A SUMMARY ABOUT THE QUIVER OF TILTING MODULES

HONGWEI PENG

Let $\Lambda$ be a basic connected finite-dimensional hereditary algebra over an algebraically closed field $k$ and assume the rank of $\Lambda$ is $n$.

Note that in fact some of the results, $\Lambda$ can be generalized to be Artin algebra over a commutative Artin ring. Because the results are from some other papers, I will omit the proof and if you are interested you can check the references.

**Definition.** A $\Lambda$-module $T$ is called tilting provided:

(a) $pd_\Lambda(T) \leq 1$.

(b) $\text{Ext}^1_\Lambda(T, T) = 0$.

(c) There exists a short exact sequence: $0 \rightarrow \Lambda \rightarrow T_1 \rightarrow T_1 \rightarrow 0$ with $T_0, T_1 \in \text{add}\, T$.

The quiver of tilting $\Lambda$-modules $\overrightarrow{K_\Lambda}$ was defined by C.Riedtmann and A.Schofield in [10] where vertices are defined to be tilting $\Lambda$-modules and there exists an arrow between tilting modules $T_1$ and $T_2$ if and only if $T_1 = M \oplus X$, $T_2 = M \oplus Y$ for some indecomposable $X$ and $Y$ and there exists a non-split exact sequence $0 \rightarrow X \rightarrow M' \rightarrow Y \rightarrow 0$ with $M' \in \text{add}\, M$.

It was proved by D.Happel in [3], there are only countably many non-isomorphic tilting modules. According to S.Fomin and A.Zelevinsky [1], the number of tilting modules equals $\frac{1}{n+1} \binom{2n}{n}$, $\frac{3n-4}{2n} \binom{2n-2}{n-1}$, 418, 2431,17342, when $\Lambda$ is representation finite of type $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$, respectively. And what’s more, By R.Kase [9], the corresponding number of arrows in $\overrightarrow{K_\Lambda}$ is $(2n-1)$, $(3n-4)\binom{2n-4}{n-3}$, 1140, 8008, 66976.

Let $T_\Lambda$ be the set of tilting $\Lambda$-modules, a partial order $\leq$ is defined on $T_\Lambda$ by $T_1 \leq T_2$ provided $\text{Fac}\, T_1 \subseteq \text{Fac}\, T_2$.

**Example.**

(a) Let $\Lambda$ be the path algebra of the Kronecker quiver $\Gamma_r$ with $r \geq 2$, then $\overrightarrow{K_\Lambda}$ would be as

```
  o ----> o ----> o ----> o ----> o
```

(b) Let $\Lambda$ be the path algebra of the quiver $o ----> o ----> o$, then $\overrightarrow{K_\Lambda}$ would be as

```
  o ----> o ----> o ----> o
  |   |   |   |
  v   v   v   v
  o ----> o ----> o
```

(c) Let $\Lambda$ be the path algebra of the quiver $o ----> o ----> o$, then $\overrightarrow{K_\Lambda}$ would be as

```
  o ----> o ----> o
  |   |   |   |
  v   v   v   v
  o ----> o ----> o
  |   |   |   |
  v   v   v   v
  o ----> o ----> o
```
Remark. According to example (a), it may happen that different algebras $\Lambda$ and $\Lambda_0$ share the same quiver of tilting modules.

Examples (b) and (c) show that for path algebras of the same underlying graph but different orientations, their quivers of tilting modules might be different.

Theorem 1. [6, 2.1] $\overrightarrow{K_\Lambda}$ is the Hasse diagram of the partial ordered set $(T_\Lambda, \leq)$.

Note that when $\Lambda$ is hereditary, $D(\Lambda^\perp)$ is also tilting module and then $\Lambda_\Lambda$ and $D(\Lambda^\perp)$ would be the maximal and minimal elements of $(T_\Lambda, \leq)$, respectively. So they would be the unique source and sink of $\overrightarrow{K_\Lambda}$, respectively.

Theorem 2. [5, 6.4] Let $\Lambda = k\Delta$, if $\Delta$ has no multiple arrows and no isolated vertices, $\Delta$ is uniquely determined by $T_\Lambda$.

