Some 3-manifolds with 2-generated fundamental group

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Abstract

We describe a family of graph manifolds that have 2-generated fundamental group but do not admit a Heegaard splitting of genus 2.

1 Introduction

Any orientable closed 3-manifold $M$ can be decomposed as the union of two handlebodies $V_1$ and $V_2$ such that $V_1 \cap V_2 = \partial V_1 = \partial V_2$. Such a decomposition is called a Heegaard splitting. The genus of the splitting is the genus of the orientable surface $V_1 \cap V_2$ which is the genus of $V_1$ and $V_2$. The Heegaard genus $h(M)$ of $M$ is the smallest integer $h$ such that $M$ admits a Heegaard splitting of genus $h$.

It is easy to see that the fundamental group $\pi_1(M)$ of a manifold $M$ can be generated by $h(M)$ elements, i.e. $h(M) \geq r(M)$ where $r(M)$ denotes the rank of $\pi_1(M)$. F. Waldhausen [H], [W3] asked whether the converse is also true, i.e. whether $h(M) = g(M)$ for all $M$. In [BZ] M. Boileau and H. Zieschang compute the ranks of the fundamental groups of Seifert manifolds and show that there exists a family of Seifert manifolds for which $h(m) = 3 > 2 = g(M)$. In [MS] J. Schultens and Y. Moriah show that the Seifert manifolds whose Heegaard genus was not decided in [BZ] provide families of manifolds for which $h(M) = r(M) + 1$.

In this note we describe a family of 3-dimensional graph manifolds which have 2-generated fundamental group but are not of Heegaard genus 2; it is not difficult to see that they are of genus 3. In [BW] it is shown that this is the only class of graph manifolds with 2-generated fundamental group that is not of Heegaard genus 2.

2 The examples

Suppose that $M$ is a Seifert fibered manifold. We denote its fundamental group by $G_M$, its base Orbifold by $O_M$ and the fibre as well as the element of $G_M$ corresponding to the regular fibre by $f_M$. In particular we have the following exact sequence

$$1 \to \langle f_M \rangle \to G_M \to \pi_1(O_M) \to 1$$

if $G_M$ is infinite. We denote the base orbifolds simply by the topological type with a list of orders of the cone points, so $D(p,q)$ stands for the disk with two cone points of order $p$ and $q$. The other occurring types are the Möbius band

1
which we denote by \( M_\hat{\alpha} \), the sphere \( \mathbb{S}^2 \) and the projective space \( \mathbb{P}^2 \). Since we are only dealing with orientable Seifert manifolds we can denote the signature in the form \( F(b; \beta_1/\alpha_1, \ldots, \beta_k/\alpha_k) \), where \( b \) denotes the Seifert invariant. If the Seifert manifold has boundary we omit \( b \) in the signature.

The only Seifert manifolds with boundary that admit non-isotopic Seifert fibrations are \( S^1 \times S^1 \times [0, 1] \), \( D^2 \times S^1 \) and the orientable circle bundle over the Möbius band, see Theorem VI.18 of [J]. Following Waldhausen [W1] we denoted the last manifold by \( Q \). If \( Q \) is fibered over the Möbius band we denote the fibre by \( f_Q \) and if \( Q \) is fibered over \( D(2, 2) \) we denote the fibre by \( f_Q' \).

The manifolds we are studying in this note are graph manifolds that are obtained from gluing two orientable Seifert manifolds \( M_1 \) and \( M_2 \) with \( O_{M_1} = M_\hat{\alpha} \) or \( O_{M_1} = M_\hat{\alpha}(p) \) and \( O_{M_2} = D(2, 2l + 1) \) for some integer \( l \geq 1 \).

We need to understand precisely which manifolds of this type are of genus 2. This follows from Kobayashi’s [K] classification of 3-manifolds with non-trivial torus decomposition that are of genus 2. An immediate consequence of the main theorem of [K] is the following:

**Proposition 1 (Kobayashi)** Suppose that \( M \) is a closed orientable 3-manifold with \( h(M) = 2 \) whose JSJ-decomposition consists of two pieces identified along a (separating) torus. Then one of the following holds:

1. \( M \) is obtained from a Seifert manifold \( M_1 \) with \( O_{M_1} = D(p, q) \) and the exterior of a 1-bridge knot in a lens space \( M_2 \) where the regular fibre of \( M_1 \) is identified with the meridian of \( M_2 \).

2. \( M \) is obtained from a Seifert manifold \( M_1 \) with \( O_{M_1} \) of type \( M_\hat{\alpha}, M_\hat{\alpha}(p), M_\hat{\alpha}(p, q), D(p, q), D(p, q, r) \) and the exterior of 2-bridge knot in \( \mathbb{S}^3 \) where the regular fibre of \( M_1 \) is identified with the meridian of \( M_2 \).

