A rank formula for amalgamated products with finite amalgam
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1 Introduction

Grushko’s theorem [4] says that the rank of a group is additive under free products, i.e. that

\[ \text{rank } A \ast B = \text{rank } A + \text{rank } B. \]

For amalgamated products no such formula exists unless we make strong assumptions on the amalgamation [11]; in fact for any \( n \) there exists an amalgamated product \( G_n = A_n \ast F_2 B_n \) such that rank \( G_n = 2 \) and rank \( A_n, \text{rank } B_n \geq n \) [10]. Of course \( F_2 \), the free group of rank 2, is a very large group. Thus it makes sense to ask whether the situation is better if the amalgam is a small group. In this note we give a positive answer to this question in the case that the amalgam is finite.

For a finite group \( C \) denote by \( l(C) \) the length of the longest strictly ascending chain of non-trivial subgroups. Thus \( l(C) = 1 \) iff \( C \) is a cyclic group of prime order. We show the following:

**Theorem** Suppose that \( G = A \ast_C B \) with \( C \) finite. Then

\[ \text{rank } G \geq \frac{1}{2l(C) + 1} (\text{rank } A + \text{rank } B). \]

The theorem also holds for amalgamated products with infinite amalgam provided that there is an upper bound for the length of any strictly ascending chain of subgroups of \( C \). Such groups must be torsion groups; examples are the Tarski monsters exhibited by A. Yu Ol’shanskii [7].

The proof is related to the proof of Dunwoody’s version [3] of Linnell’s [6] accessibility result for finitely generated groups. However it is more involved as we not only keep track of the size of edge groups but also of the (relative) rank of vertex groups.

A careful analysis of the proof suggests that a better bound might be possible namely that \( \frac{1}{2l(C) + 1} \) can be replaced by \( \frac{1}{2l(C) + 1} \). The proof applies a folding sequence that terminates with the graph of groups corresponding to the amalgamated product. A similar formula for HNN-extensions can be found in the same way.
2 Approximating graphs of groups

In this section we briefly fix notations for graphs of groups and describe how a graph of groups can be approximated by a sequence of graphs of groups. What we describe follows immediately from the discussion of $\mathcal{A}$-graphs in [5] which essentially is an alternative formulation of the combination of foldings and vertex morphisms discussed in [3] where M. Dunwoody builds on earlier work of J. Stallings [8], [9] and M. Bestvina and M. Feighn [1]. We will not describe the whole theory but only state some consequences. A trusting reader should find this sufficient to follow the arguments later on.

A graph $A$ consists of a set of vertices $V_A$, a set of edges $E_A$, an inversion $-1 : E_A \to E_A$ and maps $\alpha : E_A \to V_A$ and $\omega : E_A \to V_A$ such that $\alpha(e) = \omega(e^{-1})$ for all $e \in E_A$. We denote the Betti number of $A$ by $b(A)$.

Recall that the Betti number of $A$ is the number of edge pairs outside a maximal subtree of $A$.

A graph of groups $\mathcal{A}$ consists of a graph $A$, vertex groups $G_v$ for every $v \in V_A$, edge groups $G_e$ for every $e \in E_A$ such that $G_e = G_{e^{-1}}$ and boundary monomorphisms $\alpha_e : G_e \to G_{\alpha(e)}$ and $\omega_e : G_e \to G_{\omega(e)}$ satisfying $\alpha_e = \omega^{-1}_e$; thus only the maps $\alpha_e$ need to be specified.

An approximation of a graph of groups $\mathcal{A}$ is a pair $(\bar{A}, \bar{p})$ where $\bar{A}$ is a graph of groups with underlying graph $\bar{A}$ and $\bar{p} : \bar{A} \to A$ is graph morphism such that $G_v \subset G_{\bar{p}(v)}$ for all $v \in V_A$ and $G_e \subset G_{\bar{p}(e)}$ for all $e \in E_A$.

Suppose now that $A$ is a graph of groups with the following properties

1. $A$ is a finite graph without loop edges, i.e. $\alpha(e) \neq \omega(e)$ for all $e \in E_A$.
2. $n := \text{rank } \pi_1(A) < \infty$.
3. At least one vertex group is non-trivial and all edge groups are Noetherian.