1. Local structure and components of the quiver

For an almost complete tilting module $M$ (defined to be partial tilting modules with $n-1$ non isomorphic indecomposable direct summands), the Bongartz’s lemma asserts the existence of complements to $M$. It was proved in [8, 1.2] that an almost complete tilting module $M$ has at most two different complements and it has exactly two non isomorphic complements if and only if $M$ is sincere.

Lemma 3. [5, 3.2] For any $T \in T_\Lambda$, the degree $\delta(T) \leq n$. And $\delta(T) = n$ if and only if $(\dim T)_i \geq 2$ for all $1 \leq i \leq n$.

Theorem 4. [12, 3.5] Each connected component of $\overrightarrow{K_\Lambda}$ contains a tilting $T$ such that $\delta(T) < n$.

It was conjectured in [7] that each connected component of $\overrightarrow{K_\Lambda}$ contains only finite such vertices.

Proposition 5. [7, 3.9] If $\Lambda$ is wild hereditary with at least three simple modules and $T$ is a regular tilting module, then there exist regular $T_1, T_2 \in T_\Lambda$ with $T_1 \rightarrow T \rightarrow T_2$ in $\overrightarrow{K_\Lambda}$.

Lemma 6. [5, 5.1] Let $S_i$ be a simple $\Lambda$-module which is not injective. Then $T_{S_i} = \oplus_{j \neq i} P(j) \oplus \tau_\Lambda^{-1} S_i$ is a tilting module. In this way we obtain all immediate successors of $\overrightarrow{K_\Lambda}$. Dually, we obtain all immediate predecessors of $\overrightarrow{K_\Lambda}$.

Let $M$ be a multiplicity free partial tilting $\Lambda$-module, $\text{lk}(M)$ is defined to be the set of tilting modules having $M$ as a direct summand and $\overleftarrow{\text{lk}}(M)$ is defined to be the full subquiver of $\overrightarrow{K_\Lambda}$ consisting of $\text{lk}(M)$. According to the Bongartz’s Lemma and the dual there exist $C^\alpha$ and $C^\omega$ such that $T^\alpha = M \oplus C^\alpha$ and $T^\beta = M \oplus C^\omega$ are tilting modules.

Proposition 7. [6, 2.2] If $\overrightarrow{K_\Lambda}$ has a finite component $C$, then $\overrightarrow{K_\Lambda} = C$.

Remark. By this proposition, the quiver $\overrightarrow{K_\Lambda}$ would be connected if $\Lambda$ is representation finite. But note that the converse of the proposition doesn’t hold.

Proposition 8. [4, 2.3] Let $M$ be a multiplicity free partial tilting module with $n - r$ non-isomorphic indecomposable direct summand. Then the following hold.

(a) $\overleftarrow{\text{lk}}(M)$ is a convex subquiver of $\overrightarrow{K_\Lambda}$.

(b) $T^\alpha$ is the unique source of $\overleftarrow{\text{lk}}(M)$ and $T^\omega$ is the unique sink of $\overleftarrow{\text{lk}}(M)$.

(c) Let $T \in T_\Lambda$. Then $T \in \text{lk}(M)$ if and only if $T^\omega \leq T \leq T^\alpha$.

(d) If $M$ is faithful, then $\text{add} C^\alpha \cap \text{add} C^\omega = 0$.

(e) If $M$ is faithful, then there exists a path of length $r$ from $T^\alpha$ to $T^\omega$ in $\text{lk}(M)$.
Let $M$ be a multiplicity free partial tilting module, $^\text{perp}M$ is defined to be the full subcategory of $\text{mod}\Lambda$ consisting of modules $Y$ such that $\text{Hom}_\Lambda(Y, X) = 0$ and $\text{Ext}^1_\Lambda(X, Y) = 0$. Then according to [2], there exists a hereditary finite dimensional $k$-algebra $\Gamma$ such that $^\text{perp}M \simeq \text{mod}\Gamma$.

**Theorem 9.** \[5, 2.8\] There is an isomorphism between $\overleftarrow{lk}(M)$ and $\overrightarrow{K}\Gamma$.

**Theorem 10.** \[4, 4.1\] Let $M$ be a faithful partial tilting module. Then $\overrightarrow{lk}(M)$ is connected.

**Theorem 11.** \[12\] Let $M$ be a partial tilting module with $n - 2$ summands, then
(a) when $\text{lk}(M)$ is finite
(i) if $M$ is sincere then $\overleftarrow{lk}(M)$ is either

\[ \begin{array}{c}
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ
\end{array} \]

or

\[ \begin{array}{c}
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ
\end{array} \]

(ii) if $M$ is not sincere, then $\overrightarrow{lk}(M)$ is $\circ$ or $\circ \rightarrow \circ$ or $\circ \rightarrow \circ \rightarrow \circ$.

