THE TAME INFINITESIMAL GROUPS OF ODD CHARACTERISTIC

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Introduction

Much of the early work in the representation theory of associative algebras focused on the understanding of group algebras of finite groups over fields of positive characteristic. The remarkable success in this context primarily rests on powerful techniques such as the Mackey decomposition theorem and Green's theory of vertices and sources, which effectively link module theoretic properties to structural features of the underlying group.

One of the distinctive aspects of the representation theory of finite groups is the possibility of taking tensor products of modules. Over the years other classes of finite-dimensional algebras with this property have emerged in the representation theories of algebraic groups and modular Lie algebras. Although these associative algebras share many important properties with group algebras of finite groups, they usually afford neither a comprehensive block theory nor methods of descent. In fact, recent work on enveloping algebras of restricted Lie algebras [14] has shown that blocks and subalgebras may behave more erratically than one would expect.

One central question in the modern representation theory of algebras is the determination of the representation type. By Drozd's fundamental result (see [6, 10]) finite dimensional algebras over an algebraically closed field k may be subdivided into three disjoint classes. For the class of *representation-finite algebras*, which have only finitely many isoclasses of indecomposable modules, the representation theory is well understood. The second class, called *tame algebras*, consists of representation-infinite algebras for which the indecomposable modules occur in each dimension in a finite number of discrete and a finite number of one-parameter families. The third class is formed by the *wild algebras*, whose representation theory comprises the representation theories of all finite dimensional algebras over k. Accordingly, a classification of the indecomposable finite dimensional modules is feasible only for representation-finite and tame algebras.

We shall be concerned with the representation type of finite algebraic groups over an algebraically closed field k of characteristic $p > 0$. By general theory, such a group scheme $\mathcal{G}$ decomposes into a semidirect product

$$\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\rm red},$$

where $\mathcal{G}_{\rm red}$ is a reduced group, and $\mathcal{G}^0$ is an infinitesimal, normal subgroup. Since $\mathcal{G}_{\rm red}$ is completely determined by the finite group $\mathcal{G}(k)$ of rational points of $\mathcal{G}$, the representation theory of finite algebraic groups contains that of finite groups as a special case.

The category of finite algebraic groups is equivalent to the category of finite-dimensional cocommutative Hopf algebras. More precisely, one associates to a finite algebraic k-group $\mathcal{G} = \mathrm{Spec}_k(\mathcal{O}(\mathcal{G}))$ with coordinate algebra $\mathcal{O}(\mathcal{G})$ its algebra $H(\mathcal{G}) = \mathcal{O}(\mathcal{G})^*$ of measures on $\mathcal{G}$. In view of the above decomposition the Hopf algebra $H(\mathcal{G})$ can be written as a skew group algebra

$$H(\mathcal{G}) = H(\mathcal{G}^0)[\mathcal{G}(k)].$$

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In particular, the reduced groups correspond to the group algebras of finite groups, while the infinitesimal groups give rise to cocommuative Hopf algebras with local dual algebras.

The representation-finite infinitesimal groups are well understood (cf. [13, 15, 16]). Accordingly, this paper is devoted to the determination of those Hopf algebras of tame representation type that are associated to infinitesimal groups. The treatment of these algebras necessitates an approach that differs completely from the methods employed in the modular representation theory of finite groups. Geometric techniques involving rank varieties and schemes of tori, combined with methods from abstract representation theory, provide a good understanding of the two special cases given by semisimple groups and groups of height 1. We show in this article that the main results concerning these cases (cf. [14, 17]) in conjunction with the interpretation of Galois extensions as Galois coverings set forth in [1] allow the determination of those infinitesimal groups of characteristic ≥ 3, whose Hopf algebras possess a tame principal block. Our results of [14] suggest that it is expedient to address this slightly more general problem: Contrary to the modular representation theory of finite groups, the principal block of an arbitrary Hopf algebra is not necessarily the most complicated block.

In default of the classical descent techniques our basic problem is the transfer of the relevant structural information from subgroups and factor groups of $\mathcal{G}$ to the group $\mathcal{G}$. Since subgroups of tame infinitesimal groups may be wild, only certain types of subgroups are useful. In our case, the situation can be described schematically

$$\begin{array}{ccc} \mathcal{G}_1 & \longrightarrow & \mathcal{G} \\ & & \Big\downarrow \\ & & \mathcal{G}/\mathcal{R}(\mathcal{G}), \end{array}$$

where $\mathcal{G}_1$ and $\mathcal{R}(\mathcal{G})$ denote the first Frobenius kernel and the solvable radical of $\mathcal{G}$, respectively. In this context the canonical action of the character group $X(\mathcal{G}/\mathcal{G}_1)$ of the factor group $\mathcal{G}/\mathcal{G}_1$ on the Hopf algebra $H(\mathcal{G})$ plays a crucial rôle. More specifically, we prove the following recognition theorem, which, in view of the equivalence of the module categories of $\mathcal{G}_1$ and the Lie algebra $\mathrm{Lie}(\mathcal{G})$, enables us to detect tameness by looking at the latter (cf. [14, (7.4)]):

Theorem A. *Let $\mathcal{G}$ be an infinitesimal group of odd characteristic. Then the following statements are equivalent:*

(1) *The principal block $\mathcal{B}_0(\mathcal{G})$ of $H(\mathcal{G})$ is tame.*

(2) *$\mathcal{B}_0(\mathcal{G}_1)$ is tame and $\mathcal{G}/\mathcal{G}_1$ is multiplicative.*

In either case, we have $\mathcal{B}_0(\mathcal{G}_1) = \mathcal{B}_0(\mathcal{G})^{X(\mathcal{G}/\mathcal{G}_1)}$.

The principal block $\mathcal{B}_0(\mathcal{G})$ of the Hopf algebra $H(\mathcal{G})$ does not change under passage to the factor group $\mathcal{G}/\mathcal{M}(\mathcal{G})$ of $\mathcal{G}$ by its multiplicative center $\mathcal{M}(\mathcal{G})$. Tame groups with trivial multiplicative center are often completely determined by the Morita type of $\mathcal{B}_0(\mathcal{G})$. For $r \geq 1$ and $n \geq 0$, we construct a certain central extension

$$e_k \longrightarrow (\mathcal{W}_n)_1 \longrightarrow \mathcal{Q}_{[r,n]} \longrightarrow \mathrm{SL}(2)_1 T_r \longrightarrow e_k$$

of the product $\mathrm{SL}(2)_1 T_r$ (with T_r being the r-th Frobenius kernel of the standard maximal torus $T \subset \mathrm{SL}(2)$) by the first Frobenius kernel of the group $\mathcal{W}_n$ of Witt vectors of length n.

The Morita type of $H(\mathcal{G})$ turns out to be largely determined by algebras arising in the classification of tame hereditary algebras (see [8]). To an extended Dynkin diagram of type $\tilde{A}_{2p^{r-1}-1}$

we associate a symmetric algebra $\mathcal{N}^2(r,n)$ which is obtained from the path algebra of $\tilde{A}_{2p^{r-1}-1}$ by lengthening certain relations of a trivial extension.

Theorem B. *Let $\mathcal{G}$ be an infinitesimal group of odd characteristic of height r and with unipotent center of length n.*

(1) *If $r \geq 2$, or $n \leq 1$, then $\mathcal{B}_0(\mathcal{G})$ is tame if and only if $\mathcal{G} \cong \mathcal{Q}_{[r,n]}$.*

(2) *If $r = 1$, $n \geq 2$, and $\mathcal{B}_0(\mathcal{G})$ is tame, then there are infinitely many isomorphism classes of infinitesimal groups giving rise to a principal block of the same Morita type as $\mathcal{B}_0(\mathcal{G})$.*

(3) *If $\mathcal{B}_0(\mathcal{G})$ is tame, then each block of $H(\mathcal{G})$ is either Morita equivalent to $\mathcal{N}^2(r,n)$ or to $k[X]/(X^{p^n})$. There are $\frac{p-1}{2}$ blocks of the former type and p^{r-1} of the latter.*

While the foregoing result determines the Morita equivalence class of the tame principal block of an arbitrary infinitesimal group, the presence of a multiplicative center may significantly complicate the representation theory of $H(\mathcal{G})$. There do exists infinitesimal groups with tame principal blocks that also possess wild blocks (cf. [14, (8.10)]).

Our paper is organized as follows. After a preliminary section, in which we also summarize the relevant results from [14, 17], we show in Section 2 that most of the structure of an infinitesimal group $\mathcal{G}$ with tame principal block is encoded in its first Frobenius kernel $\mathcal{G}_1$. More precisely, in the relevant cases the factor group $\mathcal{G}/\mathcal{G}_1$ of such a group is isomorphic to a multiplicative group of type μ_{p^r} (the r-th Frobenius kernel of $\mu_k = \mathrm{Spec}_k(k[X, X^{-1}])$). With this information in hand, we show in Theorem 3.4 and Corollary 3.5 that the groups in question are central extensions of a Frobenius kernel of $\mathrm{SL}(2)_1 T$, with the unipotent part of the center being $(\mathcal{W}_n)_1$.

In Section 4 we study basic algebras with prescribed Gabriel quiver whose invariants relative to certain Galois actions are constructed from the trivial extension of the Kronecker algebra by lengthening relations. We prove that the relevant algebras are special biserial and hence in particular tame. This abstract set-up is then verified in Section 5 for those infinitesimal groups $\mathcal{G}$, whose first Frobenius kernels possess tame principal blocks and whose factor groups $\mathcal{G}/\mathcal{G}_1$ are multiplicative of type μ_{p^r}. More precisely, for such a group $\mathcal{G}$ the quiver of $\mathcal{B}_0(\mathcal{G})$ can be read off from the principal block of its largest semisimple factor group (cf. [17]), and $\mathcal{B}_0(\mathcal{G}) : \mathcal{B}_0(\mathcal{G}_1)$ is a Galois extension, with the character group $X(\mathcal{G}/\mathcal{G}_1) \cong \mathbb{Z}/(p^r)$ as Galois group. Since the order of $X(\mathcal{G}/\mathcal{G}_1)$ is divisible by p, the standard methods of invariant theory do not apply. However, as $X(\mathcal{G}/\mathcal{G}_1)$ acts freely on the isomorphism classes of simple $\mathcal{B}_0(\mathcal{G})$-modules, the results of Section 4 only allow two types of (special biserial) algebras. This information is exploited in the proof of Theorem A, which enables us in Section 6 to single out the tame groups $\mathcal{Q}_{[r,n]}$ among the central extensions given in §3.

The concluding section presents the block structure of the Hopf algebra $H(\mathcal{G})$ given in Theorem B. All tame blocks of $H(\mathcal{G})$ turn out to be Morita equivalent to the special biserial algebras $\mathcal{N}^2(r,n)$ constructed in Section 4; the remaining blocks are Nakayama algebras with one simple module. Representation-infinite special biserial algebras form a distinguished class of tame algebras, whose representation theory is rather well understood. Important examples of such algebras are provided by blocks of group algebras with dihedral defect groups (see [11, 30]) as well as the algebras appearing in the Gelfand-Ponomarev classification of the singular Harish-Chandra modules over the Lorentz group [21]. However, contrary to these classical cases the number of simple modules belonging to tame blocks of $H(\mathcal{G})$ grows exponentially with the height of the underlying infinitesimal group $\mathcal{G}$.

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1. Preliminaries

Throughout this paper we will be working over an algebraically closed field k of positive characteristic $p \geq 3$. Unless mentioned otherwise, a k-vector space is assumed to be finite-dimensional. Modules over an associative k-algebra Λ are always understood to be left modules on which the identity element of Λ operates via the identity transformation.

Given an infinitesimal k-group $\mathcal{G}$, we denote by $\mathcal{R} := \mathcal{R}(\mathcal{G})$ and $\mathcal{M} := \mathcal{M}(\mathcal{G})$ its *solvable radical* and *multiplicative center*, respectively. By general theory, the multiplicative center $\mathcal{M}(\mathcal{G})$ is trivial if and only if the *center* $\mathrm{Cent}(\mathcal{G})$ of $\mathcal{G}$ is unipotent. We write $\mathcal{G}_r$ for the *r-th Frobenius kernel* of $\mathcal{G}$. The minimum number s for which $\mathcal{G}_s = \mathcal{G}$ is called the *height* $\mathrm{ht}(\mathcal{G})$ of $\mathcal{G}$.

By general theory, the infinitesimal k-groups correspond to cocommutative Hopf algebras with local dual algebras. More precisely, the *algebra of measures* associated to the infinitesimal group $\mathcal{G} := \mathrm{Spec}_k(\mathcal{O}(\mathcal{G}))$ is defined to be $H(\mathcal{G}) := \mathcal{O}(\mathcal{G})^*$. Since the representation theories of $\mathcal{G}$ and $H(\mathcal{G})$ are equivalent, we will use the terms "$\mathcal{G}$-module" and "$H(\mathcal{G})$-module" interchangeably. The reader is referred to [7, 27, 37] for general facts on algebraic k-groups.

The associative algebra $H(\mathcal{G})$ decomposes into blocks. By definition, the *principal block* $\mathcal{B}_0(\mathcal{G})$ of $H(\mathcal{G})$ is the block belonging to the trivial $H(\mathcal{G})$-module $k = k_\varepsilon$, which is given by the co-unit $\varepsilon : H(\mathcal{G}) \longrightarrow k$ of the Hopf algebra $H(\mathcal{G})$ (cf. [3]).

As mentioned in the introduction, we are concerned with those infinitesimal groups $\mathcal{G}$, whose principal blocks are of tame representation type. For the time being, we shall be content with the following informal description of tameness: An associative algebra Λ tame if it has infinitely many isoclasses of indecomposable modules which occur in each dimension in a finite number of discrete and a finite number of continuous one-parameter families. The precise definition will be provided in §4 below.

For easy reference we collect in this section a few basic results from [17, 14], some of which we have tailored to our needs. Recall that a finite algebraic k-group is *multiplicative* or *diagonalizable* if its function algebra $\mathcal{O}(\mathcal{G})$ is a group algebra of a finite group. We say that $\mathcal{G}$ is *supersolvable* if it possesses a composition series with each factor being isomorphic to $\alpha_p := \mathrm{Spec}_k(k[T]/(T^p))$ or $\mu_p := \mathrm{Spec}_k(k[T]/(T^p - 1))$.

Lemma 1.1. *Let $\mathcal{G}$ be an infinitesimal group, $\mathcal{N} \lhd \mathcal{G}$ a normal subgroup.*

(1) *If $\mathcal{N}$ is multiplicative, then $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{N})$.*
(2) *If $\mathcal{B}_0(\mathcal{G})$ is tame, and $\mathcal{G}/\mathcal{N}$ is not supersolvable, then $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ is tame.*

Proof. See [17, (1.1)]. □

In view of (1.1(1)) we will often assume that our groups $\mathcal{G}$ have a trivial multiplicative center. While this is adequate for our present purposes, one should observe that, owing to [14, §8], the presence of a multiplicative center may significantly complicate the representation theory of the algebra $H(\mathcal{G})$.

We also require the following result concerning solvable groups, which is not valid at even characteristic:

Theorem 1.2. *Let $\mathcal{G}$ be a solvable infinitesimal group. Then $\mathcal{B}_0(\mathcal{G})$ is not tame.*

Proof. See [17, (2.4)]. □

We continue by considering the complementary case of semisimple groups. Given a natural number $r \geq 1$, we let $\mathcal{Q}_{[r]}$ be the closed subgroup of SL(2) defined by

$$\mathcal{Q}_{[r]}(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)(R) \ ; \ a^{p^r} = 1 = d^{p^r}, \ b^p = 0 = c^p \right\}$$

for every commutative k-algebra R. Note that $\mathcal{Q}_{[r]}$ is the product $\mathrm{SL}(2)_1\mathrm{T}_r$ of the first Frobenius kernel of SL(2) with the r-th Frobenius kernel of standard torus $\mathrm{T} \subset \mathrm{SL}(2)$ of diagonal matrices of determinant 1.

Theorem 1.3. *Let $\mathcal{G}$ be a semisimple infinitesimal group. Then the following statements hold:*

(1) *The principal block $\mathcal{B}_0(\mathcal{G})$ is tame if and only if $\mathcal{G} \cong \mathcal{Q}_{[r]}$ for some $r \geq 1$.*

(2) *The algebra $H(\mathcal{Q}_{[r]})$ possesses $\frac{p-1}{2}$ Morita equivalent tame blocks $\mathcal{B}_0, \dots, \mathcal{B}_{\frac{p-3}{2}}$ and p^{r-1} simple blocks. The block $\mathcal{B}_i$ has p^{r-1} simple modules of dimension $i+1$ and p^{r-1} simple modules of dimension $p-i-1$. The projective simple $H(\mathcal{Q}_{[r]})$-modules are p-dimensional.*

Proof. Both statements follow directly from [17, (5.5)] and its proof. □

Since the factor group $\mathcal{Q}_{[r]}/(\mathcal{Q}_{[r]})_1$ is isomorphic to the multiplicative group

$$\mu_{p^{r-1}} = \mathrm{Spec}_k(k[T]/(T^{p^{r-1}} - 1)),$$

we have $\mathcal{G}/\mathcal{G}_1 \cong \mu_{p^{r-1}}$ for any semisimple group $\mathcal{G}$ of height r admitting a tame block $\mathcal{B}_0(\mathcal{G})$.

The Lie algebra of the group $\mathcal{G}$ will be denoted $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$. Note that $\mathfrak{g}$ has the structure of a restricted Lie algebra, given by the p-map $\mathfrak{g} \longrightarrow \mathfrak{g} \ ; \ x \mapsto x^{[p]}$. The reader is referred to [34] concerning standard facts of restricted Lie algebras. According to [7, (II,§7,n°4)] the category of restricted Lie algebras is equivalent to the category of infinitesimal groups of height ≤ 1. At the level of Hopf algebras this equivalence is induced by an isomorphism

$$H(\mathcal{G}) \cong u(\mathrm{Lie}(\mathcal{G}))$$

between the algebra of measures of an infinitesimal group $\mathcal{G}$ of height ≤ 1 and the restricted enveloping algebra of its Lie algebra. Given any restricted Lie algebra $(\mathfrak{g}, [p])$, its *restricted enveloping algebra*

$$u(\mathfrak{g}) := \mathcal{U}(\mathfrak{g})/(x^p - x^{[p]} \ ; \ x \in \mathfrak{g})$$

is a factor algebra of the ordinary enveloping algebra $\mathcal{U}(\mathfrak{g})$. We denote by $\mathcal{B}_0(\mathfrak{g})$ the principal block of $u(\mathfrak{g})$.

According to [14] the structure of the restricted Lie algebras with tame principal block is essentially determined by a certain factor algebra (cf. (1.4) below). In this context, a special rôle is played by a one-dimensional non-split central extension

$$\mathfrak{sl}(2)_s := \mathfrak{sl}(2) \oplus kv_0,$$

whose multiplication and p-map are given by

$$[x+v, y+w] = [x,y] \quad ; \quad e^{[p]} = 0 = f^{[p]} \ , \ h^{[p]} = h + v_0 \ , \ v_0^{[p]} = 0$$

for all $x, y \in \mathfrak{sl}(2)$ and $v, w \in kv_0$. Here, $\{e, h, f\}$ denotes the standard basis of $\mathfrak{sl}(2)$.

The *center* of a restricted Lie algebra $(\mathfrak{g}, [p])$ will be denoted $C(\mathfrak{g})$. We say that $\mathfrak{g}$ is *p-nilpotent* if and only if there exists an $n \in \mathbb{N}$ such that $x^{[p]^n} = 0$ for every $x \in \mathfrak{g}$. The p-nilpotent restricted Lie algebras correspond to the unipotent infinitesimal groups of height ≤ 1 and are therefore occasionally also referred to as unipotent.

Theorem 1.4. *Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Then the following statements hold:*

(1) *The principal block $\mathcal{B}_0(\mathfrak{g})$ is tame if and only if $\mathfrak{g}/C(\mathfrak{g})^{[p]} \cong \mathfrak{sl}(2),\ \mathfrak{sl}(2)_s$.*

(2) *If $\mathcal{B}_0(\mathfrak{g})$ is tame and $C(\mathfrak{g})$ is p-nilpotent, then $u(\mathfrak{g})$ possesses $\frac{p-1}{2}$ Morita equivalent tame blocks $\mathcal{B}_0, \dots, \mathcal{B}_{\frac{p-3}{2}}$ and one representation-finite block. The block $\mathcal{B}_i$ has one simple module of dimension $i+1$ and one simple module of dimension $p-i-1$. The representation-finite block is a primary Nakayama algebra, whose simple module is p-dimensional.*

Proof. (1) This was shown in [14, (7.4)].

(2) A consecutive application of [14, (1.2)] and [14, (1.3)] shows that $u(\mathfrak{g})$ has blocks $\mathcal{B}_0, \dots, B_{\frac{p-3}{2}}$ whose simple modules have the desired dimensions. Moreover, the remark following [14, (1.3)] proves the statement concerning the representation-finite block. Our result now follows directly from [14, (7.1)]. □

Given a restricted Lie algebra $(\mathfrak{g}, [p])$, we recall that its *nullcone* is the affine, conical variety defined by

$$\mathcal{V}_{\mathfrak{g}} := \{x \in \mathfrak{g}\ ;\ x^{[p]} = 0\}.$$

In the sequel we shall repeatedly use the important fact that the Lie algebra of an infinitesimal group with tame principal block has a two-dimensional nullcone. This result is stated in this way in [17, (2.5)], but it should be noted that [29, Thm.2] and [26, Satz] are its main ingredients.

We conclude this section by recording a technical subsidiary result. We let $\mathcal{W}_n$ denote the smooth abelian unipotent group of Witt vectors of length n (cf. [7, (V,§1,1.6)]). The first Frobenius kernels of these groups are the infinitesimal unipotent uniserial groups of height ≤ 1 (cf. [13, (5.3)]). The Lie algebra of $\mathcal{W}_n$ is the *nil-cyclic* restricted Lie algebra $\mathfrak{n}_n$ of dimension n, that is, the n-dimensional Lie algebra generated by a p-nilpotent element. A restricted Lie algebra $(\mathfrak{g}, [p])$ is a *torus* if $\mathcal{V}_{\mathfrak{g}} = \{0\}$. Tori are necessarily abelian. Moreover, they are the Lie algebras of the multiplicative infinitesimal groups. Following Seligman [31] we say that $\mathfrak{g} \neq (0)$ is *characteristic semisimple* if it does not possess any non-zero solvable p-ideals that are invariant with respect to the connected component of the automorphism scheme of $(\mathfrak{g}, [p])$.

As will be shown in §5, the technical conditions (a) and (b) of the following result are in fact equivalent to the tameness of the principal block $\mathcal{B}_0(\mathcal{G})$ of an infinitesimal group $\mathcal{G}$ of height r with trivial multiplicative center.

