{Dist}(\hat{\mathcal{G}}_2)$ has a tame block. $\square$

Lemma 4.3. *Suppose that* $\mathcal{H} = \mathrm{SL}(2), \mathrm{PSL}(2)$ *is an almost simple group of rank* 1. *If* $\mathcal{G}$ *is a smooth group such that* $\mathcal{G}/\mathcal{U} \cong \mathcal{H}$ *and* $\mathcal{U} \cong \alpha_k$*, then* $\mathcal{G} \cong \mathcal{H} \times \alpha_k$.

Proof. Since $\mathcal{U}$ is abelian, the conjugation action of $\mathcal{G}$ on $\mathcal{U}$ factors through to $\mathcal{H}$, so that we obtain a homomorphism $\mathcal{H}(k) \longrightarrow \mathrm{Aut}(\alpha_k)$. The latter group coincides with $\mu_k(k)$, implying that $\mathcal{H}(k)$ acts trivially on $\mathcal{U}$. As $\mathcal{H}$ is smooth, we may apply [31, (I.2.6(12))] to see that $\mathcal{U}$ is contained in the center of $\mathcal{G}$. Accordingly, we have a central extension

$$(*) \qquad e_k \longrightarrow \alpha_k \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow e_k.$$

Owing to [33, §8] or [9, (III,§6,2.4)] such an extension is a Hochschild extension (cf. [9, (II,§3,n°2)]), and hence corresponds to an element of the second Hochschild cohomology group $H^2(\mathcal{H}, k)$.

We consider the Lyndon-Hochschild-Serre spectral sequence (cf. [31, (I.6.6(3))])

$$(**) \qquad H^m(\mathrm{PSL}(2), H^n(\mathcal{C}_2, k)) \Rightarrow H^{m+n}(\mathrm{SL}(2), k),$$

where $\mathcal{C}_2 = \mathrm{Cent}(\mathrm{SL}(2))$ is the finite algebraic group of order 2. Thanks to [9, (II,§3,4.1)] the groups $H^n(\mathcal{C}_2, k)$ are isomorphic to the cohomology groups $H^n(\mathbb{Z}/(2), k)$, which, in view of $p \geq 3$, vanish for $n > 1$. Consequently, the spectral sequence $(**)$ degenerates to isomorphisms

$$H^n(\mathrm{SL}(2), k) \cong H^n(\mathrm{PSL}(2), k) \quad \forall\, n \geq 0.$$

By virtue of [33, (9.6)] the second cohomology group $H^2(\mathrm{SL}(2), k)$ of the simply connected group $\mathrm{SL}(2)$ is trivial. We thus have $H^2(\mathcal{H}, k) = (0)$, and the exact sequence $(*)$ splits. $\square$

Given a restricted Lie algebra $(\mathfrak{g}, [p])$, we recall that the *rank variety* $\mathcal{V}_{\mathfrak{g}}(M)$ of a $U_0(\mathfrak{g})$-module M is defined via

$$\mathcal{V}_{\mathfrak{g}}(M) := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0 \text{ and } M|_{U_0(kx)} \text{ is not projective}\} \cup \{0\}.$$

Thanks to work by Jantzen [30] and Friedlander-Parshall [23], we have

$$\mathrm{cx}_{U_0(\mathfrak{g})}(M) = \dim \mathcal{V}_{\mathfrak{g}}(M)$$

for every $U_0(\mathfrak{g})$-module M.

Proposition 4.4. *Suppose that* $\mathcal{B} \subset \mathrm{Dist}(\mathcal{G}_r)$ *is a tame block such that image* $\mathcal{B}' \subset \mathrm{Dist}((\mathcal{G}/\mathcal{U})_r)$ *of* $\mathcal{B}$ *under the canonical projection* $\mathrm{Dist}(\mathcal{G}_r) \longrightarrow \mathrm{Dist}((\mathcal{G}/\mathcal{U})_r)$ *is tame. Then* $\mathcal{G}$ *is reductive.*

Proof. Assume for a contradiction that $\mathcal{U} \neq e_k$ and put $\mathcal{G}' := \mathcal{G}/\mathcal{U}$. Thanks to (4.2) we have $r \leq 2$. By the same token, there exists a (smooth) factor group $\hat{\mathcal{G}}$ of $\mathcal{G}$ by a smooth normal subgroup $\mathcal{N} \subset \mathcal{U}$ such that

(a) $\mathcal{U}(\hat{\mathcal{G}}) \cong \alpha_k$ for $r = 2$, and

(b) $\mathcal{U}(\hat{\mathcal{G}}) \cong \alpha_k$, $\alpha_k \times \alpha_k$ for $r = 1$.

