Support Spaces and Auslander-Reiten Components

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Abstract. Let $\mathcal{G}$ be a finite group scheme over an algebraically closed field k of characteristic $p > 0$. In this paper, we discuss the defining properties of the representation-theoretic support space $P(\mathcal{G})$ of p-points of $\mathcal{G}$, introduced by Friedlander-Pevtsova [**23**]. We then employ $P(\mathcal{G})$ to obtain information on the representation theory of $\mathcal{G}$.

0. Introduction

Since its inception in the early 1980's, the theory of support varieties and rank varieties has played a prominent rôle in the modular representation theory of finite groups and related classes of algebras. Aside from Quillen's fundamental work on the spectrum of the cohomology ring [**36, 37**], Chouinard's projectivity detection theorem [**9**] may be considered the starting point of the many fruitful investigations concerning the representation-theoretic realization of cohomological support varieties. In recent work [**23**], E. Friedlander and J. Pevtsova have unified and generalized the previous approaches concerning finite groups and infinitesimal group schemes by introducing the notion of a p-point of a finite group scheme $\mathcal{G}$. In view of the differences between the earlier theories for constant groups and infinitesimal groups, this unification is perhaps somewhat surprising.

The notion of a p-point takes up early work by J. Carlson [**6**], who recognized the importance of subalgebras of group algebras that do not inherit the Hopf structure from the ambient algebra. On the other hand, tensor products do play an important rôle in the theory, so one needs to create tools that are flexible enough to capture the relevant representation-theoretic features without discarding too much information encoded in the coalgebra structure. The clarification of the relevant aspects of this interplay is one of the two main objectives of the present note.

In Section 1 we review elementary properties of the space of p-points of a finite group scheme $\mathcal{G}$, by focussing on the two defining properties, namely flatness and factorization through abelian unipotent subgroups. By way of examples, we illustrate how p-points interact with the block- and tensor product structure of the relevant module categories. Two important features of p-points are their amenability to computation, and the possibility of representing them locally by homomorphisms of Hopf algebras. The former aspect is exploited in Section 2, where

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homological properties are investigated. We shall show that the flatness property of p-points follows from their cohomological non-triviality, thereby rendering p-points a natural model for the projective space of the cohomology ring $H^\bullet(\mathcal{G}, k)$. In this fashion p-points detect important properties of the module category of $\mathcal{G}$ that are reflected by the ideal structure of the cohomology ring $H^\bullet(\mathcal{G}, k)$, which a priori only seems to provide information on the principal block of the "group algebra" $k\mathcal{G}$.

In Section 3 we turn to applications concerning the representation theory of $k\mathcal{G}$. To each component $\Theta \subset \Gamma_s(\mathcal{G})$ of the stable Auslander-Reiten quiver of the self-injective algebra $k\mathcal{G}$ we associate a support space and study structural features of Θ via this correspondence. It turns out that only components with support of dimension ≤ 1 are of interest, with those of finite Dynkin class and Euclidean class being related to group schemes of finite and tame representation type, respectively.

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1. The space of p-points

Throughout, we will be working over an algebraically closed field k of characteristic $\mathrm{char}(k) = p > 0$. Unless mentioned otherwise, all algebras and modules are assumed to be finite dimensional k-spaces. We denote by

$$A_p := k[X]/(X^p) = k[x] \quad ; \quad x := X + (X^p)$$

the p-truncated polynomial ring in one variable.

Given an associative k-algebra A, we let $\mathrm{mod}\, A$ be the category of (finite dimensional) left A-modules. Any homomorphism $\varphi : A \longrightarrow B$ of k-algebras, induces, via pull-back, an exact functor

$$\varphi^* : \mathrm{mod}\, B \longrightarrow \mathrm{mod}\, A.$$

We will be mainly concerned with those homomorphisms, whose pull-back functors send projectives to projectives.

Definition. Let A be an associative k-algebra, $\alpha, \beta : A_p \longrightarrow A$ be homomorphisms.

(1) α is a *flat point* of A if α is left flat, that is, if $\alpha^*(A)$ is projective.

(2) The homomorphisms α, β are *equivalent* ($\alpha \sim \beta$) if for every $M \in \mathrm{mod}\, A$ we have

$$\alpha^*(M) \text{ is projective} \Leftrightarrow \beta^*(M) \text{ is projective}.$$

We let $\mathrm{Fl}(A)$ be the set of equivalence classes of flat points of A.

If $\varphi : A \longrightarrow B$ is a homomorphism of finite dimensional k-algebras such that B is a projective A-module, then φ induces a map

$$\varphi_* : \mathrm{Fl}(A) \longrightarrow \mathrm{Fl}(B) \quad ; \quad [\alpha] \mapsto [\varphi \circ \alpha].$$

The following elementary Lemma studies the behaviour of flat points relative to the block decomposition of an algebra.

LEMMA 1.1. *If B_1, B_2 are finite dimensional k-algebras, then the canonical projections $p_i : B_1 \times B_2 \longrightarrow B_i$ induce a bijection*

$$\Theta : \mathrm{Fl}(B_1 \times B_2) \longrightarrow \mathrm{Fl}(B_1) \times \mathrm{Fl}(B_2) \ \ ; \ \ [\alpha] \mapsto ([p_1 \circ \alpha], [p_2 \circ \alpha]).$$

PROOF. Let $\alpha : A_p \longrightarrow B_1 \times B_2$ be a flat point. Defining $\alpha_i := p_i \circ \alpha$, we have

$$\alpha(r) = (\alpha_1(r), \alpha_2(r)) \qquad \forall\ r \in A_p.$$

Setting $e_1 := (1, 0)$ and $e_2 := (0, 1)$, we see that the e_i are central orthogonal idempotents of $B := B_1 \times B_2$ and that every B-module M decomposes into a direct sum $M = e_1M \oplus e_2M$. Considering e_iM a B_i-module, we obtain

$$r.(e_1m_1 + e_2m_2) = \alpha(r)e_1m_1 + \alpha(r)e_2m_2 = \alpha_1(r)e_1m_1 + \alpha_2(r)e_2m_2,$$

so that there is a decomposition

$$(*) \qquad\qquad \alpha^*(M) = \alpha_1^*(e_1M) \oplus \alpha_2^*(e_2M).$$

Since $B_i = e_iB$, it follows that $\alpha_i : A_p \longrightarrow B_i$ is a flat point. Consequently, $[p_i \circ \alpha] \in \mathrm{Fl}(B_i)$ for $1 \leq i \leq 2$.

Let α, β be flat points of B with $\alpha \sim \beta$. Let N be a B_i-module. Then $p_i^*(N)$ is a B-module such that $e_ip_i^*(N) = N$. Thus, $\alpha^*(p_i^*(N)) = \alpha_i^*(N)$ and similarly $\beta^*(p_i^*(N)) = \beta_i^*(N)$. This readily implies $p_i \circ \alpha \sim p_i \circ \beta$. As a result, the map Θ is well-defined.

Given flat points $\alpha_i : A_p \longrightarrow B_i$, we consider the algebra homomorphism

$$\alpha : A_p \longrightarrow B_1 \times B_2 \ \ ; \ \ r \mapsto (\alpha_1(r), \alpha_2(r)).$$

In view of $(*)$, α is a flat point of B. Since $\Theta([\alpha]) = ([\alpha_1], [\alpha_2])$, we conclude that Θ is surjective.

Now let α, β be flat points of B such that $\Theta([\alpha]) = \Theta([\beta])$. Let M be a B-module and write $M_i := e_iM$. If $\alpha^*(M)$ is projective, then $(*)$ implies that $\alpha_i^*(M_i)$ is projective for $i = 1, 2$. By assumption, we conclude the projectivity of each $\beta_i^*(M_i)$, and another application of $(*)$ shows that $\beta^*(M)$ is projective. Thus $\alpha \sim \beta$, so that Θ is injective. □

Let $\mathcal{G}$ be a finite group scheme (a finite algebraic group). By definition, $\mathcal{G} = \mathrm{Spec}_k(k[\mathcal{G}])$ is an affine group scheme, whose coordinate ring $k[\mathcal{G}]$ is finite dimensional. Following [**23**], we denote by $k\mathcal{G} := k[\mathcal{G}]^*$ its *algebra of measures*, cf. [**33**, (I.8.4)]. Recall that $k\mathcal{G}$ coincides with the group algebra $k\mathcal{G}(k)$ of the finite group of k-rational points of $\mathcal{G}$ in case $\mathcal{G}$ is reduced, and with the algebra $\mathrm{Dist}(\mathcal{G})$ of distributions on $\mathcal{G}$ in case $\mathcal{G}$ is infinitesimal, see [**33**, (I.8.4),(I.8.5)]. In view of the equivalence between the categories of $\mathcal{G}$-modules and $k\mathcal{G}$-modules (cf. [**33**, (I.8.6)]), we shall henceforth use these two terms interchangeably.

Let $X(\mathcal{G}) := \mathrm{Alg}_k(k\mathcal{G}, k)$ be the character group of $\mathcal{G}$, whose identity element is the counit $\varepsilon : k\mathcal{G} \longrightarrow k$ of the cocommutative Hopf algebra $k\mathcal{G}$. The unique block $\mathcal{B}_0(\mathcal{G}) \subset k\mathcal{G}$ with $\varepsilon(\mathcal{B}_0(\mathcal{G})) \neq (0)$ is called the *principal block* of $k\mathcal{G}$.

If the finite group scheme $\mathcal{D}$ is *diagonalizable* (that is, when $k[\mathcal{D}] = kX(\mathcal{D})$ is the group algebra of a finite abelian group), then there are orthogonal primitive idempotents $\{e_\lambda \ ; \ \lambda \in X(\mathcal{D})\}$ such that

$$k\mathcal{D} = \bigoplus_{\lambda \in X(\mathcal{D})} ke_\lambda \ \ \text{and} \ \ xe_\lambda = \lambda(x)e_\lambda \quad \forall\ x \in k\mathcal{D}.$$

Recall that every abelian group scheme is a direct product

$$\mathcal{C} = \mathcal{U} \times \mathcal{D}$$

with a *unipotent* subgroup $\mathcal{U}$ (whose algebra of measures is local) and a diagonalizable subgroup $\mathcal{D}$, cf. [**33**, (I.2.5)], [**45**, (9.5)]. We let $p_{\mathcal{U}} : \mathcal{C} \longrightarrow \mathcal{U}$ be the canonical projection and label the corresponding homomorphism $k\mathcal{C} \longrightarrow k\mathcal{U}$ in the same way.

LEMMA 1.2. *Let $\mathcal{C} = \mathcal{U} \times \mathcal{D}$ be a finite abelian group scheme.*

(1) *The decomposition $k\mathcal{C} = \bigoplus_{\lambda \in X(\mathcal{D})} k\mathcal{C}e_\lambda$ is the block decomposition of $k\mathcal{C}$, with $k\mathcal{C}e_\varepsilon$ being the principal block.*

(2) *For each $\lambda \in X(\mathcal{D})$, the map $\iota_\lambda : k\mathcal{U} \longrightarrow k\mathcal{C}e_\lambda$; $x \mapsto xe_\lambda$ is an isomorphism of k-algebras.*

(3) *Let $\mathrm{pr}_0 : k\mathcal{C} \longrightarrow B_0(\mathcal{C})$ be the projection onto the principal block. Then we have*

$$\mathrm{pr}_0 = \iota_\varepsilon \circ p_{\mathcal{U}}.$$

In particular, $p_{\mathcal{U}}$ is flat.

PROOF. (1),(2) Since $\mathcal{C} = \mathcal{U} \times \mathcal{D}$, the map

$$\mu : k\mathcal{U} \otimes_k k\mathcal{D} \longrightarrow k\mathcal{C} \quad ; \quad a \otimes b \mapsto ab$$

is an isomorphism of k-algebras. There results a decomposition

$$k\mathcal{C} = \bigoplus_{\lambda \in X(\mathcal{D})} k\mathcal{U}e_\lambda$$

of $k\mathcal{C}$ by ideals, with each summand being isomorphic to $k\mathcal{U}$ via $x \mapsto xe_\lambda$. Thus, each constituent is a local algebra, and the decomposition is the block decomposition of $k\mathcal{C}$. Moreover, $k\mathcal{C}e_\lambda = k\mathcal{U}k\mathcal{D}e_\lambda \subset k\mathcal{U}e_\lambda$, and $\varepsilon(e_\varepsilon) = 1$, implying that $k\mathcal{C}e_\varepsilon = \mathcal{B}_0(\mathcal{C})$ is the principal block.

(3) Let $a \in k\mathcal{U}$, $b \in k\mathcal{D}$. By definition of $p_{\mathcal{U}}$, we have $p_{\mathcal{U}}(ab) = a\varepsilon(b)$, so that for $x = \sum_{\lambda \in X(\mathcal{D})} u_\lambda e_\lambda$ with $u_\lambda \in k\mathcal{U}$ we obtain $p_{\mathcal{U}}(x) = u_\varepsilon$, whence

$$\mathrm{pr}_0(x) = u_\varepsilon e_\varepsilon = (\iota_\varepsilon \circ p_{\mathcal{U}})(x).$$

Since pr_0 is flat, this also implies the flatness of $p_{\mathcal{U}}$. □

Let $\mathcal{G}$ be a finite algebraic group. In the sequel, we write $\mathrm{Fl}(\mathcal{G})$ for the set of flat points of the algebra $k\mathcal{G}$. By the foregoing result, the projection $p_{\mathcal{U}} : \mathcal{C} \longrightarrow \mathcal{U}$ induces a map

$$p_{\mathcal{U},*} : \mathrm{Fl}(\mathcal{C}) \longrightarrow \mathrm{Fl}(\mathcal{U}).$$

In view of (1.1) and (1.2(3)) this map will in general not be injective. The following example illustrates this point.

