A Grushko theorem for 1-acylindrical splittings

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Abstract

In [10] we gave a bound of the complexity of a 1-acylindrical splitting of a group in terms of the rank of the group, we showed that rank $G \geq N$ if $G$ can be written as a 1-acylindrical splitting with $N$ vertex groups. This result however is not satisfying in the sense that it does not take into account the size of the vertex groups the way Grushko’s Theorem does for free products. In this paper we will prove such a rank formula for amalgamated products with malnormal amalgam. Analogous results hold for 1-acylindrical splittings in general.

Introduction

Grushko’s theorem states that the rank of groups is additive under free products, i.e. that

$$\operatorname{rank} \bigoplus_{i=1}^{n} G_i = \sum_{i=1}^{n} \operatorname{rank} G_i.$$ 

Whether a similar formula holds for amalgamated products $G = A \ast_C B$ has been investigated in the 1970’s and for example asked in [7], but no satisfying answer has been given. G. Rosenberger [8] remarked that for a class of Fuchsian groups the naïve formula $\operatorname{rank} G \geq \operatorname{rank} A + \operatorname{rank} B - \operatorname{rank} C$ fails. Later R. Kaufmann and H. Zieschang [6] exhibit a family of Coxeter group $G_n = A_n \ast_{\mathbb{Z}_2} B_n$ where $\operatorname{rank} A_n = \operatorname{rank} B_n = \operatorname{rank} G_n$ for arbitrary $n$. In [10] a family of two-generated groups $G_n = A_n \ast_C B_n$ has been constructed with $\operatorname{rank} G_n = 2$, $C$ the free group of rank 2 and $\operatorname{rank} A_n, \operatorname{rank} B_n \geq n$ and we think that such examples could even be constructed with an infinite cyclic amalgamated subgroup. This clearly shows that for free products with amalgamation $G = A \ast_C B$ there exists no general formula that relates the rank of $G$ with the ranks of $A$, $B$ and $C$. For a given $n \in \mathbb{N}$ it is further easy to construct a two-generated group that can be decomposed into an amalgamated product with $n$ factors, i.e. in general no bound on the complexity of a splitting of a group can be given in terms of its rank. The examples of inaccessible groups given by M. Bestvina and M. Feighn [1] and M. Dunwoody [3] show that even for a given finitely generated group the complexity of a splitting cannot be bounded.

It turns out that in the important case that the amalgam $C$ of an amalgamated product $G = A \ast_C B$ is malnormal, i.e. that $gCg^{-1} \cap C = 1$ for all
$g \in G - C$, the situation is much better. This case occurs naturally in many situations, for example in 3-manifold theory or in the theory of hyperbolic groups. A. Karrass and D. Solitar [5] showed that an amalgamated product $A \ast_C B$ with malnormal $C \neq 1$ cannot be generated by two elements. In [11] this result has been extended to amalgamated products of more than two factors, i.e. it has been shown that rank $\ast_n^G G_i \geq n + 1$ if $C$ is malnormal. In [10] this has been extended to k-acylindrical graphs of groups, strengthening Z. Sela’s result on acylindrical accessibility [9]. Note that the malnormality of the amalgam is equivalent to the amalgamated product being 1-acylindrical. (Following [9] a splitting of a group $G$ is called $k$-acylindrical if no non-trivial element of $G$ fixes a segment of length greater than $k$ in the Bass-Serre tree associated to this splitting.)

These results are however not satisfying in the sense that they do not take into account the rank of the factor groups of the amalgamated product. In this paper we will show that the rank of the factor groups does have an impact on the rank of the whole group if the amalgam is malnormal. We show the following:

**Theorem 1** Let $G = A \ast_C B$ be a proper amalgamated product with malnormal amalgam $C$. Then $\text{rank } G \geq 1/3(\text{rank } A + \text{rank } B - 2\text{rank } C + 5)$.

It should be noted that the ideas used in the proof of Theorem 1 can also be applied in the case of arbitrary 1-acylindrical splittings. Since the proof of the general case is much more technical, we have however opted for restricting ourselves to the case of amalgamated products; one way of handling the technicalities would be to consider *controlled sets* in the sense of [4] instead of good sets (see below). The ideas used in this article also generalize in a straightforward manner to $k$-acylindrical splittings.

This paper is organized as follows: In section 1 we introduce some notations and give some definitions. In section 2 we give the proof of Theorem 1. In the last section we discuss why we believe that the formula given is at least close to being the best possible.