Thus, the canonical projection $\mathrm{Dist}(\mathcal{G}_r) \longrightarrow \mathrm{Dist}((\mathcal{G}/\mathcal{U})_r)$ factors through $\mathrm{Dist}(\mathcal{G}_r) \longrightarrow \mathrm{Dist}(\hat{\mathcal{G}}_r)$, so that a twofold application of [12, (I.4.7)] ensures the tameness of the image $\hat{\mathcal{B}} \subset \mathrm{Dist}(\hat{\mathcal{G}}_r)$ of $\mathcal{B}$. Accordingly, we may assume that $\mathcal{U} = \alpha_k$, or $r = 1$ and $\mathcal{U} = \alpha_k \times \alpha_k$.

Let $\mathcal{C}' \subset \mathcal{B}'$ be a tame block. Owing to [14, (7.1)] the group $\mathcal{G}'$ is an almost direct product

$$\mathcal{G}' = \mathcal{K}'\mathcal{H}'$$

with $\mathcal{K}' \cong \mathrm{SL}(2)$, $\mathrm{PSL}(2)$. Since $p \geq 3$, the intersection of the two factors is finite and reduced, so that passage to Frobenius kernels yields an isomorphism

$$\mathcal{G}'_r \cong \mathrm{SL}(2)_r \times \mathcal{H}'_r.$$

In particular, our block $\mathcal{C}'$ is of the form

$$\mathcal{C}' \cong \mathcal{C}'_1 \otimes_k \mathcal{C}'_2,$$

where $\mathcal{C}'_1$ is a tame block of $\mathrm{Dist}(\mathrm{SL}(2)_r)$ and $\mathcal{C}'_2$ is a simple block of $\mathrm{Dist}(\mathcal{H}'_r)$ (cf. [14, (7.1)]). Let S be a simple $\mathcal{C}'$-module. By general theory (cf. [7, §10.E]), there exist simple $\mathcal{C}'_i$-modules S_i with $S \cong S_1 \otimes_k S_2$. Moreover, we have $\mathrm{cx}_{\mathrm{SL}(2)_r}(S_1) = 2$.

Let $\mathcal{Q} \subset \mathcal{G}$ the inverse image of $\mathcal{K}'$. According to (1.2) the exact sequence

$$(*) \qquad e_k \longrightarrow \mathcal{U} \longrightarrow \mathcal{Q} \stackrel{\pi}{\longrightarrow} \mathcal{K}' \longrightarrow e_k$$

induces an extension

$$(**) \qquad e_k \longrightarrow \mathcal{U}_r \longrightarrow \mathcal{Q}_r \longrightarrow \mathrm{SL}(2)_r \longrightarrow e_k.$$

Since $S|_{\mathcal{K}'_r} \cong S_1^n$ for $n := \dim_k S_2$, we have

$$S|_{\mathcal{Q}_r} \cong S_1^n.$$

Rickard's Theorem [41, Thm.2] now implies

$$(\dagger) \qquad 2 \geq \mathrm{cx}_{\mathcal{G}_r}(S) \geq \mathrm{cx}_{\mathcal{Q}_r}(S) = \mathrm{cx}_{\mathcal{Q}_r}(S_1^n) = \mathrm{cx}_{\mathcal{Q}_r}(S_1).$$

(i) *We have* $\mathcal{U} \neq \alpha_k \times \alpha_k$.

The assumption $\mathcal{U} = \alpha_k \times \alpha_k$ implies $r = 1$. We may thus consider the restricted Lie algebras $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$, $\mathfrak{q} := \mathrm{Lie}(\mathcal{Q})$ and $\mathfrak{u} := \mathrm{Lie}(\mathcal{U})$. Owing to [9, (II,§7,n°4)] the sequence $(**)$ corresponds to an exact sequence

$$(***) \qquad (0) \longrightarrow \mathfrak{u} \longrightarrow \mathfrak{q} \stackrel{d\pi}{\longrightarrow} \mathfrak{sl}(2) \longrightarrow (0)$$

of restricted Lie algebras. Since $\mathfrak{u}$ is strongly abelian (that is, abelian with trivial p-map) an application of [27, (3.3)] shows that the equivalence classes of these extensions are given by the elements of the second restricted cohomology group $H^2_*(\mathfrak{sl}(2), \mathfrak{u}) := \mathrm{Ext}^2_{U_0(\mathfrak{sl}(2))}(k, \mathfrak{u})$.