Given $r, n \in \mathbb{N}$, we denote by $\mathbb{G}_{a(r)}$ and $\mu_{(n)}$ the r-th Frobenius kernel of the additive group and the group of n-th roots of unity, respectively. The latter group is diagonalizable, and for $p \nmid n$ its algebra of measures coincides (as a Hopf algebra) with the group algebra $k(\mathbb{Z}/(n))$.

EXAMPLE. Consider the abelian group scheme

$$\mathcal{C} = (\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}) \times \mu_{(2)},$$

so that $\mathcal{U} = \mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$ and $\mathcal{D} = \mu_{(2)}$. By (1.2) we have

$$k\mathcal{C} = k(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)})e_0 \oplus k(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)})e_1,$$

with the first summand denoting the principal block. Let $k(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}) = k[X_0, X_1]/(X_0^p, X_1^p)$, and denote the generators by x_0 and x_1, respectively. We define flat points

$$\alpha_i : A_p \longrightarrow k(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}) \ \ ; \ \ x \mapsto x_i,$$

and consider the homomorphisms

$$\alpha : A_p \longrightarrow k\mathcal{C} \ \ ; \ \ r \mapsto \alpha_0(r)e_0 + \alpha_1(r)e_1$$

as well as

$$\beta : A_p \longrightarrow k\mathcal{C} \ \ ; \ \ r \mapsto \alpha_0(r)e_0 + \alpha_0(r)e_1.$$

By the proof of (1.1) the maps α and β are flat points of $\mathcal{C}$.

We now consider the $k\mathcal{U}$-module $M := k[X_1]/(X_1^p)$ on which x_0 and x_1 act via 0 and left multiplication, respectively. In addition, we let $\mu_{(2)}$ act via its non-trivial character, so that $M = e_1M$. For the module $\alpha^*(M)$ we have

$$x.x_1^i = \alpha(x)x_1^i = \alpha_1(x)x_1^i = x_1^{i+1} \qquad 0 \le i \le p-1,$$

while for $m \in \beta^*(M)$ we obtain

$$x.m = \beta(x)m = \alpha_0(x)m = x_0m = 0.$$

As a result, $\alpha^*(M)$ is free while $\beta^*(M)$ is the trivial module. Consequently, $\alpha \not\sim \beta$. Since

$$p_{\mathcal{U}} \circ \alpha = \alpha_0 = p_{\mathcal{U}} \circ \beta,$$

the map $p_{\mathcal{U},*}$ is not injective.

In view of the example above, the "space" of flat points cannot serve as a representation-theoretic support space that reflects the properties of cohomological support varieties. Since a diagonalizable group scheme has trivial cohomology, the Künneth formula yields the bijectivity of the canonical map $p_{\mathcal{U}}^* : H^*(\mathcal{U}, k) \longrightarrow H^*(\mathcal{C}, k)$. Consequently, $\mathrm{Fl}(\mathcal{C})$ will usually be too big to describe the variety associated to the even cohomology ring.

The following definition, which is motivated by earlier work on finite groups [**6**] and infinitesimal groups [**32, 21, 22, 44**], rectifies this defect.

DEFINITION ([**23**]). Let $\mathcal{G}$ be a finite group scheme. A flat homomorphism $\alpha : A_p \longrightarrow k\mathcal{G}$ is said to be a *p-point* if there exists an abelian, unipotent subgroup $\mathcal{U} \subset \mathcal{G}$ such that $\operatorname{im}\alpha \subset k\mathcal{U}$. Let $P(\mathcal{G})$ be the set of equivalence classes of p-points.

Since images of abelian unipotent groups are abelian and unipotent, a flat morphism $f : \mathcal{G} \longrightarrow \mathcal{H}$ of finite group schemes induces a map

$$f_* : P(\mathcal{G}) \longrightarrow P(\mathcal{H}) \ \ ; \ \ [\alpha] \mapsto [f \circ \alpha].$$

LEMMA 1.3. *Let $\mathcal{C} = \mathcal{U} \times \mathcal{D}$ be a finite abelian group scheme with $\mathcal{U}$ unipotent and $\mathcal{D}$ diagonalizable. Then the projection $p_{\mathcal{U}} : \mathcal{C} \longrightarrow \mathcal{U}$ induces a bijection*

$$p_{\mathcal{U},*} : P(\mathcal{C}) \longrightarrow P(\mathcal{U}) \ \ ; \ \ [\alpha] \mapsto [p_{\mathcal{U}} \circ \alpha].$$

PROOF. Owing to (1.2(3)) the map $p_{\mathcal{U}} : k\mathcal{C} \longrightarrow k\mathcal{U}$ is flat, so that $p_{\mathcal{U},*}$ is well-defined. Let $\iota_{\mathcal{U}} : k\mathcal{U} \longrightarrow k\mathcal{C}$ be the canonical inclusion. According to [**33**, (I.8.16)] the homomorphism $\iota_{\mathcal{U}}$ is flat, so that we obtain a map $\iota_{\mathcal{U},*} : P(\mathcal{U}) \longrightarrow P(\mathcal{C})$. Since $p_{\mathcal{U},*} \circ \iota_{\mathcal{U},*} = \mathrm{id}_{P(\mathcal{U})}$, the map $p_{\mathcal{U},*}$ is surjective.

Now let α be a representative of an element of $P(\mathcal{C})$. Then there exists a unipotent subgroup $\mathcal{U}_\alpha \subset \mathcal{C}$ such that $\alpha(A_p) \subset k\mathcal{U}_\alpha$. Since $\mathcal{D}$ is diagonalizable, the projection $p_{\mathcal{D}} : \mathcal{C} \longrightarrow \mathcal{D}$ coincides with the trivial map on $\mathcal{U}_\alpha$. Thus, $\mathcal{U}_\alpha \subset \ker p_{\mathcal{D}} = \mathcal{U} \times e_k$, so that $\mathrm{im}\, \alpha \subset \mathrm{im}\, \iota_{\mathcal{U}}$. As a result, $\iota_{\mathcal{U},*}$ is surjective and $p_{\mathcal{U},*}$ is also injective. □

We take a closer look at finite abelian unipotent group schemes and their p-points. Given such a group $\mathcal{U}$, its algebra

$$k\mathcal{U} = k[\mathcal{U}^\sharp]$$

is the coordinate ring of its Cartier dual $\mathcal{U}^\sharp$, cf. [**45**, (2.4)]. Since $k\mathcal{U}$ is local, the group scheme $\mathcal{U}^\sharp$ is connected, and [**45**, (14.4)] provides $r := (r_1, \ldots, r_n) \in \mathbb{N}^n$ such that the algebra

$$k\mathcal{U} \cong k[T_1, \ldots, T_n]/(T_1^{p^{r_1}}, \ldots, T_n^{p^{r_n}})$$

is a truncated polynomial ring. We denote the residue class of T_i by t_i, and consider the subalgebra

$$kE := k[t_1^{p^{r_1-1}}, \ldots, t_n^{p^{r_n-1}}],$$

which assumes the rôle of a p-elementary abelian subgroup. For $a, b \in \mathbb{N}_0^n$, we write

- $a * b := (a_1 b_1, \ldots, a_n b_n)$,
- $a \le b :\Leftrightarrow a_i \le b_i$ for all i,
- $\tau(a) := (p^{a_1} - 1, \ldots, p^{a_n} - 1)$,
- $\underline{\mathbf{1}} := (1, 1, \ldots, 1)$.

With this notation, we have

$$kE = \bigoplus_{0 \le a \le \tau(\underline{\mathbf{1}})} k\, t^{a*(\tau(r-\underline{\mathbf{1}})+\underline{\mathbf{1}})}.$$

Given a finite dimensional k-algebra A, we denote by $\mathrm{Rad}(A)$ its nilpotent radical.

LEMMA 1.4. (1) *We have $k\mathcal{U} = \bigoplus_{0 \le b \le \tau(r-\underline{\mathbf{1}})} (kE)\, t^b$.*

(2) *We have $\{u \in k\mathcal{U} \ ; \ u^p = 0\} = \bigoplus_{0 \le b \le \tau(r-\underline{\mathbf{1}})} \mathrm{Rad}(kE)\, t^b$.*

PROOF. (1) This follows from the fact that every element $0 \le q \le \tau(r)$ has a unique decomposition

$$q = a * (\tau(r - \underline{\mathbf{1}}) + \underline{\mathbf{1}}) + b,$$

with $0 \le a \le \tau(\underline{\mathbf{1}})$ and $0 \le b \le \tau(r - \underline{\mathbf{1}})$.

(2) Note that $v^p = 0$ for every $v \in \mathrm{Rad}(kE)$, so that the left-hand side contains the right-hand side. For the other inclusion, we observe that (1) yields

$$k\mathcal{U} = \bigoplus_{0 \le b \le \tau(r-\underline{\mathbf{1}})} \mathrm{Rad}(kE)\, t^b \oplus \bigoplus_{0 \le b \le \tau(r-\underline{\mathbf{1}})} k\, t^b,$$

with the p-power map being injective on the right-hand summand. □

The equivalence relation of flat points is not very tractable in general. For abelian unipotent groups, however, we have the following criterion, which is an immediate consequence of [**23**, (2.2)]:

Lemma 1.5. (1) *Let $\alpha, \beta : A_p \longrightarrow k\mathcal{U}$ be flat points. If*

$$\alpha(x) \equiv \beta(x) \mod (\mathrm{Rad}(kE)\,\mathrm{Rad}(k\mathcal{U})),$$

then $\alpha \sim \beta$.

(2) *Let A be a k-algebra. If $f, g : A \longrightarrow k\mathcal{U}$ are flat homomorphisms such that $f(a) \equiv g(a) \mod(\mathrm{Rad}(kE)\,\mathrm{Rad}(k\mathcal{U}))$ for all $a \in A$ with $a^p = 0$, then the induced maps $f_*, g_* : \mathrm{Fl}(A) \longrightarrow P(\mathcal{U})$ are equal.* □

We let

$$\mathrm{pr}_E : \begin{cases} k\mathcal{U} & \longrightarrow & kE \\ \sum_{0 \le b \le \tau(r-\underline{1})} v_b t^b & \mapsto & v_0 \end{cases}$$

be the projection along $t^0 = 1$. (In the above presentation, we tacitly assume $v_b \in kE$.)

In view of (1.4(1)), the canonical inclusion

$$\iota_E : kE \hookrightarrow k\mathcal{U}$$

is flat and $\mathrm{pr}_E \circ \iota_E = \mathrm{id}_E$. Given a homomorphism $\alpha : A_p \longrightarrow k\mathcal{U}$, we let

$$\alpha_{(E)} : A_p \longrightarrow kE$$

be the unique homomorphism such that $\alpha_{(E)}(x) := \mathrm{pr}_E(\alpha(x))$. (Strictly speaking, $\alpha_{(E)}$ also depends on the choice of the generator x of A_p.)

One salient feature of abelian unipotent group schemes resides in their support varieties being independent of the coalgebra structure (cf. [**3**, (5.7.1)]). The third part of the following Lemma provides a consequence of such a degree of freedom within our context.

Lemma 1.6. *Let $\mathcal{U}$ be a finite abelian unipotent group scheme.*

(1) *The projection $\mathrm{pr}_E : k\mathcal{U} \longrightarrow kE$ induces a map*

$$\mathrm{pr}_{E,*} : P(\mathcal{U}) \longrightarrow \mathrm{Fl}(kE) \quad ; \quad [\alpha] \mapsto [\alpha_{(E)}].$$

(2) *The maps $\iota_{E,*}$ and $\mathrm{pr}_{E,*}$ are bijective, with $\mathrm{pr}_{E,*} = \iota_{E,*}^{-1}$.*

(3) *There are Hopf algebra structures on A_p and $k\mathcal{U}$ such that every equivalence class $x \in P(\mathcal{U})$ is represented by a homomorphism $\alpha_x : A_p \longrightarrow k\mathcal{U}$ of Hopf algebras.*

Proof. (1) Let $\beta : A_p \longrightarrow kE$ be a homomorphism and set $\alpha := \iota_E \circ \beta$. Lemma 1.4(1) provides an isomorphism

$$\iota_E^*(k\mathcal{U}) \cong (kE)^m$$

for some $m \in \mathbb{N}$. Consequently,

$$\alpha^*(k\mathcal{U}) = \beta^*(\iota_E^*(k\mathcal{U})) \cong \beta^*(kE)^m,$$

so that α is a p-point of $k\mathcal{U}$ if and only if β is a flat point of kE.

Now let $\alpha : A_p \longrightarrow k\mathcal{U}$ be a homomorphism. By applying (1.4(2)) to the element $u = \alpha(x)$, we obtain elements $v_b \in \mathrm{Rad}(kE)$ such that $\alpha(x) = \iota_E(\alpha_{(E)}(x)) + \sum_{0\neq b \leq \tau(r-\underline{\mathbf{1}})} v_b\, t^b$. Thus, (1.5(1)) implies

$$(*) \qquad \alpha \sim \iota_E \circ \alpha_{(E)}.$$

Consequently, $\alpha \mapsto \alpha_{(E)}$ sends p-points to flat points.

Let N be a kE-module. Directly from (1.4(1)) we obtain

$$\iota_E^*(k\mathcal{U} \otimes_{kE} N) \cong N^m,$$

for some $m \in \mathbb{N}$. Thus, if α and α' are equivalent p-points of $k\mathcal{U}$, then $\iota_E \circ \alpha_{(E)} \sim \iota_E \circ \alpha'_{(E)}$, and the projectivity of $\alpha^*_{(E)}(N)$ is equivalent to that of $(\iota_E \circ \alpha_{(E)})^*(k\mathcal{U} \otimes_{kE} N)$. Since this also holds for α', we conclude $\alpha_{(E)} \sim \alpha'_{(E)}$. As a result, the map $\mathrm{pr}_{E,*}$ is well-defined.