Before proceeding we need to clarify which Seifert manifolds are homeomorphic to exteriors of 1-bridge knots in lens spaces and to exteriors of 2-bridge knots in \( \mathbb{S}^3 \). The following is a simple refinement and the converse of Lemma 5.2 of [K]:

**Lemma 1** A Seifert manifold \( M \) with a boundary curve \( \gamma \) is the exterior a 1-bridge knot in a lens space with a meridian iff one of the following holds (we consider \( Q \) as the Seifert manifold with base \( D(2, 2) \)):

1. \( O_M = D(p, q) \) and \( \gamma \) has intersection number one with the fibre.

2. \( M \) has signature \( M_\hat{\alpha}(1/p) \), \( G_M \) then has the presentation \( G_M = \langle s, x, f \mid [x, f], sfs^{-1}f, x^p = f \rangle \) and \( \gamma \) corresponds to the element \( s^2x \) (or \( s^2xf^{-1} \) if \( p = 2 \)).

**Proof** It follows from Lemma 5.2 of [K] that a Seifert manifold that is the exterior of a 1-bridge knot in a lens space has either base space \( D(p, q) \) or base space \( M_\hat{\alpha}(p) \) and the meridian has intersection number one with the fibre or the base space is \( M_\hat{\alpha} \) in which case we can rewrite \( M \) as a Seifert manifold with base of type \( D(2, 2) \).

If \( O_M = D(p, q) \) and \( \gamma \) has intersection number one with the fibre then the Dehn filling with respect to \( \gamma \) gives a Seifert manifold with base \( \mathbb{S}^2(p, q) \).
This manifold is a lens space and the knot corresponds to the regular fibre. In particular it lies on the torus of a genus 1 Heegaard splitting of $M$; it follows that the knot is 1-bridge.

If $O_M = M\tilde{O}(p)$ then $M$ has signature $M\tilde{O}(q/p)$ and $G_M = \langle s, x, f | x, x^f, sf^{-1}f, x^f = f^p \rangle$ and a boundary curve that has intersection number one with the fibre corresponds to the element $s^2f^k$ for some $k \in \mathbb{Z}$. Dehn filling along this curve yields a Seifert manifold with signature $\mathbb{RP}^2(k; q/p)$. It follows from the classification of small Seifert manifolds, see [O], that this manifold is a prism manifold (and not a lens space) unless $q = 1$ and either $k = 0$ or $k = -1$ and $p = 2$ in which cases they are lens spaces. In order to verify that the knots of this type are 1-bridge knots, it suffices to inspect the genus 1 Heegaard splitting of the Seifert manifolds with signatures $\mathbb{RP}^2(0; 1/p)$ and $\mathbb{RP}^2(-1; 1/2)$. This splitting can be visualized using the Reidemeister construction as discussed in [BZ], Section 1.10. □

The classification of Seifert fibered 2-bridge knots is well known. The only knots that have Seifert fibered exteriors are the torus knots [J] and among the torus knots the 2-bridge knots are the knots $t(2, 2l + 1)$ [RZ].

**Lemma 2** A Seifert Manifold $M$ is the exterior of a 2-bridge knot iff $M$ has signature $D(1/2, l/(2l+1))$. $G_M$ then has the presentation $G_M = \langle a, b, f | [a, f], [b, f], a^2 = f, b^{2l+1} = f^l \rangle$ and the meridian corresponds to the element $abf^{-1}$. □

We now easily deduce the following:

**Lemma 3** Suppose that $M$ is a graph manifold obtained from gluing Seifert manifolds $M_1$ and $M_2$ with $O_{M_1} = M\tilde{O}(p)$ or $O_{M_1} = M\tilde{O}$ and $O_{M_2} = D(2, 2l + 1)$ along their boundary components. Then $M$ is of Heegaard genus 2 iff we are in one of the following situations:

1. $M_1 = Q$, i.e. $O_{M_1} = M\tilde{O}$, and $M_1$ and $M_2$ are glued such that $f_Q$ (the regular fibre when fibered over $D(2, 2)$) and $f_{M_2}$ have intersection number ±1.

2. $M_1$ is a Seifert space with signature $M\tilde{O}(1/p)$. $M_1$ and $M_2$ are glued such that $f_{M_1}$ gets identified with the meridian of $M_1$ when considered as the exterior of a 1-bridge knot in a lens space.

3. $M_2$ is a Seifert manifold with signature $D(1/2, l/(2l+1))$ and $f_{M_1}$ is identified with the meridian of $M_2$ when considered as the exterior of a 2-bridge knot.