If $\mathfrak{u}$ is the two-dimensional simple $\mathfrak{sl}(2)$-module, then the cohomology group $H^2_*(\mathfrak{sl}(2), \mathfrak{u})$ vanishes (cf. [22, p.45]), so that the above sequence splits. The assumption $p \geq 3$ in conjunction with Jacobson's formula for the p-map implies $\mathcal{V}_{\mathfrak{q}}(k) = \mathcal{V}_{\mathfrak{sl}(2)}(k) \times \mathfrak{u}$. Accordingly, the $U_0(\mathfrak{q})$-module S_1 has support variety $\mathcal{V}_{\mathfrak{q}}(S_1) = \mathcal{V}_{\mathfrak{sl}(2)}(k) \times \mathfrak{u}$, so that

$$\mathrm{cx}_{U_0(\mathfrak{q})}(S_1) = 4.$$

In view of $(\dagger)$ we have reached a contradiction.

Alternatively, $\mathfrak{u}$ is a self-extension of the trivial $\mathfrak{sl}(2)$-module. Since $\mathrm{Ext}^1_{U_0(\mathfrak{sl}(2))}(k, k) = (0)$, we see that $\mathfrak{u}$ is the two-dimensional trivial $\mathfrak{sl}(2)$-module. Consequently, $(***)$ is a central extension of $\mathfrak{sl}(2)$, and the arguments of [17, §1] provide a p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow \mathfrak{u}$ such that $\mathfrak{q}$ is isomorphic to the Lie algebra $\mathfrak{sl}(2)_\psi := \mathfrak{sl}(2) \oplus \mathfrak{u}$, whose bracket and p-map are given by

$$[(x,u),(y,v)] = ([x,y],0) \ \text{ and } \ (x,u)^{[p]} = (x^{[p]}, \psi(x)) \qquad x, y \in \mathfrak{sl}(2),\ u, v \in \mathfrak{u},$$

respectively.

Thanks to [18, (3.3)] (which obviously holds for arbitrary algebraic groups), the linear map $\psi : \mathfrak{sl}(2)^{(1)} \longrightarrow \mathfrak{u}$ is a homomorphism of $\mathcal{Q}$-modules. Here $\mathcal{Q}$ acts on $\mathfrak{sl}(2) = \mathrm{Lie}(\mathcal{K}')$ via the adjoint representation

$$q.d\pi(x) := d\pi(\mathrm{Ad}(q)(x)) \quad \forall\, q \in \mathcal{Q},\ x \in \mathfrak{sl}(2).$$

From the basic formula $d\pi(\mathrm{Ad}(q)(x)) = \mathrm{Ad}(\pi(q))(d\pi(x))$ (cf. [9, (II,§4,1.3)]) we conclude that $\mathcal{U}$ operates trivially on $\mathfrak{sl}(2)^{(1)}$, so that the Q-action induces the adjoint action of $\mathcal{K}'$ on $\mathfrak{sl}(2)^{(1)}$. Thus, $\mathfrak{sl}(2)^{(1)}$ is a simple $\mathcal{Q}$-module and since $\dim_k \mathfrak{u} = 2$, it follows that $\psi = 0$. The Künneth Formula now implies $\mathrm{cx}_{U_0(\mathfrak{q})}(S_1) = 4$, which contradicts (†). ◇

(ii) *We have* $\mathcal{U} \neq \alpha_k$.

If $\mathcal{U} = \alpha_k$, then we apply (4.3) to see that the extension (∗) splits. In particular, we have $\mathcal{Q}_r \cong \mathrm{SL}(2)_r \times \alpha_{p^r}$, and the Künneth formula implies

$$\mathrm{cx}_{\mathcal{Q}_r}(S_1) = \mathrm{cx}_{\mathrm{SL}(2)_r}(S_1) + \mathrm{cx}_{\alpha_{p^r}}(k) = 2 + r.$$

As this contradcts (†), we obtain $\mathcal{U} \neq \alpha_k$. ◇

Having ruled out the two possiblities $\mathcal{U} = \alpha_k, \alpha_k \times \alpha_k$, we conclude that $\mathcal{G}$ is a reductive group. □

Lemma 4.5. *Let* $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ *be the Lie algebra of a smooth reductive algebraic group* $\mathcal{G}$. *Suppose that* V *is two-dimensional* $\mathcal{G}$*-module, which is irreducible when considered a* $\mathfrak{g}$*-module. Then the following statements hold:*

(1) *There exists a decomposition* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ *of* $\mathfrak{g}$ *into* p*-ideals such that* $\mathfrak{n} \cong \mathfrak{sl}(2)$ *and* $\mathfrak{h} = \mathrm{Lie}(\mathcal{H})$ *is the Lie algebra of a smooth reductive group* $\mathcal{H}$.