(2) The equivalence $(*)$ immediately yields $\iota_{E,*} \circ \mathrm{pr}_{E,*} = \mathrm{id}_{P(\mathcal{U})}$. Given a flat point $\beta : A_p \longrightarrow kE$, we have

$$\mathrm{pr}_E((\iota_E \circ \beta)(x)) = \beta(x),$$

so that $\mathrm{pr}_{E,*} \circ \iota_{E,*} = \mathrm{id}_{\mathrm{Fl}(kE)}$.

(3) We endow A_p and $k\mathcal{U}$ with Hopf algebra structures by postulating that the generators x and t_i are primitive. Thus, A_p and $k\mathcal{U}$ are Hopf algebras of abelian unipotent infinitesimal groups of height 1 with kE being a Hopf subalgebra of $k\mathcal{U}$.

Let $\alpha : A_p \longrightarrow k\mathcal{U}$ be a p-point. Thanks to $(*)$, we have $[\alpha] = [\iota_E \circ \alpha_{(E)}]$. In view of (1.5(1)) there exists $(\zeta_1, \dots, \zeta_n) \in k^n \setminus \{0\}$ such that the flat point $\alpha_{(E)}$ is equivalent to the flat point $\beta : A_p \longrightarrow kE$ given by

$$\beta(x) = \sum_{i=1}^{n} \zeta_i t_i^{p^{r_i-1}}.$$

Consequently, α is equivalent to the p-point $\iota_E \circ \beta$. Since the element $(\iota_E \circ \beta)(x) = \sum_{i=1}^{n} \zeta_i t_i^{p^{r_i-1}}$ is primitive, the map $\iota_E \circ \beta$ has the requisite properties. □

The Hopf algebra structure defined in (1.6(3)) endows $k\mathcal{U}$ with the structure of a restricted enveloping algebra. Recall that for every restricted Lie algebra $(\mathfrak{g}, [p])$, the *restricted enveloping algebra* is the factor algebra

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\})$$

of the ordinary enveloping algebra $U(\mathfrak{g})$ (cf. [**11**, (II,§7, n°2-4)] and [**43**, (5.3)] for more details).

In the situation of (1.6(3)) we see that

$$\mathfrak{g} := \bigoplus_{i=1}^{n} \bigoplus_{j_i=0}^{r_i-1} k t_i^{p^{j_i}}$$

is the restricted Lie algebra of primitive elements of $k\mathcal{U}$, and $k\mathcal{U} \cong U_0(\mathfrak{g})$. Thanks to work by Jantzen [**32**] and Friedlander-Parshall [**21**] the *nullcone*

$$\mathcal{V}_{\mathfrak{g}} := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}$$

of any restricted Lie algebra is a good representation-theoretic support space. In our case, we have

$$\mathcal{V}_{\mathfrak{g}} = \bigoplus_{i=1}^{n} kt_i^{p^{r_i-1}}.$$

Owing to (1.5(1)) any equivalence class $c \in \mathrm{Fl}(kE)$ is represented by a map $\gamma_c : A_p \longrightarrow kE$ such that $\gamma_c(x) \in \mathcal{V}_{\mathfrak{g}}$. Since for any $v, w \in \mathcal{V}_{\mathfrak{g}} \setminus \{0\}$, the kE-module $M_v := kE \otimes_{k[v]} k$ is $k[w]$-projective if and only if $w \notin kv$, the line $k\gamma_c(x)$ only depends on c, so that we obtain a bijection $\mathrm{Fl}(kE) \xrightarrow{\sim} \mathrm{Proj}(\mathcal{V}_{\mathfrak{g}})$. Hence there also is a bijection $P(\mathcal{U}) \xrightarrow{\sim} \mathrm{Proj}(\mathcal{V}_{\mathfrak{g}})$. (See also [**22**, (2.7)] for the relationship with support varieties in this context.)

We conclude this section by briefly discussing the interaction of p-points with tensor products. Following Friedlander-Pevtsova [**23**], we associate to every $\mathcal{G}$-module M the set

$$P(\mathcal{G})_M := \{[\alpha] \in P(\mathcal{G}) \ ; \ \alpha^*(M) \text{ is not projective}\}.$$

To illustrate the flexibility of p-points we provide a proof of the following formula [**23**, (3.9)] concerning tensors products as well as an example showing that flat points do not have such a property.

Lemma 1.7. *Let $\mathcal{G}$ be a finite group scheme. Then we have*

$$P(\mathcal{G})_{M\otimes_k N} = P(\mathcal{G})_M \cap P(\mathcal{G})_N$$

for all $M, N \in \mathrm{mod}\, k\mathcal{G}$.

Proof. Suppose first that $\mathcal{G} = \mathcal{U}$ is abelian and unipotent. We denote by $\Delta_0, \Delta_1 : k\mathcal{U} \longrightarrow k\mathcal{U} \otimes_k k\mathcal{U}$, the two comultiplications, with Δ_0 being the originally given one, and Δ_1 the one defined in (1.6(3)). Since tensor products of projective modules over Hopf algebras are projective, both maps are flat. The well-known identity $\Delta_0(u) \equiv u \otimes 1 + 1 \otimes u \mod(\mathrm{Rad}(k\mathcal{U}) \otimes_k \mathrm{Rad}(k\mathcal{U}))$, in conjunction with $k\mathcal{U}$ being generated by Δ_1-primitive elements, yields

$$\Delta_0(u) \equiv \Delta_1(u) \mod(\mathrm{Rad}(k\mathcal{U}) \otimes_k \mathrm{Rad}(k\mathcal{U})) \quad \forall\, u \in k\mathcal{U}.$$

In view of (1.4(2)), Lemma 1.5(2) therefore applies to Δ_0 and Δ_1. Consequently,

$$P(\mathcal{U})_{M\otimes N} = \Delta_{0,*}^{-1}(P(k\mathcal{U} \otimes_k k\mathcal{U})_{M\otimes_k N}) = \Delta_{1,*}^{-1}(P(k\mathcal{U} \otimes_k k\mathcal{U})_{M\otimes_k N}),$$

so that we may consider the tensor product relative to the Hopf structure given in (1.6(3)).

Let $[\alpha]$ be an element of $P(\mathcal{U})$. According to (1.6(3)) we may assume that $\alpha : A_p \longrightarrow k\mathcal{U}$ is a homomorphism of Hopf algebras, where $A_p = U_0(kv)$ is the restricted enveloping algebra of the one-dimensional nil-cyclic Lie algebra kv. Thus, the functor α^* commutes with tensor products, and our result follows from the representation theory of $U_0(kv)$.

In the general case, a p-point $\alpha : A_p \longrightarrow k\mathcal{G}$ factors through an abelian, unipotent subgroup $\mathcal{U} \subset \mathcal{G}$. Hence there exists a p-point $\beta : A_p \longrightarrow k\mathcal{U}$ such that $\iota_{\mathcal{U}} \circ \beta = \alpha$. Here $\iota_{\mathcal{U}}$ denotes the canonical inclusion, whose associated pull-back functor commutes with tensor products. □

The following example shows that (1.7) does not hold for flat points.

EXAMPLE. Assuming $p \neq 2$, we consider the abelian group scheme

$$\mathcal{C} = (\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}) \times \mu_{(2)}.$$

By (1.2) we have

$$k\mathcal{C} = k(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)})e_0 \oplus k(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)})e_1,$$

with the first summand denoting the principal block. Let $k(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}) = k[X_0, X_1]/(X_0^p, X_1^p)$, and denote the generators by x_0 and x_1, respectively. We define p-points

$$\alpha_i : A_p \longrightarrow k(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}) \ \ ; \ \ x \mapsto x_i,$$

and consider the homomorphism

$$\alpha : A_p \longrightarrow k\mathcal{C} \ \ ; \ \ r \mapsto \alpha_0(r)e_0 + \alpha_1(r)e_1.$$

By the proof of (1.1) the map α is a p-point of $\mathcal{C}$.

Recall that $k\mu_{(2)} = k1 \oplus kg$ is the group algebra of $\mathbb{Z}/(2) = \langle g \rangle$. Direct computation shows

$$e_0 = \frac{1}{2} + \frac{1}{2}g \quad ; \quad e_1 = \frac{1}{2} - \frac{1}{2}g,$$

which implies

$$\Delta(e_0) = e_0 \otimes e_0 + e_1 \otimes e_1 \ \ ; \ \ \Delta(e_1) = e_0 \otimes e_1 + e_1 \otimes e_0.$$

Let η be the antipode of $k\mathcal{C}$. Since $g = g^{-1}$ we have

$$\eta(e_i) = e_i \qquad 0 \leq i \leq 1.$$

We now consider the $k(\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)})$-module $N = k[X_0]/(X_0^p)$ on which x_0 and x_1 act via left multiplication and 0, respectively. In addition, we let g act via -1, so that $N = e_1 N$. In particular, $\alpha^*(N)$ is the p-dimensional trivial A_p-module.

Since

$$e_1.(w \otimes w') = e_0.w \otimes e_1.w' + e_1.w \otimes e_0.w' \qquad \forall\, w, w' \in N$$

the module $N \otimes_k N$ belongs to the principal block of $k\mathcal{C}$. Consequently,

$$\alpha(r).v = \alpha_0(r)e_0.v \qquad \forall\, v \in N \otimes_k N,\ r \in A_p,$$

whence

$$\begin{aligned} \alpha(x).(w \otimes w') &= \Delta(x_0)(e_1 \otimes e_1)(w \otimes w') = (x_0 \otimes 1 + 1 \otimes x_0)(w \otimes w') \\ &= x_0.w \otimes w' + w \otimes x_0.w' \end{aligned}$$

for $w, w' \in N$. As a result, the A_p-module $\alpha^*(N \otimes_k N)$ is the 2-fold tensor power of the principal indecomposable module of $A_p \cong k\mathbb{G}_{a(1)}$. Hence $\alpha^*(N \otimes_k N)$ is projective, so that $[\alpha] \in \mathrm{Fl}(k\mathcal{C})_N \setminus \mathrm{Fl}(k\mathcal{C})_{N \otimes_k N}$. (Here $\mathrm{Fl}(k\mathcal{C})_N$ is defined analogously to $P(\mathcal{C})_N$.)

The conceptual importance of (1.7) resides in the following result by Friedlander and Pevtsova [**23**, (3.10)], that justifies the notion of a support space:

THEOREM 1.8. *Let $\mathcal{G}$ be a finite group scheme. Then*

$$\{P(\mathcal{G})_M \ ; \ M \in \operatorname{mod} k\mathcal{G}\}$$

is the set of closed subsets of a Noetherian topology on $P(\mathcal{G})$. □

In subsequent work [**24**], the authors have endowed $P(\mathcal{G})$ with the structure of a ringed space that turns out to be isomorphic to the variety $\mathrm{Proj}(|\mathcal{G}|)$ of the even cohomology ring of $\mathcal{G}$ (see also [**8, 25**] for related work). For our purposes, however, the topological structure of $P(\mathcal{G})$ suffices.

2. p-Points and non-triviality

We consider a finitely generated, commutative, graded k-algebra

$$R = \bigoplus_{n\geq 0} R_n \quad ; \quad R_0 = k$$

and denote by $R^\dagger := \bigoplus_{n\geq 1} R_n$ the augmentation ideal of R. Let $\mathcal{M} := \{I \lhd R \ ; \ I \subsetneq R^\dagger \ , \ I = \sqrt{I} \text{ and } I \text{ is homogeneous}\}$. Then

$$\mathrm{Proj}(R) := \{I \in \mathcal{M} \ ; \ I \text{ maximal}\}$$

is the projective space of R.

Let $k[T]$ be the polynomial ring in one variable. We call a degree zero algebra homomorphism $\Phi : R \longrightarrow k[T]$ *non-trivial* if $\Phi(R) \neq k$, and denote by $\mathrm{Hom}_{\mathrm{gr}}(R, k[T])$ the set of all these homomorphisms. The canonical action

$$\alpha.(\sum_{i\geq 0} \gamma_i T^i) := \sum_{i\geq 0} \alpha^i \gamma_i T^i$$

of the multiplicative group $k^\times$ on $k[T]$ induces an action on $\mathrm{Hom}_{\mathrm{gr}}(R, k[T])$, such that $\ker \Phi = \ker(\alpha.\Phi)$ for all $\alpha \in k^\times$ and $\Phi \in \mathrm{Hom}_{\mathrm{gr}}(R, k[T])$.

We recall the following basic result:

Proposition 2.1. *Suppose that R is a finitely generated k-algebra. The map*

$$\Theta : \mathrm{Hom}_{\mathrm{gr}}(R, k[T])/k^\times \longrightarrow \mathrm{Proj}(R) \quad ; \quad [\Phi] \longrightarrow \ker \Phi$$

is a bijection. □

Given an augmented k-algebra (A, ε), and an A-module M, we put

$$H^\bullet(A, M) := \begin{cases} \bigoplus_{n\geq 0} H^{2n}(A, M) & \text{for } p \neq 2 \\ \bigoplus_{n\geq 0} H^{n}(A, M) & \text{for } p = 2 \end{cases}.$$

Since a homomorphism $\alpha : A_p \longrightarrow A$ is compatible with the augmentations, we have an induced homomorphism

$$\alpha^\bullet : H^\bullet(A, k) \longrightarrow H^\bullet(A_p, k)$$

of graded k-algebras. Here the augmentation of A_p is defined by sending the generator x to 0.

Since $H^\bullet(A_p, k) \cong k[T]$, with T having degree 2 for $p \neq 2$, certain flat points give rise to elements of the projective space of the cohomology ring. This of course requires finite generation, which, in the relevant case $A = k\mathcal{G}$, follows from the Friedlander-Suslin Theorem [**26**].