Proposition 1.5. *Let $\mathcal{G}$ be an infinitesimal group with trivial multiplicative center such that*

(a) *the principal block $\mathcal{B}_0(\mathcal{G}_1)$ is tame, and*

(b) *the group $\mathcal{G}/\mathcal{G}_1$ is multiplicative.*

Then the following statements hold:

(1) *There exists $n \geq 0$ such that $\mathcal{R}(\mathcal{G}) \cong (\mathcal{W}_n)_1$.*

(2) *If $r = \mathrm{ht}(\mathcal{G})$, then $\mathcal{G}/\mathcal{R}(\mathcal{G}) \cong \mathcal{Q}_{[r]}$ and $\mathcal{G}/\mathcal{G}_1 \cong \mu_{p^{r-1}}$.*

Proof. (1) We let $\mathcal{R}_1$ denote the first Frobenius kernel of $\mathcal{R}(\mathcal{G})$. It follows from [7, (IV,§3,1.1)] that $\mathcal{M}(\mathcal{G}_1)$ is a normal subgroup of $\mathcal{G}$. Since $\mathcal{M}(\mathcal{G}) = e_k$, the group $\mathcal{M}(\mathcal{G}_1)$ is also trivial. Condition (a) and [17, (6.3)] now imply that $\mathcal{R}_1 \subset \mathcal{R}(\mathcal{G}_1)$ is unipotent. Induction on the height of $\mathcal{R}(\mathcal{G})$ in conjunction with [7, (IV,§2,2.3)] yields the unipotence of $\mathcal{R}(\mathcal{G})$. The canonial map $\mathcal{R}(\mathcal{G}) \longrightarrow \mathcal{G}/\mathcal{G}_1$ gives rise to a closed embedding $\mathcal{R}(\mathcal{G})/\mathcal{R}_1 \hookrightarrow \mathcal{G}/\mathcal{G}_1$ (cf. [37, (15.3)]). Consequently, (b) implies that the group $\mathcal{R}(\mathcal{G})/\mathcal{R}_1$ is, as a multiplicative and unipotent group, trivial. Thus, $\mathcal{R}(\mathcal{G}) = \mathcal{R}_1$ has height ≤ 1, so that $\mathcal{R}(\mathcal{G}) \cong (\mathcal{W}_n)_1$ for a suitably chosen $n \in \mathbb{N}_0$ (cf. [17, (6.3)], [7, (IV,§2,2.14)]).

(2) Having already observed the inclusion $\mathcal{R}_1 \subset \mathcal{R}(\mathcal{G}_1)$, we first show that $\mathcal{R}_1 \supset \mathcal{R}(\mathcal{G}_1)$. Since $\mathcal{B}_0(\mathcal{G}_1)$ is tame, [17, (6.3)] and [7, (II,§7,n$^{\circ}$4)] imply that $\mathcal{R}(\mathcal{G}_1) = \mathrm{Cent}(\mathcal{G}_1)$ is a normal subgroup of $\mathcal{G}$. Consequently, $\mathcal{R}(\mathcal{G}_1) = \mathrm{Cent}(\mathcal{G}_1) \subset \mathcal{R}(\mathcal{G})$, as desired.

If $\mathcal{G}_1/\mathcal{R}_1$ is supersolvable, then $\mathcal{G}_1$ is solvable and [17, (2.4)] implies that $\mathcal{B}_0(\mathcal{G}_1)$ is not tame. Accordingly, $\mathcal{G}_1/\mathcal{R}_1$ is not supersolvable, and (1.1) ensures the tameness of $\mathcal{B}_0(\mathcal{G}_1/\mathcal{R}_1)$. Since $\mathcal{R}_1$ coincides with the radical $\mathcal{R}(\mathcal{G}_1)$, the group $\mathcal{G}_1/\mathcal{R}_1$ is semisimple. Thanks to [17, (4.3)] we have $\mathrm{Lie}(\mathcal{G}_1/\mathcal{R}_1) \cong \mathfrak{sl}(2)$, and the exact sequence

$$e_k \longrightarrow \mathcal{G}_1/\mathcal{R}_1 \longrightarrow \mathcal{G}/\mathcal{R}_1 \longrightarrow \mathcal{G}/\mathcal{G}_1 \longrightarrow e_k$$

(cf. [7, (III,§3,3.7)]) gives rise to an exact sequence

$$(0) \longrightarrow \mathfrak{sl}(2) \longrightarrow \mathrm{Lie}(\mathcal{G}/\mathcal{R}_1) \longrightarrow \mathrm{Lie}(\mathcal{G}/\mathcal{G}_1)$$

(cf. [7, (II,§4,1.5)]). Since the Lie algebra $\mathfrak{sl}(2)$ is complete (i.e., centerless with all derivations being inner), there results a direct sum decomposition

$$\mathrm{Lie}(\mathcal{G}/\mathcal{R}_1) \cong \mathfrak{sl}(2) \oplus \mathfrak{m}$$

of restricted Lie algebras. In view of (b) the second summand is a torus and thus coincides with the center of $\mathrm{Lie}(\mathcal{G}/\mathcal{R}_1)$. As the center is invariant under the adjoint representation of $\mathcal{G}/\mathcal{R}_1$, there exists a normal subgroup $\mathcal{N} \lhd \mathcal{G}/\mathcal{R}_1$ of height ≤ 1 such that $\mathrm{Lie}(\mathcal{N}) = C(\mathrm{Lie}(\mathcal{G}/\mathcal{R}_1))$. In particular, the normal subgroup $\mathcal{N}$ is solvable. In view of (1), the group $\mathcal{G}/\mathcal{R}_1$ is semisimple, and we conclude that $\mathcal{N} = e_k$. Accordingly, the embedding

$$\mathcal{G}_1/\mathcal{R}_1 \hookrightarrow (\mathcal{G}/\mathcal{R}_1)_1$$

induces an isomorphism $\mathrm{Lie}(\mathcal{G}_1/\mathcal{R}_1) \cong \mathrm{Lie}(\mathcal{G}/\mathcal{R}_1)$, so that [7, (II,§7,4.1)] yields

$$\mathcal{G}_1/\mathcal{R}_1 \cong (\mathcal{G}/\mathcal{R}_1)_1.$$

We thus obtain, observing [7, (III,§3,3.7)],

$$(\mathcal{G}/\mathcal{R}_1)/(\mathcal{G}/\mathcal{R}_1)_1 \cong (\mathcal{G}/\mathcal{R}_1)/(\mathcal{G}_1/\mathcal{R}_1) \cong \mathcal{G}/\mathcal{G}_1.$$

Consequently, conditions (a) and (b) apply to the semisimple group $\mathcal{G}' := \mathcal{G}/\mathcal{R}_1$. Accordingly, $\mathrm{Lie}(\mathcal{G}')$ is characteristic semisimple with two-dimensional nullcone, and a consecutive application of [17, (4.2)] and [17, (5.4)] provides $r \geq 1$ with $\mathcal{G}' \cong \mathcal{Q}_{[r]}$. Since the group

$$\mu_{p^{r-1}} \cong \mathcal{Q}_{[r]}/(\mathcal{Q}_{[r]})_1 \cong \mathcal{G}'/\mathcal{G}'_1 \cong \mathcal{G}/\mathcal{G}_1$$

has height $r-1$, we have $r = \mathrm{ht}(\mathcal{G})$, as desired. □

2. The Structure of $\mathcal{G}/\mathcal{G}_1$

Throughout this section $\mathcal{G}$ is assumed to be an infinitesimal k-group. Assuming $\mathcal{B}_0(\mathcal{G})$ to be tame, we determine the factor group $\mathcal{G}/\mathcal{G}_1$, thereby showing that most of the structure of $\mathcal{G}$ is encapsulated in its first Frobenius kernel. For future reference some immediate consequences for certain $\mathcal{G}$-modules are also provided.

Proposition 2.1. *Suppose that $\mathcal{B}_0(\mathcal{G})$ is tame. Then the following statements hold:*

(1) *The canonical projection $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{R}$ induces isomorphisms $\mathcal{G}_1/\mathcal{R}_1 \cong (\mathcal{G}/\mathcal{R})_1$ as well as $\mathrm{Lie}(\mathcal{G}/\mathcal{R}) \cong \mathrm{Lie}(\mathcal{G})/\mathrm{Lie}(\mathcal{R})$.*

(2) *The factor group $\mathcal{G}/\mathcal{G}_1$ is multiplicative.*

(3) *If $\mathcal{M} = e_k$, then $\mathcal{G}/\mathcal{G}_1 \cong (\mathcal{G}/\mathcal{R})/(\mathcal{G}/\mathcal{R})_1 \cong \mu_{p^{r-1}}$, where r denotes the height of $\mathcal{G}$.*

Proof. (1) We set $\mathcal{G}' := \mathcal{G}/\mathcal{R}$ and consider the canonical projection $\pi : \mathcal{G} \longrightarrow \mathcal{G}'$. Since the Frobenius homomorphism is natural (cf. [7, (II,§7,1.4)]), the morphism π induces a morphism $\omega : \mathcal{G}_1 \longrightarrow \mathcal{G}'_1$, whose kernel coincides with $\mathcal{G}_1 \cap \mathcal{R} = \mathcal{R}_1$. By [37, (15.3)] there results a closed embedding $\bar{\omega} : \mathcal{G}_1/\mathcal{R}_1 \hookrightarrow \mathcal{G}'_1$ of algebraic groups of height ≤ 1. Thanks to [7, (II,§7,4.3)] the map $\bar{\omega}$ corresponds to an injective homomorphism $\varrho : \mathrm{Lie}(\mathcal{G}_1/\mathcal{R}_1) \hookrightarrow \mathrm{Lie}(\mathcal{G}'_1)$ of restricted Lie algebras. According to (1.2) the group $\mathcal{G}$ is not solvable. Consequently, $\mathcal{G}'$ is not supersolvable, and (1.1) implies that $\mathcal{B}_0(\mathcal{G}')$ is tame. We may now apply [17, (4.3)] to see that $\mathrm{Lie}(\mathcal{G}') \cong \mathfrak{sl}(2)$. Since $\mathrm{Lie}(\mathcal{G}') = \mathrm{Lie}(\mathcal{G}'_1)$ we conclude that $\mathrm{im}\,\varrho \subset \mathfrak{sl}(2)$ has a nullcone of dimension ≤ 2.

In view of [15, (2.1)] the assumption $\dim \mathcal{V}_{\mathrm{im}\,\varrho} \leq 1$ implies the supersolvability of the Lie algebra $\mathrm{Lie}(\mathcal{G}_1/\mathcal{R}_1)$. By [7, (IV,§4, 2.6)] the group $\mathcal{G}_1/\mathcal{R}_1$ is solvable. Thus, $\mathcal{G}_1 = \mathcal{R}_1$ is solvable, and induction on the height of $\mathcal{G}$ shows that the infinitesimal group $\mathcal{G}$ also enjoys this property. As this contradicts (1.2), we conclude that $\mathrm{im}\,\varrho$ has a two-dimensional nullcone. This readily entails $\mathrm{im}\,\varrho = \mathfrak{sl}(2)$, so that ϱ and $\bar{\omega}$ are isomorphisms (cf. [7, (II,§7,4.2)]). The above reasoning also shows the surjectivity of the map $\mathrm{Lie}(\mathcal{G}) \longrightarrow \mathrm{Lie}(\mathcal{G}/\mathcal{R})$, thereby proving that the latter Lie algebra is isomorphic to $\mathrm{Lie}(\mathcal{G})/\mathrm{Lie}(\mathcal{R})$ (cf. [7, (II,§4,1.5)]).

(2) The restriction $\pi_{\mathcal{R}} : \mathcal{G}_1\mathcal{R} \longrightarrow \mathcal{G}'$ of π to $\mathcal{G}_1\mathcal{R}$ factors through to a closed embedding $\bar{\pi}_{\mathcal{R}} : (\mathcal{G}_1\mathcal{R})/\mathcal{R} \hookrightarrow \mathcal{G}'$. Since composition of $\bar{\pi}_{\mathcal{R}}$ with the canonical isomorphism $\mathcal{G}_1/\mathcal{R}_1 \cong (\mathcal{G}_1\mathcal{R})/\mathcal{R}$ gives the canonical map $\mathcal{G}_1/\mathcal{R}_1 \hookrightarrow \mathcal{G}'$, it follows from (1) that $\bar{\pi}_{\mathcal{R}}$ factors through $\mathcal{G}'_1$. By the same token, the resulting morphism $(\mathcal{G}_1\mathcal{R})/\mathcal{R} \longrightarrow \mathcal{G}'_1$ is in fact an isomorphism. An application of [7, (III,§3,3.7)] now yields isomorphisms

$$(*) \qquad \mathcal{G}'/\mathcal{G}'_1 \cong (\mathcal{G}/\mathcal{R})/[(\mathcal{G}_1\mathcal{R})/\mathcal{R}] \cong \mathcal{G}/(\mathcal{G}_1\mathcal{R}).$$

According to (1.3) the group $\mathcal{G}'/\mathcal{G}'_1$ is multiplicative. Since the left-hand term of the exact sequence

$$e_k \longrightarrow (\mathcal{G}_1\mathcal{R})/\mathcal{G}_1 \longrightarrow \mathcal{G}/\mathcal{G}_1 \longrightarrow \mathcal{G}/(\mathcal{G}_1\mathcal{R}) \longrightarrow e_k$$

is isomorphic to $\mathcal{R}/(\mathcal{R} \cap \mathcal{G}_1) = \mathcal{R}/\mathcal{R}_1$, we may apply [17, (6.2)] to see that the extreme terms of the sequence are multiplicative. As all groups involved are connected, our assertion is now a consequence of [7, (IV,§1,4.5)].

(3) As multiplicative centers are characteristic subgroups (cf. [7, (IV,§3,1.1)]), it follows from the rigidity of multiplicative groups (cf. [37, (7.7)]) that $\mathcal{M}(\mathcal{R}) = \mathcal{M} = e_k$. By combining this fact with [17, (6.1),(6.2)] and [7, (IV,§4,1.10)], we conclude that the solvable radical $\mathcal{R}$ is unipotent and of height ≤ 1. Consequently, the isomorphism $(*)$ simplifies to

$$\mathcal{G}'/\mathcal{G}'_1 \cong \mathcal{G}/\mathcal{G}_1.$$

In view of (1.3) the former group is isomorphic to $\mu_{p^{r'-1}}$ in case $\mathcal{G}'$ has height r'. Let r be the height of $\mathcal{G}$. Then the above isomorphism implies $r - 1 = \mathrm{ht}(\mathcal{G}/\mathcal{G}_1) = r' - 1$, as desired. □

We let $X(\mathcal{G})$ be the character group of $\mathcal{G}$. By definition, $X(\mathcal{G})$ is the set of algebra homomorphisms $H(\mathcal{G}) \longrightarrow k$ with product given by the convolution

$$(\lambda * \gamma)(h) := \sum_{(h)} \lambda(h_{(1)})\gamma(h_{(2)}) \qquad \forall\ h \in H(\mathcal{G}),$$

where $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$ is the comultiplication of $H(\mathcal{G})$. Since $\mathcal{G}$ is infinitesimal, its character group $X(\mathcal{G})$ is a p-group.

The canonical surjection $H(\mathcal{G}) \longrightarrow H(\mathcal{G}/\mathcal{G}_1)$ induces an injection $X(\mathcal{G}/\mathcal{G}_1) \hookrightarrow X(\mathcal{G})$ of the corresponding character groups. Accordingly, we will henceforth identify $X(\mathcal{G}/\mathcal{G}_1)$ with its image in $X(\mathcal{G})$. This amounts to interpreting the elements of $X(\mathcal{G}/\mathcal{G}_1)$ as those homomorphisms of $X(\mathcal{G})$ that vanish on the augmentation ideal $H(\mathcal{G}_1)^\dagger := \ker \varepsilon$ of the Hopf algebra $H(\mathcal{G}_1)$. If $\mathcal{G}$ has trivial multiplicative center, and $\mathcal{B}_0(\mathcal{G})$ is tame, then (2.1) implies $X(\mathcal{G}/\mathcal{G}_1) \cong X(\mu_{p^{r-1}}) \cong \mathbb{Z}/(p^{r-1})$, where $r := \mathrm{ht}(\mathcal{G})$.

Given a $\mathcal{G}$-module M, we will write $M|_{\mathcal{H}}$ for its restriction to a closed subgroup $\mathcal{H} \subset \mathcal{G}$. Moreover, we put $M_\lambda := M \otimes_k k_\lambda$ for every $\lambda \in X(\mathcal{G})$. Here k_λ denotes the one-dimensional $H(\mathcal{G})$-module on which $H(\mathcal{G})$ acts via λ.

In view of our projected applications, the following Lemma is formulated under a set of hypotheses that will eventually be seen to imply the tameness of $\mathcal{B}_0(\mathcal{G})$.

Lemma 2.2. *Let $\mathcal{G}$ be an infinitesimal group with trivial multiplicative center such that*

(a) *the principal block $\mathcal{B}_0(\mathcal{G}_1)$ is tame, and*

(b) *there exists $r \geq 1$ such that $\mathcal{G}/\mathcal{G}_1 \cong \mu_{p^{r-1}}$.*

Then the following statements hold:

(1) *If V and W are simple $H(\mathcal{G})$-modules such that $\mathrm{Hom}_{H(\mathcal{G}_1)}(V, W) \neq (0)$, then $W \cong V_\lambda$ for some $\lambda \in X(\mathcal{G}/\mathcal{G}_1)$.*

(2) *If V is a simple $H(\mathcal{G})$-module, then $V|_{\mathcal{G}_1}$ is also simple.*

(3) *If V is a simple $H(\mathcal{G})$-module such that $V \cong V_\lambda$ for some $\lambda \in X(\mathcal{G}/\mathcal{G}_1)$, then $\lambda = \varepsilon$.*

(4) *If P is a principal indecomposable $H(\mathcal{G})$-module, then $P|_{\mathcal{G}_1}$ is a principal indecomposable $H(\mathcal{G}_1)$-module.*

Proof. (1) This follows by adopting the arguments of the proof of [17, (5.1(1))] verbatim.

(2) Recall from (1.5) that the canonical projection $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{R} =: \mathcal{G}'$ induces an isomorphism $\mathcal{G}_1/\mathcal{R}_1 \cong \mathcal{G}'_1$. In view of (1.5) the solvable radical $\mathcal{R}$ is unipotent, so that V carries the structure of an $H(\mathcal{G}')$-module. Thanks to [17, (5.1(2))] the module $V|_{\mathcal{G}'_1}$ is simple, and the above observation entails the simplicity of the $H(\mathcal{G}_1/\mathcal{R}_1)$-module V. Consequently, $V|_{\mathcal{G}_1}$ is also simple.

(3) This follows as in [17, (5.1(3))].

(4) Since $\mathcal{R}$ is unipotent, the ideal $I := H(\mathcal{G})H(\mathcal{R})^\dagger$ is nilpotent and $\hat{P} := P/IP$ is a principal indecomposable $H(\mathcal{G}')$-module. It was shown in [17, (5.5)] that $\hat{P}|_{\mathcal{G}'_1}$ is a principal indecomposable $H(\mathcal{G}'_1)$-module. Consequently, $\hat{P}$ is a principal indecomposable $H(\mathcal{G}_1/\mathcal{R}_1)$-module. Since the tops of the pull-back of $\hat{P}$ along $H(\mathcal{G}_1) \longrightarrow H(\mathcal{G}_1/\mathcal{R}_1)$ and $P|_{\mathcal{G}_1}$ coincide, we conclude that the latter module has a simple top. Owing to [35, (2.6)] the algebra $H(\mathcal{G})$ is a free $H(\mathcal{G}_1)$-module. Consequently, $P|_{\mathcal{G}_1}$ is projective. □

Remark. It follows from [1, (5.1)] and (2.2(3)) that the action of $X(\mathcal{G}/\mathcal{G}_1)$ on $H(\mathcal{G})$ is a Galois action in the sense of [1], a fact be crucial for our determination of the basic algebra of $\mathcal{B}_0(\mathcal{G})$ (cf. Section 5).

3. The Structure of Lie($\mathcal{G}$) and $\mathcal{G}$

The second step in our classification consists of the structural analysis of the first Frobenius kernel of $\mathcal{G}$. In view of [7, (II,§7,n°4)] this is equivalent to studying the restricted Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$. For groups of height ≤ 1 with tame principal block the structure of the Lie algebra $\mathfrak{g}$ is well-understood (cf. [14]), so that our task is the extension of earlier results to groups of greater height. The main difficulty is the absence of a theory of descent allowing us to derive the tameness of $\mathcal{B}_0(\mathcal{G}_1)$ from the corresponding property of $\mathcal{B}_0(\mathcal{G})$.

If a restricted Lie algebra $\mathfrak{n}$ has a trivial bracket and a trivial p-map, then we say that $\mathfrak{n}$ is *strongly abelian.* By work of Hochschild [23] the extensions of a restricted Lie algebra $\mathfrak{g}$ with strongly abelian kernel $\mathfrak{n}$ correspond to the restricted cohomology group $H^2_*(\mathfrak{g}, \mathfrak{n}) := \mathrm{Ext}^2_{u(\mathfrak{g})}(k, \mathfrak{n})$.

Proposition 3.1. *Let $\mathcal{G}$ be an infinitesimal group with trivial multiplicative center and tame principal block $\mathcal{B}_0(\mathcal{G})$. Then* $\mathrm{Lie}(\mathcal{G})/C(\mathrm{Lie}(\mathcal{G})) \cong \mathfrak{sl}(2)$, *and* $C(\mathrm{Lie}(\mathcal{G}))$ *is a p-nilpotent ideal. Moreover, we have* $\dim \mathcal{V}_{C(\mathrm{Lie}(\mathcal{G}))} \le 2$.

Proof. As before, a consecutive application of [17, (6.1),(6.2)] and [7, (IV,§4,1.10)] shows that $\mathcal{R}$ is a unipotent normal subgroup of $\mathcal{G}$ of height ≤ 1. We set $\mathfrak{r} := \mathrm{Lie}(\mathcal{R})$ and consider the p-ideal $\mathfrak{n} := [\mathfrak{r}, \mathfrak{r}] + \langle \mathfrak{r}^{[p]} \rangle$ of $\mathfrak{g}$. Proceeding in several steps, we first show

(a) $\dim_k \mathfrak{r}/\mathfrak{n} \le 2$.