(2) *There exists a character* $\lambda : U_0(\mathfrak{h}) \longrightarrow k$ *such that the* $U_0(\mathfrak{g})$*-module* $V \cong L(1) \otimes_k k_\lambda$ *is the outer tensor product of the standard* $\mathfrak{sl}(2)$*-module* $L(1)$ *and a one-dimensional* $\mathfrak{h}$*-module.*

(3) *We have* $H^2_*(\mathfrak{g}, V) = (0)$.

Proof. By general theory (cf. [4, (14.2)]), the connected reductive group

$$\mathcal{G} = \mathrm{Cent}(\mathcal{G})^0 \cdot \mathcal{G}_{(1)} \cdot \dots \cdot \mathcal{G}_{(n)}$$

is an almost direct product of its connected center (a torus) and normal, almost simple subgroups $\mathcal{G}_{(i)}$. Let $\varrho_i : \mathcal{G}_{(i)} \longrightarrow \mathrm{GL}(2)$ be the representation afforded by V. Since $\mathcal{G}_{(i)} = (\mathcal{G}_{(i)}, \mathcal{G}_{(i)})$ we actually have homomorphisms $\varrho_i : \mathcal{G}_{(i)} \longrightarrow \mathrm{SL}(2)$. If $\ker \varrho_i = \mathcal{G}_{(i)}$ for every $i \in \{1, \dots, n\}$, then V is a direct sum of two one-dimensional $\mathcal{G}$-modules, and the same applies to the $\mathfrak{g}$-module V. As this contradicts the irreducibility of V, there exists $i_0 \in \{1, \dots, n\}$ such that $\ker \varrho_{i_0} \neq \mathcal{G}_{(i_0)}$. Thus, $\ker \varrho_{i_0}$ is finite, so that

$$\dim \mathcal{G}_{(i_0)} = \dim \mathcal{G}_{(i_0)} / \ker \varrho_{i_0} \leq 3.$$

As $\mathcal{G}_{(i_0)}$ is almost simple, we obtain equality. Consequently, the p-ideal $\mathfrak{n} := \mathrm{Lie}(\mathcal{G}_{(i_0)})$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2)$, and $V|_{\mathfrak{n}}$ is the standard $\mathfrak{sl}(2)$-module $L(1)$. We set $\mathcal{H} = \mathrm{Cent}(\mathcal{G})^0 \prod_{i \neq i_0} \mathcal{G}_{(i)}$ and obtain a decomposition

$$\mathcal{G} = \mathcal{G}_{(i_0)} \mathcal{H}$$

of $\mathcal{G}$ as an almost direct product of two smooth reductive groups. Since $p \geq 3$, the center of $\mathcal{G}_{(i_0)}$ is étale, so $\mathfrak{g}$ decomposes into p-ideals

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h},$$

with $\mathfrak{h} := \mathrm{Lie}(\mathcal{H})$. Thanks to Schur's Lemma, the algebra $U_0(\mathfrak{h})$ acts on V via a character $\lambda : U_0(\mathfrak{h}) \longrightarrow k$, so that

$$V \cong L(1) \otimes_k k_\lambda$$

is an outer tensor product. The Künneth formula now yields

$$H^2_*(\mathfrak{g}, V) \cong \bigoplus_{i=0}^{2} (H^i_*(\mathfrak{sl}(2), L(1)) \otimes_k H^{2-i}_*(\mathfrak{h}, k_\lambda)) \cong H^1_*((\mathfrak{sl}(2), L(1)) \otimes_k H^1_*(\mathfrak{h}, k_\lambda),$$

as $H^i_*(\mathfrak{sl}(2), L(1)) = (0)$ for $i = 0, 2$. For $p \neq 3$, we also have $H^1_*(\mathfrak{sl}(2), L(1)) = (0)$. As this fails for $p = 3$, we consider the second term. Thanks to [27, (2.1)] $H^1_*(\mathfrak{h}, k_\lambda)$ is the subspace of the Chevalley-Eilenberg cohomology group $H^1(\mathfrak{h}, k_\lambda)$ which is represented by those Lie-1-cocycles satisfying $f(x^{[p]}) = x^{p-1}.f(x)$ for every $x \in \mathfrak{h}$. Let

$$\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{h}_\alpha$$

be the root space decomposition of $\mathfrak{h}$ relative to a maximal torus of $\mathcal{H}$. Given $\alpha \in R$, the p-subalgebra

$$\mathfrak{t} \oplus \mathfrak{h}_\alpha \oplus \mathfrak{h}_{-\alpha} \cong \mathfrak{t}' \oplus \mathfrak{sl}(2)$$