Our first Lemma elaborates on [**23**, (3.4)]:

Lemma 2.2. *Let $\mathcal{G}$ be a finite group scheme. Then the following statements hold:*

(1) *If $\alpha : A_p \longrightarrow k\mathcal{G}$ is a flat point, then $\operatorname{im}\alpha^\bullet \neq k$.*

(2) *If $\operatorname{Fl}(\mathcal{G}) \neq \emptyset$ and $\alpha : A_p \longrightarrow \mathcal{B}_0(\mathcal{G})$ is a flat point, then $\operatorname{im}\alpha^\bullet \neq k$.*

Proof. (1) By assumption $\alpha^*(k\mathcal{G})$ is a projective A_p-module, so that we have an isomorphism

$$H^\bullet(A_p, k) \cong H^\bullet(\mathcal{G}, \operatorname{Hom}_{A_p}(k\mathcal{G}, k))$$

of $H^\bullet(\mathcal{G}, k)$-modules, with $H^\bullet(\mathcal{G}, k)$ operating on $H^\bullet(A_p, k)$ via $\alpha^\bullet$. Thanks to the Friedlander-Suslin Theorem [**26**, (1.1)] (and its proof), $H^\bullet(A_p, k) \cong k[T]$ is a finitely generated $H^\bullet(\mathcal{G}, k)$-module, so that $\operatorname{im}\alpha^\bullet \neq k$.

(2) Let $\operatorname{pr} : k\mathcal{G} \longrightarrow \mathcal{B}_0(\mathcal{G})$ be the canonical projection. Owing to Lemma 1.1 we can find a flat point $\beta : A_p \longrightarrow k\mathcal{G}$ such that $\operatorname{pr} \circ \beta = \alpha$. Since the $\mathcal{G}$-module k belongs to the principal block $\mathcal{B}_0(\mathcal{G})$, the map $\operatorname{pr}^\bullet : H^\bullet(\mathcal{B}_0(\mathcal{G}), k) \longrightarrow H^\bullet(\mathcal{G}, k)$ is an isomorphism. Thus, (1) implies $\operatorname{im}\alpha^\bullet = \operatorname{im}(\beta^\bullet \circ \operatorname{pr}^\bullet) = \operatorname{im}\beta^\bullet \neq k$. □

The following example shows that converse of (2.2(1)) does not hold.

Example. Consider the abelian group scheme

$$\mathcal{C} = \mathbb{G}_{a(1)} \times \mu_{(2)},$$

whose block decomposition is given by $k\mathcal{C} = k\mathbb{G}_{a(1)}e_0 \oplus k\mathbb{G}_{a(1)}e_1$, with the first summand being $\mathcal{B}_0(\mathcal{C})$. Let $\alpha : A_p \longrightarrow \mathcal{B}_0(\mathcal{C})$ be a flat point. As before, we let $\operatorname{pr} : k\mathcal{C} \longrightarrow \mathcal{B}_0(\mathcal{C})$ be the canonical projection. Since $r \mapsto \varepsilon(r)e_1$ is not a flat point of $k\mathbb{G}_{a(1)}e_1$, Lemma 1.1 ensures that the homomorphism

$$\beta : A_p \longrightarrow k\mathcal{C} \ \ ; \ \ r \mapsto \alpha(r) + \varepsilon(r)e_1$$

is not a flat point, while $\alpha = \operatorname{pr} \circ \beta$ is a flat point of $\mathcal{B}_0(\mathcal{C})$. Using Lemma 2.2 we conclude that $k \neq \operatorname{im}\alpha^\bullet = \operatorname{im}(\beta^\bullet \circ \operatorname{pr}^\bullet) = \operatorname{im}\beta^\bullet$.

Let (B_1, ε_1), (B_2, ε_2) be two finite dimensional augmented k-algebras and consider their tensor product $B := B_1 \otimes_k B_2$ with the induced augmentation $\varepsilon : B \longrightarrow k$. There are canonical algebra homomorphisms

$$\iota_1 : B_1 \longrightarrow B \ \ ; \ \ b_1 \mapsto b_1 \otimes 1 \quad \text{and} \quad \pi_1 : B \longrightarrow B_1 \ \ ; \ \ b_1 \otimes b_2 \mapsto b_1\varepsilon_2(b_2)$$

which satisfy

$$\pi_1 \circ \iota_1 = \operatorname{id}_{B_1}.$$

There are analogous maps

$$\iota_2 : B_2 \longrightarrow B \quad \text{and} \quad \pi_2 : B \longrightarrow B_2.$$

Since all maps involved are homomorphisms of augmented algebras, we obtain algebra homomorphisms

$$\pi_i^* : H^*(B_i, k) \longrightarrow H^*(B, k),$$

which, by virtue of $\operatorname{id}_{H^*(B_i,k)} = \iota_i^* \circ \pi_i^*$, are injective. Consequently,

$$\pi_1^* \otimes \pi_2^* : H^*(B_1, k) \otimes_k H^*(B_2, k) \longrightarrow H^*(B, k)$$

is an injective homomorphism of graded-commutative k-algebras. By the Künneth formula, the finite dimensional homogeneous subspaces of the domain and the range have the same dimension, so that $\pi_1^* \otimes \pi_2^*$ is bijective.

LEMMA 2.3. *Let $\alpha : A_p \longrightarrow B$ be a homomorphism such that* $\mathrm{im}(\alpha^\bullet \circ \pi_i^\bullet) = k$ *for $i = \{1,2\}$. Then we have* $\mathrm{im}\,\alpha^\bullet = k$.

PROOF. We only consider the case $p \neq 2$. Suppose that $\mathrm{im}(\alpha^\bullet \circ \pi_i^\bullet) = k$ for $i \in \{1,2\}$. Setting $C^j := \pi_1^*(H^j(B_1,k))$ and $D^j := \pi_2^*(H^j(B_2,k))$, the above observations imply

$$H^{2n}(B,k) = \bigoplus_{i+j=2n} C^i D^j \qquad \forall\, n \geq 0.$$

Let $n \geq 1$. If i and j are odd, then an element $y := c\,d$, with $c \in C^i$ and $d \in D^j$ satisfies

$$y^2 \in \pi_1^\bullet(H^\bullet(B_1,k))\,\pi_2^\bullet(H^\bullet(B_2,k)),$$

so that

$$\alpha^\bullet(y)^2 = \alpha^\bullet(y^2) \in \alpha^\bullet(\pi_1^\bullet(H^{2i}(B_1,k))\,\alpha^\bullet(\pi_2^\bullet(H^{2j}(B_2,k)) = (0).$$

Since the algebra $H^\bullet(A_p,k) = k[T]$ is reduced, it follows that $y \in \ker\alpha^\bullet$. As a result, we obtain

$$H^\bullet(B,k) = \pi_1^\bullet(H^\bullet(B_1,k))\pi_2^\bullet(H^\bullet(B_2,k)) + \ker\alpha^\bullet \subset H^0(B,k) + \ker\alpha^\bullet,$$

whence $\mathrm{im}\,\alpha^\bullet = \alpha^\bullet(H^0(B,k)) = k$. □

THEOREM 2.4. *Let $\mathcal{G}$ be a finite group scheme, $\alpha : A_p \longrightarrow k\mathcal{G}$ be a homomorphism which factors through an abelian, unipotent subgroup $\mathcal{U} \subset \mathcal{G}$. Then the following statements are equivalent:*

(1) *α is a p-point.*
(2) $\mathrm{im}\,\alpha^\bullet \neq k$.

PROOF. (1) ⇒ (2). This follows directly from (2.2).

(2) ⇒ (1). Let $\iota_{\mathcal{U}} : k\mathcal{U} \longrightarrow k\mathcal{G}$ be the canonical inclusion. By assumption, there exists a homomorphism $\tilde{\alpha} : A_p \longrightarrow k\mathcal{U}$ such that $\alpha = \iota_{\mathcal{U}} \circ \tilde{\alpha}$. Consequently,

$$\alpha^\bullet = \tilde{\alpha}^\bullet \circ \iota_{\mathcal{U}}^\bullet,$$

so that $k \subsetneq \mathrm{im}\,\alpha^\bullet \subset \mathrm{im}\,\tilde{\alpha}^\bullet$. According to [**34**, (2.6)] there exists $n \in \mathbb{N}$ such that $\iota_{\mathcal{U}}^*(k\mathcal{G}) \cong k\mathcal{U}^n$. Application of $\tilde{\alpha}^*$ thus gives

$$\alpha^*(k\mathcal{G}) = \tilde{\alpha}^*(\iota_{\mathcal{U}}^*(k\mathcal{G})) \cong \tilde{\alpha}^*(k\mathcal{U})^n,$$

so that α is flat whenever $\tilde{\alpha}$ enjoys this property. We may therefore assume that $\mathcal{G} = \mathcal{U}$ is an abelian, unipotent group scheme.

As before, we can find $r \in \mathbb{N}^n$ with

$$k\mathcal{U} = k[T_1,\ldots,T_n]/(T_1^{p^{r_1}},\ldots,T_n^{p^{r_n}}).$$

We begin by considering the case where $n = 1$ and write $k\mathcal{U} := k[T]/(T^{p^r}) = k[t]$ where $t := T + (T^{p^r-1})$. Then

$$k\mathcal{U}^*: \quad \cdots \longrightarrow k\mathcal{U} \xrightarrow{d_n} k\mathcal{U} \xrightarrow{d_{n-1}} k\mathcal{U} \longrightarrow \cdots \longrightarrow k\mathcal{U} \xrightarrow{d_1} k\mathcal{U} \xrightarrow{\varepsilon} k \longrightarrow (0)$$

with

$$d_n(u) := \begin{cases} tu & n \text{ odd} \\ t^{p^r-1}u & n \text{ even} \end{cases}$$

is a minimal projective resolution of the trivial $k\mathcal{U}$-module k. We denote the differentials of the corresponding resolution A_p^* for the trivial A_p-module by $\partial_n : A_p \longrightarrow$

A_p. Upon applying the exact functor α^* to $k\mathcal{U}^*$, we obtain an exact complex of A_p-modules. The comparison theorem [**41**, (6.9)] thus provides a chain map $(\alpha_n : A_p \longrightarrow \alpha^*(k\mathcal{U}))_{n\geq 0}$ with $\alpha_0 = \alpha$. By definition, we have

$$\alpha^\bullet(f_{2n}) = f_{2n} \circ \alpha_{2n} \qquad \forall\ f_{2n} \in \mathrm{Hom}_{k\mathcal{U}}(k\mathcal{U}, k) = H^{2n}(k\mathcal{U}, k).$$

In the following, we define an explicit chain map.

Setting $kE := k[t^{p^{r-1}}]$, we obtain $\mathrm{Rad}(kE) = t^{p^{r-1}}(kE)$, and (1.4(2)) provides an element $u \in k\mathcal{U}$ such that

$$\alpha(x) = t^{p^{r-1}}u.$$

(i) *There exists a chain map* $(\alpha_n : A_p \longrightarrow \alpha^*(k\mathcal{U}))_{n\geq 0}$ *with* $\alpha_{2n}(1) = u^{np}$ *for all* $n \geq 0$.

We proceed by induction on n, the case $n = 0$ being trivial. Assuming $n \geq 1$ and $\alpha_{2n-2}(1) = u^{(n-1)p}$, we define an A_p-linear map $\alpha_{2n-1} : A_p \longrightarrow \alpha^*(k\mathcal{U})$ via $\alpha_{2n-1}(1) = t^{p^{r-1}-1}u^{(n-1)p+1}$. This implies

$$\begin{aligned}\alpha_{2n-2}(\partial_{2n-1}(1)) &= \alpha_{2n-2}(x) = \alpha(x)\alpha_{2n-2}(1) = t^{p^{r-1}}uu^{(n-1)p} = t\alpha_{2n-1}(1)\\ &= d_{2n-1}(\alpha_{2n-1}(1)),\end{aligned}$$

where we have identified d_{2n-1} with $\alpha^*(d_{2n-1})$. Hence the definition $\alpha_{2n}(1) := u^{np}$ gives rise to

$$\begin{aligned}\alpha_{2n-1}(\partial_{2n}(1)) &= \alpha_{2n-1}(x^{p-1}) = \alpha(x)^{p-1}\alpha_{2n-1}(1)\\ &= t^{(p-1)p^{r-1}}u^{p-1}t^{p^{r-1}-1}u^{(n-1)p+1}\\ &= t^{p^r-1}u^{np} = d_{2n}(u^{np}) = d_{2n}(\alpha_{2n}(1)),\end{aligned}$$

as desired. $\diamond$

(ii) *Let* $\alpha : A_p \longrightarrow k[T]/(T^{p^r})$ *be a homomorphism such that* $\mathrm{im}\,\alpha^\bullet \neq k$. *Then* $\alpha_{(E)}(x) \notin \mathrm{Rad}(kE)^2$.