Since $\mathfrak{n} \subset \mathfrak{r}$ is invariant under the adjoint representation of $\mathcal{G}$, there exists a normal subgroup $\mathcal{N} \lhd \mathcal{R}$ of $\mathcal{G}$ of height ≤ 1 such that $\mathrm{Lie}(\mathcal{N}) = \mathfrak{n}$. There results an exact sequence

$$(0) \longrightarrow \mathfrak{n} \longrightarrow \mathfrak{g} \longrightarrow \mathrm{Lie}(\mathcal{G}/\mathcal{N})$$

of restricted Lie algebras (cf. [7, (II,§4,1.5)]). As $\mathcal{N} \subset \mathcal{R}$ is unipotent and $\mathcal{B}_0(\mathcal{G})$ is tame, a consecutive application of (1.2) and (1.1) implies that $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ is tame. Owing to [17, (2.5)] we thus have $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G}/\mathcal{N})} = 2$. As $\mathfrak{r}/\mathfrak{n} \subset \mathcal{V}_{\mathfrak{g}/\mathfrak{n}} \subset \mathcal{V}_{\mathrm{Lie}(\mathcal{G}/\mathcal{N})}$, we obtain $\dim_k \mathfrak{r}/\mathfrak{n} \le 2$, as desired. ◇

We put $\mathfrak{r}^1 := \mathfrak{r},\ \mathfrak{r}^2 := \mathfrak{n},\ \mathfrak{r}^3 := [\mathfrak{r}, \mathfrak{n}] = [\mathfrak{r}, [\mathfrak{r}, \mathfrak{r}]]$ as well as $\mathfrak{r}^i := [\mathfrak{r}, \mathfrak{r}^{i-1}]$ for $i \ge 4$. Thus, except for the second term, the $\mathfrak{r}^i$ coincide with the corresponding terms of the descending central series of $\mathfrak{r}$.

(b) *The Lie algebra* $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{sl}(2)$ *operates trivially on* $\mathfrak{r}^i/\mathfrak{r}^{i+1}$ *for every* $i \ge 1$.

By applying (2.1(1)) and [17, (4.3)] consecutively, we obtain $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{sl}(2)$. In view of (a) we know that $\dim_k \mathfrak{r}^1/\mathfrak{r}^2 \le 2$. If $\dim_k \mathfrak{r}^1/\mathfrak{r}^2 \le 1$, then our assertion obviously holds. Alternatively, the exact sequence

$$(0) \longrightarrow \mathfrak{r}^1/\mathfrak{r}^2 \longrightarrow \mathfrak{g}/\mathfrak{r}^2 \longrightarrow \mathfrak{sl}(2) \longrightarrow (0)$$

is an extension of a restricted Lie algebra by a strongly abelian, two-dimensional ideal. If $\mathfrak{r}^1/\mathfrak{r}^2$ is the standard $\mathfrak{sl}(2)$-module, then [28, Thm.1] (see [18] for $p = 3$) implies that $H^2_*(\mathfrak{sl}(2), \mathfrak{r}^1/\mathfrak{r}^2) = (0)$. In view of [23, (3.3)] the above sequence therefore splits. Since $p \ge 3$ and $\dim \mathcal{V}_{\mathfrak{r}^1/\mathfrak{r}^2} = 2$, this readily implies $\dim \mathcal{V}_{\mathfrak{g}/\mathfrak{n}} \ge 3$, which contradicts the equality $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G}/\mathcal{N})} = 2$ that we observed in (a). Consequently, $\mathfrak{r}^1/\mathfrak{r}^2$ is an extension of two one-dimensional modules, which, in view of $\mathrm{Ext}^1_{u(\mathfrak{sl}(2))}(k, k) = (0)$, splits. As a result, $\mathfrak{r}^1/\mathfrak{r}^2$ is a direct sum of trivial $\mathfrak{g}/\mathfrak{r}$-modules.

Proceeding inductively, we assume for some $i \ge 2$ that $\mathfrak{g}/\mathfrak{r}$ acts trivially on $\mathfrak{r}^{i-1}/\mathfrak{r}^i$. Then we have $[\mathfrak{g}, \mathfrak{r}^{i-1}] \subset \mathfrak{r}^i$. Given $x \in \mathfrak{g},\ a \in \mathfrak{r},\ b \in \mathfrak{r}^{i-1}$ we obtain

$$[x, [a, b]] = [[x, a], b] + [a, [x, b]] \in [\mathfrak{r}^2, \mathfrak{r}^{i-1}] + [\mathfrak{r}, \mathfrak{r}^i] \subset \mathfrak{r}^{i+1},$$

as well as

$$[x, a^{[p]}] = (\mathrm{ad}\, a)^{p-1}([x, a]) \in (\mathrm{ad}\, a)^{p-1}(\mathfrak{r}) \subset \mathfrak{r}^3,$$

whence $[\mathfrak{g}, \mathfrak{r}^i] \subset \mathfrak{r}^{i+1}$, as desired. ◇

(c) *The exact sequence*

$$(0) \longrightarrow \mathfrak{r}/\mathfrak{r}^i \longrightarrow \mathfrak{g}/\mathfrak{r}^i \xrightarrow{\sigma_i} \mathfrak{g}/\mathfrak{r} \longrightarrow (0)$$

of ordinary Lie algebras splits for every $i \ge 1$.

We use induction on i, and note that the case $i = 1$ is trivial. Assume that for some $i \ge 2$ the exact sequence

$$(0) \longrightarrow \mathfrak{r}/\mathfrak{r}^{i-1} \longrightarrow \mathfrak{g}/\mathfrak{r}^{i-1} \xrightarrow{\sigma_{i-1}} \mathfrak{g}/\mathfrak{r} \longrightarrow (0)$$

splits. Then there exists a subalgebra $\mathfrak{u} \subset \mathfrak{g}/\mathfrak{r}^{i-1}$ such that $\sigma_{i-1}|_{\mathfrak{u}} : \mathfrak{u} \longrightarrow \mathfrak{g}/\mathfrak{r}$ is an isomorphism. Let $\mathfrak{v} \subset \mathfrak{g}/\mathfrak{r}^i$ be the inverse image of $\mathfrak{u}$ under the canonical projection $\mathfrak{g}/\mathfrak{r}^i \longrightarrow \mathfrak{g}/\mathfrak{r}^{i-1}$. In view of (b), the algebra $\mathfrak{u}$ acts trivially on $\mathfrak{r}^{i-1}/\mathfrak{r}^i$, and there thus results a central extension

$$(0) \longrightarrow \mathfrak{r}^{i-1}/\mathfrak{r}^i \longrightarrow \mathfrak{v} \longrightarrow \mathfrak{u} \longrightarrow (0)$$

of ordinary Lie algebras. As noted in [14, §1], the second Chevalley-Eilenberg cohomology group $H^2(\mathfrak{sl}(2), k)$ is trivial, and [22, (VII.3.3)] shows that the above extension splits. In particular, $\mathfrak{v}$ contains a subalgebra $\mathfrak{w}$ that is isomorphic to $\mathfrak{sl}(2)$. Since $p \geq 3$, the algebra $\mathfrak{sl}(2)$ is simple, proving that the solvable ideal $\mathfrak{w} \cap \ker \sigma_i$ vanishes. Hence $\sigma_i|_{\mathfrak{w}}$ is injective, and thus bijective for dimension reasons. Accordingly, the above exact sequence splits. $\diamond$

Since the Lie algebra $\mathfrak{r}$ is p-nilpotent, Engel's Theorem provides a number $i \in \mathbb{N}$ with $\mathfrak{r}^i = (0)$. Now (c) implies that the sequence

$$(0) \longrightarrow \mathfrak{r} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{r} \longrightarrow (0)$$

splits. Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra which is isomorphic to $\mathfrak{g}/\mathfrak{r}$ via the canonical projection. In view of (b) the $\mathfrak{h}$-module $\mathfrak{r}$ has an ascending filtration $(X_i)_{i\geq 0}$ with trivial factors. Since $H^1(\mathfrak{sl}(2), k) = (0)$, we obtain $\mathfrak{r} \cong \bigoplus_{i\geq 0} X_{i+1}/X_i$, so that $\mathfrak{r}$ is a direct sum of trivial $\mathfrak{h}$-modules.

The general theory of restricted Lie algebras (cf. [34, (II.2.1)]) now provides a p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow C(\mathfrak{r})$ such that $\mathfrak{g}$ is isomorphic to the restricted Lie algebra, whose underlying Lie algebra is the direct product $\mathfrak{sl}(2) \oplus \mathfrak{r}$, and whose p-map is given by

$$(x, r)^{[p]} := (x^{[p]}, \psi(x) + r^{[p]}) \qquad \text{for all } \ x \in \mathfrak{sl}(2), \ r \in \mathfrak{r}.$$

Since $C(\mathfrak{r}) = C(\mathfrak{g})$ is invariant under the adjoint representation, there exists a normal subgroup $\mathcal{C} \subset \mathcal{G}$ with $\mathrm{Lie}(\mathcal{C}) = C(\mathfrak{r})$. As $\mathcal{C}$ is solvable, (1.2) and (1.1) yield the tameness of the principal block $\mathcal{B}_0(\mathcal{G}/\mathcal{C})$. The arguments of (a) then show $\dim \mathcal{V}_{\mathfrak{g}/C(\mathfrak{r})} \leq \dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G}/\mathcal{C})} \leq 2$. Since $\mathrm{im}\, \psi \subset C(\mathfrak{r})$, the Lie algebra $\mathfrak{g}/C(\mathfrak{r}) \cong \mathfrak{sl}(2) \oplus \mathfrak{r}/C(\mathfrak{r})$ is a direct sum of restricted Lie algebras, so that

$$2 \geq \dim \mathcal{V}_{\mathfrak{g}/C(\mathfrak{r})} = \dim \mathcal{V}_{\mathfrak{sl}(2)} + \dim \mathcal{V}_{\mathfrak{r}/C(\mathfrak{r})}.$$

Thus, $\mathcal{V}_{\mathfrak{r}/C(\mathfrak{r})} = \{0\}$, and we may apply [34, (II.3.9),(II.3.6)] to see that $\mathfrak{r}/C(\mathfrak{r})$ is a torus. Since $\mathfrak{r}/C(\mathfrak{r})$ is also p-nilpotent, we finally obtain $\mathfrak{r} = C(\mathfrak{r}) = C(\mathfrak{g})$. As $\mathcal{B}_0(\mathcal{G})$ is tame, [17, (2.5)] implies $\dim \mathcal{V}_{C(\mathfrak{g})} \leq \dim \mathcal{V}_{\mathfrak{g}} \leq 2$. $\square$

To obtain further information on the structure of $\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$ we have to analyze its center more closely. Recall that the dimension $\dim_k H(\mathcal{G})$ of the algebra of measures of a finite algebraic k-group $\mathcal{G}$ is also referred to as the *order* of $\mathcal{G}$. We say that $\mathfrak{g}$ is *representation-finite* if the algebra $u(\mathfrak{g})$ possesses only finitely many isoclasses of indecomposable modules.

Lemma 3.2. *Suppose that $\mathcal{G}$ has minimal order subject to the following conditions:*
(a) *$\mathcal{B}_0(\mathcal{G})$ is tame,*
(b) *$\mathcal{M} = e_k$, and*
(c) *$\dim \mathcal{V}_{C(\mathrm{Lie}(\mathcal{G}))} = 2$.*
Then $C(\mathrm{Lie}(\mathcal{G}))$ is a two-dimensional, strongly abelian p-ideal.

Proof. We put $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ as well as $\mathfrak{n} := C(\mathfrak{g})^{[p]}$. Then $\mathfrak{n} \lhd \mathfrak{g}$ is a $\mathcal{G}$-invariant p-ideal of $\mathfrak{g}$, and there exists a normal subgroup $\mathcal{N} \lhd \mathcal{G}$ of height ≤ 1 such that $\mathrm{Lie}(\mathcal{N}) = \mathfrak{n}$. Thanks to (3.1) the algebra $H(\mathcal{N}) \cong u(\mathfrak{n})$ is local, hence $\mathcal{N}$ is a unipotent subgroup of $\mathcal{G}$.

We consider the factor group $\mathcal{G}' := \mathcal{G}/\mathcal{N}$. If $\mathcal{G}'$ is solvable, then $\mathcal{G}$ is solvable, which, in view of (1.2), contradicts the tameness of $\mathcal{B}_0(\mathcal{G})$. Thus (1.1) applies, and $\mathcal{G}'$ satisfies condition (a).

Let $\mathcal{H} \subset \mathcal{G}$ be the pre-image of $\mathcal{M}(\mathcal{G}')$ under the quotient map $\mathcal{G} \longrightarrow \mathcal{G}'$. Then $\mathcal{M}(\mathcal{G}') \cong \mathcal{H}/\mathcal{N}$, so that $\mathcal{H}$ is an extension of two solvable groups. Thus, $\mathcal{H} \subset \mathcal{R}$, and we have observed before that (a) and (b) force the solvable radical to be unipotent. Accordingly, $\mathcal{M}(\mathcal{G}') \cong \mathcal{H}/\mathcal{N}$ enjoys the same property, whence $\mathcal{M}(\mathcal{G}') = e_k$.

Since $\mathcal{R}(\mathcal{G}') \cong \mathcal{R}/\mathcal{N}$ and $\mathrm{Lie}(\mathcal{R}) = C(\mathfrak{g})$ (3.1), we have an exact sequence

$$(0) \longrightarrow \mathfrak{n} \longrightarrow C(\mathfrak{g}) \longrightarrow \mathrm{Lie}(\mathcal{R}(\mathcal{G}')).$$

¿From the tameness of $\mathcal{B}_0(\mathcal{G}')$ we obtain, observing [17, (2.5)],

$$\dim_k C(\mathfrak{g})/\mathfrak{n} \leq \dim \mathcal{V}_{\mathrm{Lie}(\mathcal{R}(\mathcal{G}'))} \leq 2.$$

If $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{R}(\mathcal{G}'))} \leq 1$, then [23, (2.1)] implies that $\dim_k H^1_*(C(\mathfrak{g}), k) = \dim_k C(\mathfrak{g})/\mathfrak{n} \leq 1$. As the enveloping algebra $u(C(\mathfrak{g}))$ of the p-nilpotent Lie algebra $C(\mathfrak{g})$ is local, $C(\mathfrak{g})$ is representation-finite. It follows that $\dim \mathcal{V}_{C(\mathfrak{g})} \leq 1$ (see for instance [16, (1.3)]), which contradicts condition (c). Accordingly, the nullcone of $\mathrm{Lie}(\mathcal{R}(\mathcal{G}')) = C(\mathrm{Lie}(\mathcal{G}'))$ is two-dimensional, and the group $\mathcal{G}'$ also satisfies (a) through (c). The minimality of the order of $\mathcal{G}$ now shows that $\mathcal{N} = e_k$. Consequently, $\mathfrak{n} = (0)$, and we obtain

$$2 = \dim \mathcal{V}_{C(\mathfrak{g})} = \dim_k C(\mathfrak{g}),$$

as asserted. □

Let $\mathcal{G}$ be an infinitesimal k-group with Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$, M a $\mathcal{G}$-module. Following [27, (I.2.16)], we let $M^{(1)}$ be the $\mathcal{G}$-module on which a scalar $\alpha \in k$ acts via $\alpha^{\frac{1}{p}}$. In our next Lemma we are going to apply this construction to factor modules of the adjoint representation $\mathrm{Ad} : \mathcal{G} \longrightarrow \mathrm{GL}(\mathfrak{g})$.

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra such that $\mathfrak{g}/C(\mathfrak{g}) \cong \mathfrak{sl}(2)$. As noted earlier, the corresponding extension of ordinary Lie algebras splits and the general theory of restricted Lie algebras (cf. [34, (II.2.1)]) guarantees the existence of a p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow C(\mathfrak{g})$, which induces the p-map on $\mathfrak{g} = \mathfrak{sl}(2) \oplus C(\mathfrak{g})$, i.e.,

$$(x, c)^{[p]} = (x^{[p]}, \psi(x) + c^{[p]}) \qquad \forall\ x \in \mathfrak{sl}(2),\ c \in C(\mathfrak{g}).$$

Since ψ is p-semilinear, it gives rise to a linear map $\psi : \mathfrak{sl}(2)^{(1)} \longrightarrow C(\mathfrak{g})$.

Lemma 3.3. *Let $\mathcal{G}$ be an infinitesimal group such that $\mathfrak{g}/C(\mathfrak{g}) \cong \mathfrak{sl}(2)$. Then the map $\psi : \mathfrak{sl}(2)^{(1)} \longrightarrow C(\mathfrak{g})$, induced by the p-map on $\mathfrak{g}$, is a homomorphism of $\mathcal{G}$-modules.*

Proof. Since taking centers commutes with base change, $C(\mathfrak{g})$ is a $\mathcal{G}$-submodule of $\mathfrak{g}$, and our assumption provides an exact sequence

$$(0) \longrightarrow C(\mathfrak{g}) \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{sl}(2) \longrightarrow (0)$$

of $\mathcal{G}$-modules. As $\mathcal{G}$ acts on $\mathfrak{g}$ via automorphisms of restricted Lie algebras, the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ is a $\mathcal{G}$-submodule of $\mathfrak{g}$. Since $\mathfrak{g} = \mathfrak{sl}(2) \oplus C(\mathfrak{g})$ is a direct sum of ideals (but not necessarily p-ideals), we have, observing $p \geq 3$, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}(2)$, so that $\pi|_{[\mathfrak{g},\mathfrak{g}]}$ is bijective. Consequently, the above sequence of $\mathcal{G}$-modules splits.

Let R be a commutative k-algebra. The p-map on $\mathfrak{g} \otimes_k R = (\mathfrak{sl}(2) \otimes_k R) \oplus (C(\mathfrak{g}) \otimes_k R)$ is given by

$$(x \otimes r, c \otimes r)^{[p]} = (x^{[p]} \otimes r^p, \psi_R(x \otimes r) + c^{[p]} \otimes r^p) \qquad \forall\ x \in \mathfrak{sl}(2),\ c \in C(\mathfrak{g}),\ r \in R,$$

where $\psi_R : \mathfrak{g} \otimes_k R \longrightarrow C(\mathfrak{g}) \otimes_k R$ is the p-R-semilinear map defined via $\psi_R(x \otimes r) = \psi(x) \otimes r^p$. Let g be an element of $\mathcal{G}(R)$. Since g acts on $\mathfrak{g} \otimes_k R$ and $\mathfrak{sl}(2) \otimes_k R$ via automorphisms of restricted R-Lie algebras, we obtain, observing the $\mathcal{G}$-module decomposition $\mathfrak{g} = \mathfrak{sl}(2) \oplus C(\mathfrak{g})$,

$$\begin{aligned}
((g \cdot (x \otimes 1))^{[p]}, \psi_R((g \cdot (x \otimes 1)) &= (g \cdot (x \otimes 1), 0)^{[p]} = g \cdot (x \otimes 1, 0)^{[p]} \\
&= g \cdot (x^{[p]} \otimes 1, \psi_R(x \otimes 1)) \\
&= (g \cdot (x \otimes 1)^{[p]}, g \cdot \psi_R(x \otimes 1)) \\
&= ((g \cdot (x \otimes 1))^{[p]}, g \cdot \psi_R(x \otimes 1)).
\end{aligned}$$

Consequently, $\psi_R((g \cdot (x \otimes 1)) = g \cdot \psi_R(x \otimes 1))$, as desired. □

Let $(\mathfrak{g},[p])$ be a one-dimensional central extension of $\mathfrak{sl}(2)$ with strongly abelian center. According to [14, (2.2)] there exist exactly three isomorphism classes of such algebras, whose representatives $\mathfrak{sl}(2)_0$, $\mathfrak{sl}(2)_s$, and $\mathfrak{sl}(2)_n$, correspond to p-semilinear maps $0, \psi_s, \psi_n : \mathfrak{sl}(2) \longrightarrow k$, respectively. Directly from the definition of $\mathfrak{sl}(2)_s$ given in Section 1 we obtain $\psi_s(e) = 0 = \psi_s(f)$, $\psi_s(h) = 1$.

We let $\mathrm{Cent}(\mathcal{G})$ denote the *center* of the infinitesimal k-group $\mathcal{G}$ (cf. [7, (II,§1,3.9)]). The *complexity* of $\mathcal{G}$ is the rate of growth of a minimal projective resolution of the trivial $H(\mathcal{G})$-module (cf. [4, (5.1)]). Thanks to [29, Thm2] and [26, Satz] this number coincides with the dimension of the nullcone $\mathcal{V}_{\mathrm{Lie}(\mathcal{G})}$ for infinitesimal groups of height ≤ 1.

Theorem 3.4. *Let $\mathcal{G}$ be an infinitesimal k-group with tame principal block and trivial multiplicative center. Then there exist $n \geq 0$, $r \geq 1$ such that*

(1) $C(\mathrm{Lie}(\mathcal{G})) \cong \mathrm{Lie}(\mathcal{W}_n)$ *and* $\mathrm{Lie}(\mathcal{G})/C(\mathrm{Lie}(\mathcal{G}))^{[p]} \cong \mathfrak{sl}(2), \mathfrak{sl}(2)_s$,

(2) $\mathcal{G}/\mathrm{Cent}(\mathcal{G}) \cong \mathcal{Q}_{[r]}$ *and* $\mathrm{Cent}(\mathcal{G}) \cong (\mathcal{W}_n)_1$.

Proof. In view of [14, (7.4)] and [17, (6.4)] we may assume that $\mathcal{G}$ has height $r \geq 2$. Thanks to (3.1) the Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ has the form $\mathfrak{g} = \mathfrak{sl}(2) \oplus C(\mathfrak{g})$, with p-map induced by a p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow C(\mathfrak{g})$.

(a) *If* $\dim_k C(\mathfrak{g}) = \dim \mathcal{V}_{C(\mathfrak{g})} = 2$, *then* $\langle \psi(\mathcal{V}_{\mathfrak{sl}(2)}) \rangle = C(\mathfrak{g})$, *and ψ is surjective.*

Since $\mathcal{B}_0(\mathcal{G})$ is tame, [17, (2.5)] implies $\dim \mathcal{V}_{\mathfrak{g}} = 2$. On the other hand, the nullcone of $\mathfrak{g}$ is given by $\mathcal{V}_{\mathfrak{g}} = (\mathcal{V}_{\mathfrak{sl}(2)} \cap \ker \psi) \times C(\mathfrak{g})$, so that $\dim(\mathcal{V}_{\mathfrak{sl}(2)} \cap \ker \psi) = 0$. Since this variety is conical, we obtain $\mathcal{V}_{\mathfrak{sl}(2)} \cap \ker \psi = \{0\}$, and the morphism $\psi|_{\mathcal{V}_{\mathfrak{sl}(2)}} : \mathcal{V}_{\mathfrak{sl}(2)} \longrightarrow C(\mathfrak{g})$ of irreducible varieties satisfies $\psi^{-1}(0) = \{0\}$. Upper semicontinuity of fibre dimension now shows that its generic fibre has dimension zero. Since $\mathcal{V}_{\mathfrak{sl}(2)}$ and $C(\mathfrak{g})$ both have dimension 2, we conclude that the morphism $\psi|_{\mathcal{V}_{\mathfrak{sl}(2)}}$ is dominant. As $\langle \psi(\mathcal{V}_{\mathfrak{sl}(2)}) \rangle$ is a closed subset of $C(\mathfrak{g})$ containing $\mathrm{im}\, \psi|_{\mathcal{V}_{\mathfrak{sl}(2)}}$, our first assertion follows. As a result, ψ is surjective. $\diamond$

(b) $\dim \mathcal{V}_{C(\mathfrak{g})} \leq 1$.