It then follows immediately from the discussion in [5] that there exists a sequence of approximations

$$(A_0, p_0), (A_1, p_1), \ldots, (A_m, p_m)$$

of $\mathcal{A}$ (where $A_i$ has underlying graph $A_i$) such that the following hold:

1. $A_0$ is a subdivision of the wedge of $n-1$ circles with base vertex $v_0$. All vertex and edge groups are trivial except $G_{v_0}$ which is a cyclic group.
2. $p_m$ is a graph isomorphism, $G_v = G_{p_m(v)}$ for all $v \in VA_m$ and $G_e = G_{p_m(e)}$ for all $e \in EA_m$.

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1Note that we make the assumption that $A$ has no loop edges as we would otherwise need to consider 6 moves instead of 3. The assumption that edge groups are Noetherian guarantees that the folding process can be done in such a way that all possible folds of type II are applied before any fold of type I or III. As a consequence folds of type I and III only identify edges with the same edge group. Finally the fact that at least one vertex group is non-trivial guarantees that any generating set of $\pi_1(A)$ is Nielsen equivalent to a another generating set containing an element from one of the vertex groups. This guarantees the existence of the first approximation in the sequence of approximations.
3. The approximation $(A_i, p_i)$ is obtained from $(A_{i-1}, p_{i-1})$ by on the three moves described below for $1 \leq i \leq m$. Furthermore there exists a surjective graph morphism $f_i : A_{i-1} \to A_i$ such that $p_{i-1} = p_i \circ f_i$.

In the following we will use the superscript $i$ to indicate that the object in question belongs to the graph of groups $A_i$.

A move of type I: $x, y, z \in VA_{i-1}$ are distinct vertices with $p_{i-1}(y) = p_{i-1}(z)$ and $e_1, e_2 \in EA_{i-1}$ are edges such that $\alpha(e_1) = \alpha(e_2) = x$, $\omega(e_1) = y$, and $\omega(e_2) = z$. Furthermore $\alpha_e^{-1}(G_{i-1}^e)$ and $\alpha_{e_2}^{-1}(G_{i-1}^{e_2})$ are conjugate in $G_{i-1}^x$.

A move is obtained from $A_{i-1}$ by identifying $e_1$ with $e_2$ and $y$ with $z$ and $f_i : A_{i-1} \to A_i$ is the quotient map. Put $[y] = f_i(y) = f_i(z)$ and $[e_1] = f_i(e_1) = f_i(e_2)$. Thus we have $VA_i = (VA_{i-1} - \{y, z\}) \cup \{[y]\}$ and $EA_i = (EA_{i-1} - \{e_1, e_2\}) \cup \{[e_1]\}$.

The vertex and edge groups of $A_i$ are as follows: $G_{[y]}^i = \langle G_{y}^{i-1}, hG_{z}^{i-1}h^{-1} \rangle$ for some $h \in G_{p_{i-1}(y)}^{i-1}$, $G_{[e]}^i = G_{e_1}^{i-1}$, $G_e^i = G_{e_2}^{i-1}$ for all $v \neq y, z$, and $G_{f}^i = G_{f}^{i-1}$ for all $f \neq e_1, e_2$.

The boundary monomorphisms are as follows: $\alpha_{[e]}^i = \alpha_{e_1}^{-1}, \alpha_{[e]}^{-1} = \omega_{[e]}^i = \omega_{e_1}^{-1}$. For all edges $f$ with $\alpha(f) = z$ we define $\alpha_f^i : G_f^i \to G_{[y]}^i$ as $\alpha_f^i(g) = h\alpha_f^{-1}(g)h^{-1}$. For all remaining edges $e$ we put $\alpha_e^i = \omega_e^{-1}$. Furthermore $U := \omega_{[e_1]}^i(G_{[e_1]}^i) \subset hG_{z}^{i-1}h^{-1}$ and $U$ is in $hG_{z}^{i-1}h^{-1}$ conjugate to $h(\omega_{e_1}^{-1}(G_{e_2}^{i-1})h^{-1}$.

A move of type II: There exists an edge $e \in A_{i-1}$ with $\alpha(e) = x$ and $\omega(e) = y$.

We have $A_i = A_{i-1}$, $f_i$ is the identity and $p_{i+1} = p_i$.

We have $G_e^i = \langle G_e^{i-1}, h' \rangle$ for some element $h' \in G_{p_{i-1}(e)}^{i-1}$ and $G_f^i = G_f^{i-1}$ for all other edges. Furthermore $G_y^i = \langle G_y^{i-1}, h \rangle$ for some $h \in G_{p_{i-1}(y)}^{i-1}$ and $G_e^i = G_e^{i-1}$ for all other vertices.