is a direct product of $\mathfrak{sl}(2)$ with a torus $\mathfrak{t}'$ (cf. [28, (26.2,32.3)]). Thus, if f is a Lie-1-cocycle defining an element of $H^1_*(\mathfrak{h}, k_\lambda)$, then its restriction to $\mathfrak{sl}(2) \subset \mathfrak{h}$ defines an element of $H^1_*(\mathfrak{sl}(2), k) = (0)$. Consequently, f vanishes on $\mathfrak{h}_\alpha$. We conclude that $f(\sum_{\alpha \in R} \mathfrak{h}_\alpha) = (0)$. Since the cohomology group $H^1_*(\mathfrak{t}, k_\lambda)$ vanishes, the foregoing arguments show that there exists $q \in k$ with $f|_{\mathfrak{t}} = q\lambda|_{\mathfrak{t}}$. Consequently,

$$f = q\lambda$$

is an inner derivation, and $H^1_*(\mathfrak{h}, k_\lambda) = (0)$. □

We denote by $T(\mathrm{Kr}) := \mathrm{Kr}^* \rtimes \mathrm{Kr}$ the trivial extension of the path algebra Kr of the Kronecker quiver $\Delta : \bullet \rightrightarrows \bullet$. It is well-known that $T(\mathrm{Kr})$ has a bound quiver presentation with quiver

$$0 \;\begin{array}{c} \xrightarrow{\alpha_0} \\ \xrightarrow{\beta_0} \\ \xleftarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array}\; 1,$$

and relations defining the ideal $J \subset k[\Delta]$ generated by

$$\{\beta_{i+1}\alpha_i - \alpha_{i+1}\beta_i, \quad \alpha_{i+1}\alpha_i, \quad \beta_{i+1}\beta_i \quad ; \quad i \in \mathbb{Z}/(2)\}.$$

In the early eighties Drozd [11], Fischer [22] and Rudakov [42] independently showed that $T(\mathrm{Kr})$ is the basic algebra of every non-simple block of $\mathrm{Dist}(\mathrm{SL}(2)_1) \cong U_0(\mathfrak{sl}(2))$.

We are now in a position to classify the tame blocks of our distribution algebra.

Theorem 4.6. *Let $\mathcal{G}$ be a smooth algebraic group of characteristic $p \geq 3$. If $\mathcal{B} \subset \mathrm{Dist}(\mathcal{G}_r)$ is a tame block, then $\mathcal{G}$ is reductive, and there exists a (tame) block $\mathcal{C} \subset \mathrm{Dist}(\mathrm{SL}(2)_1)$ such that*

$$\mathcal{B} \cong \mathrm{Mat}_{p^{\ell_r(\mathcal{G})-1}}(\mathcal{C}).$$

In particular, $\mathcal{B}$ is Morita equivalent to $T(\mathrm{Kr})$.

Remark. For arbitrary infinitesimal groups more complicated tame blocks can occur. These arise as Galois extensions of generalizations of $T(\mathrm{Kr})$ and include the trivial extensions of the radical square zero tame hereditary algebras of type $\tilde{A}_{2p^n-1}$ (cf. [18, (7.1)]).

Proof. We begin by showing that $\mathcal{G}$ is reductive. Let $\mathcal{U} \subset \mathcal{G}$ be the unipotent radical, and put $\mathcal{G}' := \mathcal{G}/\mathcal{U}$. As before, we let $\mathcal{B}' \subset \mathrm{Dist}(\mathcal{G}'_r)$ denote the image of $\mathcal{B}$ under the canonical map $\mathrm{Dist}(\mathcal{G}_r) \longrightarrow \mathrm{Dist}(\mathcal{G}'_r)$. As observed earlier, the block ideal $\mathcal{B}'$ is of tame or finite representation type. In the former case the reductivity of $\mathcal{G}$ follows from (4.4). Hence we may assume that $\mathcal{B}'$ is representation-finite. Thanks to [14, (7.1)] we conclude that $\mathcal{B}'$ is in fact semisimple.

Assume that $\mathcal{U} \neq e_k$. By (4.2) we have $r \leq 2$, and if $r = 2$, then there exists a factor group $\hat{\mathcal{G}}$ of $\mathcal{G}$ with $\mathcal{U}(\hat{\mathcal{G}}) \cong \alpha_k$ and such that the image $\hat{\mathcal{B}} \subset \mathrm{Dist}(\hat{\mathcal{G}}_r)$ is tame. As $\mathrm{Aut}(\alpha_{p^2})$ is solvable (2.3) ensures the applicability of (2.2). Accordingly, $\hat{\mathcal{B}}$ is not tame, a contradiction. We therefore have $r = 1$.