If this is not the case, then (3.1) yields $\dim \mathcal{V}_{C(\mathfrak{g})} = 2$, and we may assume the order of $\mathcal{G}$ to be minimal subject to this property. Thanks to (3.2) this forces $C(\mathfrak{g})$ to be two-dimensional and strongly abelian.

Thanks to [17, (6.1)] the solvable radical $\mathcal{R}$ of $\mathcal{G}$ is nilpotent. In view of [7, (IV,§4,1.11)] $\mathcal{M}(\mathcal{R})$ is normal in $\mathcal{G}$, and the factor group $\mathcal{R}/\mathcal{M}(\mathcal{R})$ is unipotent. As $\mathcal{M} = e_k$, [17, (6.2)] implies that $\mathcal{R}$ has height ≤ 1, and from the structure of $\mathfrak{g}$ we obtain $\mathcal{R} \subset \mathrm{Cent}(\mathcal{G}_1)$. In particular, $\mathcal{R}$ operates trivially on $\mathfrak{sl}(2)$ and $C(\mathfrak{g})$ via the adjoint representation. By (a) and (3.3) the map $\psi : \mathfrak{sl}(2)^{(1)} \longrightarrow C(\mathfrak{g})$ is a surjective homomorphism of $\mathcal{G}/\mathcal{R}$-modules. According to (1.3) the factor group $\mathcal{G}' := \mathcal{G}/\mathcal{R}$ is defined over the Galois field $\mathbb{F}_p$. As the module $\mathfrak{sl}(2)$ enjoys the same property, [27, (I.9.10)] implies that $\mathfrak{sl}(2)^{(1)}$ is isomorphic to the Frobenius twist of $\mathfrak{sl}(2)$. In particular, $\mathcal{G}'_1$ operates trivially on $\mathfrak{sl}(2)^{(1)}$, and the weights of $\mathfrak{sl}(2)^{(1)}$ relative to the standard torus of $\mathcal{G}'$ are just the p-fold multiples of those for $\mathfrak{sl}(2)$. Since the Frobenius kernel $\mathcal{G}_1$ also operates trivially on $C(\mathfrak{g})$, we conclude that $\psi : \mathfrak{sl}(2)^{(1)} \longrightarrow C(\mathfrak{g})$ is a homomorphism of $\mathcal{G}/\mathcal{G}_1$-modules. In view of (2.1(3)) we have $\mathcal{G}/\mathcal{G}_1 \cong \mathcal{G}'/\mathcal{G}'_1 \cong \mu_{p^{r-1}}$, so we will view ψ as a morphism for the latter group.

Let $\mathfrak{sl}(2) = \mathfrak{sl}(2)_\alpha \oplus \mathfrak{sl}(2)_0 \oplus \mathfrak{sl}(2)_{-\alpha}$ be the root space decomposition of $\mathfrak{sl}(2)$ relative to the standard maximal torus of the group $\mathcal{G}'$. Since $p \neq 2$, the character α has order p^r. Let $\beta := p\alpha$. Then β has order p^{r-1}, and the weight space decomposition of $\mathfrak{sl}(2)^{(1)}$ is given by

$$\mathfrak{sl}(2)^{(1)} = \mathfrak{sl}(2)^{(1)}_\beta \oplus \mathfrak{sl}(2)^{(1)}_0 \oplus \mathfrak{sl}(2)^{(1)}_{-\beta}.$$

Since $\ker\psi \cap \mathcal{V}_{\mathfrak{sl}(2)} = \{0\}$, we have $\ker\psi \cap \mathfrak{sl}(2)^{(1)}_{\pm\beta} = \{0\}$. Consequently, (a) gives rise to $C(\mathfrak{g}) = C(\mathfrak{g})_\beta \oplus C(\mathfrak{g})_{-\beta}$. By the same token, we have $(0) \neq (\mathfrak{sl}(2)^{(1)}_\beta)^{[p]} \subset C(\mathfrak{g})_{p\beta}$, so that $p\beta \in \{\beta, -\beta\}$, and $\beta = 0$. As this contradicts our assumption $r \geq 2$, we conclude that $\dim \mathcal{V}_{C(\mathfrak{g})} \leq 1$. $\diamond$

Let $\mathcal{C} \subset \mathcal{G}_1$ be the normal subgroup of $\mathcal{G}$ such that $\mathrm{Lie}(\mathcal{C}) = C(\mathfrak{g})$. By (b) and (3.1) $\mathcal{C}$ is a unipotent infinitesimal subgroup of complexity ≤ 1 and height ≤ 1. Thanks to [13, (5.3)] this implies $\mathcal{C} = e_k$ or $\mathcal{C} \cong (\mathcal{W}_n)_1$ for some $n \geq 1$. This proves the first statement of (1).

Assuming $C(\mathfrak{g}) \neq (0)$, we have $\dim_k C(\mathfrak{g})/C(\mathfrak{g})^{[p]} = 1$. We consider the four-dimensional Lie algebra $\mathfrak{g}' := \mathfrak{g}/C(\mathfrak{g})^{[p]}$, whose p-map is given by the composition ψ' of ψ with the canonical map $\mathfrak{g} \longrightarrow \mathfrak{g}'$. Since $C(\mathfrak{g})^{[p]}$ is $\mathcal{G}$-invariant, $\psi' : \mathfrak{sl}(2)^{(1)} \longrightarrow C(\mathfrak{g}') = C(\mathfrak{g})/C(\mathfrak{g})^{[p]}$ is a homomorphism of $\mathcal{G}$-modules. The assumption $\psi'(\mathfrak{sl}(2)^{(1)}_{\pm\beta}) \neq (0)$ implies $C(\mathfrak{g}') = C(\mathfrak{g}')_{\pm\beta}$, while $(\mathfrak{sl}(2)^{(1)}_{\pm\beta})^{[p]} \subset C(\mathfrak{g}')_{\pm p\beta}$ gives $\beta = 0$, a contradiction. Consequently, ψ' annihilates $\mathfrak{sl}(2)^{(1)}_\beta \oplus \mathfrak{sl}(2)^{(1)}_{-\beta}$, and ψ' is a multiple of the p-semilinear form ψ_s defining $\mathfrak{sl}(2)_s$. Thus, $\mathfrak{g}/C(\mathfrak{g})^{[p]} \cong \mathfrak{sl}(2)_s$ (cf. [14, (2.2)]).

Since the subgroups $\mathcal{C} \subset \mathcal{R}$ of $\mathcal{G}_1$ satisfy $\mathrm{Lie}(\mathcal{R}) = C(\mathfrak{g}) = \mathrm{Lie}(\mathcal{C})$, we have $\mathcal{R} = \mathcal{C} \cong (\mathcal{W}_n)_1$. The last paragraph also shows that the multiplicative group $\mathcal{G}/\mathcal{G}_1$ operates trivially on $C(\mathfrak{g})/C(\mathfrak{g})^{[p]}$. As $C(\mathfrak{g})$ is a completely reducible $\mathcal{G}/\mathcal{G}_1$-module, there thus exists $v_0 \in C(\mathfrak{g})^{\mathcal{G}}$ such that $k(v_0 + C(\mathfrak{g})^{[p]}) = C(\mathfrak{g})/C(\mathfrak{g})^{[p]}$. Consequently, v_0 generates the nil-cyclic restricted Lie algebra $C(\mathfrak{g})$. Observing that $C(\mathfrak{g})^{\mathcal{G}}$ is a p-subalgebra of $C(\mathfrak{g})$ containing v_0, we obtain $C(\mathfrak{g})^{\mathcal{G}} = C(\mathfrak{g})$. Consequently, $\mathcal{G}$ operates trivially on $C(\mathfrak{g})$, and the identity $\mathrm{Lie}(\mathcal{R}) = C(\mathfrak{g})$ implies that $\mathcal{R} = \mathrm{Cent}(\mathcal{G})$. By virtue of (1.3) the group $\mathcal{G}$ is a central extension of $\mathcal{Q}_{[r]}$ by $(\mathcal{W}_n)_1$, as desired. □

Corollary 3.5. *Let $\mathcal{G}$ be an infinitesimal group with tame principal block. Then there exist $r \geq 1$ and $n \geq 0$ such that $\mathcal{G}/\mathrm{Cent}(\mathcal{G}) \cong \mathcal{Q}_{[r]}$ and $\mathrm{Cent}(\mathcal{G}) \cong \mathcal{M}(\mathcal{G}) \times (\mathcal{W}_n)_1$.*

Proof. We consider the factor group $\mathcal{G}' := \mathcal{G}/\mathcal{M}$ and the canonical quotient map $\pi : \mathcal{G} \longrightarrow \mathcal{G}'$. In virtue of (3.4) there exists $n \geq 0$ such that $\mathrm{Cent}(\mathcal{G}') \cong (\mathcal{W}_n)_1$. Setting $\mathcal{C} := \pi^{-1}(\mathrm{Cent}(\mathcal{G}'))$ we obtain an exact sequence

$$e_k \longrightarrow \mathcal{M} \longrightarrow \mathcal{C} \longrightarrow (\mathcal{W}_n)_1 \longrightarrow e_k.$$

Thanks to [15, (3.3)] this sequence splits, so that

$$\mathcal{C} \cong \mathcal{M} \times (\mathcal{W}_n)_1$$

is a commutative, normal subgroup of $\mathcal{G}$. In view of [7, (III,§3,3.7)] and (3.4) there exists $r \geq 1$ such that

$$\mathcal{G}/\mathcal{C} \cong (\mathcal{G}/\mathcal{M})/(\mathcal{C}/\mathcal{M}) \cong \mathcal{Q}_{[r]}.$$

The above split exact sequence induces a sequence

$$(0) \longrightarrow \mathrm{Lie}(\mathcal{M}) \longrightarrow \mathrm{Lie}(\mathcal{C}) \longrightarrow \mathrm{Lie}(\mathcal{W}_n) \longrightarrow (0)$$

of $\mathcal{G}/\mathcal{C}$-modules whose extreme terms are trivial modules. Owing to [17, (5.2)] this sequence therefore splits, so that the adjoint action of $\mathcal{G}$ on $\mathrm{Lie}(\mathcal{C})$ is also trivial. In particular,

$$(\mathcal{W}_n)_1 \subset \mathcal{C}_1 \subset \mathrm{Cent}(\mathcal{G}),$$

which readily implies $\mathcal{C} \subset \mathrm{Cent}(\mathcal{G})$. As $\mathcal{G}/\mathcal{C}$ is semisimple, the reverse inclusion also holds. □

4. Double Nakayama Algebras

In this section we are concerned with two families of basic algebras; the members of one of them will turn out to be Morita equivalent to tame blocks of infinitesimal groups. As will be shown in Proposition 4.1 below, the algebras are completely determined by their bound quivers and their invariants relative to a cyclic subgroup of their automorphism group. Our method of recovering these algebras from their invariants rests on the interpretation of the associated bounded categories of the latter as quotient categories, with the passage corresponding to Galois coverings. We begin by recalling a few basic notions concerning locally bounded categories and their modules. The reader is referred to [1, 5, 20] for further details.

Given any locally finite quiver Q (i.e., each vertex is the starting and endpoint of only finitely many arrows), the *path category* $k[Q]$ of Q has as objects the vertices of Q, and as morphisms between two objects x, y the space $k[Q](x, y)$ of k-linear combinations of paths from x to y. For $n \in \mathbb{N}$ and objects x, y of $k[Q]$, we denote by $k[Q](x, y)_n$ the subspace of $k[Q](x, y)$ generated by all paths of length $\geq n$. An ideal I of the path category $k[Q]$ is called *admissible* if

(a) $I(x, y) \subset k[Q](x, y)_2 \quad \forall\, x, y \in Q$, and

(b) for every $x \in Q$ there exists $n_x \in \mathbb{N}$ such that $k[Q](x, y)_{n_x} \subset I(x, y)$ and $k[Q](y, x)_{n_x} \subset I(y, x) \quad \forall\, y \in Q$.

In that case (Q, I) is called a *bound quiver* and the residue category $R = k[Q]/I$ is a *locally bounded* k-category, that is,

(i) distinct objects are non-isomorphic, and

(ii) $R(x, x)$ is a local algebra for every $x \in \mathrm{Ob}(R)$, and

(iii) $\sum_{x\in \mathrm{Ob}(R)}(\dim_k R(x, y) + \dim_k R(y, x)) < \infty$ for every $y \in \mathrm{Ob}(R)$.

An *R-module* over a locally bounded category is a covariant functor $M : R \longrightarrow \mathrm{Vec}_k$ into the category of k-vector spaces. We say that M is *finite-dimensional* if $\dim M := \sum_{x\in \mathrm{Ob}(R)} \dim_k M(x) < \infty$. The category of finite-dimensional R-modules will be denoted $\mathrm{mod}\, R$. If R is *bounded* (R has only finitely many objects), then $\mathrm{mod}\, R$ is equivalent to the category $\mathrm{mod}\, \Lambda$ of finite dimensional modules over the algebra $\Lambda := \oplus R$, which is defined to be the space of those matrices $(a_{yx})_{x,y\in R}$ with $a_{yx} \in R(x, y)$.

If Q is a quiver with finitely many vertices, we will denote the path algebra of Q also by $k[Q]$. An admissible ideal I of the algebra $k[Q]$ corresponds to an admissible ideal I in the path category $k[Q]$, and the factor algebra $k[Q]/I$ is the algebra associated to the residue category $k[Q]/I$. We shall henceforth identify a bound quiver algebra $k[Q]/I$ with its bounded category $k[Q]/I$.

Given an algebra Λ, we let $\mathrm{ind}_\Lambda(d)$ be the set of isoclasses of d-dimensional indecomposable Λ-modules. We say that Λ is *tame*, if it is not representation-finite, and if for every $d > 0$ there exist $(\Lambda, k[X])$-bimodules $M_1, \dots, M_{m(d)}$ that are finitely generated free right $k[X]$-modules, such that all but finitely many elements of $\mathrm{ind}_\Lambda(d)$ are isoclasses of modules of the form $M_i \otimes_{k[X]} S$ for some simple $k[X]$-module S and $i \in \{1, \dots, m(d)\}$. It is well-known that an algebra is tame if and only if its basic algebra enjoys this property. Following [32, p.174] we say that Λ is *special biserial* if Λ is Morita equivalent to a bound quiver algebra $k[Q]/I$ satisfying

(SB1) each vertex of Q is the starting point and end point of at most two arrows, and

(SB2) for any arrow $\alpha \in Q$, there is at most one arrow β and one arrow γ with $\alpha\beta \notin I$ and $\gamma\alpha \notin I$.

Representation-infinite special biserial algebras form an important class of tame algebras.

Assume that R is a locally bounded category and G is a group of k-linear automorphisms of R acting freely on the objects of R. According to [20, Proposition 3.1] the quotient category R/G

exists. Its objects are the G-orbits of the objects of R. Moreover, we have

$$(R/G)(a,b) := \left\{ (f_{yx}) \in \prod_{(x,y)\in a\times b} R(x,y) \ ; \ g\cdot f_{yx} = f_{g(y)g(x)} \quad \forall\, g\in G,\, x\in a,\, y\in b \right\},$$

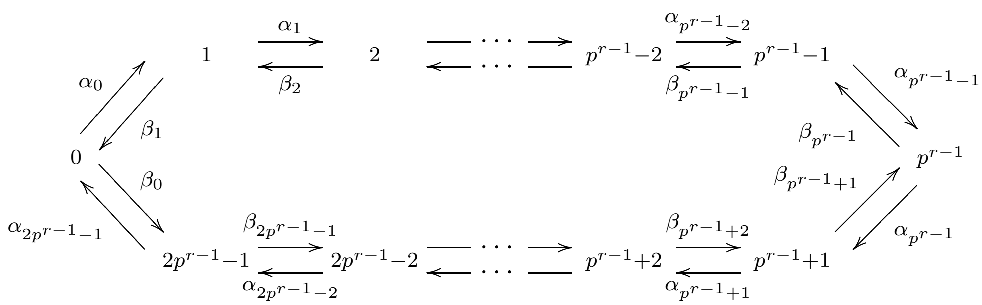
and the composition of $e \in (R/G)(b,c)$ with $f \in (R/G)(a,b)$ is given by $(ef)_{zx} := \sum_{y\in b} e_{zy}f_{yx}$. The canonical functor $F : R \longrightarrow R/G$ which assigns to each object of R its G-orbit and to each $\zeta \in R(x,y)$ the family $F(\zeta)$ given by $F(\zeta)_{h(y)g(x)} = \delta_{gh}g\zeta$ is called the *Galois covering* of R/G with Galois group G.

Suppose that R is bounded, and consider the associated basic finite dimensional algebras $A := \oplus R$ and $\Lambda := \oplus(R/G)$. The group G acts on A via

$$g((a_{yx})_{x,y}) := (g\cdot a_{g^{-1}(y)g^{-1}(x)})_{x,y},$$

and the invariant algebra A^G consists of all matrices $(a_{yx})_{x,y}$ satisfying $g\cdot a_{yx} = a_{g(y)g(x)}$ for all $x,y \in R$ and $g \in G$. In view of [1, (6.2)] we have an isomorphism $\Lambda \cong A^G$ of k-algebras. Moreover, the push-down functor (cf. [5, (3.2)]) $F_\lambda : \operatorname{mod} R \longrightarrow \operatorname{mod} R/G$ associated to the Galois covering $F : R \longrightarrow R/G$ sends the indecomposable projective R-module $\operatorname{Hom}_R(x,\cdot)$ to the indecomposable projective R/G-module $\operatorname{Hom}_{R/G}(F(x),\cdot)$.

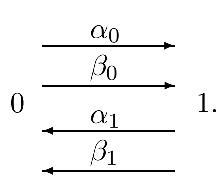
We fix a prime number $p \geq 3$. Given a natural number $r \geq 1$, we denote by Δ_r the quiver with underlying set of vertices $\mathbb{Z}/(2p^{r-1})$ and arrows $\alpha_i : i \to i+1 \ ; \ \beta_i : i \to i-1$ for $i \in \mathbb{Z}/(2p^{r-1})$:



The map

$$g : \mathbb{Z}/(2p^{r-1}) \longrightarrow \mathbb{Z}/(2p^{r-1}) \ ; \ i \mapsto i+2$$

is an automorphism of Δ_r of order p^{r-1}, so that the subgroup $G_r \subset \operatorname{Aut}(\Delta_r)$ generated by g is isomorphic to $\mathbb{Z}/(p^{r-1})$. Note that Δ_1 is the quiver



For $n \geq 0$, let $J_{r,n} \subset k[\Delta_r]$ be the ideal generated by

$$\{(\beta_{i+1}\alpha_i)^{p^n} - (\alpha_{i-1}\beta_i)^{p^n}, \ \ \alpha_{i+1}\alpha_i, \ \ \beta_i\beta_{i+1} \ ; \ i \in \mathbb{Z}/(2p^{r-1})\}.$$

In case $r = 1$, we put $I_n := J_{1,n}$. The bound quiver algebras $k[\Delta_r]/J_{r,n}$ are special biserial.

Definition. Given $r \geq 1$ and $n \geq 0$, we call $\mathcal{N}^2(r,n) := k[\Delta_r]/J_{r,n}$ the *double Nakayama algebra associated to r and n.*

Note that $\mathcal{N}^2(r,0)$ is the trivial extension of the hereditary radical square zero algebra of type $\tilde{A}_{2p^{r-1}-1}$ by its dual module. According to [14, (7.1)] the algebras $\mathcal{N}^2(1,n)$ are the basic algebras of the tame principal blocks associated to infinitesimal groups of height ≤ 1.

In the following result we employ the foregoing remarks to recover certain bound quiver algebras of the quiver Δ_r from their invariants relative to the group $G_r \subset \mathrm{Aut}(k[\Delta_r])$. The reader is referred to [5] for undefined terminology.

Proposition 4.1. *Let $I \subset k[\Delta_r]$ be an admissible G_r-invariant ideal such that $(k[\Delta_r]/I)^{G_r} \cong \mathcal{N}^2(1,n)$. Then $k[\Delta_r]/I$ is isomorphic to $k[\Delta_r]/J$, where*

(a) *$J = J_{r,n}$, or*

(b) *$r \geq 2$, $n \geq r-1$, and J is generated by*

$$\{\alpha_{i+2p^n-1}\cdots\alpha_{i+1}\alpha_i - \beta_{i-2p^n+1}\cdots\beta_{i-1}\beta_i,\quad \alpha_i\beta_{i+1},\quad \beta_i\alpha_{i-1}\quad ;\quad i \in \mathbb{Z}/(2p^{r-1})\}.$$

Conversely, in each of these cases we have $(k[\Delta_r]/J)^{G_r} \cong \mathcal{N}^2(1,n)$.

Proof. We may assume $r \geq 2$. Let $\Lambda := k[\Delta_r]/I$ and $A(n) := k[\Delta_1]/I_n$ be the bounded categories associated to I and I_n, respectively. Then G_r acts freely on the objects of Λ, and our current assumption implies the existence of a Galois covering $F : \Lambda \longrightarrow \Lambda/G_r \cong A(n)$ with Galois group G_r. Being a covering functor, F induces isomorphisms of vector spaces

$$(*)\qquad \bigoplus_{h\in G_r}\Lambda(x,hy) \xrightarrow{\sim} A(n)(a,b),\qquad \bigoplus_{h\in G_r}\Lambda(hx,y) \xrightarrow{\sim} A(n)(a,b)$$

for all objects a, b of $A(n)$ and all objects x, y of Λ such that $a = F(x)$, $b = F(y)$ (see [20, Sections 1 and 3]). Without loss of generality, we may assume that $F : \mathbb{Z}/(2p^{r-1}) \longrightarrow \mathbb{Z}/(2)$ coincides with the canonical projection. By the same token we find $u_1 \in \mathrm{Rad}\,\Lambda(1,2)\setminus \mathrm{Rad}^2\,\Lambda(1,2)$ and $v_1 \in \mathrm{Rad}\,\Lambda(1,0)\setminus\mathrm{Rad}^2\,\Lambda(1,0)$, such that

$$F(u_1) = \alpha_1 + I_n \quad;\quad F(v_1) = \beta_1 + I_n.$$

Thanks to $(*)$ there exist $u_0 \in \mathrm{Rad}\,\Lambda(0,1)\setminus\mathrm{Rad}^2\,\Lambda(0,1)$ and $v_2 \in \mathrm{Rad}\,\Lambda(2,1)\setminus\mathrm{Rad}^2\,\Lambda(2,1)$ such that either

(1) $F(u_0) = \alpha_0 + I_n$; $F(v_2) = \beta_0 + I_n$, or

(2) $F(u_0) = \beta_0 + I_n$; $F(v_2) = \alpha_0 + I_n$.