For all $f \neq e, e^{-1}$ we have $\alpha_f^i = \alpha_f^{i-1}$ and $\alpha_e^i$ and $\omega_e^i = \alpha_e^{i-1}$ are such that the restriction of $\alpha_e^i$ to $G_{x}^{i-1}$ is $\alpha_e^{i-1}$, the restriction of $\omega_e^i$ to $G_{y}^{i-1}$ is $\omega_e^{i-1}$ and $\omega_e^i$ maps $h'$ to $h$.

A move of type III: In this move there exist edges $e_1, e_2 \in EA_{i-1}$ such that $\alpha(e_1) = \alpha(e_2) = x$ and $\omega(e_1) = \omega(e_2) = y$. $A_i$ is obtained from $A_i$ by identifying $e_1$ with $e_2$ and $f_i$ is the quotient map. Denote $f_i(e_1) = f_i(e_2)$ by $[e_1]$. Thus we
have \( VA_i = VA_{i-1} \) and \( EA_i = (EA_{i-1} - \{e_1^{\pm 1}, e_2^{\pm 1}\}) \cup \{[e_1]^{\pm 1}\} \). The map \( p_i \) is simply the restriction of \( p_{i-1} \) to \( A_i \).

We have \( G_y^i = (G_y^{i-1}, h) \) for some \( h \in G_{p_i(y)} \) satisfying \( h\omega_{[e_1]}(G_{x_1}^{i-1})h^{-1} = \omega_{[e_2]}(G_{x_2}^{i-1}) \) and \( G_v^i = G_v^{i-1} \) for all other vertices. Also \( G_{[e_1]}^i = G_{[e_1]}^{i-1} \) and \( G_f^i = G_f^{i-1} \) for all other edges.

We further have \( \alpha_{[e_1]}^i = \alpha_{e_1}^{i-1}, \omega_{[e_1]}^i = \omega_{e_1}^{i-1} = \omega_{[e_1]}^{i-1} \) and \( \alpha_f^i = \alpha_f^{i-1} \) for all \( f \in EA_i - \{[e_1], [e_1]^{-1}\} \).

\[ \begin{array}{c}
G_{x_1}^{i-1} \\
\downarrow \\
G_x^i \\
\downarrow \\
G_y^i \\
\downarrow \\
G_{x_2}^{i-1}
\end{array} \quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array} \quad \begin{array}{c}
G_{[e_1]}^i = G_{[e_1]}^{i-1} \\
\downarrow \\
G_x^i = G_x^{i-1} \\
\downarrow \\
G_y^i = (G_y^{i-1}, h)
\end{array} \]

3 Generation complexities

Let \( A \) be a graph of groups. Let \( \mathcal{P}(EA) \) be the power set of \( EA \). Suppose we have functions

\[ r : VA \to \mathbb{N}_0, \quad p : VA \to \mathbb{N}_0 \text{ and } E : VA \to \mathcal{P}(EA) \]

such that the following hold:

1. \( E(v) \subset EA \) is a set of cardinality \( p(v) \) with \( \alpha(e) = v \) for all \( e \in E(v) \).

2. There exist elements \( g_1, \ldots, g_{r(v)} \in G_v \) and elements \( h_e \in G_v \) for each \( e \in E(v) \) such that

\[ G_v = \langle \{g_1, \ldots, g_{r(v)}\} \cup \{h_e \alpha_{e_1}(G_e)h_e^{-1} \mid e \in E(v)\} \rangle \, . \]

Put \( w = (r, p) : VA \to \mathbb{N}_0 \times \mathbb{N}_0 \). We call the pair \( (w, E) \) a generation complexity\(^2\) and

\[ \bar{w}(v) = r(v) + \max(0, p(x) - 1) \text{ the weight of } v. \]

It is clear that \( G_v = 1 \) for any vertex \( v \) with \( w(v) = (0, 0) \) and that any vertex \( v \) with \( w(v) = (0, 1) \) is inessential, i.e. there exists an edge \( e \) with \( \alpha(e) = v \) such that \( \alpha_e \) is surjective. However the opposite is not necessarily true as we make no minimality assumption. In particular no essential vertex lies in the set

\[ S = S(A, w) := \{ v \in VA \mid w(v) = (0, 0) \text{ or } w(v) = (0, 1) \} \, . \]