## 1 Preliminaries

Suppose that we have a proper amalgamated product $G = A \ast_C B$ with a malnormal subgroup $C$. We consider the action $G \times T \to T$ of $G$ on the Bass-Serre tree $T$ associated to this splitting. By $d : T \times T \to \mathbb{R}$ we denote the standard metric on $T$, i.e. the path metric obtained by making every edge isometric to the unit interval. Let $e_C$ be the edge fixed under the action of $C$ and $f$ be an arbitrary edge. It follows from the construction of the Bass-Serre tree $T$ all edges are $G$-equivalent and that any element $g$ satisfying $e_C = gf$ must have a reduced form of length $d(e_C, f) + 1$, i.e. it can be written as a product of $d(e_C, f) + 1$ elements of $A \cup B$. It is further clear that for an arbitrary vertex $v \in T$ there exists an element $g$ with reduced form of length $d(e_C, v)$ such that $gv \in e_C$. Conjugation with an appropriate elements gives the following simple but useful fact:
Lemma 2 Suppose that $e = [x, y]$ and $f$ are edges of $T$ and that $v$ is a vertex of $T$. Then

(i) There exists a product $p$ of at most $d(e, f) + 1$ elements of $\text{Stab } x \cup \text{Stab } y$ such that $pf = e$.

(ii) There exists a product $p$ of at most $d(e, v)$ elements of $\text{Stab } x \cup \text{Stab } y$ such that $pv \subseteq e$.

The malnormality of $C$ further guarantees that the action of $G$ on $T$ is 1-acylindrical, i.e. that no non-trivial element fixes a segment of length greater than $1$. To an element $g \in G$ we associate the set $T_g = \{x \in T | gx = x\}$. The set $T_g$ is clearly connected and therefore a subtree of $T$ and the 1-acylindricity of the action guarantees that $T_g$ is either empty or consists of a single vertex or of a single edge. Let further $U < G$ be a subgroup. By $TU$ we denote the minimal $U$-invariant subtree of $T$. We further associate a subtree $T_U$ to $U$ by defining

$$T_U := TU \cup \{x \in T | ux = x \text{ for some } u \in U - 1\}.$$ 

To see that $T_U$ is a tree it suffices to observe that $T_U := TU \cup (\bigcup_{u \in U - 1} T_u)$ and that $TU \cap T_u \neq \emptyset$ if $T_u \neq \emptyset$. The first part is trivial and the second follows from the fact that $TU \cap T_u = \emptyset$ and $T_u \neq \emptyset$ would imply that $TU$ and $TU$ lie in different components of $T - T_u$ which clearly contradicts the $U$-invariance of $TU$. The fact that $T_u$ consists of at most one edge further implies immediately that the distance of any point $x \in T_U$ and $TU$ is at most $1$.

If $S$ is a subset of $G$ and $U = \langle S \rangle$ the subgroup generated by $S$, then we will for simplicity denote the trees $T_U$ and $TU$ by $TS$ and $T_S$ instead of $T_{\langle S \rangle}$ and $T\langle S \rangle$.

In this article we will only be dealing with trees $T_U$ of subgroups $U$ that are generated by subsets of $\text{Stab } x \cup \text{Stab } y$ where $[x, y]$ is some edge of $T$. We call such sets good. If a good set contains only one element we say that it is tiny, if it contains two elements fixing a common vertex we call it small. In the following we have a look at the trees $TU$ and $T_U$ for subgroups of this type.

(1) In the case that $U$ is generated by a single element $g$ of $\text{Stab } x$ (or $\text{Stab } y$), the tree $T_U$ either consists of the single vertex $x$ or a single edge $f = [x, z]$ and $U < \text{Stab } f$. By the remark above it suffices to show that $T_U = T_g$. Assuming that $T_u \neq T_g$ would imply that $T_U$ contains an edge $e = [a, b]$ such that $a$ (or $b$) but not $e$ is fixed under the action of $g$ but that $e$ is fixed under the action of some power $g^k$ of $g$ with $k \geq 2$. This however contradicts the malnormality of $C$ (and therefore every edge-stabilizer) since $ff^k f^{-1} = f^k$.

(2) If $U$ is generated by a subset of $\text{Stab } x$, the minimal $U$-invariant subtree $TU$ is clearly $x$ and it follows that $T_U$ is the union of $x$ and a collection of edges emanating from $x$. In particular $T_U$ is of diameter at most $2$.