For $j \in \{0,1\}$ and $\ell \in \{1,2\}$ we define $u_{2i+j} := g^i u_j$ and $v_{2i+\ell} := g^i v_\ell$ $(0 \leq i \leq p^{r-1}-1)$. As F is a Galois covering, we obtain

$$F(u_{2i+1}) = \alpha_1 + I_n \quad;\quad F(v_{2i+1}) = \beta_1 + I_n \qquad 0 \leq i \leq p^{r-1}-1.$$

Moreover, we either have

(1) $F(u_{2i}) = \alpha_0 + I_n$; $F(v_{2i+2}) = \beta_0 + I_n$ $\quad 0 \leq i \leq p^{r-1}-1$, or

(2) $F(u_{2i}) = \beta_0 + I_n$; $F(v_{2i+2}) = \alpha_0 + I_n$ $\quad 0 \leq i \leq p^{r-1}-1$.

Thanks to [3, (1.2.8)] the assignment $\alpha_i \mapsto u_i$, $\beta_i \mapsto v_i$, $0 \leq i \leq 2p^{r-1}-1$, is readily seen to induce an algebra epimorphism $\Phi : k[\Delta_r] \longrightarrow k[\Delta_r]/I$. We have the inclusion $J_{r,n} \subset \ker\Phi$ in the former case. If (2) applies then, invoking the facts that Δ_r has $2p^{r-1}$ vertices with p being odd, we conclude that $r-1 \leq n$ as well as $J \subset \ker\Phi$. In view of $(*)$ we also have $\dim_k k[\Delta_r]/I = p^{r-1}\dim_k A(n) = \dim_k k[\Delta_r]/J$, rendering the resulting epimorphism $k[\Delta_r]/J \longrightarrow k[\Delta_r]/I$ an isomorphism.

Finally, if J is given as in (a) or (b), then the canonical map $\Delta_r \longrightarrow \Delta_1$ gives rise to a Galois covering $k[\Delta_r]/J \longrightarrow k[\Delta_1]/I_n$ of bounded k-categories with Galois group G_r. In view of [1, (6.2)] we thus have $(k[\Delta_r]/J)^{G_r} \cong \mathcal{N}^2(1,n)$. □

For future reference we record an immediate consequence of (4.1). Recall that a Λ-module M is *sincere* if every simple Λ-module occurs as a composition factor of M.

Corollary 4.2. *Suppose that $r \geq 2$. Let $I \subset k[\Delta_r]$ be an admissible G_r-invariant ideal such that $(k[\Delta_r]/I)^{G_r} \cong \mathcal{N}^2(1,n)$. Then $k[\Delta_r]/I$ is symmetric and one of the following statements applies:*

(1) *We have $k[\Delta_r]/I \cong \mathcal{N}^2(r,n)$, and every principal indecomposable $k[\Delta_r]/I$-module has (up to isomorphism) exactly 3 simple composition factors. In particular, the principal indecomposable $k[\Delta_r]/I$-modules are not sincere.*

(2) *Every principal indecomposable $k[\Delta_r]/I$-module is sincere, and thus has (up to isomorphism) $2p^{r-1}$ simple composition factors.*

Proof. By (4.1) the algebra $k[\Delta_r]/I$ is isomorphic to $k[\Delta_r]/J$, where J is given by (a) or (b). The relevant symmetrizing forms are induced by the linear forms $\psi_{(a)}, \psi_{(b)} \in k[\Delta_r]^*$ satisfying

$$\psi_{(a)}(q) = \begin{cases} 1 & q = (\alpha_{i-1}\beta_i)^{p^n},\, (\beta_{i+1}\alpha_i)^{p^n} \\ 0 & \text{otherwise} \end{cases} \quad ; \quad \psi_{(b)}(q) = \begin{cases} 1 & q = p_i,\, p'_i \\ 0 & \text{otherwise,} \end{cases}$$

where $p_i = \alpha_{i+2p^n-1} \cdots \alpha_{i+1}\alpha_i$ and $p'_i = \beta_{i-2p^n+1} \cdots \beta_{i-1}\beta_i$. □

We conclude this section by studying the behaviour of invariants under the passage to basic algebras. Let Λ be a finite dimensional k-algebra, $\mathrm{Aut}_k(\Lambda)$ its automorphism group. For a Λ-module M and $g \in \mathrm{Aut}_k(\Lambda)$ we let $M^{(g)}$ be the Λ-module with underlying k-space M and action given by

$$a \cdot m := g^{-1}(a)m \quad \forall\, a \in \Lambda,\ m \in M.$$

In particular, the assignment $(g, M) \mapsto M^{(g)}$ induces an operation of $\mathrm{Aut}_k(\Lambda)$ on the set of isoclasses of simple Λ-modules.

Proposition 4.3. *Let Λ be a finite dimensional k-algebra, $G \subset \mathrm{Aut}_k(\Lambda)$ a finite group of k-algebra automorphisms such that the induced action of G on the isoclasses of simple Λ-modules is free. Then there is an idempotent e of Λ^G such that*

(a) *$e\Lambda e$ is the basic algebra of Λ, and*

(b) *$e\Lambda e$ is a G-submodule of Λ, and*

(c) *$(e\Lambda e)^G = e\Lambda^G e$ is the basic algebra of Λ^G.*

Proof. Since [1, (6.5)] also holds for not necessarily basic algebras, there exists a complete set of orthogonal primitive idempotents of Λ which is (freely) permuted by G. Hence there are orthogonal primitive idempotents $e_1, \ldots, e_r$ of Λ such that $(\Lambda g(e_i))_{g\in G,\, 1\leq i\leq r}$ is a complete set of principal indecomposable Λ-modules. Consequently, the algebra $e\Lambda e$, defined by the idempotent $e := \sum_{i=1}^{r} \sum_{g\in G} g(e_i)$, is the basic algebra of Λ. Since e belongs to Λ^G, the algebra $e\Lambda e$ is G-submodule of Λ. Moreover, there exists a bounded category R with $\oplus R \cong e\Lambda e$ and G being a group of k-linear automorphisms of R acting freely on the objects (cf. [1, (6.3)]), and $\oplus(R/G) \cong (e\Lambda e)^G = e\Lambda^G e$. Since the push-down functor sends principal indecomposables to principal indecomposables the elements $\tilde{e}_i := \sum_{g\in G} g(e_i)$ form a complete set of primitive orthogonal idempotents of Λ^G. In view of $e = \sum_{i=1}^{r} \tilde{e}_i$, we conclude that $e\Lambda^G e = (e\Lambda e)^G$ is the basic algebra of Λ^G. □

Corollary 4.4. *Let Λ be a finite dimensional k-algebra with Gabriel quiver Δ_r for some $r \geq 2$. Suppose that $G \subset \mathrm{Aut}_k(\Lambda)$ is a finite group of k-algebra automorphisms such that the induced action of G on the isoclasses of simple Λ-modules is free. If Λ^G is Morita equivalent to $\mathcal{N}^2(1,n)$, then Λ is special biserial. If, in addition, Λ has a principal indecomposable module which is not sincere, then Λ is Morita equivalent to $\mathcal{N}^2(r,n)$.*

Proof. Let B and C be the basic algebras of Λ and Λ^G, respectively. Since Λ has Gabriel quiver Δ_r, B is the algebra associated to a residue category R of the path category $k[\Delta_r]$. Moreover, it follows from (4.3) and [1, (6.2),(6.3)] that G is a group of k-linear automorphisms of R acting freely on its objects, and $C = B^G$ is the algebra associated to the quotient category R/G. Since $C \cong \mathcal{N}^2(1,n)$ the group $G \cong \mathbb{Z}/(p^{r-1})$ is cyclic. As $r \geq 2$, Δ_r is a square-free quiver (i.e., a quiver with at most one arrow $a \to b$ between any two vertices a and b). Hence the operation of G on R is induced by the action of the group G_r on the quiver Δ_r. Consequently, $R \cong k[\Delta_r]/I$ for an admissible G_r-invariant ideal I in the path category $k[\Delta_r]$. Thus, $B \cong k[\Delta_r]/I$ for an admissible G_r-invariant ideal I of $k[\Delta_r]$ and $(k[\Delta_r]/I)^{G_r} \cong \mathcal{N}^2(1,n)$. Applying (4.1) we conclude that B is special biserial, so that Λ also has this property.

The last statement follows directly from (4.2). □

5. Ascent via Galois Extensions

Returning to groups, we recall our general assumption that $\mathcal{G}$ is an infinitesimal group, defined over an algebraically closed field k of characteristic $p \geq 3$. In this section we are going to show that, under certain hypotheses, the invariants $\mathcal{B}^{X(\mathcal{G}/\mathcal{G}_1)}$ of a block $\mathcal{B} \subset H(\mathcal{G})$ with quiver Δ_r form a tame block of the algebra of measures on the first Frobenius kernel of $\mathcal{G}$. In view of the results of [14] this will place us into the situation discussed in §4.

Given an $H(\mathcal{G})$-module M, we denote by $H^n(\mathcal{G}, M) := \mathrm{Ext}^n_{H(\mathcal{G})}(k, M)$ the n-th cohomology group of the supplemented algebra $(H(\mathcal{G}), \varepsilon)$ with coefficients in M.

Proposition 5.1. *Let $\mathcal{G}$ be an infinitesimal group with trivial multiplicative center such that*

(a) *the principal block $\mathcal{B}_0(\mathcal{G}_1)$ is tame, and*

(b) *there exists $r \geq 1$ such that $\mathcal{G}/\mathcal{G}_1 \cong \mu_{p^{r-1}}$.*

Then $H(\mathcal{G})$ has $\frac{p-1}{2}$ blocks $\mathcal{B}_0, \dots, \mathcal{B}_{\frac{p-3}{2}}$ having Gabriel quiver Δ_r. The block $\mathcal{B}_i$ has p^{r-1} simple modules of dimension $i+1$, and p^{r-1} simple modules of dimension $p-i-1$. There are exactly p^{r-1} simple $H(\mathcal{G})$-modules that do not belong to $\bigoplus_{i\geq 0} \mathcal{B}_i$, each having dimension p.

Proof. Owing to (1.5) we have $\mathcal{R}(\mathcal{G}) \cong (\mathcal{W}_n)_1$ for some $n \geq 0$ as well as $\mathcal{G}/\mathcal{R}(\mathcal{G}) \cong \mathcal{Q}_{[r]}$. We consider the corresponding projection $\pi : H(\mathcal{G}) \longrightarrow H(\mathcal{Q}_{[r]})$, whose kernel $I := H(\mathcal{G})H(\mathcal{R}(\mathcal{G}))^\dagger$ is nilpotent. In particular, $H(\mathcal{G})$ and $H(\mathcal{Q}_{[r]})$ have the same simple modules. According to (1.3(2)) the algebra $H(\mathcal{Q}_{[r]})$ has $\frac{p-1}{2}$ tame blocks $\mathcal{B}'_0, \dots, \mathcal{B}'_{\frac{p-3}{2}}$, whose simple modules satisfy the conditions of our Proposition. By the same token, the other p^{r-1} simple $H(\mathcal{Q}_{[r]})$-modules are projective and of dimension p. If $[j] \in \mathbb{Z}/(2)$ denotes the residue class of $j \in \mathbb{Z}$, then [17, (5.5)] shows that the map

$$\psi : \{0, \dots, 2p^{r-1}-1\} \longrightarrow \mathbb{Z}/(p^{r-1}) \times \mathbb{Z}/(2) \quad ; \quad \psi(j) = \begin{cases} (j, [j]) & 0 \leq j \leq p^{r-1}-1 \\ (j-p^{r-1}, [j]) & p^{r-1} \leq j \leq 2p^{r-1}-1 \end{cases}$$

induces an isomorphism between Δ_r and the Gabriel quiver of $\mathcal{B}'_i$. Thus, our result holds for $\mathcal{R}(\mathcal{G}) = e_k$ and we may assume that $\mathcal{R}(\mathcal{G}) \neq e_k$.

Now suppose that $\mathcal{B} \subset H(\mathcal{G})$ is a block having a simple module of dimension $\neq p$. Then there exists $i \in \{0, \dots, \frac{p-3}{2}\}$ such that $\mathcal{B}'_i$ is a direct summand of the block ideal $\pi(\mathcal{B}) \subset H(\mathcal{Q}_{[r]})$.

Let S, T be simple $H(\mathcal{G})$-modules. The cohomology five term sequence associated to the spectral sequence

$$\mathrm{Ext}^n_{H(\mathcal{Q}_{[r]})}(S, H^m(\mathcal{R}(\mathcal{G}), T)) \Rightarrow \mathrm{Ext}^{n+m}_{H(\mathcal{G})}(S, T)$$

[27, (I.6.6)] yields the exactness of

$$(0) \longrightarrow \mathrm{Ext}^1_{H(\mathcal{Q}_{[r]})}(S,T) \longrightarrow \mathrm{Ext}^1_{H(\mathcal{G})}(S,T) \longrightarrow \mathrm{Hom}_{H(\mathcal{Q}_{[r]})}(S, H^1(\mathcal{R}(\mathcal{G}),T)).$$

Since $\mathcal{R}(\mathcal{G}) \cong (\mathcal{W}_n)_1$, we have $H(\mathcal{R}(\mathcal{G})) \cong k[X]/(X^{p^n})$, and the cohomology group $H^1(\mathcal{R}(\mathcal{G}),k)$ is one-dimensional. As $\mathcal{B}_0(\mathcal{G}_1)$ is tame, a consecutive application of (1.5) and [17, (6.3)] implies that $\mathcal{R}(\mathcal{G})$ is contained in the center of $\mathcal{G}_1$. Accordingly, $\mathcal{G}_1$ acts trivially on $H^1(\mathcal{R}(\mathcal{G}),k)$, so that this space is $H(\mathcal{Q}_{[r]})$-isomorphic to k_λ for some $\lambda \in X(\mathcal{G}/\mathcal{G}_1)$. From the triviality of $T|_{\mathcal{R}(\mathcal{G})}$ we now obtain isomorphisms

$$H^1(\mathcal{R}(\mathcal{G}),T) \cong H^1(\mathcal{R}(\mathcal{G}),k) \otimes_k T \cong T_\lambda$$

of $H(\mathcal{Q}_{[r]})$-modules. From Schur's Lemma we conclude that

$$\mathrm{Ext}^1_{H(\mathcal{G})}(S,T) \cong \mathrm{Ext}^1_{H(\mathcal{Q}_{[r]})}(S,T)$$

whenever T does not belong to the $X(\mathcal{G}/\mathcal{G}_1)$-orbit of S (cf. §2). The structure of the quiver of $H(\mathcal{Q}_{[r]})$ entails $\mathrm{Ext}^1_{H(\mathcal{Q}_{[r]})}(S,T) = (0)$ for $\dim_k S + \dim_k T \neq p$. Consequently, each simple $\mathcal{B}$-module has dimension $i+1$ or $p-i-1$.

Let S be a simple $\mathcal{B}$-module. It remains to compute $\mathrm{Ext}^1_{H(\mathcal{G})}(S,S_\lambda)$ for $\lambda \in X(\mathcal{G}/\mathcal{G}_1)$. According to (b) the group $\mathcal{G}/\mathcal{G}_1$ is multiplicative, and [27, (I.6.9)] provides an isomorphism

$$\mathrm{Ext}^1_{H(\mathcal{G})}(S,S_\lambda) \cong \mathrm{Ext}^1_{H(\mathcal{G}_1)}(S,S_\lambda)^{\mathcal{G}/\mathcal{G}_1}.$$

Thanks to (2.2(2)) the modules $S|_{\mathcal{G}_1} \cong S_\lambda|_{\mathcal{G}_1}$ are simple. In view of (1.4(2)) the module $S|_{\mathcal{G}_1}$ belongs to a tame block of $H(\mathcal{G}_1)$, and we conclude from the shape of the quiver of $H(\mathcal{G}_1)$ (cf. [14, (7.1)]) that $\mathrm{Ext}^1_{H(\mathcal{G}_1)}(S|_{\mathcal{G}_1}, S|_{\mathcal{G}_1}) = (0)$. Accordingly, we have

$$\mathrm{Ext}^1_{H(\mathcal{G})}(S,T) \cong \mathrm{Ext}^1_{H(\mathcal{Q}_{[r]})}(S,T)$$

for any two simple $\mathcal{B}$-modules S,T. As an upshot of the above, $\mathcal{B}$ and $\mathcal{B}'_i$ have the same simple modules and the same quiver. The remaining statements follow from (1.3(2)). □

Given $\lambda \in X(\mathcal{G}/\mathcal{G}_1)$, we denote by $\omega_\lambda := \lambda * \mathrm{id}_{H(\mathcal{G})}$ the corresponding automorphism of $H(\mathcal{G})$, i.e.,

$$\omega_\lambda(h) = \sum_{(h)} \lambda(h_{(1)})h_{(2)} \qquad \forall\ h \in H(\mathcal{G}).$$

Note that $\lambda \mapsto \omega_\lambda$ is a homomorphism of groups, so that $X(\mathcal{G}/\mathcal{G}_1)$ acts on $H(\mathcal{G})$ via algebra automorphisms. Since $\lambda|_{H(\mathcal{G}_1)} = \varepsilon$, we have $\omega_\lambda|_{H(\mathcal{G}_1)} = \mathrm{id}_{H(\mathcal{G}_1)}$, proving that $H(\mathcal{G}_1)$ is contained in the algebra $H(\mathcal{G})^{X(\mathcal{G}/\mathcal{G}_1)}$ of $X(\mathcal{G}/\mathcal{G}_1)$-invariants.

Proposition 5.2. *Let $\mathcal{G}$ be an infinitesimal group with trivial multiplicative center and such that*
(a) *the principal block $\mathcal{B}_0(\mathcal{G}_1)$ is tame, and*
(b) *there is $r \geq 1$ such that $\mathcal{G}/\mathcal{G}_1 \cong \mu_{p^{r-1}}$.*
If $\mathcal{B} \subset H(\mathcal{G})$ is a block with Gabriel quiver Δ_r, then $\mathcal{B}$ is an $X(\mathcal{G}/\mathcal{G}_1)$-submodule of $H(\mathcal{G})$, and $\mathcal{B}^{X(\mathcal{G}/\mathcal{G}_1)}$ is a tame block of $H(\mathcal{G}_1)$. In particular, $\mathcal{B}_0(\mathcal{G})^{X(\mathcal{G}/\mathcal{G}_1)} = \mathcal{B}_0(\mathcal{G}_1)$.

Proof. Let λ be an element of $X(\mathcal{G}/\mathcal{G}_1)$. We first show that $\omega_\lambda(\mathcal{B}) = \mathcal{B}$. Since ω_λ is an automorphism, $\omega_\lambda(\mathcal{B})$ is a block of $H(\mathcal{G})$ whose simple modules have the same dimension as those of $\mathcal{B}$. In view of (5.1) these data determine a block of $H(\mathcal{G})$ with Gabriel quiver Δ_r, so that $\omega_\lambda(\mathcal{B}) = \mathcal{B}$.

Using the notation of (5.1), we let $\mathcal{B} = \mathcal{B}_i$. Since the principal block $\mathcal{B}_0(\mathcal{G}_1)$ of $H(\mathcal{G}_1)$ is tame and $\mathcal{G}_1$ has trivial multiplicative center, (1.4(2)) shows that there is a uniquely determined tame block $\mathcal{C}_i \subset H(\mathcal{G}_1)$ having two simple modules of dimensions $i+1$ and $p-i-1$, respectively. Now let $e_i \in H(\mathcal{G}_1)$ be the primitive central idempotent of $H(\mathcal{G}_1)$ such that $\mathcal{C}_i = H(\mathcal{G}_1)e_i$. We write

$$e_i = x_i + y_i$$

as an orthogonal sum of two idempotents of $H(\mathcal{G})$, with x_i belonging to $\mathcal{B}_i$. If $y_i \neq 0$, then there exists a simple $H(\mathcal{G})$-module V, not belonging to $\mathcal{B}_i$, such that $y_iV \neq (0)$. Thanks to (2.2(2)) the module $V|_{\mathcal{G}_1}$ is simple, and by virtue of (5.1) it does not belong to the block $\mathcal{C}_i$. Hence

$$(0) \neq y_iV = e_iV = e_iV|_{\mathcal{G}_1} = (0),$$

a contradiction. We conclude that $e_i \in \mathcal{B}_i$, whence $\mathcal{C}_i = H(\mathcal{G}_1)e_i \subset H(\mathcal{G})e_i \subset \mathcal{B}_i$. This readily implies

$$\mathcal{C}_i \subset \mathcal{B}_i^{X(\mathcal{G}/\mathcal{G}_1)}.$$

Let S and T be simple $\mathcal{B}_i$-modules of dimensions $i+1$ and $p-i-1$, respectively. We denote by $P(S)$ and $P(T)$ the corresponding projective covers. Thanks to (2.2(3)) and the fact that $\mathcal{B}_i$ is $X(\mathcal{G}/\mathcal{G}_1)$-stable, the modules S_λ, T_λ and $P(S)_\lambda$, $P(T)_\lambda$ with $\lambda \in X(\mathcal{G}/\mathcal{G}_1)$ form complete sets of representatives of the isoclasses of the simple and principal indecomposable $\mathcal{B}_i$-modules, respectively. Owing to (2.2(4)) the restriction $P|_{\mathcal{G}_1}$ of a principal indecomposable $H(\mathcal{G})$-module is the principal indecomposable $H(\mathcal{G}_1)$-module whose top is $\mathrm{Top}(P)|_{\mathcal{G}_1}$. Hence (5.1) implies that $P(S)|_{\mathcal{G}_1}$ and $P(T)|_{\mathcal{G}_1}$ actually belong to $\mathcal{C}_i$. Since $S|_{\mathcal{G}_1}$ and $T|_{\mathcal{G}_1}$ are the two simple $\mathcal{C}_i$-modules, counting the multiplicities of the principal indecomposables yields

$$\dim_k \mathcal{B}_i = \mathrm{ord}(X(\mathcal{G}/\mathcal{G}_1)) \dim_k \mathcal{C}_i.$$

Let $\mathcal{O}(\mathcal{G})$ be the function algebra of $\mathcal{G}$, and recall that $X(\mathcal{G}/\mathcal{G}_1)$ is a subgroup of the group of group-like elements of $\mathcal{O}(\mathcal{G})$. Hence the group algebra $k[X(\mathcal{G}/\mathcal{G}_1)]$ is a Hopf-subalgebra of $\mathcal{O}(\mathcal{G})$, and by [35, (2.6)] the algebra $\mathcal{O}(\mathcal{G})$ is a free left $k[X(\mathcal{G}/\mathcal{G}_1)]$-module. Since $k[X(\mathcal{G}/\mathcal{G}_1)]$ is self-injective, the dual space $H(\mathcal{G}) := \mathcal{O}(\mathcal{G})^*$ is a projective $k[X(\mathcal{G}/\mathcal{G}_1)]$-module. Direct computation shows that the action of $X(\mathcal{G}/\mathcal{G}_1)$ on $H(\mathcal{G})$ is given by $\lambda \mapsto \omega_\lambda$. By our above observation $\mathcal{B}_i$ is a $k[X(\mathcal{G}/\mathcal{G}_1)]$-direct summand of $H(\mathcal{G})$. Since $k[X(\mathcal{G}/\mathcal{G}_1)] \cong k[\mathbb{Z}/(p^{r-1})]$ is local, $\mathcal{B}_i$ is in fact free over $k[X(\mathcal{G}/\mathcal{G}_1)]$. Observing $\mathcal{B}_i^{X(\mathcal{G}/\mathcal{G}_1)} = \mathrm{Soc}_{k[X(\mathcal{G}/\mathcal{G}_1)]}(\mathcal{B}_i)$, we now obtain

$$\dim_k \mathcal{B}_i = \mathrm{ord}(X(\mathcal{G}/\mathcal{G}_1)) \dim_k \mathcal{B}_i^{X(\mathcal{G}/\mathcal{G}_1)},$$

so that $\dim_k \mathcal{B}_i^{X(\mathcal{G}/\mathcal{G}_1)} = \dim_k \mathcal{C}_i$. Thus, the inclusion $\mathcal{C}_i \subset \mathcal{B}_i^{X(\mathcal{G}/\mathcal{G}_1)}$ is in fact an equality, and $\mathcal{B}_i^{X(\mathcal{G}/\mathcal{G}_1)}$ is a tame block of $H(\mathcal{G}_1)$. □

Remark. In view of (2.2) and [1, (5.1)] the identity $\mathcal{B}_0(\mathcal{G}_1) = \mathcal{B}(\mathcal{G})^{X(\mathcal{G}/\mathcal{G}_1)}$ implies that $\mathcal{B}_0(\mathcal{G}) : \mathcal{B}_0(\mathcal{G}_1)$ is a Galois extension in the sense of [1].