\(^2\)The function \( w_G : VA \to \mathbb{N}_0 \times \mathbb{N}_0 \) defined as \( w_G(v) = (\text{rank } G_v, 0) \) clearly is a generation complexity (with \( E(v) = \emptyset \) for all \( v \in VA \). It is this function that generation complexities in general attempt to generalize by also taking into account elements that have been pushed through edge groups into a vertex group. Note that the generation complexity \( w_G \) would be the right one to prove Grushko’s theorem.
4 The proof of the theorem

Suppose now that $G = A \ast_C B$ with finite $C$. Thus $G = \pi_C(A)$ where $EA = \{e, e^{-1}\}$, $VA = \{v, w\}$ with $\alpha(e) = v$ and $\omega(e) = w$, $G_v = A$, $G_w = B$ and $G_e = C$. Clearly $A$ satisfies the conditions spelt out in Section 2. Thus we can find a sequence of approximations $(h_0, p_0), (h_m, p_m)$ and maps $f_1, \ldots, f_m$ as described in Section 2. Put $l := l(C)$.

Define $w_0 : A_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ as $w_0(v_0) = (1, 0)$ and $w_0(v) = (0, 0)$ for $v \neq v_0$ and $E_0 : VA_0 \rightarrow \mathcal{P}(EA_0)$ as $E_0(v) = \emptyset$ for all $v \in VA_0$. Clearly the pair $(w_0, E_0)$ is a generation complexity for $A_0$. The proof of the theorem follows from the following:

Proposition Let $1 \leq i \leq m$. Suppose that $(w_{i-1}, E_{i-1})$ is a generation complexity for $A_{i-1}$. Then there exists a generation complexity $(w_i, E_i)$ for $A_i$ such that
\[ c(A_i, (w_i, E_i)) \leq c(A_{i-1}, (w_{i-1}, E_{i-1})) \]
where the complexity $c(A_j, (w_j, E_j))$ of the pair $(A_j, (w_j, E_j))$ is defined as
\[ b(A_j) + \sum_{e \in E A_j} (2^{l-i(G^e_j)+1} - 1) - \sum_{v \in S_j} (2^{l-i(G^e_j)+1} - 1) + \sum_{v \in V A_j} \bar{w}_j(v). \]
with $S_j = S(A_j, w_j)$ and $\bar{w}_j$ the weight function corresponding to $w_j$.

Before we give a proof of the Proposition we show that it implies the Theorem:

Proof of the Theorem: It clearly follows from the proposition that there exists a generation complexity $(w_m, E_m)$ for $A_m$ such that
\[ c(A_m, (w_m, E_m)) \leq c(A_0, (w_0, E_0)). \] (1)
It turns out that the Theorem follows immediately from this inequality once we have computed $c(A_0, (w_0, E_0))$ and $c(A_m, (w_m, E_m))$.

Claim 1: $c(A_0, (w_0, E_0)) = (n-1)2^{l+1} + 1$.
Proof of Claim 1: Clearly $b(A_0) = n - 1$ and $\# EA_0 - \# S_0 = n - 1$. It follows that
\[ \sum_{e \in E A_0} (2^{l-i(G^e_0)+1} - 1) - \sum_{v \in S_0} (2^{l-i(G^e_0)+1} - 1) = (n - 1)(2^{l+1} - 1) \]
as all occuring edge and vertex groups are trivial and $l(1) = 0$. The last summand yields $1$ as $\bar{w}_0(v) = 1$ and $\bar{w}_0(v) = 0$ for $v \neq v_0$. Thus we have $c(A_0, (w_0, E_0)) = (n - 1) + (n - 1)(2^{l+1} - 1) + 1 = (n - 1)2^{l+1} + 1$. \qed