We set $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$, $\mathfrak{g}' := \mathrm{Lie}(\mathcal{G}')$, $\mathfrak{u} := \mathrm{Lie}(\mathcal{U})$, and note that $\mathfrak{n} := [\mathfrak{u}, \mathfrak{u}] + \langle \mathfrak{u}^{[p]} \rangle$ is a p-ideal of $\mathfrak{g}$. We consider the restricted Lie algebras $\mathfrak{m} := \mathfrak{u}/\mathfrak{n}$ and $\mathfrak{h} := \mathfrak{g}/\mathfrak{n}$ as well as the exact sequence

$$(*) \qquad (0) \longrightarrow \mathfrak{m} \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g}' \longrightarrow (0)$$

of restricted Lie algebras. Since $\mathfrak{m}$ is strongly abelian, [27, (2.1)] implies $\mathcal{V}_{\mathfrak{m}}(k) = \mathfrak{m} \cong H^1_*(\mathfrak{u}, k)$. Moreover, the arguments of (1.1) now yield $\dim_k \mathfrak{m} \le 2$.

If $\dim_k \mathfrak{m} = 1$, then $U_0(\mathfrak{u})$ is a local Nakayama algebra, so that $\mathfrak{u}$ is a nil-cyclic restricted Lie algebra (cf. [13, (4.3)]). As noted earlier, the conditions of (2.2) apply to $\mathfrak{u}$ and $\mathfrak{g}'$, so that (2.2) yields a contradiction.

Suppose that $\dim_k \mathfrak{m} = 2$. Since $\mathfrak{n}$ is a $\mathcal{G}$-submodule of $\mathfrak{u}$, the group $\mathcal{G}$ also acts on $\mathfrak{m}$. If $\mathfrak{m}$ is an irreducible $\mathfrak{g}'$-module, then it is also an irreducible $\mathfrak{g}$-module, so that $\mathcal{G}$ operates irreducibly on $\mathfrak{m}$. Thus, $\mathcal{U}$ acts trivially on $\mathfrak{m}$, and the $\mathfrak{g}'$-action coincides with the differential of the $\mathcal{G}'$-action. Thanks to (4.5(1),(2)) we have a decomposition

$$\mathfrak{g}' = \mathfrak{p} \oplus \mathfrak{q},$$

with $\mathfrak{p} \cong \mathfrak{sl}(2)$. Moreover, the $\mathfrak{g}'$-module $\mathfrak{m}$ is the outer tensor product $\mathfrak{m} \cong L(1) \otimes_k k_\lambda$ for a suitably chosen character $\lambda : U_0(\mathfrak{q}) \longrightarrow k$. Now let S, T be simple $\mathcal{B}$-modules. Since $\mathcal{B}'$ is semisimple and $H^1_*(\mathfrak{m}, k) \cong \mathfrak{m}$ as a $U_0(\mathfrak{g}')$-module, the spectral sequence (see [31, (I.6.6(2))]) $\mathrm{Ext}^m_{U_0(\mathfrak{g}')}(S, H^n_*(\mathfrak{m}, k) \otimes_k T) \Rightarrow \mathrm{Ext}^{m+n}_{U_0(\mathfrak{h})}(S, T)$ collapses and yields an isomorphism

$$\mathrm{Ext}^1_{U_0(\mathfrak{h})}(S, T) \cong \mathrm{Hom}_{U_0(\mathfrak{g}')}(S, \mathfrak{m} \otimes_k T).$$

By general theory, there exist simple $U_0(\mathfrak{p})$-modules S_1, T_1 and simple $U_0(\mathfrak{q})$-modules S_2, T_2 such that

$$S \cong S_1 \otimes_k S_2 \ \text{ and } \ T \cong T_1 \otimes_k T_2.$$

Since S and T are projective $U_0(\mathfrak{g}')$-modules, S_1 and T_1 are simple projective $U_0(\mathfrak{sl}(2))$-modules. Accordingly, we have $S_1 \cong L(p-1) \cong T_1$. We may now apply the modular Clebsch-Gordan formula [22, Satz,p.59] to see that

$$L(1) \otimes_k L(p-1) \cong P(p-2)$$

is the projective cover of the (p-1)-dimensional simple $\mathfrak{sl}(2)$-module $L(p-2)$. Consequently, $\mathrm{Hom}_{U_0(\mathfrak{p})}(S_1, L(1) \otimes_k T_1) = (0)$, so that

$$\mathrm{Hom}_{U_0(\mathfrak{g}')}(S, \mathfrak{m} \otimes_k T) \cong \mathrm{Hom}_{U_0(\mathfrak{p})}(S_1, L(1) \otimes_k T_1) \otimes_k \mathrm{Hom}_{U_0(\mathfrak{q})}(S_2, T_2 \otimes_k k_\lambda) = (0).$$

As a result, the block $\mathcal{B}$ is simple, a contradiction.