Given a restricted Lie algebra $(\mathfrak{g}, [p])$, we denote by $T(\mathfrak{g})$ the *toral radical* of $\mathfrak{g}$. By definition $T(\mathfrak{g})$ is the largest toral p-ideal of $\mathfrak{g}$. If $\mathfrak{n} \subset \mathfrak{g}$ is a p-subalgebra, then we let $C_{\mathfrak{g}}(\mathfrak{n}) := \{x \in \mathfrak{g} \ ; \ [x, \mathfrak{n}] = (0)\}$ be the *centralizer* of $\mathfrak{n}$ in $\mathfrak{g}$.

The main result of this section provides a recognition criterion for infinitesimal groups with tame principal block.

Theorem 5.3. *Let $\mathcal{G}$ be an infinitesimal group. Then the following statements are equivalent:*

(1) *The principal block $\mathcal{B}_0(\mathcal{G})$ is tame.*
(2) *$\mathcal{B}_0(\mathcal{G}_1)$ is tame and $\mathcal{G}/\mathcal{G}_1$ is muliplicative.*

Proof. (1) ⇒ (2). Thanks to (2.1) the group $\mathcal{G}/\mathcal{G}_1$ is multiplicative. Corollary 3.5 gives rise to a commutative diagram

$$\begin{array}{ccccccccc} e_k & \longrightarrow & \mathrm{Cent}(\mathcal{G}) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{Q}_{[r]} & \longrightarrow & e_k \\ & & \downarrow & & \downarrow \pi & & \downarrow \mathrm{id} & & \\ e_k & \longrightarrow & \mathrm{Cent}(\mathcal{G})/\mathcal{M}(\mathcal{G}) & \longrightarrow & \mathcal{G}/\mathcal{M}(\mathcal{G}) & \longrightarrow & \mathcal{Q}_{[r]} & \longrightarrow & e_k \end{array}$$

with exact rows (cf. [7, (III,§3,3.7)]). Owing to (1.2) the group $\mathcal{G}$ is not solvable. Hence $\mathcal{G}_1$ is not either, and the above diagram induces a commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathrm{Lie}(\mathrm{Cent}(\mathcal{G})) & \longrightarrow & \mathrm{Lie}(\mathcal{G}) & \longrightarrow & \mathfrak{sl}(2) & \longrightarrow & (0) \\ & & \downarrow & & \downarrow d\pi & & \downarrow \mathrm{id} & & \\ (0) & \longrightarrow & \mathrm{Lie}(\mathrm{Cent}(\mathcal{G})/\mathcal{M}(\mathcal{G})) & \longrightarrow & \mathrm{Lie}(\mathcal{G}/\mathcal{M}(\mathcal{G})) & \longrightarrow & \mathfrak{sl}(2) & \longrightarrow & (0) \end{array}$$

with exact rows. Corollary 3.5 ensures the surjectivity of the left-hand vertical arrow. Consequently, the map $d\pi$ is also surjective, so that $\mathrm{Lie}(\mathcal{G}/\mathcal{M}(\mathcal{G})) \cong \mathrm{Lie}(\mathcal{G})/\mathrm{Lie}(\mathcal{M}(\mathcal{G}))$. Applying [7, (II,§7,n°4)] we obtain $\mathcal{G}_1/\mathcal{M}(\mathcal{G})_1 \cong (\mathcal{G}/\mathcal{M}(\mathcal{G}))_1$. By (3.4) the principal block $\mathcal{B}_0((\mathcal{G}/\mathcal{M}(\mathcal{G}))_1)$ is tame and (1.1) now yields the tameness of

$$\mathcal{B}_0(\mathcal{G}_1) \cong \mathcal{B}_0(\mathcal{G}_1/\mathcal{M}(\mathcal{G})_1) \cong \mathcal{B}_0((\mathcal{G}/\mathcal{M}(\mathcal{G}))_1),$$

as desired.

(2) ⇒ (1). Setting $\mathcal{G}' := \mathcal{G}/\mathcal{M}(\mathcal{G})$, $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$ and $\mathfrak{g}' := \mathrm{Lie}(\mathcal{G}')$ we begin by showing that $\mathcal{B}_0(\mathcal{G}'_1)$ is tame.

Since $\mathcal{M}(\mathcal{G}_1)$ is a characteristic subgroup of $\mathcal{G}$, it coincides with $\mathcal{M}(\mathcal{G}) \cap \mathcal{G}_1$. In view of [7, (III,§3,3.7)] we have

$$\mathcal{G}_1/\mathcal{M}(\mathcal{G}_1) = \mathcal{G}_1/(\mathcal{G}_1 \cap \mathcal{M}(\mathcal{G})) \cong \mathcal{G}_1\mathcal{M}(\mathcal{G})/\mathcal{M}(\mathcal{G}). \qquad (*)$$

The quotient map $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}_1\mathcal{M}(\mathcal{G})$ factors through to a morphism $\mathcal{G}/\mathcal{G}_1 \longrightarrow \mathcal{G}/\mathcal{G}_1\mathcal{M}(\mathcal{G})$, which is necessarily also a quotient map. Consequently, $\mathcal{G}/\mathcal{G}_1\mathcal{M}(\mathcal{G})$ is multiplicative (cf. [7, (IV,§1,2.4)]), and, in virtue of $(*)$, the exact sequence

$$e_k \longrightarrow \mathcal{G}_1\mathcal{M}(\mathcal{G})/\mathcal{M}(\mathcal{G}) \longrightarrow \mathcal{G}/\mathcal{M}(\mathcal{G}) \longrightarrow \mathcal{G}/\mathcal{G}_1\mathcal{M}(\mathcal{G}) \longrightarrow e_k$$

induces an exact sequence

$$(0) \longrightarrow \mathrm{Lie}(\mathcal{G}_1/\mathcal{M}(\mathcal{G}_1)) \longrightarrow \mathrm{Lie}(\mathcal{G}') \longrightarrow \mathfrak{t} \longrightarrow (0),$$

where $\mathfrak{t} \subset \mathrm{Lie}(\mathcal{G}/\mathcal{G}_1\mathcal{M}(\mathcal{G}))$ is a torus. Moreover, since $\mathrm{Lie}(\mathcal{M}(\mathcal{G}_1)) = T(\mathfrak{g})$ is the toral radical of the Lie algebra $\mathfrak{g}$, the first term is isomorphic to $\mathfrak{g}/T(\mathfrak{g})$. Setting $\mathfrak{n} := \mathfrak{g}/T(\mathfrak{g})$ we thus obtain an exact sequence

$$(0) \longrightarrow \mathfrak{n} \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{t} \longrightarrow (0) \qquad (**)$$

of restricted Lie algebras. By definition, the Lie algebra $\mathfrak{n}$ has a trivial toral radical. Similarly, $\mathcal{M}(\mathcal{G}') = e_k$ implies $\mathcal{M}(\mathcal{G}'_1) = \mathcal{M}(\mathcal{G}') \cap \mathcal{G}'_1 = e_k$, so that $T(\mathfrak{g}') = (0)$. As a result, the centers of $\mathfrak{n}$ and $\mathfrak{g}'$ are p-nilpotent (cf. [34, (II.3.4)]).

A derivation $D : \mathfrak{q} \longrightarrow \mathfrak{q}$ of a restricted Lie algebra $(\mathfrak{q}, [p])$ is called a *p-derivation* if $D(x^{[p]}) = (\mathrm{ad}\, x)^{p-1}(D(x))$ for every $x \in \mathfrak{q}$. Note that every inner derivation of $\mathfrak{q}$ is a p-derivation.

(†) Let $\mathfrak{q} := \mathfrak{sl}(2)$, $\mathfrak{sl}(2)_s$. Then every p-derivation $D : \mathfrak{q} \longrightarrow \mathfrak{q}$ is inner.

It is well-known that every derivation of $\mathfrak{sl}(2)$ is inner. Hence we assume that $\mathfrak{q} = \mathfrak{sl}(2)_s$. Since

$$\mathfrak{q} = [\mathfrak{q}, \mathfrak{q}] \oplus C(\mathfrak{q})$$

is a direct sum of two D-stable ideals, there exist derivations $D_1 \in \mathrm{Der}_k([\mathfrak{q},\mathfrak{q}])$ and $D_2 \in \mathrm{Der}_k(C(\mathfrak{q}))$ such that

$$D(x,c) = (D_1(x), D_2(c)) \qquad \forall\, x \in \mathfrak{sl}(2),\ c \in C(\mathfrak{q}).$$

Recall that $C(\mathfrak{q}) = kv_0$ is one-dimensional and p-nilpotent, and that the element h of the standard basis of $\mathfrak{sl}(2)$ satisfies $(h,0)^{[p]} = (h,v_0)$. As D is a p-derivation, we have

$$(D_1(h), D_2(v_0)) = D((h,0)^{[p]}) = (\mathrm{ad}\,(h,0))^{p-1}(D_1(h),0) = ((\mathrm{ad}\,h)^{p-1}(D_1(h)),0),$$

so that $D_2 = 0$. Since every derivation of $[\mathfrak{q},\mathfrak{q}] = \mathfrak{sl}(2)$ is inner, we see that $D = \mathrm{ad}\,(x,0)$ for a suitable element $x \in \mathfrak{sl}(2)$.

According to (1.1) the tameness of $B_0(\mathfrak{g})$ implies that $B_0(\mathfrak{n})$ is tame, and (1.4) shows that $\mathfrak{q} := \mathfrak{n}/C(\mathfrak{n})^{[p]} \cong \mathfrak{sl}(2), \mathfrak{sl}(2)_s$. We put $\mathfrak{h} := \mathfrak{g}'/C(\mathfrak{n})^{[p]}$ and note that the p-nilpotent p-ideal $C(\mathfrak{n})^{[p]}$ belongs to the kernel of $\mathfrak{g}' \longrightarrow \mathfrak{t}$. Accordingly, the exact sequence $(**)$ induces an exact sequence

$$(0) \longrightarrow \mathfrak{q} \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{t} \longrightarrow (0)$$

of restricted Lie algebras. Let $\mathfrak{m} \subset \mathfrak{g}'$ be the pre-image of $T(\mathfrak{h})$ under the canonical projection $\mathfrak{g}' \longrightarrow \mathfrak{h}$. There results an exact sequence

$$(0) \longrightarrow C(\mathfrak{n})^{[p]} \longrightarrow \mathfrak{m} \longrightarrow T(\mathfrak{h}) \longrightarrow (0).$$

By choosing a maximal torus of $\mathfrak{m}$, applying [34, (II.4.5)], and observing the p-nilpotence of $C(\mathfrak{n})^{[p]}$, we conclude that the sequence splits. Since $C(\mathfrak{n}) \subset \mathfrak{g}'$ is an abelian p-ideal, we have $C(\mathfrak{n})^{[p]} \subset C(\mathfrak{g}')$. Consequently, $\mathfrak{m} \cong T(\mathfrak{h}) \oplus C(\mathfrak{n})^{[p]}$ is a direct sum of restricted Lie algebras, and $T(\mathfrak{h}) \cong T(\mathfrak{m}) \subset T(\mathfrak{g}') = (0)$.

In view of (†) we can write

$$\mathfrak{h} = \mathfrak{q} + C_{\mathfrak{h}}(\mathfrak{q})$$

as a sum of two p-ideals, whose intersection is p-nilpotent and of dimension ≤ 1. Moreover,

$$\mathfrak{t} \cong \mathfrak{h}/\mathfrak{q} \cong C_{\mathfrak{h}}(\mathfrak{q})/(C_{\mathfrak{h}}(\mathfrak{q}) \cap \mathfrak{q})$$

is a torus, so that the above arguments yield

$$C_{\mathfrak{h}}(\mathfrak{q}) \cong (C_{\mathfrak{h}}(\mathfrak{q}) \cap \mathfrak{q}) \rtimes \mathfrak{t}.$$

As $C_{\mathfrak{h}}(\mathfrak{q}) \cap \mathfrak{q}$ lies in the center of $C_{\mathfrak{h}}(\mathfrak{q})$, the above decomposition is in fact a direct sum. Consequently,

$$\mathfrak{t} \cong T(C_{\mathfrak{h}}(\mathfrak{q})) \subset T(\mathfrak{h}) = (0),$$

and the exact sequence $(**)$ now implies $\mathfrak{n} \cong \mathfrak{g}'$. As a result, the principal block $\mathcal{B}_0(\mathcal{G}_1') \cong \mathcal{B}_0(\mathfrak{g}') \cong \mathcal{B}_0(\mathfrak{n})$ is tame.

Since $\mathcal{G}/\mathcal{G}_1 \longrightarrow \mathcal{G}'/\mathcal{G}_1'$ is a quotient map, [7, (IV,§1,2.4)] implies that the group $\mathcal{G}'/\mathcal{G}_1'$ is multiplicative. We may now apply (1.5) to see that $\mathcal{G}'/\mathcal{G}_1' \cong \mu_{p^{r-1}}$, where $r := \mathrm{ht}(\mathcal{G}')$ is the height of $\mathcal{G}'$.

We put $G := X(\mathcal{G}'/\mathcal{G}'_1)$ and apply (5.2) and (2.2(3)) to see that the group G operates freely on the set of isoclasses of the simple $\mathcal{B}_0(\mathcal{G}')$-modules. Moreover, $\mathcal{B}_0(\mathcal{G}_1') = \mathcal{B}_0(\mathcal{G}')^G$ and [14, (7.1)] now shows that $\mathcal{B}_0(\mathcal{G}')^G$ is Morita equivalent to $\mathcal{N}^2(1,n)$ for some $n \geq 0$.

If $r = 1$, then $\mathcal{G}' = \mathcal{G}_1'$, and we are done. Alternatively, (5.1) implies that $\mathcal{B}_0(\mathcal{G}')$ has Gabriel quiver Δ_r, and (4.4) ensures that $\mathcal{B}_0(\mathcal{G}')$ is special biserial, which, by [36, (2.4)] or [9, (5.2)], implies that $\mathcal{B}_0(\mathcal{G}')$ is representation-finite or tame. Since the separated quiver of Δ_r consists of two copies of $\tilde{A}_{2p^{r-1}-1}$, it follows from [2, (X.2.6)] that the basic algebra $k[\Delta_r]/I$ of $\mathcal{B}_0(\mathcal{G}')$ has infinite representation type. Hence $\mathcal{B}_0(\mathcal{G}')$ is tame and the proof may now be completed by applying (1.1). □

Our first application concerns a similarity with the classification of representation-finite infinitesimal groups. As observed in [16, (2.7)] the principal block of an infinitesimal group scheme is representation-finite if and only if $\mathcal{B}_0(\mathcal{G}_2)$ enjoys this property. While subgroups of infinitesimal tame groups are not necessarily tame (cf. [14, §6]), the following result shows that descent does work for Frobenius kernels.

Corollary 5.4. *Let $\mathcal{G}$ be an infinitesimal group. Then $\mathcal{B}_0(\mathcal{G})$ is tame if and only if $\mathcal{B}_0(\mathcal{G}_2)$ is tame.*

Proof. Let $r := \mathrm{ht}(\mathcal{G})$. If $r \le 2$, then $\mathcal{G}_2 = \mathcal{G}$ and there is nothing to be shown. Hence we shall assume $r \ge 3$.

Suppose that $\mathcal{B}_0(\mathcal{G})$ is tame. Then (5.3) implies that $\mathcal{G}/\mathcal{G}_1$ is multiplicative as well as the tameness of $\mathcal{B}_0(\mathcal{G}_1)$. Thus, $\mathcal{G}_2/\mathcal{G}_1 \hookrightarrow (\mathcal{G}/\mathcal{G}_1)_1$ is, as a closed subgroup of a multiplicative group, multiplicative (cf. [7, (IV,§1,2.4)]). The tameness of $\mathcal{B}_0(\mathcal{G}_2)$ now follows from (5.3).

Conversely, assume $\mathcal{B}_0(\mathcal{G}_2)$ to be tame. In view of (5.3) the principal block $\mathcal{B}_0(\mathcal{G}_1)$ is tame and $\mathcal{G}_2/\mathcal{G}_1$ is multiplicative. Let $\mathcal{H}$ be the inverse image of $(\mathcal{G}/\mathcal{G}_1)_1$ under the canonical quotient map

$$\pi : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}_1.$$

Then we have $\mathcal{G}_2 \subset \mathcal{H}$.

We consider the Frobenius homomorphism $F_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{G}^{(1)}$ (cf. [27, (I.9.2)]). Since $F_{\mathcal{G}/\mathcal{G}_1} \circ \pi = \pi^{(1)} \circ F_{\mathcal{G}}$ (cf. [7, (II,§7,1.4)]), an element $h \in \mathcal{H}$ satisfies

$$\pi^{(1)} \circ F_{\mathcal{G}}(h) = F_{\mathcal{G}/\mathcal{G}_1} \circ \pi(h) = 1,$$

so that

$$F_{\mathcal{G}}(h) \in \ker \pi^{(1)} = \mathcal{G}_1^{(1)}.$$

Consequently, $F^2_{\mathcal{G}}(h) = 1$, implying $h \in \mathcal{G}_2$. From the sheaf property of quotient maps (cf. [37, (15.5)]) we conclude that $\mathcal{G}_2/\mathcal{G}_1 = \mathcal{H}/\mathcal{G}_1 \cong (\mathcal{G}/\mathcal{G}_1)_1$, so that the latter group is multiplicative. Owing to [7, (IV,§3,3.7)] the group $\mathcal{G}/\mathcal{G}_1$ is also multiplicative. As a result, (5.3) applies, and we conclude that $\mathcal{B}_0(\mathcal{G})$ is tame. □

Corollary 5.5. *Let $\mathcal{G}$ be a smooth, connected algebraic k-group. If $\mathcal{B}_0(\mathcal{G}_r)$ is tame, then $r = 1$, and $\mathcal{G} \cong \mathcal{H}\mathcal{T}$ is an almost direct product of a torus $\mathcal{T}$ and a group $\mathcal{H} \cong \mathrm{SL}(2), \mathrm{PSL}(2)$. Moreover, $\mathcal{B}_0(\mathcal{G}_r) \cong \mathcal{B}_0(\mathrm{SL}(2)_1)$ is Morita equivalent to $\mathcal{N}^2(1,0)$.*

Proof. Thanks to [27, (I.9.5)] and (5.3) the group $(\mathcal{G}^{(1)})_{r-1} \cong \mathcal{G}_r/\mathcal{G}_1$ is multiplicative. If $r > 1$, then [7, (IV,§3,3.7)] implies that $\mathcal{G}^{(1)}$ is multiplicative. Hence $\mathcal{G}$ and $\mathcal{G}_r$ also have this property. This, however, means that the block $\mathcal{B}_0(\mathcal{G}_r)$ is simple, a contradiction. As a result, we have $r = 1$.

Let $\mathfrak{g} := \mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G}_r)$ be the Lie algebra of $\mathcal{G}$, and denote by $\mathrm{Ad} : \mathcal{G}(k) \longrightarrow \mathrm{GL}(\mathfrak{g})(k)$ the adjoint representation of $\mathcal{G}(k)$. Since $\mathcal{G}(k)$ acts on $\mathfrak{g}$ via automorphisms of restricted Lie algebras, the spaces $C(\mathfrak{g})$ and $C(\mathfrak{g})^{[p]}$ are $\mathcal{G}(k)$-invariant p-ideals of $\mathfrak{g}$. Consequently, $\mathcal{G}(k)$ also operates on $\mathfrak{g}/C(\mathfrak{g})^{[p]}$ via automorphisms of restricted Lie algebras.