Claim 2: $c(A_m, (w_m, E_m)) \geq \text{rank } A + \text{rank } B - 2 \text{rank } C + 1$.
Proof of Claim 2: We have $b(A_m) = 0$. Clearly $v, w \in VA_m$ are essential; thus $S_m = \emptyset$. We have \[ \sum_{v \in S_m} (2^{l-i(G^v_m)+1} - 1) = 1 \] as the only edge group is $C$ itself. Furthermore \[ \sum_{v \in S_m} 2^{l-i(G^v_m)} = 0 \] as $S_m = \emptyset$. As $v$ and $w$ are vertices of valence $1$ we have $p_m(v), p_m(w) \leq 1$ and it is clear that $\bar{w}_m(v) = r_m(v) \geq \text{rank } A - \text{rank } C$ as at least $\text{rank } A - \text{rank } C$ elements of $A$ needed to generate $A$ together with a homomorphic image of $C$; the same argument shows that $\bar{w}(w) \geq \text{rank } B - \text{rank } C$. This implies that $c(A_m, (w_m, E_m)) \geq 1 + (\text{rank } A - \text{rank } C) + (\text{rank } B - \text{rank } C) = \text{rank } A + \text{rank } B - 2 \text{rank } C + 1$. \qed
The claims together with (1) gives \((n - 1)2^{l+1} + 1 \geq \text{rank } A + \text{rank } B - 2\text{rank } C + 1\) which implies \(n \geq \frac{A + \text{rank } B - 2\text{rank } C + 2}{2}\). As \(C \leq l(C)\) this yields \(\text{rank } G = n \geq \frac{1}{2}\text{rank } A + \text{rank } B - 2l + 2^{l+1}\). The assertion of the theorem follows as \(-2l + 2^{l+1} \geq 0\) for \(l \geq 0\).

**Proof of the Proposition:** We will distinguish a variety of cases. In each situation we will define the pair \((w_i, E_i)\) and show that it is a generating complexity. We distinguish the cases by the type of move that is applied to \((A_{i-1}, p_{i-1})\) to obtain \((A_i, p_i)\). Recall that \(S_j := \{v \in V A_j | w_j(v) = (0, 0)\text{ or } w_j(v) = (0, 1)\}\) and that \(w_j(v) = (r_j(v), p_j(v))\).

Before we deal with the different cases we explain how to define the maps \(w_i\) and \(E_i\) for a vertex that is not affected by the move, i.e. a vertex different from \(x, y\) and \(z\) if the move is of type I and different from \(x\) and \(y\) if the move is of type II or III. For such a vertex we put \(w_i(v) := w_i-1(v)\) and \(E_i(v) := E_i-1(v)\). It is obvious that this is permitted by the definition of the generation complexity as around such vertices the graph of groups is unchanged. It follows in particular that for such a vertex \(v \in S_i\) iff \(v \in S_{i-1}\).

For simplicity we will write

\[
c_j = c(A_j, (w_j, E_j)) = b(A_j) + \Sigma_1 + \Sigma_2 + \Sigma_3
\]

where

\[
\Sigma_1 = \sum_{e \in E A_j} (2^{l(G_i) + 1} - 1), \quad \Sigma_2 = -\sum_{v \in S_j} (2^{l(G_i) + 1} - 1) \quad \text{and} \quad \Sigma_3 = \sum_{v \in V A_j} \tilde{w}_j(v).
\]

Note that for moves of type I or II we need to show that \(\Sigma_1 + \Sigma_2 + \Sigma_3 \leq \Sigma_1^{(i-1)} + \Sigma_2^{(i-1)} + \Sigma_3^{(i-1)}\) as \(b(A_i) = b(A_{i-1})\).

**Moves of type I:** We need to explain \(w_i\) and \(E_i\) for the vertices \(x\) and \(y\).

We first deal with the vertex \(x\): If \(x \in S_{i-1}\), i.e. if \(w_i(x) = (0, 0)\text{ or } w_i(x) = (0, 1)\) we can clearly put \(w_i(x) = w_{i-1}(x)\) and \(E_i(x) = f_i(E_{i-1}(x))\). If \(x \notin S_{i-1}\) and \(\{e_1, e_2\} \not\subset E_{i-1}(x)\) we can again put \(w_i(x) = w_{i-1}(x)\) and \(E_i(x) = f_i(E_{i-1}(x))\).

If \(x \notin S_{i-1}\) and \(\{e_1, e_2\} \subset E_{i-1}(x)\) we have to work a little harder. Again we put \(E_i(x) = f_i(E_{i-1}(x))\) but contrary to the above case we have that the cardinality of \(E_i(x)\) is by one smaller than the cardinality of \(E_{i-1}(x)\), thus we have one less edge group to generate the vertex group. As \(\alpha_{e_1} G_{e_1}^{(i)}\) and \(\alpha_{e_2} G_{e_2}^{(i)}\) where in \(G_x\) conjugate, it is possible to make up for this missing edge group by adding one element. Thus we can put \(w_i(x) = (r_{i-1}(x) + 1, p_{i-1}(x) - 1)\).