**Proof** If $O_{M_1} = M\tilde{O}(p)$ this is an immediate consequence of Poposition 1 with Lemma 1 and Lemma 2. If $O_{M_1} = M\tilde{O}$ and we are in situation 1 of Proposition [K] this puts us into situation 3 of Lemma 3. Otherwise we see that after rewriting $Q$ as the Seifert manifold over $D(2, 2)$ that we are in situation 1. □

Lemma 3 implies in particular that there are instances where $M_1$ is a Seifert space over $M\tilde{O}(p)$ and $M_2$ is a Seifert space over $D(2, 2l + 1)$ such that there is no way of gluing the boundaries such that the resulting manifold is of Heegaard genus 2. This is the case if $M_1$ is not the exterior of a 1-bridge knot in a lens space and $M_2$ is not the exterior of a 2-bridge knot in $S^3$. The following lemma
however implies that there is a multitude of gluing maps such that the resulting manifold has 2-generated fundamental group. Most of these gluing maps are not of the types above, i.e. the resulting manifolds are not of Heegaard genus 2. Also in the cases where there is a gluing map such that the resulting manifold has genus 2 there are many more such that the fundamental group is 2-generated.

Lemma 4 Let $M_1$ be a Seifert space over $\tilde{M}\tilde{o}$ or $\tilde{M}\tilde{o}(p)$ and $M_2$ be a Seifert manifold over the base $D(2, 2l+1)$. Then any 3-manifold $M$ obtained by gluing $M_1$ and $M_2$ such that $f_{M_1}$ and $f_{M_2}$ have intersection number $\pm 1$ has 2-generated fundamental group.

Proof. As before we have $G_{M_1} = \langle s, f_{M_1}, s^{-1}f_{M_1} \rangle$ if $O_{M_1} = \tilde{M}\tilde{o}$, $G_{M_2} = \langle s, x, f_{M_1}, [x, f_{M_1}], s^{-1}f_{M_1}, x^p = f_{x}^{p} \rangle$ if $O_{M_1} = \tilde{M}\tilde{o}(p)$ and $G_{M_2} = \langle a, b, f_{M_1} \rangle$. By assumption we have $f_{M_1} = ab^k f_{M_1}^k$, for some $k \in \mathbb{Z}$. We verify that the group $\langle abf_{M_1}^k, a^{-1}abf_{M_1}^k a = abf_{M_1}^k, ba f_{M_1}^k \rangle \subset G_{M_1}$ maps surjectively onto $\pi_1(O_{M_1}) = \langle a, b \rangle$. It follows that $f_{M_1}^m a \in \langle abf_{M_1}^k, a^{-1}abf_{M_1}^k a \rangle$ for some $m \in \mathbb{Z}$. To see this it clearly suffices to show that $\langle a, b \rangle$ is generated by $ab$ and $ba$. This holds since $ba \cdot ab = b^2$ which implies that $b^{2l+2} = b \in \langle ab, ba \rangle$ and therefore also $a = ab \cdot b^{-1} \in \langle ab, ba \rangle$.

Choose $g = f_{M_1}$ if $O_{M_1} = \tilde{M}\tilde{o}$ and $g$ to be a generator of the cyclic subgroup $\langle x, f_{M_1} \rangle \subset G_{M_1}$ if $O_{M_1} = \tilde{M}\tilde{o}(p)$. Choose $h = s f_{M_1}^n a$. We show that $G = \langle g, h \rangle$. It is clear that $f_{M_1} \in \langle g, h \rangle$. We therefore have $h^{-1}f_{M_1}^{-1}h = \langle a^{-1} f_{M_1}^{-1} s^{-1} f_{M_1}^{-1} s^{-1} f_{M_1}^{-1} s^{-1} f_{M_1}^{-1} a = a^{-1} f_{M_1}^{-1} a = a^{-1}abf_{M_1}^k a = ba f_{M_1}^k \rangle \in \langle g, h \rangle$. By the argument above this implies that $f_{M_1}^m a \in \langle g, h \rangle$ and therefore $h(f_{M_1}^m a)^{-1} = s f_{M_1}^n a(f_{M_1}^m a)^{-1} = s \in \langle g, h \rangle$. It follows that $G_{M_1} \subset \langle g, h \rangle$ since $G_{M_1}$ is generated by $g$ and $s$. This implies that $f_{M_1} \in \langle g, h \rangle$. It follows that $a, b \in \langle g, h \rangle$ and therefore $G_{M_1} \subset \langle g, h \rangle$ which shows that $G = \langle g, h \rangle$. □

References


[MS] Y. Moriah and J. Schultens Heegaard splittings of Seifert fibered spaces are either vertical or horizontal, Topology 37, 1998, 1089-1112.