In view of (i), the assumption $\mathrm{im}\,\alpha^\bullet \neq k$ implies the existence of $n \geq 1$ and $f_{2n} \in \mathrm{Hom}_{k\mathcal{U}}(k\mathcal{U}, k)$ such that

$$0 \neq f_{2n}(\alpha_{2n}(1)) = f_{2n}(u^{np}) = \varepsilon(u)f_{2n}(u^{np-1}).$$

Consequently, (1.4(1)) yields $\mathrm{pr}_E(u) = \zeta 1 + v$ with $\zeta \in k \setminus \{0\}$ and $v \in \mathrm{Rad}(kE)$. As a result,

$$\alpha_{(E)}(x) = \mathrm{pr}_E(\alpha(x)) = t^{p^{r-1}}\mathrm{pr}_E(u) \equiv \zeta t^{p^{r-1}} \mod(\mathrm{Rad}(kE)^2),$$

as desired. $\diamond$

Returning to the general case, we set $k\mathcal{U}_i := k[T_i]/(T_i^{p^{r_i}})$ and put $t_i := T_i + (T_i^{p^{r_i}})$. By abuse of notation, we denote the canonical generators $T_i + (T_1^{p^{r_1}}, \dots, T_n^{p^{r_n}})$ in the same way. The map

$$f : k\mathcal{U} \longrightarrow k\mathcal{U}_1 \otimes_k \cdots \otimes_k k\mathcal{U}_n \ \ ; \ \ t_i \mapsto 1 \otimes \cdots \otimes 1 \otimes t_i \otimes 1 \otimes \cdots \otimes 1$$

is an isomorphism of augmented algebras. For $i \in \{1, \dots, n\}$, we consider the algebra homomorphism

$$\gamma_i : k\mathcal{U} \longrightarrow k\mathcal{U}_i \ \ ; \ \ t_j \mapsto \delta_{ij}t_i.$$

By definition, we have

$$(\gamma_i \circ f^{-1})(a_1 \otimes \cdots \otimes a_n) = \varepsilon(a_1)\cdots\varepsilon(a_{i-1})a_i\varepsilon(a_{i+1})\cdots\varepsilon(a_n).$$

By applying Lemma 2.3 repeatedly, starting with the map $f \circ \alpha$, we find an index $i \in \{1, \dots, n\}$ such that

$$k \neq \operatorname{im}[(f \circ \alpha)^\bullet \circ (\gamma_i \circ f^{-1})^\bullet] = \operatorname{im}(\alpha^\bullet \circ \gamma_i^\bullet).$$

As before, we define $kE := k[t_1^{p^{r_1-1}}, \dots, t_n^{p^{r_n-1}}]$ and set $kE_i := k[t_i^{p^{r_i-1}}]$. Using (1.4(2)), we write

$$\alpha(x) = \sum_{0 \leq b \leq \tau(r-\underline{\mathbf{1}})} v_b t^b,$$

where $v_b \in \operatorname{Rad}(kE)$. Setting $\alpha_i := \gamma_i \circ \alpha$, we obtain

$$\alpha_i(x) = \sum_{0 \leq b_i \leq p^{r_i-1}-1} v'_{b_i} t_i^{b_i},$$

where $v'_{b_i} = \gamma_i(v_{(0,\dots,0,b_i,0,\dots,0)})$. By virtue of (ii), the element $v'_0 = \gamma_i(v_0)$ does not belong to $\operatorname{Rad}(kE_i)^2$. Hence v_0 does not belong to $\operatorname{Rad}(kE)^2$, and [**23**, (2.2)] identifies $\alpha_{(E)}$ as a flat point of kE. We may now apply Lemma 1.6 to see that $\alpha \sim \iota_E \circ \alpha_{(E)}$ is a p-point of $k\mathcal{U}$. □

Given $M \in \operatorname{mod} k\mathcal{G}$, we consider the algebra homomorphism

$$\Phi_M : H^\bullet(\mathcal{G}, k) \longrightarrow \operatorname{Ext}^\bullet_{\mathcal{G}}(M, M) \quad ; \quad [f] \mapsto [f \otimes \operatorname{id}_M].$$

The *cohomological support variety* is the zero locus

$$\mathcal{V}_{\mathcal{G}}(M) := Z(\ker \Phi_M)$$

associated to the ideal $\ker \Phi_M$ of the finitely generated commutative k-algebra $H^\bullet(\mathcal{G}, k)$, cf. [**23**, (1.3)]. Since Φ_M is a homomorphism of degree 0, the variety $\mathcal{V}_{\mathcal{G}}(M)$ is conical. By the work of Friedlander-Pevtsova the corresponding projective varieties can be characterized via support spaces:

THEOREM 2.5 ([**23**],(4.11)). *The map*

$$\Psi_{\mathcal{G}} : \begin{cases} P(\mathcal{G}) & \longrightarrow & \operatorname{Proj}(\mathcal{V}_{\mathcal{G}}(k)) \\ \alpha & \mapsto & \ker \alpha^\bullet \end{cases}$$

is a homeomorphism such that

$$\Psi_{\mathcal{G}}^{-1}(\operatorname{Proj}(\mathcal{V}_{\mathcal{G}}(M))) = P(\mathcal{G})_M$$

for every $M \in \operatorname{mod} k\mathcal{G}$. □

In particular, the singletons of the topological space $P(\mathcal{G})$ are closed.

3. Auslander-Reiten components for finite group schemes

Given a finite group scheme $\mathcal{G}$, we denote by $\Gamma_s(\mathcal{G})$ the *stable Auslander-Reiten quiver* of the self-injective algebra $k\mathcal{G}$. By definition, $\Gamma_s(\mathcal{G})$ is a valued quiver, whose vertices are the isomorphism classes of the non-projective indecomposable $\mathcal{G}$-modules, and whose arrows are induced by irreducible morphisms $M \longrightarrow N$. In addition, $\Gamma_s(\mathcal{G})$ is fitted with an automorphism $\tau_{\mathcal{G}} : \Gamma_s(\mathcal{G}) \longrightarrow \Gamma_s(\mathcal{G})$, which reflects homological properties of $\operatorname{mod} k\mathcal{G}$. As $k\mathcal{G}$ is self-injective,

$$\tau_{\mathcal{G}} = \mathcal{N}_{\mathcal{G}} \circ \Omega^2_{\mathcal{G}} = \Omega^2_{\mathcal{G}} \circ \mathcal{N}_{\mathcal{G}}$$

is the composite of the square of the Heller operator $\Omega_{\mathcal{G}}$ with the Nakayama functor $\mathcal{N}_{\mathcal{G}}$, cf. [**1**, (IV.3.7)]. For Frobenius algebras, such as $k\mathcal{G}$, $\mathcal{N}_{\mathcal{G}}$ is the twist by a Nakayama automorphism ν of $k\mathcal{G}$, so that

$$\tau_{\mathcal{G}} = \nu^* \circ \Omega^2_{\mathcal{G}} = \Omega^2_{\mathcal{G}} \circ \nu^*.$$

If $\zeta : k\mathcal{G} \longrightarrow k$ denotes the modular function of the Hopf algebra $k\mathcal{G}$, then the convolution

$$\nu := \zeta * \mathrm{id}_{k\mathcal{G}}$$

is a Nakayama automorphism of the Frobenius algebra $k\mathcal{G}$ (see [**20**, (1.5)]). In particular, the Nakayama functor has order a divisor of $\dim_k k\mathcal{G}$. We refer the reader to [**1, 2**] for further details on Auslander-Reiten theory.

Recall that

$$P(\mathcal{G})_M = \{[\alpha] \in P(\mathcal{G}) \ ; \ \alpha^*(M) \text{ is not projective}\}$$

is the support space of M. Our first subsidiary result shows that $M \mapsto P(\mathcal{G})_M$ is constant on the connected components of the quiver $\Gamma_s(\mathcal{G})$.

LEMMA 3.1. *Let $\Theta \subset \Gamma_s(\mathcal{G})$ be a connected component. Then we have*

$$P(\mathcal{G})_M = P(\mathcal{G})_N$$

for every $[M], [N] \in \Theta$.

PROOF. We begin by showing that $P(\mathcal{G})_M = P(\mathcal{G})_{\tau_{\mathcal{G}}(M)}$. To that end, let $\alpha : A_p \longrightarrow k\mathcal{G}$ be a p-point. Then there exists an abelian, unipotent subgroup $\mathcal{U} \subset \mathcal{G}$ with $\operatorname{im} \alpha \subset k\mathcal{U}$. Since $\mathcal{U}$ is unipotent, we have $\zeta|_{k\mathcal{U}} = \varepsilon|_{k\mathcal{U}}$, so that

$$\nu \circ \alpha = \alpha.$$

We therefore obtain

$$P(\mathcal{G})_{\tau_{\mathcal{G}}(M)} = P(\mathcal{G})_{\Omega^2_{\mathcal{G}}(M)} = P(\mathcal{G})_M,$$

as desired.

Consider the almost split sequence

$$(0) \longrightarrow \tau_{\mathcal{G}}(N) \longrightarrow X_N \longrightarrow N \longrightarrow (0)$$

terminating in N. By the above and [**23**, (5.6)], every direct summand M of X_N gives rise to inclusions

$$P(\mathcal{G})_M \subset P(\mathcal{G})_{X_N} \subset P(\mathcal{G})_{\tau_{\mathcal{G}}(N)} \cup P(\mathcal{G})_N = P(\mathcal{G})_N.$$

Consequently,

$$P(\mathcal{G})_M \subset P(\mathcal{G})_N$$

whenever there is an arrow $[M] \to [N]$ in $\Gamma_s(\mathcal{G})$. Owing to [**1**, (V.1.12,V.5.3)] there also is an arrow $[\tau_{\mathcal{G}}(N)] \to [M]$, so that

$$P(\mathcal{G})_N = P(\mathcal{G})_{\tau_{\mathcal{G}}(N)} \subset P(\mathcal{G})_M.$$

Our claim now follows from the connectedness of Θ. □

In view of Lemma 3.1, we define for a component $\Theta \subset \Gamma_s(\mathcal{G})$

$$P(\mathcal{G})_\Theta := P(\mathcal{G})_M \qquad \forall\ [M] \in \Theta.$$

The quiver $\Gamma_s(\mathcal{G})$ is a stable representation quiver in the sense of Riedtmann. Thanks to [**39**, Struktursatz] a connected component $\Theta \subset \Gamma_s(\mathcal{G})$ is of the form

$$\Theta \cong \mathbb{Z}[T_\Theta]/\Pi,$$

where Π is a directed tree and $\Pi \subset \mathrm{Aut}(\mathbb{Z}[T_\Theta])$ is an "admissible group". Moreover, the underlying graph $\bar{T}_\Theta$ of T_Θ is uniquely determined by Θ, and is customarily referred to as the *tree class* of Θ. If the underlying module category has enough periodic modules, then the list of possible tree classes can be narrowed down considerably. The following extension of Webb's Theorem [**46**] to finite group schemes, which can also be found in [**12**, (5.7)], illustrates this point.

Proposition 3.2. *Let $\Theta \subset \Gamma_s(\mathcal{G})$ be a component of the stable Auslander-Reiten quiver. Then the tree class $\bar{T}_\Theta$ is either a finite or infinite Dynkin diagram, or a Euclidean diagram.*

Proof. Since $P(\mathcal{G})_\Theta \neq \emptyset$, we can find a p-point α such that $\alpha^*(M)$ is not projective for every $[M] \in \Theta$. Consider the module $M_\alpha := k\mathcal{G} \otimes_{A_p} k$. Since α factors through an abelian, unipotent subgroup $\mathcal{U} \subset \mathcal{G}$, and $k\mathcal{G}$ is a free right $k\mathcal{U}$-module (cf. [**34**, (2.6)]), $k\mathcal{G}$ is a projective right A_p-module. By general properties of the Heller operator, we thus have

$$M_\alpha \cong \Omega^2_{\mathcal{G}}(M_\alpha) \oplus (\mathrm{proj}).$$

Observing $\nu \circ \alpha = \alpha$, we obtain $\nu^*(M_\alpha) \cong M_\alpha$, whence

$$M_\alpha = \tau_{\mathcal{G}}(M_\alpha) \oplus (\mathrm{proj}).$$

Let $[M]$ be an element of Θ. Frobenius reciprocity in conjunction with our choice of α implies

$$\mathrm{Ext}^1_{\mathcal{G}}(M_\alpha, M) \cong \mathrm{Ext}^1_{A_p}(k, M) \neq (0),$$

rendering our result a consequence of [**13**, (3.2)]. □

In view of (3.2) one would like to know, which components actually appear, and how the occurrence of certain types of components is related to structural features of the group scheme $\mathcal{G}$. For finite groups, Euclidean tree classes only occur for $p = 2$ [**35**], and in that case the corresponding block $\mathcal{B} \subset kG$ is Morita equivalent to the group algebra of the Klein 4-group [**5**]. Among the finite Dynkin diagrams, only A_n occurs (see [**40**] and [**4**, (3.7)]), while all of the infinite tree classes ($A_\infty, A_\infty^\infty$, D_∞) actually appear, cf. also [**13**, (4.2),(4.3)].

Before addressing these issues within the context of finite group schemes, let us characterize certain components via their support spaces. Recall that an indecomposable $\mathcal{G}$-module is said to be *periodic* if there exists $n \in \mathbb{N}$ such that

$$\Omega^n_{\mathcal{G}}(M) \cong M.$$

For instance, if the algebra $k\mathcal{G}$ has finite representation type (that is, when $\operatorname{mod} k\mathcal{G}$ possesses up to isomorphism only finitely many indecomposable objects), then every non-projective indecomposable $\mathcal{G}$-module is periodic. Since the Nakayama functor of $\operatorname{mod} k\mathcal{G}$ has finite order, periodic modules are also periodic with respect to the Auslander-Reiten translation $\tau_{\mathcal{G}}$. The infinite periodic components consisting of periodic modules are called *infinite tubes*. In view of [**28**], they are of the form

$$\mathbb{Z}[A_\infty]/(\tau^m) \qquad m \geq 1,$$

where τ denotes the translation of the stable representation quiver $\mathbb{Z}[A_\infty]$.

Theorem 3.3. *Let M be a non-projective indecomposable $\mathcal{G}$-module, $\Theta \subset \Gamma_s(\mathcal{G})$ be a component.*

(1) *The topological space $P(\mathcal{G})_M$ is connected.*

(2) *If $|P(\mathcal{G})_M| = 1$, then M is periodic.*

(3) *$|P(\mathcal{G})_\Theta| = 1$ if and only if either $\bar{T}_\Theta$ is a finite Dynkin diagram, or Θ is an infinite tube.*

(4) *If $\Theta \not\cong \mathbb{Z}[A_\infty]$, then $|P(\mathcal{G})_\Theta| = 1$ or $\dim P(\mathcal{G})_\Theta = 1$.*

Proof. (1) If $[\zeta] \in H^n(\mathcal{G}, k)$ is a homogeneous element of $H^\bullet(\mathcal{G}, k)$ and L_ζ is the kernel of the corresponding map $\Omega^n_{\mathcal{G}}(M) \longrightarrow k$, then [**18**, (2.1)] yields

$$(*) \qquad Z(\zeta) = \mathcal{V}_{\mathcal{G}}(L_\zeta).$$

Thus, [**7**, (2.3)] also holds for finite group schemes, and the arguments of [**7**, §3] show that $\operatorname{Proj}(\mathcal{V}_{\mathcal{G}}(M))$ is connected. Owing to Theorem 2.5 the spaces $\operatorname{Proj}(\mathcal{V}_{\mathcal{G}}(M))$ and $P(\mathcal{G})_M$ are homeomorphic, so that $P(\mathcal{G})_M$ is also connected.