(b) when $\text{lk}(M)$ is infinite,
(i) if $M$ is sincere then $\overleftarrow{lk}(M)$ is of the form

\[ \begin{array}{c}
\circ \\
\downarrow \\
\circ
\end{array} \]

\[ \begin{array}{c}
\circ \\
\downarrow \\
\circ
\end{array} \]

(ii) if $M$ is not sincere, then $\overrightarrow{lk}(M)$ is

\[ \begin{array}{c}
\circ \rightarrow \circ \rightarrow \circ
\end{array} \]

\[ \begin{array}{c}
\circ \\
\downarrow \\
\circ
\end{array} \]

\[ \begin{array}{c}
\circ \\
\downarrow \\
\circ
\end{array} \]

2. **Filtration induced by indexed chains of torsion classes**

**Definition.** Let $\mathcal{A}$ be an abelian category and consider the closed set $[0, 1]$. The chain of torsion classes $\eta$ indexed by $[0, 1]$ is a set of torsion classes $\eta := \{T_s : T_0 = \mathcal{A}, T_1 = \{0\}, T_s \subset T_r \text{ if } r \leq s\}$.
Definition. A chain of torsion classes $\eta$ indexed by $[0, 1]$ in $A$ is called quasi-Noetherian if for every $(a, b) \subset [0, 1]$ there exists $s \in (a, b)$ such that the torsion object $t_s M$ of $M$ with respect to $T_0$ contains the torsion object $t_r M$ of $M$ with respect to $T_r$ for all $r \in (a, b)$ and all non zero object $M$ of $A$.

Dually, weakly-Artinian chains of torsion classes are defined and $CT(A)$ is defined to be the set of all chains of torsion classes that are quasi-Noetherian and weakly-Artinian.

For every $t \in [0, 1]$, $F_t$ is the torsion free class corresponds to $T_t$ and $P_t$ is defined by the following

\[ P_t = \begin{cases} \bigcap_{s \in (s, t]} F_s & t = 0 \\ \bigcap_{s \in (s, t]} T_s \cap \bigcap_{s \in (s, t]} F_s & t \in (0, 1) \\ \bigcap_{s \in (s, t]} T_s & t = 1 \end{cases} \]

Theorem 12. [11, 2.9] Let $A$ be an abelian category and $\eta$ be an indexed chain of torsion classes in $CT(A)$. Then every non zero object $M$ of $A$ admits a Harder-Narasimhan filtration. That is a filtration

\[ M_0 \subset M_1 \subset \ldots \subset M_n \]

such that:
(a) $0 = M_0$ and $M = M_n$;
(b) there exists $r_k \in [0, 1]$ such that $M_k/M_{k-1} \in P_{r_k}$ for all $1 \leq k \leq n$;
(c) $r_1 > r_2 > \ldots > r_n$

Moreover, this filtration is unique up to isomorphism.

Example. In example (b), if we take the shortest path $P(1) \oplus P(2) \oplus P(3) \to P(1) \oplus P(3) \oplus I(1) \to P(1) \oplus I(2) \oplus I(1)$ into consideration, we can obtain a chain of torsion class indexed by $[0, 1]$

\[ T_t = \begin{cases} \text{mod}kQ \\ \text{add}\{P(1) \oplus P(3) \oplus I(1) \oplus I(2)\} & t \in (0, 1/3] \\ \text{add}\{P(1) \oplus I(1) \oplus I(2)\} & t \in (1/3, 2/3] \\ \{0\} & t \in (2/3, 1] \end{cases} \]

Then $P_0 = \text{add}\{S(2)\}$, $P_{1/3} = \text{add}\{P(3)\}$, $P_{2/3} = \text{add}\{P(1) \oplus I(1) \oplus I(2)\}$ and $P_t = \{0\}$ otherwise. And the filtration for $P(2)$ is : $\{0\} \subset P(3) \subset P(2)$.

References