(3) The most interesting case is the case where $U$ is generated by a subset $S$ of $\text{Stab } x \cup \text{Stab } y$ that is not contained in $\text{Stab } x$ or $\text{Stab } y$. An easy normal form argument shows that $U = \tilde{A} \ast \tilde{B}$ where $\tilde{C} = U \cap \text{Stab } e$, $\tilde{A} = \langle \tilde{C}, S \cap \text{Stab } x \rangle$ and $\tilde{B} = \langle \tilde{C}, S \cap \text{Stab } y \rangle$. The minimal $U$-invariant subtree $TU$ is the canonical embedding of the Bass-Serre tree of $U$ with respect to the splitting $U = \tilde{A} \ast \tilde{C} \tilde{B}$, which consists of the union of the edges $ue$ with $u \in U$. This implies in particular
that every vertex of the tree $T_U$ is $U$-equivalent to another vertex of $T_U$ that has distance at most 1 from $x$ or $y$.

This last remark implies that after an inner automorphism of $U$ we can assume that $S$ lies in $\text{Stab} \ p \cup \text{Stab} \ q$ where $[p, q]$ is any given edge of $T_U$, i.e. we get the following:

**Lemma 3** Let $S$ be a good set that is not contained in the stabilizer of any vertex of $T$. Suppose that $e = [x, y]$ is an edge of $T_U$. Then there exists an element $u \in \langle S \rangle$ such that $uS u^{-1} \subset \text{Stab} x \cup \text{Stab} y$. [□]

If a good set $S$ is a generating set, the above discussion implies in particular that $(S \cap \text{Stab} \ x) \cup \text{Stab} \ e$ is a generating set of $\text{Stab} \ x$ and that $(S \cap \text{Stab} \ y) \cup \text{Stab} \ e$ is a generating set of $\text{Stab} \ y$. This implies that $S$ contains at least rank $\text{Stab} \ x - \text{rank} \text{Stab} \ e$ elements of $\text{Stab} \ x - \text{Stab} \ e$ and rank $\text{Stab} \ y - \text{rank} \text{Stab} \ e$ elements of $\text{Stab} \ y - \text{Stab} \ e$. By Lemma 3 we can assume that after conjugation in $G$ we can assume that the vertex $x$ is fixed under the action of $A$ and that $y$ is fixed under the action of $B$ (or the other way around). This clearly gives the following which has been independently observed by Inna Bumagina for cyclic amalgams:

**Lemma 4** Suppose that $S$ is a good generating set of $G$. Then
$$\# S \geq \text{rank} A + \text{rank} B - 2 \text{rank} C.$$ [□]

It is clear that we could also state Lemma 4 saying that $\# S \geq \text{rank} A \text{ rel } C + \text{rank} B \text{ rel } C$ if we define $\text{rank} G \text{ rel } U$, the rank of a group $G$ relative to a subgroup $U$, to be the minimal number elements that have to be added to $U$ to generate $G$.

We end this section by recalling some definitions and stating the main result of [10] which is integral to the proof of Theorem 1. We call a finite set $M \subset G$ a **partitioned set** with partition $(S_1, \ldots, S_p, H)$ if

(i) $M = S_1 \cup \cdots \cup S_p \cup H$ and

(ii) $\#H < \infty$.

We further say that two partitioned sets $M$ and $\tilde{M}$ with partitions $(S_1, \ldots, S_p, H)$ and $(\tilde{S}_1, \ldots, \tilde{S}_p, \tilde{H})$ are **equivalent** if the following are fulfilled:

(i) The underlying sets $M$ and $\tilde{M}$ are Nielsen equivalent.

(ii) $\tilde{S}_i = g_i S_i g_i^{-1}$ for some $g_i \in G$ for $1 \leq i \leq p$.

(iii) $\# H = \# H$.

The main result of [10] is the following:

**Theorem 5** Let $M$ be a set with partition $(S_1, \ldots, S_p, H)$. Then either

$$\langle M \rangle = \langle S_1 \rangle * \cdots * \langle S_p \rangle * F(H)$$

and the induced splitting of $(M)$ contains trivial edge stabilizers unless $p = 1$ and $H = \emptyset$ or $(S_1, \ldots, S_p, H) \sim (\tilde{S}_1, \ldots, \tilde{S}_p, \tilde{H})$ such that one of the following holds:

1. $T_{\tilde{S}_i} \cap T_{\tilde{S}_j} \neq \emptyset$ for some $i, j \in \{1, \ldots, p\}$ with $i \neq j$. [□]
(2) \( hT_{S_i} \cap T_{S_i} \neq \emptyset \) for some \( i \in \{1, \ldots, p\} \) and \( h \in \tilde{H} \).
(3) There exists \( \tilde{h} \in \tilde{H} \) such that \( \tilde{h} \) acts with a fixed point.