Note that our discussion implies that \(\tilde{w}_i(x) = \tilde{w}_{i-1}(x)\) and that \(x \in S_{i-1}\) iff \(x \in S_i\). As further \(G_x^{(i-1)} = G_x^{(i)}\) it follows that the contribution of \(x\) to the complexity is unchanged.

We next deal with the vertex \(y\). We distinguish three situations:

**Case 1:** Suppose that \(y \in S_{i-1}\) and \(y \in S_{i-1}\), i.e. that \(r_{i-1}(y) = r_{i-1}(z) = 0\) and that \(E_{i-1}(y)\) and \(E_{i-1}(z)\) contain at most one edge. We distinguish three subcases:

(a) Suppose that \(E_{i-1}(y) = \{e_1^{-1}\}\) or \(E_{i-1}(y) = \emptyset\) (the case \(E_{i-1}(z) = \{e_2^{-1}\}\) or \(E_{i-1}(z) = \emptyset\) is analogous). In this case we put \(w_i([y]) = (0, 1)\) and \(E_i([y]) = f_i(E_{i-1}(z))\). Note that this implies that \([y] \in S_i\). We clearly have \(\Sigma_1 = \Sigma_1^{(i)} - (2^{l(G_x^{(i-1)}) + 1} - 1), \quad \Sigma_2 = \Sigma_2^{(i)} + (2^{l(G_x^{(i)}) + 1} - 1) \quad \text{and} \quad \Sigma_3 = \Sigma_3^{(i)}\). As \(G_x^{(i-1)} \cong\)
$G_{c}^{-1}$ it follows that $2^{l(G_{c}^{-1})+1} - 1 = 2^{l(G_{y}^{-1})+1} - 1$. This implies that \( \Sigma_{1} + \Sigma_{2} + \Sigma_{3} = \Sigma_{1}^{-1} + \Sigma_{2}^{-1} + \Sigma_{3}^{-1} \).

(b) Suppose now that \( e_{2}^{-1} \notin \{ e_{3} \} = E_{1}-1(y) \), \( e_{2}^{-1} \notin \{ e_{4} \} = E_{1}-1(z) \) and that \( \omega_{e_{2}}^{-1} \) is surjective (the case that \( \omega_{e_{2}}^{-1} \) is surjective is analogous). We put \( w_{1}([y]) = (0,1) \) and \( E_{i}([y]) = \{ e_{3} \} \). As in (a) we get \( \Sigma_{1} + \Sigma_{2} + \Sigma_{3} = \Sigma_{1}^{-1} + \Sigma_{2}^{-1} + \Sigma_{3}^{-1} \).

(c) Suppose now that \( e_{1}^{-1} \notin \{ e_{3} \} = E_{1}-1(y) \), \( e_{1}^{-1} \notin \{ e_{4} \} = E_{1}-1(z) \) and that neither \( \omega_{e_{1}}^{-1} \) nor \( \omega_{e_{2}}^{-1} \) is surjective. Put \( w_{1}([y]) = (0,2) \) and \( E_{i}([y]) = \{ e_{3}, e_{4} \} \).

We have \( \Sigma_{1} = \Sigma_{1}^{-1} - (2^{l(G_{e_{1}}^{-1})+1}) - 1), \Sigma_{2} = \Sigma_{2}^{-1} + (2^{l(G_{y}^{-1})+1}) - 1) + (2^{l(G_{c}^{-1})+1}) - 1) \) and \( \Sigma_{1} = \Sigma_{1}^{-1} + 1 \).

The non-surjectivity of \( \omega_{e_{1}}^{-1} \) and \( \omega_{e_{2}}^{-1} \) imply that \( l(G_{e_{1}}^{-1}) > l(G_{c}^{-1}) \) and \( l(G_{e_{1}}^{-1}) > l(G_{c}^{-1}) \) and therefore \( (2^{l(G_{c}^{-1})+1}) - 1) \geq \Sigma_{1} + \Sigma_{2} + \Sigma_{3} \leq \Sigma_{1}^{-1} + \Sigma_{2}^{-1} + \Sigma_{3}^{-1} \).