According to (5.3) we have $\mathfrak{g}/C(\mathfrak{g})^{[p]} \cong \mathfrak{sl}(2), \mathfrak{sl}(2)_s$. We assume the latter alternative to hold. By the above, $\mathcal{G}(k)$ also acts on $\mathfrak{sl}(2) \cong \mathfrak{sl}(2)_s/C(\mathfrak{sl}(2)_s)$ and the canonical projection

$$\pi : \mathfrak{sl}(2)_s \longrightarrow \mathfrak{sl}(2)$$

is $\mathcal{G}(k)$-equivariant. Recall that $\mathfrak{sl}(2)_s = \mathfrak{sl}(2) \oplus kv_0$ has center kv_0 and p-map induced by the p-semilinear map $\psi_s : \mathfrak{sl}(2) \longrightarrow kv_0$ sending the canonical basis vectors e, f, h to $0, 0$, and v_0, respectively (cf. §1). Direct computation shows that

$$\mathcal{V}_{\mathfrak{sl}(2)_s} = (ke \oplus kv_0) \cup (kf \oplus kv_0).$$

Since $\mathcal{G}(k)$ is connected, both irreducible components of $\mathcal{V}_{\mathfrak{sl}(2)_s}$ are $\mathcal{G}(k)$-invariant. Consequently,

$$ke = \pi(ke \oplus kv_0) \subset \mathfrak{sl}(2)$$

also enjoys this property. Differentiation yields the $\mathfrak{g}$-invariance of ke, implying that ke is an ideal of $\mathfrak{sl}(2)$, a contradiction.

It follows that $\mathfrak{g}/C(\mathfrak{g})^{[p]} \cong \mathfrak{sl}(2)$, so that $C(\mathfrak{g}) = C(\mathfrak{g})^{[p]}$ is a torus. Consequently, the p-nilpotent radical of $\mathfrak{g}$ is trivial, and an application of [24, (11.8)] yields the reductivity of the group $\mathcal{G}$.

Thanks to [12, (5.1)], which also holds for $p = 3$, the group $\mathcal{G}$ is an almost direct product $\mathcal{G} = \mathcal{H}{\cdot}\mathcal{K}$ of two connected normal subgroups, with $\mathcal{H}$ being almost simple and such that $\mathrm{Lie}(\mathcal{H}) \cong \mathfrak{sl}(2)$. In particular, $\mathcal{H}$ has rank 1 so that [33, (8.2.4)] implies $\mathcal{H} \cong \mathrm{SL}(2), \mathrm{PSL}(2)$. By the same token, $\mathcal{V}_{\mathrm{Lie}(\mathcal{K})} = \{0\}$, and the first Frobenius kernel $\mathcal{K}_1$ is multiplicative. Thanks to [7, (IV,§3,3.7)] the group $\mathcal{K}$ is multiplicative. Consequently, $\mathcal{K}$ is a torus.

¿From the structure of $\mathfrak{g}$ we obtain $\mathcal{G}_1/\mathcal{M}(\mathcal{G}_1) \cong \mathrm{SL}(2)_1$, and (1.1) now implies

$$\mathcal{B}_0(\mathcal{G}_r) \cong \mathcal{B}_0(\mathrm{SL}(2)_1).$$

The basic algebra of the latter block has been known to be isomorphic to $\mathcal{N}^2(1,0)$ for quite some time (cf. [18]). □

Remark. Under the assumptions of (5.5) the algebra $H(\mathcal{G}_1)$ is a direct product of copies of $H(\mathrm{SL}(2)_1)$, and is thus in particular tame.

6. The Isomorphism Type of Tame Infinitesimal Groups

By Corollary 3.5 an infinitesimal group $\mathcal{G}$ with tame principal block is a central extension of $\mathcal{Q}_{[r]}$ by a group of type $\mathcal{M} \times (\mathcal{W}_n)_1$ with multiplicative part $\mathcal{M}$. In this section we shall show which of these extensions actually do give rise to tame blocks. In view of (1.1) we may assume $\mathcal{M} = e_k$.

Following our general philosophy, we begin by studying groups of height ≤ 1. Let $\mathfrak{n}_n := \bigoplus_{i=0}^{n-1} kv_0^{[p]^i}$ be the n-dimensional nil-cyclic restricted Lie algebra. Given a p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow \mathfrak{n}_n$, we consider the restricted Lie algebra $\mathfrak{sl}(2)^n_\psi := \mathfrak{sl}(2) \oplus \mathfrak{n}_n$, whose product and p-map are given by

$$[(x,v),(y,w)] = ([x,y],0) \quad \text{ and } \quad (x,v)^{[p]} = (x^{[p]}, \psi(x) + v^{[p]})$$

for $x, y \in \mathfrak{sl}(2)$ and $v, w \in \mathfrak{n}_n$. By general theory (cf. [14, §1]), the algebras $\mathfrak{sl}(2)^n_\psi$ are just the central extensions of $\mathfrak{sl}(2)$ by $\mathfrak{n}_n$. Our next result shows that their isomorphism classes are orbits relative to the canonical action of the product of the automorphism groups of the restricted Lie algebras $\mathfrak{sl}(2)$ and $\mathfrak{n}_n$.

Lemma 6.1. *The restricted Lie algebras $\mathfrak{sl}(2)^n_\psi$ and $\mathfrak{sl}(2)^n_{\psi'}$ are isomorphic if and only if there exist $\gamma \in \mathrm{Aut}_p(\mathfrak{n}_n)$ and $\eta \in \mathrm{Aut}_p(\mathfrak{sl}(2))$ such that $\psi' = \gamma \circ \psi \circ \eta^{-1}$.*

Proof. Suppose that $\mu : \mathfrak{sl}(2)^n_\psi \longrightarrow \mathfrak{sl}(2)^n_{\psi'}$ is an isomorphism. By considering μ an isomorphism of ordinary Lie algebras, we obtain, observing $\mathfrak{sl}(2) = [\mathfrak{sl}(2)^n_\psi, \mathfrak{sl}(2)^n_\psi]$ and $C(\mathfrak{sl}(2)^n_\psi) = \mathfrak{n}_n$, two automorphisms $\eta : \mathfrak{sl}(2) \longrightarrow \mathfrak{sl}(2)$ and $\gamma : \mathfrak{n}_n \longrightarrow \mathfrak{n}_n$ such that

$$\mu(x,v) = (\eta(x), \gamma(v)) \qquad \forall\ x \in \mathfrak{sl}(2),\ v \in \mathfrak{n}_n.$$

Since μ is an automorphism of restricted Lie algebras, γ and η also have this property and $\psi' \circ \eta = \gamma \circ \psi$. Conversely, if γ and η satisfy these conditions, then, defining μ by the above identity, we obtain an isomorphism $\mathfrak{sl}(2)^n_\psi \cong \mathfrak{sl}(2)^n_{\psi'}$ of restricted Lie algebras. □

In the sequel we let $S(\mathfrak{sl}(2),\mathfrak{n}_n)$ denote the space of p-semilinear maps from $\mathfrak{sl}(2)$ to $\mathfrak{n}_n$ and identify $S(\mathfrak{sl}(2),\mathfrak{n}_1)$ with the space of p-semilinear forms on $\mathfrak{sl}(2)$. The group $\mathrm{SL}(2)(k)$ acts on $\mathfrak{sl}(2)$ via conjugation and we denote this action by $(g,x)\mapsto g\cdot x$. According to [25, p.281ff] every element of $\mathrm{Aut}_p(\mathfrak{sl}(2))$ is of the form $x\mapsto g\cdot x$ for a suitably chosen element $g\in \mathrm{SL}(2)(k)$. Consequently, (6.1) implies that the isomorphism classes of the central extensions of $\mathfrak{sl}(2)$ by $\mathfrak{n}_n$ correspond to the $(\mathrm{SL}(2)(k)\times \mathrm{Aut}_p(\mathfrak{n}_n))$-orbits of $S(\mathfrak{sl}(2),\mathfrak{n}_n)$.

Let

$$(\, , \,) : \mathfrak{sl}(2)\times\mathfrak{sl}(2)\longrightarrow k \quad ; \quad (x,y) := \mathrm{tr}(x\circ y)$$

be the trace form of $\mathfrak{sl}(2)$ on its standard module, and recall that $(\, , \,)$ is a non-degenerate $\mathrm{SL}(2)(k)$-invariant form. Given $x\in\mathfrak{sl}(2)$, we define $\psi_x\in S(\mathfrak{sl}(2),\mathfrak{n}_1)$ by $\psi_x(y)=(x,y)^p$. In this fashion the bilinear form $(\, , \,)$ induces a bijective map

$$\mathfrak{sl}(2)\longrightarrow S(\mathfrak{sl}(2),\mathfrak{n}_1) \quad ; \quad x\mapsto\psi_x$$

which satisfies the following identity relative to the canonical actions of the reductive group $G(1) := \mathrm{SL}(2)(k)\times k^\times$ on $\mathfrak{sl}(2)$ and $S(\mathfrak{sl}(2),\mathfrak{n}_1)$:

$$\psi_{(g,\alpha)\cdot x} = (g,\alpha^p)\cdot\psi_x .$$

Let $\{e,h,f\}$ be the standard basis of $\mathfrak{sl}(2)$. Then we have $(g,\alpha^p)\in\mathrm{Stab}_{G(1)}(\psi_h)$ if and only if $(g,\alpha)\in\mathrm{Stab}_{G(1)}(h)$. Consequently, $g\in\mathrm{Nor}_{\mathrm{SL}(2)(k)}(kh)$, and since $p\geq 3$, this group coincides with the normalizer $\mathrm{Nor}_{\mathrm{SL}(2)(k)}(T(k))$ of the standard maximal torus $T(k)\subset\mathrm{SL}(2)(k)$. Thus, we also have $g\cdot h=\alpha h$ with $\alpha\in\{1,-1\}$.

Let $n\in\{0,1\}$. In view of [14, (7.1),(7.4)] there exists, up to isomorphism, exactly one infinitesimal group of height ≤ 1 with trivial multiplicative center and principal block Morita equivalent to $\mathcal{N}^2(1,n)$.

Proposition 6.2. *Let $n\geq 2$. There exist infinitely many non-isomorphic infinitesimal groups of height ≤ 1 with trivial multiplicative center, whose principal block is Morita equivalent to $\mathcal{N}^2(1,n)$.*

Proof. We begin by considering the case $n=2$, and put $\mathfrak{n}_2 := kv_0\oplus kv_0^{[p]}$. An element ψ of $S(\mathfrak{sl}(2),\mathfrak{n}_2)$ corresponds to a pair $(\psi_1,\psi_2)\in S(\mathfrak{sl}(2),\mathfrak{n}_1)\times S(\mathfrak{sl}(2),\mathfrak{n}_1)$ via

$$\psi(x)=\psi_1(x)v_0+\psi_2(x)v_0^{[p]} \qquad \forall\; x\in\mathfrak{sl}(2).$$

Each element $\gamma\in\mathrm{Aut}_p(\mathfrak{n}_2)$ is determined by its image of v_0, so that $\gamma=\gamma_{(\mu,\alpha)}$ with $\mu\in k^\times$ and $\alpha\in k$ satisfying

$$\gamma(v_0)=\mu v_0+\alpha v_0^{[p]}.$$

Accordingly, the action of the group $G(2) := \mathrm{SL}(2)(k)\times\mathrm{Aut}_p(\mathfrak{n}_2)$ on $S(\mathfrak{sl}(2),\mathfrak{n}_2)$ is given by

$$(g,\gamma_{(\mu,\alpha)})\cdot(\psi_1,\psi_2) = (\mu(g\cdot\psi_1),\alpha(g\cdot\psi_1)+\mu^p(g\cdot\psi_2)).$$

Consequently,

$$(\psi_h,\psi)\in G(2)\cdot(\psi_h,\varphi)\Leftrightarrow\psi=\alpha\mu^{-1}\psi_h+(g,\mu^p)\cdot\varphi$$

for $\alpha\in k$ and $(g,\mu)\in\mathrm{Stab}_{G(1)}(h)$, and the orbits of the elements $(\psi_h,\varphi)\in S(\mathfrak{sl}(2),\mathfrak{n}_2)$ correspond to the $\mathrm{Stab}_{G(1)}(h)$-orbits of $ke\oplus kf\subset\mathfrak{sl}(2)$. By the remarks preceding our theorem the function

$$\sigma : ke\oplus kf\longrightarrow k \quad ; \quad \alpha e+\beta f\mapsto(\alpha\beta)^2$$

is readily seen to be constant on the $\mathrm{Stab}_{G(1)}(h)$-orbits. Accordingly, there are infinitely many such orbits. Hence there are also infinitely many $G(2)$-orbits of $S(\mathfrak{sl}(2),\mathfrak{n}_2)$ containing an element $\psi_\varphi := (\psi_h,\varphi)$. The standard orthogonality relations of the form $(\, , \,)$ in conjunction with a consecutive application of [14, (7.4)] and [14, (7.1)] show that the principal block of $u(\mathfrak{sl}(2)^2_{\psi_\varphi})$ is Morita equivalent to $\mathcal{N}^2(1,2)$. We may now invoke (6.1) to obtain our assertion for $n=2$.

Suppose that $n \geq 3$. Then there exists an exact sequence

$$(0) \longrightarrow \mathfrak{n}_{n-2} \longrightarrow \mathfrak{n}_n \stackrel{\pi}{\longrightarrow} \mathfrak{n}_2 \longrightarrow (0)$$

of restricted Lie algebras, which induces a surjection

$$\eta : \mathrm{Aut}_p(\mathfrak{n}_n) \longrightarrow \mathrm{Aut}_p(\mathfrak{n}_2)$$

such that $\eta(\gamma) \circ \pi = \pi \circ \gamma$ for every $\gamma \in \mathrm{Aut}_p(\mathfrak{n}_n)$. By the first part of our proof there exists an infinite subset $\Psi \subset S(\mathfrak{sl}(2), \mathfrak{n}_2)$ such that

(a) $\mathcal{B}_0(\mathfrak{sl}(2)^2_\psi)$ is tame for every $\psi \in \Psi$, and

(b) $G(2) \cdot \psi \cap G(2) \cdot \varphi = \emptyset$ whenever $\varphi \neq \psi \in \Psi$.

Let $\hat{\pi} : S(\mathfrak{sl}(2), \mathfrak{n}_n) \longrightarrow S(\mathfrak{sl}(2), \mathfrak{n}_2)$ be the surjection induced by π. Choose a pre-image $\lambda_\psi \in S(\mathfrak{sl}(2), \mathfrak{n}_n)$ under $\hat{\pi}$ for every $\psi \in \Psi$. By applying [14, (7.4)] and [14, (7.1)] consecutively we see that $\mathcal{B}_0(\mathfrak{sl}(2)^n_{\lambda_\psi})$ is Morita equivalent to $\mathcal{N}^2(1, n)$ for every $\psi \in \Psi$. Moreover, if ψ, φ are elements of Ψ such that $\lambda_\psi = (g, \gamma) \cdot \lambda_\varphi$ for some $(g, \gamma) \in \mathrm{SL}(2)(k) \times \mathrm{Aut}_p(\mathfrak{n}_n)$, then

$$\psi = \hat{\pi}(\lambda_\psi) = \hat{\pi}((g, \gamma) \cdot \lambda_\varphi) = (g, \eta(\gamma)) \cdot \varphi,$$

so that $\psi = \varphi$. In view of (6.1) and its succeeding remark the restricted Lie algebras $(\mathfrak{sl}(2)^n_{\lambda_\psi})_{\psi \in \Psi}$ are therefore pairwise non-isomorphic. □

Turning to infinitesimal groups of height ≥ 2, we let $\psi^n_s : \mathfrak{sl}(2) \longrightarrow \mathfrak{n}_n$ be the p-semilinear map which is given by

$$\psi^n_s(e) = 0 = \psi^n_s(f) \quad ; \quad \psi^n_s(h) = v_0$$

relative to the standard basis $\{e, h, f\} \subset \mathfrak{sl}(2)$. We write $\mathfrak{sl}(2)^n_s$ for the corresponding central extension of $\mathfrak{sl}(2)$ with kernel $\mathfrak{n}_n$.

Let $T \subset \mathrm{SL}(2)$ be the standard maximal torus of diagonal matrices. In view of (6.1) the restriction of the adjoint representation $\mathrm{Ad} : \mathrm{SL}(2) \longrightarrow \mathrm{GL}(\mathfrak{sl}(2))$ induces an operation of T

$$T \times \mathfrak{sl}(2)^n_s \longrightarrow \mathfrak{sl}(2)^n_s \quad ; \quad t \cdot (x, v) := (\mathrm{Ad}(t)(x), v)$$

by automorphisms of $\mathfrak{sl}(2)^n_s$. Note that, relative to the adjoint action of T on $\mathfrak{sl}(2)$, the canonical projection

$$\mathfrak{sl}(2)^n_s \longrightarrow \mathfrak{sl}(2)$$

is a homomorphism of T-modules. We let T act on $\mathrm{SL}(2)_1$ via conjugation. According to [7, (II,§7,n°3,n°4)] the above action induces an operation of T on the infinitesimal group $\mathrm{SL}(2)^n_1$ of height ≤ 1 associated to $\mathfrak{sl}(2)^n_s$ such that the canonical quotient map

$$\pi : \mathrm{SL}(2)^n_1 \longrightarrow \mathrm{SL}(2)_1$$

is T-equivariant.

Lemma 6.3. *Let $e_k \longrightarrow (\mathcal{W}_n)_1 \longrightarrow \mathcal{G} \stackrel{\pi}{\longrightarrow} \mathcal{Q}_{[r]} \longrightarrow e_k$ be a central extension of $\mathcal{Q}_{[r]}$ by $(\mathcal{W}_n)_1$. If $r \geq 2$, then the following statements hold:*

(1) *There exists a closed embedding $\sigma : T_r \hookrightarrow \mathcal{G}$ such that $\pi \circ \sigma = \mathrm{id}_{T_r}$ and $\mathcal{G} = \mathcal{G}_1 \sigma(T_r)$.*

(2) *Let σ be as in (1). The principal block $\mathcal{B}_0(\mathcal{G})$ is tame if and only if there exists an isomorphism $\gamma : \mathrm{SL}(2)^n_1 \longrightarrow \mathcal{G}_1$ such that $\gamma(t \cdot g) = \sigma(t)\gamma(g)\sigma(t)^{-1}$ for every $t \in T_r$ and $g \in \mathrm{SL}(2)^n_1$.*

Proof. Recall that $\mathcal{Q}_{[r]} = \mathrm{SL}(2)_1 T_r$. As both statements are trivial for $n = 0$, we assume $n \geq 1$.

(1) Setting $\mathcal{H} := \pi^{-1}(T_r)$ we obtain an exact sequence

$$e_k \longrightarrow (\mathcal{W}_n)_1 \longrightarrow \mathcal{H} \stackrel{\pi|_{\mathcal{H}}}{\longrightarrow} T_r \longrightarrow e_k.$$

Since $(\mathcal{W}_n)_1$ is unipotent, an application of [7, (III,§6,6.3)] shows that this sequence splits. Accordingly, there exists a homomorphism $\sigma : T_r \longrightarrow \mathcal{G}$ such that $\pi \circ \sigma = \mathrm{id}_{T_r}$. As σ is injective, [37, (15.3)]) ensures that it is a closed embedding.

Observing [7, (II,§4,1.5)] we consider the associated exact sequence

$$(0) \longrightarrow \mathfrak{n}_n \longrightarrow \mathrm{Lie}(\mathcal{G}) \longrightarrow \mathfrak{sl}(2)$$

of restricted Lie algebras. If the right-hand arrow is not surjective, then its image is solvable, so that $\mathrm{Lie}(\mathcal{G})$ has the same property. As this implies the solvability of $\mathcal{G}$, and $\mathcal{Q}_{[r]}$ is not solvable, we have reached a contradiction (cf. [7, (IV,§4,2.2)]). It now follows that the sequence

$$e_k \longrightarrow (\mathcal{W}_n)_1 \longrightarrow \mathcal{G}_1 \xrightarrow{\pi|_{\mathcal{G}_1}} \mathrm{SL}(2)_1 \longrightarrow e_k$$

is exact. Consequently, π restricts to a quotient map $\mathcal{G}_1\sigma(T_r) \longrightarrow \mathcal{Q}_{[r]}$ with kernel $(\mathcal{W}_n)_1$. In particular, $\mathcal{G}_1\sigma(T_r)$ and $\mathcal{G}$ have the same order, so that $\mathcal{G} = \mathcal{G}_1\sigma(T_r)$.

(2) As noted in (1), we have an exact sequence

$$(0) \longrightarrow \mathfrak{n}_n \longrightarrow \mathrm{Lie}(\mathcal{G}) \xrightarrow{d\pi} \mathfrak{sl}(2) \longrightarrow (0)$$

of restricted Lie algebras such that $d\pi(\mathrm{Ad}(g)(u)) = \mathrm{Ad}(\pi(g))(d\pi(u))$ for $g \in \mathcal{G}$ and $u \in \mathrm{Lie}(\mathcal{G})$. The adjoint action of $\mathcal{G}$ leaves $\mathfrak{sl}(2) = [\mathrm{Lie}(\mathcal{G}), \mathrm{Lie}(\mathcal{G})]$ invariant. Moreover, $\mathcal{G}$ acts trivially on $\mathfrak{n}_n = C(\mathrm{Lie}(\mathcal{G})) = \mathrm{Lie}(\mathrm{Cent}(\mathcal{G}))$. Accordingly, we have

$$\mathrm{Ad}(\sigma(t))(x, v) = (\mathrm{Ad}(t)(x), v)$$

for $t \in T_r$, $x \in \mathfrak{sl}(2)$, and $v \in \mathfrak{n}_n$.

As observed earlier, there exists a p-semilinear form $\psi : \mathfrak{sl}(2) \longrightarrow \mathfrak{n}_n$ such that $\mathrm{Lie}(\mathcal{G}) = \mathfrak{sl}(2)^n_\psi$. According to (3.3) the corresponding linear map $\psi : \mathfrak{sl}(2)^{(1)} \longrightarrow \mathfrak{n}_n$ is a homomorphism of $\mathcal{G}$-modules. Since $\sigma(T_r)$ operates trivially on $\mathfrak{n}_n$ and $r \geq 2$, the map ψ annihilates the root vectors e and f of T_r (cf. part (b) of the proof of (3.4)).

If $\mathcal{B}_0(\mathcal{G})$ is tame, then Theorem 3.4 provides $\alpha \in k \setminus \{0\}$ and $c \in (\mathfrak{n}_n)^{[p]}$ such that

$$\psi(h) = \alpha v_0 + c.$$

Consequently, the automorphism λ of $\mathfrak{n}_n$ that sends v_0 to $\alpha v_0 + c$ satisfies

$$\lambda \circ \psi^n_s = \psi.$$

Observing (6.1), we obtain an isomorphism

$$\omega : \mathfrak{sl}(2)^n_s \longrightarrow \mathfrak{sl}(2)^n_\psi \quad ; \quad (x, v) \mapsto (x, \lambda(v))$$

of restricted Lie algebras such that

$$\omega(t \cdot (x, v)) = \omega(\mathrm{Ad}(t)(x), v) = (\mathrm{Ad}(t)(x), \lambda(v)) = \mathrm{Ad}(\sigma(t))(x, \lambda(v)) = \mathrm{Ad}(\sigma(t))(\omega(x, v))$$

for every $t \in T_r$, $x \in \mathfrak{sl}(2)$ and $v \in \mathfrak{n}_n$. Passage to the corresponding infinitesimal groups of height 1 yields the desired isomorphism $\mathrm{SL}(2)^n_1 \xrightarrow{\sim} \mathcal{G}_1$.