2 The proof of Theorem 1

Throughout this section the sets \( S_i \) in all partitions are assumed to be good in the sense specified above. The strategy of the proof is to start with a minimal (partitioned) generating set \( M \) with the trivial partition \( (H = M) \) of \( G \). Since the induced splitting of \( G = \langle M \rangle \) is clearly the original splitting of \( G \) and therefore has no trivial edge stabilizers, \( \langle M \rangle = \langle S_1 \rangle \cdots \langle S_p \rangle \ast F(H) \) can only hold if \( M \) has the partition \( (S_1 = M, H = \emptyset) \). We repeatedly apply Theorem 5 and change the partitioned generating set unless we are in this situation. We assign to every partitioned set a signature and a potential. At every step the set \( M \) is replaced by another partitioned set \( \tilde{M} \) such that \( \langle M \rangle \subset \langle \tilde{M} \rangle \), that the signature of \( \tilde{M} \) is smaller than the signature of \( M \) and that the potential of \( \tilde{M} \) is less or equal than the potential of \( M \). Since the ordering on the signatures is a well-ordering, this process terminates. Theorem 1 now follows from the fact that the potential of a set with partition \( (H = M) \) is defined to be \( 3\#H - 5 \) and of a set with partition \( (S_1 = M, H = \emptyset) \) is defined to be \( \#S_1 \) together with Lemma 4.

As in [10] we define the signature of a partition \( (S_1, \ldots, S_p, H) \) to be the pair \( (p, \#H) \in \mathbb{N} \times \mathbb{N} \). We further define an order on \( \mathbb{N} \times \mathbb{N} \) by saying that

\[
(n, m) \leq (n', m') \text{ if}
\]

(i) \( m < m' \)
(ii) \( m = m' \) and \( n \leq n' \).

This clearly gives a well-ordering on \( \mathbb{N} \times \mathbb{N} \), i.e. every subset of \( \mathbb{N} \times \mathbb{N} \) has a minimal element.

The definition of the potential of a partition \( (S_1, \ldots, S_p, H) \) where all the \( S_i \) are good is a little more cumbersome. Suppose now that we know of \( p_1 \) of the \( p \) good sets \( S_i \) that are tiny and that we know of \( p_2 \) of the remaining sets \( S_i \) that they are small. We further define \( \Sigma \) to be the sum of the cardinalities of the remaining \( p_3 := p - p_1 - p_2 \) sets \( S_i \). We define the potential \( p(M) \) of a set \( M \) with partition \( (S_1, \ldots, S_p, H) \) as

(i) \( 3(\#H + p_1) - 5 \) if \( p_2 = p_3 = 0 \)
(ii) \( 4p_2 + 3(\#H + p_1) - 3 \) if \( p_1 \neq p \) and \( p_3 = 0 \)
(iii) \( \Sigma + 4p_2 + 3(\#H + p_1 + p_3 - 1) \) if \( p_3 \neq 0 \).

For a set \( M \) with partition \( (H = M) \) this implies that \( p(M) = 3\#M - 5 \) and for a set with partition \( (S_1 = M) \) where \( S_1 \) does not lie in the stabilizer of a single vertex this means that \( p(M) = \#M = \#S_1 = \Sigma \).

The key to the proof of Theorem 1 is the following:

**Proposition 6** Let \( M \) be a generating set of \( G \) with partition \( (S_1, \ldots, S_p, H) \) where every set \( S_i \) is good. Then there exists a good generating set \( S \) where \( \#S \leq p(M) \).
The proof of Theorem 1 follows easily from Proposition 6 as outlined in the beginning of this section:

**Proof of Theorem 1** Let \( M \) be a minimal generating set of \( G \). Endow \( M \) with the trivial partition \( \{ H = M \} \), this implies in particular that \( p(M) = \text{rank } G - 5 \). Proposition 6 now guarantees that there exists a good generating set \( S \) with \( \# S \leq p(M) = \text{rank } G - 5 \). Lemma 4 on the other hand guarantees that \( \# S \geq \text{rank } A + \text{rank } B - \text{rank } C \). Combining these two inequalities yields \( \text{rank } A + \text{rank } B - \text{rank } C \leq 3 \text{rank } G - 5 \) which is clearly equivalent to the statement of Theorem 1. \( \square \)

In the remainder of this section we will give a proof of Proposition 6. The main tools will be the following two Lemmas.