It remains to consider the case, where the two-dimensional $\mathfrak{g}'$-module $\mathfrak{m}$ is reducible. Then there exist characters $\lambda, \mu : U_0(\mathfrak{g}') \longrightarrow k$ and an exact sequence

$$(0) \longrightarrow k_\lambda \longrightarrow \mathfrak{m} \longrightarrow k_\mu \longrightarrow (0)$$

of $U_0(\mathfrak{g}')$-modules. There results a representation of $\mathfrak{g}'$ by upper triangular matrices. As before, we conclude from general structure theory that $\bigoplus_{\alpha \in R} \mathfrak{g}'_\alpha$ acts trivially on $\mathfrak{m}$. Consequently, the above sequence splits, and $\mathfrak{m}$ is semisimple. As a result, the conditions of (2.2) apply, so that this result again yields a contradiction.

As an upshot of our discussion above, we see that $\mathcal{G}$ is in fact reductive. The proof of [14, (7.1)] now provides a decomposition

$$\mathcal{G} = \mathcal{K} \cdot \mathcal{H}$$

of $\mathcal{G}$, as an almost direct product of two normal subgroups with $\mathcal{K} \cong \mathrm{SL}(2), \mathrm{PSL}(2)$. As argued before, this implies

$$\mathcal{G}_r \cong \mathrm{SL}(2)_r \times \mathcal{H}_r.$$

Writing $S \cong S_1 \otimes_k S_2$ as an outer tensor product of a simple $\mathrm{SL}(2)_r$-module S_1 and a simple $\mathcal{H}_r$-module S_2, the arguments of [14, (7.1)] prove that S_2 is projective. We may now apply (2.1) to see that $\dim_k S_2 = p^{\ell_r(\mathcal{H})} = p^{\ell_r(\mathcal{G})-r}$. Thus, if $\mathcal{B}_1 \subset \mathrm{Dist}(\mathrm{SL}(2)_r)$ and $\mathcal{B}_2 \subset \mathrm{Dist}(\mathcal{H}_r)$ are the blocks of S_1 and S_2, then

$$\mathcal{B} \cong \mathcal{B}_1 \otimes_k \mathcal{B}_2 \cong \mathcal{B}_1 \otimes_k \mathrm{Mat}_{p^{\ell_r(\mathcal{G})-r}}(k) \cong \mathrm{Mat}_{p^{\ell_r(\mathcal{G})-r}}(\mathcal{B}_1).$$

Thanks to [37, Satz 6] and [31, (II.10.5)], there exists a tame block $\mathcal{C} \subset \mathrm{Dist}(\mathrm{SL}(2)_1)$ such that $\mathcal{B}_1 \cong \mathrm{Mat}_{p^{r-1}}(\mathcal{C})$. Consequently,

$$\mathcal{B} \cong \mathrm{Mat}_{p^{\ell_r(\mathcal{G})-r}}(\mathrm{Mat}_{p^{r-1}}(\mathcal{C})) \cong \mathrm{Mat}_{p^{\ell_r(\mathcal{G})-1}}(\mathcal{C}),$$

so that [22, 42] also yield the asserted Morita equivalence. □

Remark. In [40, Theorem] Premet classifies the indecomposable $\mathrm{Dist}(\mathrm{SL}(2)_1)$-modules. Aside from the simple and principal indecomposable modules, only Weyl modules and their duals as well as certain explicitly given submodules of Weyl-modules occur. The foregoing proof in conjunction with [31, (II.10.5)] provides a recipe for the construction of the indecomposable $\mathcal{B}$-modules. Given an indecomposable $\mathcal{C}$-module X, one first considers the $\mathrm{SL}(2)_r$-module $Y_X := \mathrm{St}_{r-1} \otimes_k X^{[r-1]}$, given by the tensor product of the Frobenius twist [31, (II.3.16)] of X with the (r-1)st Steinberg module. Tensoring Y_X with the simple module S_2, then yields an indecomposable $\mathcal{B}$-module. By the above, all indecomposable $\mathcal{B}$-modules are of the form $S_2 \otimes_k Y_X$ for a suitably chosen X.