Finally, suppose such an isomorphism to exist. Then $\mathrm{Lie}(\mathcal{G})/C(\mathrm{Lie}(\mathcal{G}))^{[p]} \cong \mathfrak{sl}(2)_s$, and $\mathcal{G}/\mathcal{G}_1 \longrightarrow \mathcal{Q}_{[r]}/(\mathcal{Q}_{[r]})_1 \cong \mu_{p^{r-1}}$ is a quotient map. From the exact sequences above we conclude

$$\mathrm{ord}(\mathcal{G}/\mathcal{G}_1) = \frac{\mathrm{ord}(\mathcal{G})}{\mathrm{ord}(\mathcal{G}_1)} = \frac{\mathrm{ord}(\mathcal{Q}_{[r]})\mathrm{ord}((\mathcal{W}_n)_1)}{\mathrm{ord}(\mathrm{SL}(2)_1)\mathrm{ord}((\mathcal{W}_n)_1)} = p^{r-1},$$

so that $\mathcal{G}/\mathcal{G}_1 \cong \mu_{p^{r-1}}$ is multiplicative. A consecutive application of (1.4) and (5.3) now implies the tameness of $\mathcal{B}_0(\mathcal{G})$. □

Since the group T_r acts on $\mathrm{SL}(2)_1^n$ by automorphisms we can form the semidirect product $\mathrm{SL}(2)_1^n \rtimes T_r$. We denote the multiplicative center of this group by $\mathcal{M}(r,n)$ and define

$$\mathcal{Q}_{[r,n]} := (\mathrm{SL}(2)_1^n \rtimes T_r)/\mathcal{M}(r,n).$$

Note that $\mathcal{Q}_{[r,0]} \cong \mathcal{Q}_{[r]}$ for every $r \geq 1$.

According to [7, (IV,§2,2.9)] a unipotent infinitesimal group $\mathcal{U}$ possesses a composition series with each composition factor being isomorphic to α_p. The number $\ell(\mathcal{U})$ of composition factors is called the *length* of $\mathcal{U}$.

Theorem 6.4. *Let $\mathcal{G}$ be an infinitesimal group of height $r := \mathrm{ht}(\mathcal{G}) \geq 2$ and with unipotent center of length n. Then $\mathcal{B}_0(\mathcal{G})$ is tame if and only if $\mathcal{G} \cong \mathcal{Q}_{[r,n]}$.*

Proof. We begin by showing that the principal block $\mathcal{B}_0(\mathcal{Q}_{[r,n]})$ is tame. By construction of $\mathrm{SL}(2)_1^n$ the morphism π of the exact sequence

$$e_k \longrightarrow (\mathcal{W}_n)_1 \longrightarrow \mathrm{SL}(2)_1^n \stackrel{\pi}{\longrightarrow} \mathrm{SL}(2)_1 \longrightarrow e_k$$

is T_r-equivariant. There thus results an exact sequence

$$e_k \longrightarrow (\mathcal{W}_n)_1 \longrightarrow \mathrm{SL}(2)_1^n \rtimes T_r \stackrel{(\pi,\mathrm{id}_{T_r})}{\longrightarrow} \mathrm{SL}(2)_1 \rtimes T_r \longrightarrow e_k.$$

Let $\zeta : \mathrm{SL}(2)_1 \rtimes T_r \longrightarrow \mathcal{Q}_{[r]}$ be the canonical map and put $\eta := \zeta \circ (\pi, \mathrm{id}_{T_r})$. Since $\ker \zeta \cong \mathrm{SL}(2)_1 \cap T_r$ is multiplicative of order p, it follows that $\ker \eta = (\pi, \mathrm{id}_{T_r})^{-1}(\ker \zeta)$ is a normal subgroup of $\mathrm{SL}(2)_1^n \rtimes T_r$ which is an extension of a multiplicative group of type μ_p by $(\mathcal{W}_n)_1$. Hence $\ker \eta$ is the direct product of its multiplicative center $\mathcal{M}$ and $(\mathcal{W}_n)_1$ (cf. [7, (III,§6,6.3)]). As multiplicative centers are characteristic (cf. [7, (IV,§3,1.1)]), $\mathcal{M} \cong \mu_p$ is a normal subgroup of $\mathrm{SL}(2)_1^n \rtimes T_r$. There results the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
e_k & \longrightarrow & (\mathcal{W}_n)_1 & \longrightarrow & \mathrm{SL}(2)_1^n \rtimes T_r & \stackrel{(\pi,\mathrm{id}_{T_r})}{\longrightarrow} & \mathrm{SL}(2)_1 \rtimes T_r & \longrightarrow & e_k \\
 & & \downarrow & & \downarrow & & \downarrow \zeta & & \\
e_k & \longrightarrow & (\mathcal{W}_n)_1 & \longrightarrow & (\mathrm{SL}(2)_1^n \rtimes T_r)/\mathcal{M} & \longrightarrow & \mathcal{Q}_{[r]} & \longrightarrow & e_k.
\end{array}$$

By the observations above, the kernel of the quotient map η is solvable. As $\mathcal{Q}_{[r]}$ is semisimple, it follows that $\ker \eta$ is the solvable radical of $\mathrm{SL}(2)_1^n \rtimes T_r$. Thus, $\mathcal{M}(r,n) \subset \ker \eta$, so that $\mathcal{M}(r,n) = \mathcal{M}$. Consequently, the middle term of the lower sequence coincides with $\mathcal{Q}_{[r,n]}$.

To see that $\mathcal{Q}_{[r,n]}$ has the requisite properties, we first note that the exact sequence

$$(0) \longrightarrow \mathfrak{n}_n \longrightarrow \mathrm{Lie}(\mathcal{Q}_{[r,n]}) \longrightarrow \mathfrak{sl}(2) \longrightarrow (0)$$

implies $\dim_k \mathrm{Lie}(\mathcal{Q}_{[r,n]}) = \dim_k \mathfrak{sl}(2)_s^n$. On the other hand, the kernel of the canonical map $\mathrm{SL}(2)_1^n \longrightarrow \mathcal{Q}_{[r,n]}$ is contained in $(\mathcal{W}_n)_1 \cap \mathcal{M}$ and is, as a multiplicative and unipotent group, trivial. Hence we obtain a T_r-equivariant isomorphism

$$\mathrm{SL}(2)_1^n \cong (\mathcal{Q}_{[r,n]})_1,$$

and Lemma 6.3 shows that the block $\mathcal{B}_0(\mathcal{Q}_{[r,n]})$ is tame. Thanks to (1.5) the group $\mathcal{Q}_{[r,n]}$ has height r and unipotent center of length n.

Now let $\mathcal{G}$ be an arbitrary infinitesimal group of height r with unipotent center of length n and tame principal block. Owing to (3.4) $\mathcal{G}$ is a central extension of $\mathcal{Q}_{[r]}$ by $(\mathcal{W}_n)_1$. Thanks to Lemma 6.3, there is a closed embedding $\sigma : T_r \hookrightarrow \mathcal{G}$ such that $\mathcal{G} = \mathcal{G}_1\sigma(T_r)$ as well as an isomorphism

$$\gamma : \mathrm{SL}(2)_1^n \longrightarrow \mathcal{G}_1$$

satisfying $\gamma(t\cdot g) = \sigma(t)\gamma(g)\sigma(t)^{-1}$ for $g \in \mathrm{SL}(2)_1^n$ and $t \in T_r$. Consequently,

$$\omega : \mathrm{SL}(2)_1^n \rtimes T_r \longrightarrow \mathcal{G} \quad ; \quad (g,t) \mapsto \gamma(g)\sigma(t)$$

is a homomorphism of group schemes, which is easily seen to be a quotient map. Moreover, γ induces a closed embedding

$$\ker\omega \hookrightarrow T_r.$$

Accordingly, $\ker\omega$ is a multiplicative normal subgroup of order p, and thus coincides with $\mathcal{M}(r,n)$. There results a quotient map

$$\bar{\omega} : \mathcal{Q}_{[r,n]} \longrightarrow \mathcal{G},$$

which, due to equality of orders, is the desired isomorphism. □

7. The Block Structure of $H(\mathcal{G})$

In this final section we determine the structure of those algebras of measures on infinitesimal groups with trivial multiplicative center, whose principal blocks are tame. It turns out that the principal block governs the entire representation theory in this case.

Theorem 7.1. *Let $\mathcal{G}$ be an infinitesimal group with $r := \mathrm{ht}(\mathcal{G}/\mathcal{M}(\mathcal{G}))$ and $n := \ell(\mathrm{Cent}(\mathcal{G})/\mathcal{M}(\mathcal{G}))$. Then the following statements are equivalent:*

(1) *$\mathcal{B}_0(\mathcal{G})$ is tame.*

(2) *Each block of $H(\mathcal{G}/\mathcal{M}(\mathcal{G}))$ is either Morita equivalent to $\mathcal{N}^2(r,n)$ or to $k[X]/(X^{p^n})$. There are $\frac{p-1}{2}$ blocks $\mathcal{B}_0,\dots,\mathcal{B}_{\frac{p-3}{2}}$ of the former type, and p^{r-1} of the latter. The block $\mathcal{B}_i$ has p^{r-1} simple modules of dimension $i+1$, and p^{r-1} simple modules of dimension $p-i-1$. Each representation-finite block has one simple module of dimension p.*

(3) *$\mathcal{B}_0(\mathcal{G})$ is Morita equivalent to $\mathcal{N}^2(r,n)$.*

(4) *$\mathcal{B}_0(\mathcal{G})$ is representation-infinite, special biserial.*

Proof. (1) ⇒ (2). According to (1.1) the principal block of the factor group $\mathcal{G}' := \mathcal{G}/\mathcal{M}(\mathcal{G})$ is tame. Since $\mathcal{R}(\mathcal{G}') \cong \mathcal{R}(\mathcal{G})/\mathcal{M}(\mathcal{G})$, (3.5) implies that $\mathcal{R}(\mathcal{G}') \cong \mathrm{Cent}(\mathcal{G})/\mathcal{M}(\mathcal{G})$ is unipotent and of length n. It thus suffices to verify (2) under the assumption that $\mathrm{ht}(\mathcal{G}) = r$ and that $\mathcal{R}(\mathcal{G}) = \mathrm{Cent}(\mathcal{G}) \cong (\mathcal{W}_n)_1$.

In virtue of (1.3) the assertion holds for $n = 0$. We thus assume $\mathcal{C} := \mathrm{Cent}(\mathcal{G}) \neq e_k$. Since $\mathcal{C}$ is unipotent, the multiplicative center of $\mathcal{G}$ is trivial. In view of (1.3) and (2.1) we have isomorphisms

$$\mathcal{G}/\mathcal{C} \cong \mathcal{Q}_{[s]} \quad \text{and} \quad \mu_{p^{r-1}} \cong \mathcal{G}/\mathcal{G}_1 \cong (\mathcal{G}/\mathcal{C})/(\mathcal{G}/\mathcal{C})_1 \cong \mathcal{Q}_{[s]}/(\mathcal{Q}_{[s]})_1 \cong \mu_{p^{s-1}},$$

so that $r = s$. Thanks to (5.3) the principal block $\mathcal{B}_0(\mathcal{G}_1)$ is tame.

We denote by $\pi : H(\mathcal{G}) \longrightarrow H(\mathcal{Q}_{[r]})$ the canonical projection. Let $\mathcal{B} \subset H(\mathcal{G})$ be a block. In view of (5.1) we either have $\mathcal{B} = \mathcal{B}_i$, or all simple $\mathcal{B}$-modules are p-dimensional.

In the latter case, the block ideal $\pi(\mathcal{B}) \subset H(\mathcal{Q}_{[r]})$ is semisimple. Given simple $\mathcal{B}$-modules S, T, the corresponding $H(\mathcal{Q}_{[r]})$-modules are therefore projective. Consequently, the spectral sequence

$$\mathrm{Ext}^q_{H(\mathcal{Q}_{[r]})}(S, H^m(\mathcal{C},T)) \Rightarrow \mathrm{Ext}^{q+m}_{H(\mathcal{G})}(S,T)$$

(cf. [27, (I.6.6)]) yields an isomorphism

$$\mathrm{Ext}^1_{H(\mathcal{G})}(S,T) \cong \mathrm{Hom}_{H(\mathcal{Q}_{[r]})}(S, H^1(\mathcal{C},T)).$$

As $\mathcal{C}$ is unipotent, $T|_{H(\mathcal{C})}$ is a trivial module, and we thus obtain an isomorphism $H^1(\mathcal{C},T) \cong H^1(\mathcal{C},k) \otimes_k T$ of $H(\mathcal{G})$-modules with $H(\mathcal{G})$ operating trivially on $H^1(\mathcal{C},k)$. The latter fact can be seen by computing $H^\bullet(\mathcal{C}, k)$ via the Hochschild complex (cf. [27, (I.4.16)]). There result isomorphisms

$$\operatorname{Ext}^1_{H(\mathcal{G})}(S,T) \cong H^1(\mathcal{C},k) \otimes_k \operatorname{Hom}_{H(\mathcal{Q}_{[r]})}(S,T).$$

Since $n \neq 0$, the local Hopf algebra $H(\mathcal{C}) \cong H((\mathcal{W}_n)_1) \cong k[X]/(X^{p^n})$ is representation-finite and not simple, so that $H^1(\mathcal{C}, k) \cong k$. Schur's Lemma now yields

$$\dim_k \operatorname{Ext}^1_{H(\mathcal{G})}(S,T) = \delta_{[S],[T]}.$$

In particular, the block $\mathcal{B}$ is a primary Nakayama algebra, and $\pi(\mathcal{B})$ is a simple block with a simple module of dimension p.

Alternatively, let us assume that $\mathcal{B} = \mathcal{B}_i$, so that $\mathcal{B}$ has Gabriel quiver Δ_r. We put $G := X(\mathcal{G}/\mathcal{G}_1)$ and apply (5.2) and [14, (7.1)] to see that $\mathcal{C} := \mathcal{B}^G$ is a tame block of $H(\mathcal{G}_1)$, which is Morita equivalent to $\mathcal{N}^2(1,m)$ for $m = \dim_k \operatorname{rad}_p(C(\operatorname{Lie}(\mathcal{G})))$. Here $\operatorname{rad}_p(C(\operatorname{Lie}(\mathcal{G})))$ is the p-nilpotent radical of the center $C(\operatorname{Lie}(\mathcal{G}))$, which, in view of (3.4) coincides with $\operatorname{Lie}(\operatorname{Cent}(\mathcal{G})) \cong \operatorname{Lie}(\mathcal{W}_n) \cong \mathfrak{n}_n$. As a result, we have $n = m$.

If $r = 1$, then $\mathcal{G} = \mathcal{G}_1$ and we are done. Assume that $r \geq 2$. Thanks to (2.2(3)) the group G operates freely on the set of isoclasses of the simple $\mathcal{B}$-modules, so that (4.4) applies. Accordingly, the algebra $\mathcal{B}$ is either Morita equivalent to $\mathcal{N}^2(r,n)$, or each principal indecomposable $\mathcal{B}$-module is sincere.

Since $H(\mathcal{C}) \cong H((\mathcal{W}_n)_1) \cong k[X]/(X^{p^n})$, there exists $v_0 \in H((\mathcal{W}_n)_1)^\dagger$ such that $H((\mathcal{W}_n)_1) = k[v_0]$. Note that v_0 is a nilpotent element of the center of $H(\mathcal{G})$ which satisfies $v_0^{p^n} = 0$. We set $I := H(\mathcal{G})v_0$ and observe that $I = H(\mathcal{G})H((\mathcal{W}_n)_1)^\dagger$. Consequently, $H(\mathcal{Q}_{[r]}) \cong H(\mathcal{G}/\mathcal{C}) \cong H(\mathcal{G})/I$.

Let P be the projective cover of a simple $H(\mathcal{G})$-module S. By our above observations, P/v_0P is the projective cover of the corresponding $H(\mathcal{Q}_{[r]})$-module. Moreover, left multiplication by v_0^i induces a surjection

$$P/v_0P \xrightarrow{\pi_i} v_0^iP/v_0^{i+1}P$$

of $H(\mathcal{Q}_{[r]})$-modules for every $i \in \{0,\dots,p^n-1\}$. Since $k[v_0] = H((\mathcal{W}_n)_1)$, it follows from [35, (2.6)] that $P|_{k[v_0]}$ is projective. As $k[v_0]$ is local, $P|_{k[v_0]}$ is actually free. Consequently, we have

$$P|_{k[v_0]} \cong k[v_0]^\ell$$

for some $\ell \geq 1$, whence

$$\dim_k v_0^iP/v_0^{i+1}P = \ell \qquad 0 \leq i \leq p^n - 1.$$

As a result, π_i is an isomorphism.

If S belongs to $\mathcal{B}$, then [17, (5.5)] ensures that the module P/v_0P has a composition series of length 4 with three non-isomorphic constituents. As a result, P, having a filtration with factors $v_0^iP/v_0^{i+1}P$, also admits only three non-isomorphic composition factors, and (4.4) shows that $\mathcal{B}$ is Morita equivalent to $\mathcal{N}^2(r,n)$.

Now assume $\mathcal{B}$ to be a Nakayama algebra, and suppose that S belongs to $\mathcal{B}$. Then $\mathcal{B}/v_0\mathcal{B}$ is simple, and $P/v_0P \cong S$ has dimension p. The above observations now show that P has length p^n, implying $\mathcal{B} \cong \operatorname{Mat}_p(k[X]/(X^{p^n}))$.

(2) ⇒ (3) By (1.1) we have $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{M}(\mathcal{G}))$. Since the latter block has a simple module of dimension $\neq p$, (2) implies that $\mathcal{B}_0(\mathcal{G})$ is Morita equivalent to $\mathcal{N}^2(r,n)$.

(3) ⇒ (4) This follows from the fact that the separated quiver of $\mathcal{B}_0(\mathcal{G})$ is not a union of Dynkin quivers (cf. the proof of (5.3)).

(4) ⇒ (1) This follows from [36, (2.4)] or [9, (5.2)]. □

Remark. The foregoing result has several applications, namely

- the computation of the Auslander-Reiten quiver of $H(\mathcal{G})$,

- the determination of the infinitesimal groups of domestic representation type, and
- the computation of the Krull-Gabriel dimension of $H(\mathcal{G})$.

We hope to return to these issues in conjunction with a discussion of the blocks of infinitesimal groups with nontrivial multiplicative center. As in [14] the treatment of this problem involves a detailed study of certain reduced enveloping algebras.

Given a k-algebra Λ with a complete set $\{S_1, \dots, S_n\}$ of representatives of the isomorphism classes of simple Λ-modules and corresponding principal indecomposable modules $\{P_1, \dots, P_n\}$, we let

$$C_\Lambda := (\dim_k \operatorname{Hom}_\Lambda(P_i, P_j))_{1\le i,j\le n}$$

be the *Cartan matrix* of Λ. By Brauer's theorem (cf. [3, (5.7.2)]) the Cartan matrix of a block of a group algebra of a finite group is nonsingular. By contrast, we have the following result concerning tame principal blocks of infinitesimal groups:

Corollary 7.2. *Let $\mathcal{G}$ be an infinitesimal group such that $\mathcal{B}_0(\mathcal{G})$ is tame. Then $H(\mathcal{G})$ is symmetric, and $\mathcal{B}_0(\mathcal{G})$ has a singular Cartan matrix.*

Proof. According to (7.1(3)) the algebra $\mathcal{B}_0(\mathcal{G})$ is Morita equivalent to $\mathcal{N}^2(r,n)$. By (4.2) the latter algebra is symmetric, so that $\mathcal{B}_0(\mathcal{G})$ also enjoys this property.

Let $\zeta : H(\mathcal{G}) \longrightarrow k$ be the modular function of the cocommutative Hopf algebra $H(\mathcal{G})$. Owing to [19, (1.5)] the convolution $\zeta * \mathrm{id}_{H(\mathcal{G})}$ is a Nakayama automorphism of the Frobenius algebra $H(\mathcal{G})$. By [27, (I.8.13)] the socle of the projective cover of the trivial module k is isomorphic to k_ζ. On the other hand, $\mathcal{B}_0(\mathcal{G})$ is symmetric, so that $k \cong k_\zeta$. Accordingly, $\zeta = \varepsilon$ is the co-unit of $H(\mathcal{G})$ and $\zeta * \mathrm{id}_{H(\mathcal{G})} = \varepsilon * \mathrm{id}_{H(\mathcal{G})} = \mathrm{id}_{H(\mathcal{G})}$ is a Nakayama automorphism of $H(\mathcal{G})$. Consequently, $H(\mathcal{G})$ is symmetric.

We put $C_{(r,n)} := C_{\mathcal{N}^2(r,n)}$. According to (7.1(3)) we have $C_{\mathcal{B}_0(\mathcal{G})} = C_{(r,n)}$ for suitably chosen elements r, n. Let P_i and S_i be the principal indecomposable and the simple $\mathcal{N}^2(r,n)$-module corresponding to the vertex $i \in \Delta_r$, respectively. Directly from the definition of $\mathcal{N}^2(r,n)$ we see that P_j has a presentation

$$[P_j] = p^n[S_{j-1}] + 2p^n[S_j] + p^n[S_{j+1}]$$

in the Grothendieck group of $\mathcal{N}^2(r,n)$. Here the indices are to be interpreted $\bmod 2p^{r-1}$. Consequently, the Cartan matrix is given by

$$C_{(r,n)} = \begin{pmatrix} 2p^n & p^n & 0 & \cdots & \cdots & 0 & p^n \\ p^n & 2p^n & p^n & 0 & \cdots & 0 & 0 \\ 0 & p^n & 2p^n & p^n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & p^n & 2p^n & p^n & 0 \\ 0 & 0 & \cdots & 0 & p^n & 2p^n & p^n \\ p^n & 0 & \cdots & \cdots & 0 & p^n & 2p^n \end{pmatrix}$$

for $r \geq 2$, and by

$$C_{(1,n)} = \begin{pmatrix} 2p^n & 2p^n \\ 2p^n & 2p^n \end{pmatrix}$$

for r = 1. Thus, if C_j denotes the j-th column vector of the $(2p^{r-1} \times 2p^{r-1})$-matrix $C_{(r,n)}$, then

$$\sum_{j=1}^{2p^{r-1}} (-1)^j C_j = 0,$$

so that $C_{(r,n)}$ is singular. □

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