**Case 2:** Suppose now that \( y \notin S_{1}-1 \) and that \( z \in S_{1}-1 \); the opposite case is analogous. If \( w_{1}([z]) = (0,0) \) we can argue as in case (a) above; thus we can assume that \( w_{1}([z]) = (0,1) \). Choose \( e_{3} \in E_{1}-1(z) \) such that \( E_{1}-1(z) = \{ e_{3} \} \).

We distinguish three subcases:

(a) Suppose that \( e_{2}^{-1} = e_{3} \). We put \( w_{1}([y]) = w_{1}([z]) \) and \( E_{i}([y]) = f_{i}(E_{i}-1(y)) \). As in (a) of case 1 it follows that \( \Sigma_{1} + \Sigma_{2} + \Sigma_{3} = \Sigma_{1}^{-1} + \Sigma_{2}^{-1} + \Sigma_{3}^{-1} \).

(b) Suppose that \( e_{2}^{-1} \neq e_{3} \) and that \( \omega_{e_{2}}^{-1} \) is surjective. We put \( w_{1}([y]) = w_{1}([z]) \) and \( E_{i}([y]) = f_{i}(E_{i}-1(y)) \). Again we get \( \Sigma_{1} + \Sigma_{2} + \Sigma_{3} = \Sigma_{1}^{-1} + \Sigma_{2}^{-1} + \Sigma_{3}^{-1} \).

(c) Suppose that \( e_{2}^{-1} \neq e_{3} \) and that \( \omega_{e_{2}}^{-1} \) is not surjective. We put \( w_{1}([y]) = (r_{1}-1(y), p_{1}-1(y) + 1) \) and \( E_{i}([y]) = f_{i}(E_{i}-1(y)) \cup \{ e_{3} \} \). We have \( \Sigma_{1} = \Sigma_{1}^{-1} - (2^{l(G_{e_{1}}^{-1})+1}) - 1), \Sigma_{2} = \Sigma_{2}^{-1} + (2^{l(G_{y}^{-1})+1}) - 1) \) and \( \Sigma_{3} = \Sigma_{3}^{-1} + 1 \). The non-surjectivity of \( \omega_{e_{2}}^{-1} \) implies that \( l(G_{e_{2}}^{-1}) < l(G_{y}^{-1}) \) and therefore \( 2^{l(G_{y}^{-1})+1} - 1) \geq \Sigma_{1} + \Sigma_{2} + \Sigma_{3} \leq \Sigma_{1}^{-1} + \Sigma_{2}^{-1} + \Sigma_{3}^{-1} \).

**Case 3:** Finally suppose that \( y, z \notin S_{1}-1 \). We put \( E_{i}([y]) = f_{i}(E_{i}-1(y) \cup E_{i}-1(z)) \) and \( w_{1}([y]) = (r_{1}-1(y), r_{1}-1(z), p_{1}([y]) \) with \( p_{1}([y]) = \#E_{i}([y]) \). Clearly \( p_{1}([y]) \leq p_{1}-1(y) + p_{1}-1(z) \) and therefore \( \Sigma_{1} \leq \Sigma_{1}^{-1} + 1 \). As further \( \Sigma_{1} < \Sigma_{1}^{-1} \) and \( \Sigma_{2} = \Sigma_{2}^{-1} \) it follows that \( \Sigma_{1} + \Sigma_{2} + \Sigma_{3} \leq \Sigma_{1}^{-1} + \Sigma_{2}^{-1} + \Sigma_{3}^{-1} \).

**Moves of type II:** Again dealing with the vertex \( x \) is simple as we can simply put \( w_{1}([x]) = w_{1}([x]) \) and \( E_{i}([x]) = E_{i}-1(x) \). The contribution of \( x \) to the complexity does not change.

**Case 1:** Suppose that \( y \notin S_{1}-1 \).

(a) Suppose that \( e^{-1} \in E_{1}-1(y) \) or that \( E_{1}-1(y) = \emptyset \). We can put \( w_{1}([y]) = (0,1) \) and \( E_{i}([y]) = \{ e^{-1} \} \). We have \( \Sigma_{1} = \Sigma_{1}^{-1} - (2^{l(G_{e}^{-1})+1}) - 1), \Sigma_{2} = \Sigma_{2}^{-1} + (2^{l(G_{e}^{-1})+1}) - 1) - (2^{l(G_{y}^{-1})+1}) - 1) \) and \( \Sigma_{3} = \Sigma_{3}^{-1} \). As \( G_{e} \cong G_{y}^{-1} \) and \( G_{e} \cong G_{y} \) this implies that \( \Sigma_{1} + \Sigma_{2} + \Sigma_{3} = \Sigma_{1}^{-1} + \Sigma_{2}^{-1} + \Sigma_{3}^{-1} \).