5. The Stable Auslander-Reiten Quiver

In the sequel, we let $\Gamma_s(\mathcal{G}_r)$ denote the *stable Auslander-Reiten quiver* of the self-injective algebra $\mathrm{Dist}(\mathcal{G}_r)$. By definition, its vertices are the isoclasses of the non-projective indecomposable $\mathrm{Dist}(\mathcal{G}_r)$-modules and the arrows correspond to the so-called irreducible morphisms. We refer the reader to [2] for more details.

Auslander-Reiten components are classified by means of their *tree classes* (cf. [3, §4.15]). According to [15, §1] the tree class of an AR-component of an infinitesimal group scheme is either a simply-laced Dynkin diagram, an infinite Dynkin diagram, or a Euclidean diagram of type $\tilde{A}_{12}$, $\tilde{D}_n$, $(\tilde{E}_i)_{6\leq i\leq 8}$. A component Θ is called *Euclidean* if its tree class is a Euclidean diagram, or if $\Theta \cong \mathbb{Z}[\tilde{A}_n]$ (in which case the tree class is A_∞^∞).

Proposition 5.1. *Let $\mathcal{G}$ be a smooth algebraic group with unipotent radical of dimension $n := \dim \mathcal{U}$. Given a component $\Theta \subset \Gamma_s(\mathcal{G}_r)$, the following statement hold:*

(1) *If $\Theta \subset \Gamma_s(\mathcal{G}_1)$ is finite, then there exists $i \in \{1, p\}$ such that $\Theta \cong \mathbb{Z}[A_{p^n}]/(\tau^i)$.*
(2) *If $r \geq 2$, then Θ is infinite.*
(3) *If $r \geq 3$ and Θ contains a simple $\mathrm{Dist}(\mathcal{G}_r)$-module, then $\Theta \cong \mathbb{Z}[A_\infty]$, $\mathbb{Z}[\tilde{A}_{12}]$.*
(4) *If $r \geq 3$ and Θ is Euclidean, then $\mathcal{G}$ is reductive, and $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$.*

Proof. (1),(2) If $\Theta \subset \Gamma_s(\mathcal{G}_r)$ is a finite component, then Auslander's Theorem [2, (VII.2.1)] shows that the vertices of Θ are the non-projective indecomposable modules of a representation-finite block $\mathcal{B} \subset \mathrm{Dist}(\mathcal{G}_r)$. If $r \geq 2$, then (3.1(1)) implies the simplicity of $\mathcal{B}$, whence $\Theta = \emptyset$, a contradiction.

Alternatively, we may apply (3.1(2)) to see that the block $\mathcal{B}$ is Morita equivalent to $k[X]/(X^{p^n})$ or $k[\tilde{A}_{p-1}]/\mathrm{Rad}(k[\tilde{A}_{p-1}])^{p^n}$. Our result now follows from the classification of stable Auslander-Reiten quivers of self-injective Nakayama algebras (cf. [2, p.253]).

(3) Let S be a simple $\mathrm{Dist}(\mathcal{G}_r)$-module belonging to Θ. If $\mathrm{cx}_{\mathcal{G}_r}(S) \leq 2$, then (1.1) yields that $\mathcal{G}$ is reductive, rendering our result a consequence of the remark following [15, (4.1)]. Alternatively, $\mathrm{cx}_{\mathcal{G}_r}(S) \geq 3$, and [15, (2.2)] implies $\Theta \cong \mathbb{Z}[A_\infty]$.

(4) Owing to [45, Thm.A] and [5, p.155] the Euclidean component Θ is attached to a principal indecomposable module. Thus, the Heller translate $\Omega(\Theta) \cong \Theta$ of Θ has the same property, and Θ contains a simple vertex. The result thus follows from (3). □

Corollary 5.2. *Let $\mathcal{G}$ be a smooth algebraic group. If $r \geq 3$, then every component $\Theta \subset \Gamma_s(\mathcal{G}_r)$ is isomorphic to $\mathbb{Z}[A_\infty]/(\tau^i)$, $\mathbb{Z}[A_\infty]$, $\mathbb{Z}[\tilde{A}_{12}]$, $\mathbb{Z}[A_\infty^\infty]$, or $\mathbb{Z}[D_\infty]$.*

Proof. This is a direct consequence of (5.1) and [15, (1.3)]. □

Remark. Presently, components of type $\mathbb{Z}[A_\infty^\infty]$ are only known to occur for distribution algebras that do not belong to Frobenius kernels of smooth groups (cf. [17, (7.2)]). Moreover, there are no examples of infinitesimal group schemes giving rise to components of type $\mathbb{Z}[D_\infty]$.

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