(2) Let N be a $\mathcal{G}$-module. According to (2.5) the dimension of the Noetherian space $P(\mathcal{G})_N$ is given by

$$(**) \qquad \dim P(\mathcal{G})_N = \dim \mathcal{V}_{\mathcal{G}}(N) - 1.$$

Thus, $\dim \mathcal{V}_{\mathcal{G}}(M) = 1$, and there exists $[\zeta] \in H^n(\mathcal{G}, k)$ such that $Z(\zeta) \cap \mathcal{V}_{\mathcal{G}}(M) = \{0\}$. In view of $(*)$ we may now adopt the arguments of [**3**, (5.10.4)].

(3) Suppose that $|P(\mathcal{G})_\Theta| = 1$ and let $[N] \in \Theta$ be a vertex. Then (2) ensures the periodicity of N. As a result, Θ contains a $\tau_{\mathcal{G}}$-periodic module and [**28**, Theorem] yields the desired conclusion.

In view of (1), we may complete the proof of (3) by showing that the stated conditions on Θ imply $\dim P(\mathcal{G})_\Theta = 0$. As in [**15**, (2.1)], it follows that Θ contains a periodic vertex $[N]$, so that $\dim \mathcal{V}_{\mathcal{G}}(N) = 1$.

(4) This statement follows by adopting the arguments of [**15**, (2.2)] mutatis mutandis. □

By the foregoing result, components of type $\mathbb{Z}[A_\infty]$ usually occur most frequently. We shall illustrate below that components of other types naturally appear for blocks of finite and tame representation type, respectively.

Let $\mathcal{G}$ be a finite group scheme. Then

$$\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}}$$

decomposes into a semidirect product with an infinitesimal, normal subgroup $\mathcal{G}^0$ and a reduced group $\mathcal{G}_{\mathrm{red}}$, cf. [**45**, (6.8)]. There results an isomorphism

$$k\mathcal{G} \cong (k\mathcal{G}^0)[\mathcal{G}(k)],$$

where the right-hand side denotes the skew group algebra defined by $k\mathcal{G}^0$ and the finite group $\mathcal{G}(k)$ of k-rational points of $\mathcal{G}$. If $\mathcal{G}^0$ has height 1, then $k\mathcal{G}^0 \cong U_0(\mathfrak{g})$ is the restricted enveloping algebra of the restricted Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$, cf. [**11**, (II,§7,4.2)].

Let $G \subset \mathrm{SL}(2)(k)$ be a finite group such that $p \nmid |G|$. These groups were classified by Klein and are known to be the binary polyhedral groups. Each such group G is completely determined by its McKay quiver Δ_G relative to its 2-dimensional standard module, with the underlying graph $\bar{\Delta}_G$ being a Euclidean diagram of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_{6,7,8}$, see [**42, 29**]. Note that adjoint representation induces an action of G on $U_0(\mathfrak{sl}(2))$ via automorphisms.

Proposition 3.4. *Suppose that $p \geq 3$. Let $G \subset \mathrm{SL}(2)(k)$ be a binary polyhedral group, and consider the skew group algebra*

$$\Lambda := U_0(\mathfrak{sl}(2))[G].$$

Then Λ is the algebra of measures of a finite group scheme $\mathcal{G}_G$, and if $\bar{\Delta}_G$ is a tree, then $\Gamma_s(\mathcal{G}_G)$ contains 2 components of tree class $\bar{\Delta}_G$. In particular, components with Euclidean tree classes $\tilde{D}_n, \tilde{E}_{6,7,8}$ occur.

Proof. Let $\hat{G} := \mathrm{Spec}(kG^*)$ be the reduced group scheme, whose group of k-rational points coincides with G. In view of [**17**, (3.2)] we may consider $\hat{G}$ a closed subgroup of SL(2). Since $\hat{G}(k) = G$,

$$\mathcal{G}_G := \mathrm{SL}(2)_1\hat{G} \cong \mathrm{SL}(2)_1 \rtimes \hat{G}$$

is the desired finite group scheme. According to [**17**, (6.2.2)], the principal block $\mathcal{B}_0(\mathcal{G}_G)$ has tame representation type, and [**17**, (7.3.1)] provides a Morita equivalence

$$\mathcal{B}_0(\mathcal{G}_G) \sim_M H_G \ltimes H_G^*,$$

between $\mathcal{B}_0(\mathcal{G}_G)$ and the trivial extension of H_G by its dual bimodule. Here $H_G = k[\tilde{\Delta}_G]$ is the path algebra of a quiver with underlying graph $\bar{\Delta}_G$, oriented such that there are no paths of length 2. (This requires n to be odd if $\bar{\Delta}_G = \tilde{A}_n$.) Owing to [**27**, (V.3.2)] the stable Auslander-Reiten quiver of the symmetric algebra $H_G \ltimes H_G^*$ has two components of type $\mathbb{Z}[\tilde{\Delta}_G]$. The tree class of this component is $\bar{\Delta}_G$, whenever Δ_G is a tree. In the exceptional cases $(\tilde{A}_n)_{n\geq 2}$, the tree class is A_∞^∞. □

We turn to components $\Theta \subset \Gamma_s(\mathcal{G})$, whose tree classes $\bar{T}_\Theta$ are finite Dynkin diagrams. In view of (3.3) such components are necessarily finite, and Auslander's Theorem (cf. [**1**, (VII.2.1)] or [**2**, (4.13.6)]) implies the existence of a representation-finite block $\mathcal{B} \subset k\mathcal{G}$ such that Θ consists of the set of non-projective, indecomposable $\mathcal{B}$-modules. In case $\mathcal{G}$ is infinitesimal of height 1, or a Frobenius kernel of a smooth group, such blocks are Nakayama algebras (cf. [**14**, (3.2)], [**16**, (3.1)]), so that $\bar{T}_\Theta \cong A_n$ for some $n \in \mathbb{N}_0$. Moreover, there exists $m \in \mathbb{N}$ such that $\Theta \cong \mathbb{Z}[A_n]/(\tau^m)$, see [**1**, p. 253] for more details.

Given a finite group scheme $\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\rm red}$, we let $\iota : k\mathcal{G}^0 \longrightarrow k\mathcal{G}$ and $j : k\mathcal{G}_{\rm red} \longrightarrow k\mathcal{G}$ be the corresponding inclusions. As noted earlier, these homomorphisms induce maps

$$\iota_* : P(\mathcal{G}^0) \longrightarrow P(\mathcal{G}) \ \text{ and } \ j_* : P(\mathcal{G}_{\rm red}) \longrightarrow P(\mathcal{G}),$$

respectively. We have

$$\iota_*(P(\mathcal{G}^0)_{\iota^*(M)}) \subset P(\mathcal{G})_M \quad \forall\ M \in \operatorname{mod} k\mathcal{G}$$

with the anologous inclusion holding for j_*. We denote by $\pi : k\mathcal{G} \longrightarrow k\mathcal{G}_{\rm red}$ the canonical projection, so that $\pi \circ j = \mathrm{id}_{k\mathcal{G}_{\rm red}}$.

Let $\varphi : k\mathcal{G} \longrightarrow k\mathcal{G}$ be an automorphism of the k-algebra $k\mathcal{G}$. Given a $k\mathcal{G}$-module M, we let $M^{(\varphi)}$ be the $k\mathcal{G}$-module with underlying k-space M, and twisted action

$$a\boldsymbol{.}m := \varphi^{-1}(a)m \quad \forall\ a \in k\mathcal{G},\ m \in M.$$

Thus, $M^{(\varphi)} = (\varphi^{-1})^*(M)$ and direct computation shows that

$$P(\mathcal{G})_{M^{(\varphi)}} = \varphi_*(P(\mathcal{G})_M) \quad \forall\ M \in \operatorname{mod} k\mathcal{G}.$$

If $M^{(\varphi)} \cong M$, then $P(\mathcal{G})_M$ is φ_*-invariant. By applying this to the automorphism of $k\mathcal{G}^0$ effected by an element $g \in \mathcal{G}(k)$, we see that $\mathcal{G}(k)$ acts on the fibres of the map ι_*.

Lemma 3.5. *The following statements hold:*
(1) *The fibres of ι_* are the $\mathcal{G}(k)$-orbits of $P(\mathcal{G}^0)$.*
(2) *The map j_* is injective.*
(3) $\iota_*(P(\mathcal{G}^0)) \cap j_*(P(\mathcal{G}_{\rm red})) = \emptyset$.

Proof. (1) This follows directly from [**23**, (5.2)].

(2) Let $\alpha, \beta : A_p \longrightarrow \mathcal{G}_{\rm red}$ be p-points such that $j_*([\alpha]) = j_*([\beta])$. Given a $\mathcal{G}_{\rm red}$-module M, we have

$$(j \circ \alpha)^*(\pi^*(M)) = \alpha^*(M).$$

Consequently, $j \circ \alpha \sim j \circ \beta$ implies $\alpha \sim \beta$, as desired.

(3) Let P be a projective $k\mathcal{G}_{\rm red}$-module. Then $k\mathcal{G}^0$ acts trivially on $\pi^*(P)$. If $\alpha : A_p \longrightarrow k\mathcal{G}^0$ and $\beta : A_p \longrightarrow k\mathcal{G}_{\rm red}$ are p-points, then $\alpha^*(\iota^*(\pi^*(P)))$ is a trivial A_p-module, while $\beta^*(j^*(\pi^*(P))) = \beta^*(P)$ is projective. Consequently, $\iota \circ \alpha \not\sim j \circ \beta$, so that $\operatorname{im} \iota_* \cap \operatorname{im} j_* = \emptyset$. □

Proposition 3.6. *Let $\mathcal{B} \subset k\mathcal{G}$ be a block.*

(1) *There exists a block $\mathcal{C}_\mathcal{B} \subset k\mathcal{G}^0$ such that $\mathcal{B}$ is Morita equivalent to a block of the skew group algebra $\mathcal{C}_\mathcal{B}[G_\mathcal{B}]$, given by the stabilizer $G_\mathcal{B} := \{g \in \mathcal{G}(k)\ ;\ g(\mathcal{C}_\mathcal{B}) = \mathcal{C}_\mathcal{B}\}$.*

(2) *For every simple $\mathcal{B}$-module S, there exists a simple $\mathcal{C}_\mathcal{B}$-submodule $T_S \subset S$, a subset $R_{T_S} \subset \mathcal{G}(k)$ and $m_S \in \mathbb{N}$ such that*

$$S|_{k\mathcal{G}^0} \cong \bigoplus_{g \in R_{T_S}} m_S T_S^{(g)}.$$

(3) *If $\pi(\mathcal{B}) \neq (0)$, then $\mathcal{C}_\mathcal{B} = \mathcal{B}_0(\mathcal{G})$ and $G_\mathcal{B} = \mathcal{G}(k)$.*

(4) *If $\mathcal{B}$ is representation-finite, then one of the following statements holds:*

(a) *$C_{\mathcal{B}} \cong \mathrm{Mat}_n(k)$ is simple, with n dividing the dimension of any simple $\mathcal{B}$-module.*

(b) *For every simple $\mathcal{B}$-module S, the restriction $j^*(S)$ is projective. In particular, if $\pi(\mathcal{B}) \neq (0)$, then the block ideal $\pi(\mathcal{B}) \lhd k\mathcal{G}_{\mathrm{red}}$ is a semi-simple algebra, and $k\mathcal{G}^0$ is representation-finite.*

Proof. (1),(2) Set $\Lambda = k\mathcal{G}^0$ and decompose Λ into its blocks

$$\Lambda = \Lambda e_1 \oplus \cdots \oplus \Lambda e_n.$$

The finite group $G := \mathcal{G}(k)$ acts on the set $\{e_1, \ldots, e_n\}$ of central, orthogonal, primitive idempotents of Λ, and we may assume without loss of generality that $e_1, \ldots, e_m$ $(m \leq n)$ are the orbit representatives. Setting $\Lambda_i := \bigoplus_{e \in Ge_i} \Lambda e$ for $i \in \{1, \ldots, m\}$, we obtain a decomposition

$$\Lambda[G] = \bigoplus_{i=1}^{m} \Lambda_i[G]$$

of $\Lambda[G]$ into two-sided ideals. Thus, there exists $i_0 \in \{1, \ldots, m\}$ such that $\mathcal{B}$ is a summand (block) of $\Lambda_{i_0}[G]$. Thanks to [**38**, (1.6)] the algebra $\Lambda_{i_0}[G]$ is Morita equivalent to $\mathcal{C}_{\mathcal{B}}[G_{\mathcal{B}}]$, where $\mathcal{C}_{\mathcal{B}}$ is the block Λe_{i_0} of $k\mathcal{G}^0$.

Given a Λ-module T, we let

$$G_T := \{g \in G \ ; \ T^{(g)} \cong T\}$$

be its inertia group.

Let S be a simple $\mathcal{B}$-module, $T \subset S$ a simple Λ-submodule. Clifford theory (cf. [**2**, (3.13)]) then provides $m \in \mathbb{N}$ such that

$$(*) \qquad S|_{\Lambda} \cong \bigoplus_{g \in R} mT^{(g)},$$

where $R \subset G$ is a complete set of left coset representatives of G_T.

Since $\Lambda_{i_0}[G].S \neq (0)$, we conclude that $\Lambda_{i_0}.S \neq (0)$. Hence there is an element $g' \in R$ with $\Lambda_{i_0} T^{(g')} \neq (0)$. This in turn provides $g \in G$ such that $g(e_{i_0})T \neq (0)$. Consequently, the simple Λ-submodule $T' := g^{-1}T \cong T^{(g^{-1})}$ of S belongs to $\mathcal{C}_{\mathcal{B}}$, so that assertion (2) follows by applying $(*)$ to T'.