(b) Suppose now that \( e^{-1} \notin \{ e_{3} \} \) and that \( \omega_{e}^{-1} \) is not surjective. We put \( w_{1}([y]) = (0,2) \) and \( E_{i}([y]) = \{ e^{-1}, e_{3} \} \). We have \( \Sigma_{1} = \Sigma_{1}^{-1} - (2^{l(G_{e})+1}) - 1), \Sigma_{2} = \Sigma_{2}^{-1} + (2^{l(G_{e})+1}) - 1) \) and \( \Sigma_{3} = \Sigma_{3}^{-1} + 1 \). As \( l(G_{e}) > l(G_{e}^{-1}) \) and \( l(G_{y}^{-1}) > l(G_{e}^{-1}) \) this implies that \( \Sigma_{1} + \Sigma_{2} + \Sigma_{3} \leq \Sigma_{1}^{-1} + \Sigma_{2}^{-1} + \Sigma_{3}^{-1} \).
(c) Suppose now that \( e^{-1} \not\in \{e_3\} = E_{i-1}(y) \) and that \( \omega_{i-1}^y \) is surjective. We put \( w_i(y) = (0,1) \) and \( E_i(y) = \{ e^{-1} \} \). The same calculation as in (a) shows that \( \Sigma_1^i + \Sigma_2^i + \Sigma_3^i = \Sigma_1^{i-1} + \Sigma_2^{i-1} + \Sigma_3^{i-1} \).

**Case 2:** Suppose that \( y \notin S_{i-1} \). We clearly have \( \Sigma_1^i < \Sigma_1^{i-1} \) and \( \Sigma_2^i = \Sigma_2^{i-1} \).

(a) Suppose that \( e^{-1} \in E_{i-1}(y) \). We put \( w_i(y) = w_{i-1}(y) \) and \( E_i(y) = E_{i-1}(y) \). It follows that \( \Sigma_1^i = \Sigma_1^{i-1} \) and therefore \( \Sigma_1^i + \Sigma_2^i + \Sigma_3^i = \Sigma_1^{i-1} + \Sigma_2^{i-1} + \Sigma_3^{i-1} \).

(b) Suppose that \( e^{-1} \not\in E_{i-1}(y) \). We put \( w_i(y) = (r_{i-1}(y), r_{i-1}(y) + 1) \) and \( E_i(y) = E_{i-1}(y) \cup \{ e^{-1} \} \). It follows that \( \Sigma_1^i \leq \Sigma_1^{i-1} + 1 \) and therefore \( \Sigma_1^i + \Sigma_2^i + \Sigma_3^i \leq \Sigma_1^{i-1} + \Sigma_2^{i-1} + \Sigma_3^{i-1} \).

**Moves of type III:** We deal with \( x \) as we did in the case of a move of type I. We only need to prove that \( \Sigma_1^i + \Sigma_2^i + \Sigma_3^i \leq \Sigma_1^{i-1} + \Sigma_2^{i-1} + \Sigma_3^{i-1} + 1 \) as \( b(A_i) = b(A_{i-1}) - 1 \). We put \( E_i(y) = f_i(E_{i-1}(y)) \) and \( w_i(y) = (f_{i-1}(y) + 1, p_i(y)) \) with \( p_i(y) = \# E_i(y) = \# E_{i-1}(y) = p_{i-1}(y) \). In particular we have \( y \notin S_i \).

We have \( \Sigma_1^i = \Sigma_1^{i-1} - (2^{l(G_i^y)} - 1) \), \( \Sigma_2^i = \Sigma_2^{i-1} \) if \( y \notin S_{i-1} \), \( \Sigma_3^i = \Sigma_3^{i-1} + (2^{l(G_i^y)} - 1) \) if \( y \in S_{i-1} \) and \( \Sigma_1^i \leq \Sigma_1^{i-1} + 1 \). As \( l(G_i^y) \geq l(G_{i-1}^y) \) this implies that \( \Sigma_1^i + \Sigma_2^i + \Sigma_3^i \leq \Sigma_1^{i-1} + \Sigma_2^{i-1} + \Sigma_3^{i-1} + 1 \).

\[ \square \]

**References**