**Lemma 7** Suppose that \( S_i \) and \( S_j \) are good sets and that \( T_{S_i} \cap T_{S_j} \neq \emptyset \). Then there exists a good set \( S_{ij} \) such that \( \langle S_i \cup S_j \rangle \subset \langle S_{ij} \rangle \) and that the following hold:

1. \( S_{ij} \) is small if \( S_i \) and \( S_j \) are tiny.
2. \( \# S_{ij} \leq 3 \) if \( S_i \) is tiny and \( S_j \) is small.
3. \( \# S_{ij} \leq 5 \) if \( S_i \) and \( S_j \) are small.
4. \( \# S_{ij} \leq \# S_j + 2 \) if \( S_i \) is tiny.
5. \( \# S_{ij} \leq \# S_j + 4 \) if \( S_i \) is small.
6. \( \# S_{ij} \leq \# S_i + \# S_j + 3 \).

**Proof** We look at the 6 cases in the same order as in the statement of Lemma 7.

1. \( S_i = \{ f \} \) and \( S_j = \{ s \} \) are tiny. Choose \( x \in T_{S_i} \cap T_{S_j} \). Since \( T_{S_i} = T_f \) and \( T_{S_j} = T_s \) we get that \( fx = x \) and \( sx = x \) and we can choose \( S_{ij} \) to be the small good set \( \{ f, s \} \).

2. \( S_i = \{ f \} \) is tiny and \( S_j = \{ s_1, s_2 \} \) is small. Choose \( x \in T_{S_i} \cap T_{S_j} \) and \( y \) such that \( S_j y = y \). As in (1) we get that \( fx = x \). Since no point of \( T_{S_j} \) is more than distance 1 from \( y \) it follows that \( d(x, y) \leq 1 \) and we can choose \( S_{ij} \) to be the good set \( S_i \cup S_j = \{ f, s_1, s_2 \} \).

3. \( S_i = \{ f_1, f_2 \} \) and \( S_j = \{ s_1, s_2 \} \) are small. Choose \( y \) and \( z \) such that \( S_j y = y \) and that \( S_j z = z \) if \( d(y, z) \leq 1 \) we can choose \( S_{ij} = S_i \cup S_j \). If \( d(y, z) \geq 2 \) then we get that \( d(y, z) = 2 \) since \( y \) and \( z \) are distance at most 1 from any point \( x \in T_{S_i} \cap T_{S_j} \). Choose an element \( g \in \text{Stab } x \) such that \( g y = z \). Such an element exists by the definition of the Bass-Serre tree. Now choose \( S_{ij} \) to be \( S_i \cup g^{-1} S_j \) and \( \{ g \} \). Since \( g^{-1} S_j g y = y \) we have that \( S_{ij} \subset \text{Stab } x \cup \text{Stab } y \), i.e. that \( S_{ij} \) is good. It is further clear that \( S_{ij} \) does contain at most 5 elements.

4. \( S_i = \{ f \} \) is tiny. If \( TS_j \) consists of a single vertex we argue as in (2), we can therefore assume that \( TS_j \) contains at least one (and therefore infinitely many) edge. Choose \( x \in T_{S_j} \cap T_{S_i} \) as before we see that \( fx = x \). Let further \( e = [y, z] \) be an edge of \( TS_j \) that has minimal distance from \( x \). We know that \( d(e, x) \leq 1 \) and by Lemma 3 we can change \( S_j \) into a new generating set of \( S_j \) (again denoted by \( S_j \)) of the same cardinality such that \( S_j \in \text{Stab } y \cup \text{Stab } z \). If \( d(e, x) = 0 \), i.e. \( x = y \) or \( x = z \) we choose \( S_{ij} = S_i \cup S_j \). If \( d(e, x) = 1 \) than either \( d(x, y) = 1 \) or \( d(x, z) = 1 \). We restrict ourselves to the first case, the second case is analogous. Let now \( g \) be an element such that \( g y = y \)
and that $g^z = x$. It is clear that $g^{-1}fg^z = z$, i.e. we can choose $S_{ij}$ as $S_j \cup \{ g^{-1}fg, g \} \subseteq \text{Stab } y \cup \text{Stab } z$.