(3) The image $\pi(e)$ of the central idempotent defining $\mathcal{B}$ is a central idempotent of $k\mathcal{G}_{\mathrm{red}}$ such that $\pi(\mathcal{B}) = k\mathcal{G}_{\mathrm{red}}\pi(e)$. Consequently, $\pi(\mathcal{B})$ is a block ideal, which is an algebra whenever $\pi(\mathcal{B}) \neq (0)$. In that case, let V be a simple $\pi(\mathcal{B})$-module. Then $S := \pi^*(V)$ is a simple Λ-module belonging to $\mathcal{B}$, whose restriction $S|_{\Lambda}$ is a direct sum of copies of the trivial Λ-module k. Consequently, we have $T_S = k$, so that $\mathcal{C}_{\mathcal{B}} = \mathcal{B}_0(\mathcal{G}^0)$ is the principal block of $k\mathcal{G}^0$. As $\mathcal{G}(k)$ acts on $k\mathcal{G}^0$ via automorphisms of Hopf algebras, the principal block $\mathcal{B}_0(\mathcal{G}^0)$ is $\mathcal{G}(k)$-invariant, whence $G_{\mathcal{B}} = \mathcal{G}(k)$.

(4) Let $\mathcal{S}(\mathcal{B})$ be a complete set of representatives for the simple $\mathcal{B}$-modules. Since $\mathcal{B}$ has finite representation type, $\mathcal{B}$ is simple or every simple module $S \in \mathcal{S}(\mathcal{B})$ is periodic [**30**]. In view of (3.3) we thus have

$$|P(\mathcal{G})_S| \leq 1 \quad \forall \ S \in \mathcal{S}(\mathcal{B}).$$

Let $S \in \mathcal{S}(\mathcal{B})$ be a simple $\mathcal{B}$-module. Basic properties of support spaces [**23**, (5.6)] in conjunction with (2) imply

$$\bigcup_{g \in R_{T_S}} \iota_*(g_*(P(\mathcal{G}^0)_{T_S})) = \iota_*(P(\mathcal{G}^0)_{\iota^*(S)}) \subset P(\mathcal{G})_S.$$

Suppose first that $P(\mathcal{G}^0)_{T_S} = \emptyset$, so that T_S is a projective $\mathcal{C}_{\mathcal{B}}$-module, cf. [**23**, (5.6)]. Wedderburn's Theorem provides an isomorphism $\mathcal{C}_{\mathcal{B}} \cong \mathrm{Mat}_n(k)$ with $n := \dim_k T_S$. Owing to (2), the number n divides the dimension of any simple $\mathcal{B}$-module.

Alternatively, we may assume that $\iota_*(P(\mathcal{G}^0)_{\iota^*(S)}) \neq \emptyset$ for every $S \in \mathcal{S}(\mathcal{B})$. Thanks to (3.3(3)), $P(\mathcal{G})_S$ is a singleton, and Lemma 3.5(3) implies $P(\mathcal{G}_{\mathrm{red}})_{j^*(S)} = \emptyset$, so that the first part of (b) holds. If T is a simple $\pi(\mathcal{B})$-module, then $S := \pi^*(T)$ is a simple $\mathcal{B}$-module, whose restriction

$$j^*(S) = (\pi \circ j)^*(T) = T$$

is projective. Consequently, $\pi(\mathcal{B})$ is semi-simple. By the arguments of (3), we have $T_S = k$, $\mathcal{C}_{\mathcal{B}} = \mathcal{B}_0(k)$ and $G_{\mathcal{B}} = \mathcal{G}(k)$. Since $\iota^*(S)$ is a trivial module, (3.5(1)) shows that $P(\mathcal{G}^0)$ is finite. By virtue of (2.5) and [**19**, (2.7)] the algebra $k\mathcal{G}^0$ has finite representation type. □

Remark. By the proof of (3.6) the block $\mathcal{C}_{\mathcal{B}}$ and the group $G_{\mathcal{B}}$ are unique up to $\mathcal{G}(k)$-conjugacy. Moreover, the $\mathcal{C}_{\mathcal{B}}$-module T_S is unique up to twist by an element of $G_{\mathcal{B}}$.

The main shortfall of the foregoing result is the lack of information concerning $\mathcal{C}_{\mathcal{B}}$ and $G_{\mathcal{B}}$. There are special cases, where more is known. If $\mathcal{G}^0$ is a group of height 1, then the existence of a simple $\mathcal{C}_{\mathcal{B}}$-module T with finite support space $P(\mathcal{G}^0)_T$ already forces $\mathcal{C}_{\mathcal{B}}$ to be a Nakayama algebra (cf. [**23**, (5.6)] and [**14**, (3.2)]). Another tractable case pertains to the principal block $\mathcal{B}_0(\mathcal{G})$, enabling us to retrieve [**19**, (3.1)]:

Corollary 3.7. *Suppose that $\mathcal{B}_0(\mathcal{G})$ has finite representation type. Then $k\mathcal{G}^0$ and $k\mathcal{G}_{\mathrm{red}}$ have finite representation type, and at least one of these algebras is semi-simple.*

Proof. Since the idempotent e_0 of $k\mathcal{G}^0$ is not contained in the augmentation ideal $(k\mathcal{G})^\dagger$ of $k\mathcal{G}$, we have $e_0 \notin \ker \pi = k\mathcal{G}(k\mathcal{G}^0)^\dagger \subset (k\mathcal{G})^\dagger$. Owing to (3.6(3)) we obtain $\mathcal{C}_{\mathcal{B}_0(\mathcal{G})} = \mathcal{B}_0(\mathcal{G}^0)$ and $G_{\mathcal{B}_0(\mathcal{G})} = \mathcal{G}(k)$. Thanks to (3.6(2)) condition (a) of (3.6(4)) implies that k is a projective $k\mathcal{G}^0$-module, so that $k\mathcal{G}^0$ is semi-simple. Nagata's Theorem [**11**, (IV,§3,3.6)] now shows that $\mathcal{G}^0$ is diagonalizable and [**17**, (1.1)] yields

$$\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}_{\mathrm{red}}).$$

According to standard defect theory (cf. [**2**, (6.1.2),(6.3.5),(4.4.4)]), the group algebra $k\mathcal{G}(k) = k\mathcal{G}_{\mathrm{red}}$ is representation-finite.

Alternatively, (3.6(4b)) ensures the semi-simplicity of $\pi(\mathcal{B}_0(\mathcal{G}))$, so that k is a projective $k\mathcal{G}_{\mathrm{red}}$-module. Consequently, the group algebra $k\mathcal{G}_{\mathrm{red}}$ is semi-simple. By the same token, the algebra $k\mathcal{G}^0$ is representation-finite. □

PROPOSITION 3.8. *Let $\mathcal{B} \subset k\mathcal{G}$ be a block. Assume there exists a simple $\mathcal{B}$-module S such that $S|_{k\mathcal{G}^0}$ is projective. Then the following statements hold:*

(1) *If $\dim_k T_S = p^m$, then T_S possesses a $k\mathcal{G}^0[G_{\mathcal{B}}]$-structure that extends the given $\mathcal{G}^0$-structure.*

(2) *If the $k\mathcal{G}_0$-structure of T_S can be extended to $k\mathcal{G}^0[G_{\mathcal{B}}]$, then $\mathcal{B}$ is Morita equivalent to the principal block of the group algebra $kG_{\mathcal{B}}$.*

PROOF. (1) By assumption, the simple $\mathcal{G}^0$-module T_S is projective, so that $\mathcal{C}_{\mathcal{B}} \cong \mathrm{Mat}_n(k)$. In view of the Noether-Skolem Theorem, any automorphism of $\mathcal{C}_{\mathcal{B}}$ is inner, thereby providing a homomorphism $\varphi : G_{\mathcal{B}} \longrightarrow \mathrm{PGL}(n)(k)$ satisfying $g.x = \varphi(g)x\varphi(g)^{-1}\ \forall\ g \in G_{\mathcal{B}},\ x \in \mathcal{C}_{\mathcal{B}}$. Our current assumption implies $n = p^m$, whence $\mathrm{GL}(n)(k) = \mathrm{SL}(n)(k) \times k^{\times}$, so that there exists a homomorphism $\psi : G_{\mathcal{B}} \longrightarrow C_{\mathcal{B}}^{\times}$ with

$$g.x = \psi(g)x\psi(g)^{-1} \qquad \forall\ g \in G_{\mathcal{B}},\ x \in \mathcal{C}_{\mathcal{B}}.$$

Consequently,

$$(\lambda g).t := \lambda\psi(g)t \qquad \forall\ \lambda \in k\mathcal{G}^0,\ g \in G_{\mathcal{B}},\ t \in T_S$$

defines an action of $k\mathcal{G}^0[G_{\mathcal{B}}]$ on T_S that extends the given one.

(2) We let $e_{\mathcal{B}} \in k\mathcal{G}$ and $e_{\mathcal{C}} \in k\mathcal{G}^0$ be the central idempotents defining the blocks $\mathcal{B} \subset k\mathcal{G}$ and $\mathcal{C}_{\mathcal{B}} \subset k\mathcal{G}^0$, respectively. Let $\mathcal{G}_{\mathcal{B}} := \mathrm{Spec}((kG_{\mathcal{B}})^*)$ be the reduced group such that $\mathcal{G}_{\mathcal{B}}(k) \cong G_{\mathcal{B}}$. Owing to [**17**, (3.2)] we may consider $\mathcal{G}_{\mathcal{B}}$ a closed subgroup of $\mathcal{G}_{\mathrm{red}}$. Thus, the finite group scheme

$$\bar{\mathcal{G}} := \mathcal{G}^0 \rtimes \mathcal{G}_{\mathcal{B}}$$

has algebra of measures $k\bar{\mathcal{G}} \cong k\mathcal{G}^0[G_{\mathcal{B}}]$, and T_S is a simple $\bar{\mathcal{G}}$-module. Consequently, $g.e_{\mathcal{C}} = e_{\mathcal{C}}$ for every $g \in G_{\mathcal{B}}$, implying that $e_{\mathcal{C}}$ is a central idempotent of $k\bar{\mathcal{G}}$.

Let $\mathrm{mod}\,\mathcal{G}_{\mathcal{B}}$ and $\mathrm{mod}(e_{\mathcal{C}}\bar{\mathcal{G}})$ be the category of $\mathcal{G}_{\mathcal{B}}$-modules and the full subcategory of $\bar{\mathcal{G}}$-modules V satisfying $e_{\mathcal{C}}V = V$, respectively. As $e_{\mathcal{C}}$ is a central idempotent of $k\bar{\mathcal{G}}$, $\mathrm{mod}(e_{\mathcal{C}}\bar{\mathcal{G}})$ is a sum of blocks of $\mathrm{mod}\,k\mathcal{G}$. In analogy with [**33**, (II.10.4)] we claim that

$$\mathrm{mod}\,\mathcal{G}_{\mathcal{B}} \longrightarrow \mathrm{mod}(e_{\mathcal{C}}\bar{\mathcal{G}}) \ \ ; \ \ M \mapsto M \otimes_k T_S$$

is an equivalence. Here, $M \in \mathrm{mod}\,\mathcal{G}_{\mathcal{B}}$ is considered a $\bar{\mathcal{G}}$-module with trivial $\mathcal{G}^0$-action. This readily implies that the above functor is well-defined.

We consider the functor

$$\mathrm{mod}(e_{\mathcal{C}}\bar{\mathcal{G}}) \longrightarrow \mathrm{mod}\,\mathcal{G}_{\mathcal{B}} \ \ ; \ \ V \mapsto \mathrm{Hom}_{\mathcal{G}^0}(T_S, V).$$

Owing to [**33**, (I.2.14(3))] the natural transformation

$$\varphi_V : \mathrm{Hom}_{\mathcal{G}^0}(T_S, V) \otimes_k T_S \longrightarrow V \ \ ; \ \ f \otimes t \mapsto f(t)$$

is an isomorphism for every $V \in \mathrm{mod}(e_{\mathcal{C}}\bar{\mathcal{G}})$. Given $M \in \mathrm{mod}\,\mathcal{G}_{\mathcal{B}}$, the map

$$\psi_M : M \longrightarrow \mathrm{Hom}_{\mathcal{G}^0}(T_S, M \otimes_k T_S) \ \ ; \ \ \psi_M(m)(t) = m \otimes t$$

is injective, and thus, for dimension reasons, bijective.

As $\mathrm{mod}(e_{\mathcal{C}}\bar{\mathcal{G}})$ is a sum of blocks of $\mathrm{mod}\,k\mathcal{G}$, the above equivalence sends the category $\mathrm{mod}\,\mathcal{B}_0(\mathcal{G}_{\mathcal{B}})$ to the block of $\mathrm{mod}\,k\bar{\mathcal{G}}$ containing the simple $\bar{\mathcal{G}}$-module T_S. By the proof of [**38**, (1.6)] we have

$$\mathcal{C}_{\mathcal{B}}[G_{\mathcal{B}}] = e_{\mathcal{C}}k\mathcal{G}e_{\mathcal{C}},$$

as well as a Morita equivalence $\mathcal{B} \sim_M e_{\mathcal{C}}\mathcal{B}e_{\mathcal{C}}$. Thus, $e_{\mathcal{C}}S$ is a simple $e_{\mathcal{C}}\mathcal{B}e_{\mathcal{C}}$-module. Since $G_{\mathcal{B}}$ is the inertia group of T_S, Proposition 3.6(2) provides an isomorphism

$e_{\mathcal{C}}S \cong m_S T_S$ of $\mathcal{G}^0$-modules, so that the $\bar{\mathcal{G}}$-module T_S belongs to $e_{\mathcal{C}}\mathcal{B}e_{\mathcal{C}}$. As a result, $\mathcal{B}$ is Morita equivalent to $\mathcal{B}_0(\mathcal{G}_{\mathcal{B}}) \cong \mathcal{B}_0(G_{\mathcal{B}})$. □

Suppose that $\mathcal{G} \subset \mathcal{H}$ is a closed finite algebraic subgroup of a smooth reductive group $\mathcal{H}$ such that $\mathcal{G}^0 = \mathcal{H}_r$ is the r-th Frobenius kernel of $\mathcal{H}$. The r-th *Steinberg module* St_r is known to be an $\mathcal{H}$-module such that $\mathrm{St}_r|_{\mathcal{H}_r}$ is simple and projective (cf. [**33**, (II.10.2)]). If $\mathcal{B} \subset k\mathcal{G}$ is the block containing $\mathrm{St}_r|_{\mathcal{G}}$, then Proposition 3.8 provides a Morita equivalence between $\mathcal{B}$ and $\mathcal{B}_0(\mathcal{G}(k))$.

Our concluding results refine Corollary 3.7 to obtain the structure of those stable Auslander-Reiten components of Dynkin type that contain a one-dimensional module k_λ, given by a character $\lambda \in X(\mathcal{G})$. We denote a block and a component containing such a module by $\mathcal{B}_\lambda(\mathcal{G})$ and Θ_λ, respectively. (Since the principal block corresponds to the neutral element ε of the abelian group $X(\mathcal{G})$, we write $B_0(\mathcal{G})$ in that case.) The reader is referred to [**2**, §4.18] for the definition and basic properties of Brauer graph algebras.

Theorem 3.9. *Let $\mathcal{G}$ be a finite group scheme, $\lambda \in X(\mathcal{G})$ be a character.*

(1) *If $\mathcal{B}_\lambda(\mathcal{G})$ has finite representation type, then $\mathcal{B}_\lambda(\mathcal{G})$ is a Brauer tree algebra or a Nakayama algebra.*

(2) *If the tree class $\bar{T}_{\Theta_\lambda}$ is a Dynkin diagram, then $\bar{T}_{\Theta_\lambda} \cong A_n$ for some $n \in \mathbb{N}$ and there exists $m \in \mathbb{N}$ with $\Theta_\lambda \cong \mathbb{Z}[A_n]/(\tau^m)$.*

Proof. Since the convolution

$$\lambda * \mathrm{id}_{k\mathcal{G}} : k\mathcal{G} \longrightarrow k\mathcal{G} \quad ; \quad h \mapsto \sum_{(h)} \lambda(h_{(1)})h_{(2)}$$

is an automorphism of $k\mathcal{G}$ such that $\varepsilon \circ (\lambda * \mathrm{id}_{k\mathcal{G}}) = \lambda$, it sends $\mathcal{B}_\lambda(\mathcal{G})$ onto $\mathcal{B}_0(\mathcal{G})$. It thus suffices to verify both statements for the case $\lambda = \varepsilon$.

(1) According to (3.7) we first assume $k\mathcal{G}^0$ to be semi-simple. Then $\mathcal{G}^0$ is diagonalizable, and

$$\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}_{\mathrm{red}})$$

is a representation-finite block of the group algebra $k\mathcal{G}(k) \cong k\mathcal{G}_{\mathrm{red}}$. In view of [**2**, (6.3.5),(6.5.5)], the block $\mathcal{B}_0(\mathcal{G})$ is a Brauer tree algebra.

Alternatively, $k\mathcal{G}(k)$ is semi-simple and the group $\mathcal{G}^0$ is representation-finite. Owing to [**19**, (2.7)] we have an isomorphism

$$\mathcal{G}^0/\mathcal{M} \cong \mathcal{U} \rtimes \mu_{(p^r)},$$

where $\mathcal{M}$ is the diagonalizable part of the center of $\mathcal{G}^0$, and $\mathcal{U}$ is abelian unipotent such that $k\mathcal{U}$ is a Nakayama algebra. Since $\mathcal{M}$ is a characteristic subgroup of $\mathcal{G}^0$ (cf. [**11**, (IV,§3,1.1)]), it is normal in $\mathcal{G}$, and the proof of [**17**, (1.1(3))] provides an isomorphism $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{M})$. We may therefore assume $\mathcal{M} = e_k$.

Thus, $\mathcal{U}$ is a unipotent normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{U} \cong \mu_{(p^r)} \rtimes \mathcal{G}_{\mathrm{red}}$ is linearly reductive. As a result, $J := k\mathcal{G}(k\mathcal{U})^\dagger = (k\mathcal{U})^\dagger k\mathcal{G}$ is the Jacobson radical of $k\mathcal{G}$. By the above, the augmentation ideal $(k\mathcal{U})^\dagger$ of the commutative algebra $k\mathcal{U}$ is generated by an element $u \in (k\mathcal{U})^\dagger$, whence

$$J = (k\mathcal{G})u = u(k\mathcal{G}).$$

Morita's Theorem [**10**, (62.26)] now implies that $k\mathcal{G}$ is a Nakayama algebra, so that $\mathcal{B}_0(\mathcal{G})$ also enjoys this property.

(2) If $\bar{T}_{\Theta_0}$ is a Dynkin diagram, then Θ_0 is finite and $\mathcal{B}_0(\mathcal{G})$ has finite representation type. The result now follows from (1), [**4**, (3.7)] and [**1**, (VII.4)]. □

Remark. There is an alternative proof of (1) that follows the methods employed in the investigation of tame principal blocks [**17**]. By general principles, the Gabriel quiver of $k(\mathcal{G}/\mathcal{M})$ coincides with the McKay quiver of the linearly reductive group $\mu_{(p^r)} \rtimes \mathcal{G}_{\rm red}$ relative to the one-dimensional module $H^1(\mathcal{U}, k)$. Hence the connected components are oriented cycles of type $\tilde{A}_n$, which implies that the algebra $\mathcal{B}_0(\mathcal{G})$ is a Nakayama algebra (cf. [**31**, Thm. 9]).

A finite group scheme is called *trigonalizable* if every simple $\mathcal{G}$-module is one-dimensional. Thus, trigonalizable groups give rise to algebras of measures that are basic. By Gabriel's Theorem such algebras are isomorphic to bound quiver algebras (cf. [**1**, (II.1.10)]).

Corollary 3.10. *Let $\mathcal{G}$ be a trigonalizable finite group scheme.*
(1) *If $\mathcal{B} \subset k\mathcal{G}$ is a representation-finite block, then $\mathcal{B}$ is a Nakayama algebra.*
(2) *If $\Theta \subset \Gamma_s(\mathcal{G})$ is a finite component, then $\Theta \cong \mathbb{Z}[A_n]/(\tau^m)$ for $n, m \in \mathbb{N}$.*

Proof. (1) As in (3.9) we may assume that $\mathcal{B} = \mathcal{B}_0(\mathcal{G})$ is the principal block of $k\mathcal{G}$. If $\mathcal{G}^0$ is diagonalizable, then

$$\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}_{\rm red})$$

is the principal block of the finite, trigonalizable group $G := \mathcal{G}(k)$. Since the Sylow-p-subgroup of G is normal, [**2**, (6.5.4)] implies that $\mathcal{B}_0(\mathcal{G}_{\rm red})$ is a Nakayama algebra.
Alternatively, the arguments of (3.9) yield the desired conclusion.
(2) This follows from a consecutive application of Auslander's Theorem [**1**, (VII.2.1)] and (1). □

References

[1] M. Auslander, I. Reiten, and S. Smalø, *Representation Theory of Artin Algebras.* Cambridge Studies in Advanced Mathematics **36**. Cambridge University Press, 1995.
[2] D. Benson, *Representations and Cohomology*, I. Cambridge Studies in Advanced Mathematics **30**. Cambridge University Press, 1991.
[3] ———, *Representations and Cohomology*, II. Cambridge Studies in Advanced Mathematics **31**. Cambridge University Press, 1991.
[4] C. Bessenrodt, *Modular representation theory for blocks with cyclic defect groups via the Auslander-Reiten quiver.* J. Algebra **140** (1991), 247–262.
[5] ———, *The Auslander-Reiten quiver of a modular group algebra revisited.* Math. Z. **206** (1991), 25–34.
[6] J. Carlson, *The varieties and the cohomology ring of a module.* J. Algebra **85** (1983), 104–143.
[7] ———, *The variety of an indecomposable module is connected.* Invent. math. **77** (1984), 291–299.
[8] J. Carlson, E. Friedlander, and J. Pevtsova, *Modules of constant Jordan type.* Preprint, 2006.
[9] L. Chouinard, *Projectivity and relative projectivity over group rings.* J. Pure Appl. Algebra **7** (1976), 287–303.
[10] C. Curtis and I. Reiner, *Methods of Representation Theory*, II. John Wiley and Sons, 1987.
[11] M. Demazure and P. Gabriel, *Groupes Algébriques.* Masson & Cie, Paris 1970.

[12] K. Erdmann, M. Holloway, R. Taillefer, N. Snashall, and Ø. Solberg, *Support varieties for selfinjective algebras.* K-Theory **33** (2004), 67–87.

[13] K. Erdmann and A. Skowroński, *On Auslander-Reiten components of blocks and self-injective special biserial algebras.* Trans. Amer. Math. Soc. **330** (1992), 165–189.

[14] R. Farnsteiner, *Periodicity and representation type of modular Lie algebras.* J. reine angew. Math. **464** (1995), 47–65.

[15] ———, *On the Auslander-Reiten quiver of an infinitesimal group.* Nagoya Math. J. **160** (2000), 103–121.

[16] ———, *Block representation type of Frobenius kernels of smooth groups.* J. reine angew. Math. **586** (2005), 45–69.

[17] ———, *Polyhedral groups, McKay quivers, and the finite algebraic groups with tame principal blocks.* Invent. math. **166** (2006), 27–94.

[18] ———, *Tameness and complexity of finite group schemes.* Bull. London. Math. Soc. (to appear).

[19] R. Farnsteiner and D. Voigt, *On cocommutative Hopf algebras of finite representation type.* Adv. Math. **155** (2000), 1–22.

[20] D. Fischman, S. Montgomery and H. Schneider, *Frobenius extensions of subalgebras of Hopf algebras.* Trans. Amer. Math. Soc. **349** (1996), 4857–4895.

[21] E. Friedlander and B. Parshall, *Support varieties for restricted Lie algebras.* Invent. math. **86** (1986), 553–562.

[22] ———, *Geometry of p-unipotent Lie algebras.* J. Algebra **109** (1987), 25–45.

[23] E. Friedlander and J. Pevtsova, *Representation-theoretic support spaces for finite group schemes.* Amer. J. Math. **127** (2005), 379–420. Erratum: Amer. J. Math. **128** (2006), 1067–1068.

[24] ———, Π-*Supports for modules for finite group schemes over a field.* Duke Math. J. (to appear)

[25] E. Friedlander, J. Pevtsova, and A. Suslin, *Generic and maximal Jordan types.* Preprint, 2006.

[26] E. Friedlander and A. Suslin, *Cohomology of finite group schemes over a field.* Invent. math. **127** (1997), 209–270.

[27] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras.* LMS Lecture Notes **119**. Cambridge University Press, Cambridge 1988.

[28] D. Happel, U. Preiser, and C. Ringel, *Vinberg's characterization of Dynkin digrams using subadditive functions with applications to DTr-periodic modules.* In: Representation Theory II. Springer Lecture Notes in Mathematics **832** (1981), 280–294.

[29] ———, *Binary polyhedral groups and Euclidean diagrams.* Manuscripta math. **31** (1980), 317–329.

[30] A. Heller, *Indecomposable representations and the loop space operation.* Proc. Amer. Math. Soc. **12** (1961), 640–643.

[31] L. Huppert, *Homological characteristics of pro-uniserial rings.* J. Algebra **69** (1981), 43–66.

[32] J. Jantzen, *Kohomologie von p-Lie Algebren und nilpotente Elemente.* Abh. Math. Sem. Univ. Hamburg **56** (1986), 191–219.

[33] ———, *Representations of Algebraic Groups.* Mathematical Surveys and Monographs **107**. American Mathematical Society, 2003.

[34] U. Oberst and H. Schneider, *Über Untergruppen endlicher algebraischer Gruppen.* Manuscripta math. **8** (1973), 217–241.

[35] T. Okuyama, *On the Auslander-Reiten quiver of a finite group.* J. Algebra **110** (1987), 425–430.

[36] D. Quillen, *The spectrum of an equivariant cohomology ring*, I. Ann. of Math. **94** (1971), 549–572.

[37] ———, *The spectrum of an equivariant cohomology ring*, II. Ann. of Math. **94** (1971), 573–602.

[38] I. Reiten and C. Riedtmann, *Skew group algebras in the representation theory of Artin algebras.* J. Algebra **92** (1985), 224–282.

[39] C. Riedtmann, *Algebren, Darstellungsköcher, Ueberlagerungen und zurück.* Comment. Math. Helvetici **55** (1980), 199–224.

[40] I. Reiten, *Almost split sequences for group algebras of finite representation type.* Trans. Amer. Math. Soc. **233** (1977), 125–136.

[41] J. Rotman, *An Introduction to Homological Algebra.* Pure and Applied Mathematics **85**. Academic Press, 1979.

[42] T. Springer, *Invariant Theory.* Lecture Notes in Mathematics **585**. Springer-Verlag, 1977.

[43] H. Strade and R. Farnsteiner, *Modular Lie Algebras and their Representations.* Pure and Applied Mathematics **116**. Marcel Dekker, New York 1988.

[44] A. Suslin, E. Friedlander, and C. Bendel, *Infinitesimal 1-parameter subgroups and cohomology.* J. Amer. Math. Soc. **10** (1997), 693–728.

[45] W. Waterhouse, *Introduction to Affine Group Schemes.*, Graduate Texts in Mathematics **66**. Springer-Verlag, 1979.

[46] P. Webb, *The Auslander-Reiten quiver of a finite group.* Math. Z. **179** (1982), 97–121.

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