(5) $S_i = \{ f_1, f_2 \}$ is small. If $TS_i$ is a single vertex we argue as in (3). Choose $x \in T_{S_i} \cap T_{S_j}$, $p$ such that $S_p = p$ and an edge $f = \langle y, z \rangle$ of $TS_i$ such that $d(x, e) \leq 1$. This implies in particular that $d(f, p) \leq 2$. As before we change $S_j$ such that $S_j \subseteq \text{Stab } y \cup \text{Stab } z$. By Lemma 2 there exists a set $S \subseteq \text{Stab } y \cup \text{Stab } z$ such that $\#S \leq 2$ and that there exists $g \in \langle S \rangle$ such that $gp \in f$, i.e. $gp \in \langle y, z \rangle$. We then choose $S_{ij}$ as $S_j \cup S \cup g^{-1}S_i g$.

(6) If $TS_i$ or $TS_j$ consists of a single vertex we argue as in the cases before, i.e. we can restrict ourselves to the case that $TS_i$ and $TS_j$ contain edges. Choose $x \in T_{S_i} \cap T_{S_j}$, and edges $e = \langle y, z \rangle$ in $TS_i$ and $f = \langle p, q \rangle$ in $TS_j$ such that $d(x, e) \leq 1$ and $d(x, f) \leq 1$. This implies in particular that $d(e, f) \leq 2$. As before we change $S_i$ and $S_j$ such that $S_i \subseteq \text{Stab } y \cup \text{Stab } z$ and $S_j \subseteq \text{Stab } p \cup \text{Stab } q$. By Lemma 2 there exists a set $S \subseteq \text{Stab } y \cup \text{Stab } z$ with $\#S \leq 3$ such that there exists a $g \in \langle S \rangle$ such that $gff = e$. This implies that $g^{-1}S_j g \subseteq \text{Stab } y \cup \text{Stab } z$ and we can choose $S_{ij}$ as $S_i \cup S \cup g^{-1}S_j g$. □

**Lemma 8** Suppose that $S$ is a good set and that $hT_S \cap T_S \neq \emptyset$ for some $h \in G$. Then there exists a good set $S'$ such that $\langle S \cup \{h\} \rangle \subseteq \langle S' \rangle$ and the following hold:

1. $S'$ is small if $S$ is tiny.
2. $\#S' \leq 4$ if $S$ is small.
3. $\#S' \leq \#S + 3$.

**Proof** Again we look at the cases in the order of the Lemma. The statement $hT_S \cap T_S \neq \emptyset$ is clearly equivalent to the existence of two vertices $p$ and $q$ of $T_S$ such that $hp = q$.

1. $S = \{ f \}$ is tiny. $T_S$ either consists of a single vertex $v$ or a single edge $e$ which is fixed under the action of $h$. Since the two endpoints of an edge are $G$-inequivalent in a Bass-Serre tree associated to an amalgamated product $hp = q$ for two vertices of $T_S$ implies that $p = q$. Since $f$ also fixes $p$ we get that $S_{ij} = \{ f, h \}$ is small.

2. $S = \{ f_1, f_2 \}$ is small. Choose a vertex $x$ such that $Sx = x$. $T_S$ consists of the union of $x$ and of edges that have $x$ as an endpoint. This implies that $x$ is $G$-inequivalent to all other vertices of $T_S$. This means that $hp = q$ for two vertices of $T_S$ implies that either $p = q = x$ or that $p \neq x$ and $q \neq x$. If $p = q = x$ or $p = q \neq x$ we choose $S' = \{ f_1, f_2, h \} \subseteq \text{Stab } p \cup \text{Stab } x$. If $p \neq q$ then there exists an element $g$ of $\text{Stab } x$ such that $gp = q$. This clearly implies that $(g^{-1}hp, p)$ and we choose $S' = S \cup \{ g, g^{-1}h \} \subseteq \text{Stab } x \cup \text{Stab } p$.

3. If $TS$ consists of a single vertex we argue as in (2), i.e. we can restrict ourselves to the case that $TS$ contains an edge $e = [x, y]$ and that $S \subseteq \text{Stab } x \cup \text{Stab } y$. Now choose $p$ and $q$ of $T_S$ such that $hp = q$. Since $p$ and $q$ are at most distance 1 from an edge of $TS$ and all edges of $TS$ are $\langle S \rangle$-equivalent to $e$, we can assume that $p$ and $q$ are at most distance 1 from $e$ after replacing $h$ by $g_1 h g_2$ for suitable $g_1, g_2 \in \langle S \rangle$. Now $p$ and $q$ lie in the union of $e$ with all edges emanating at $x$ or $y$ which is a set of diameter 3. Since $G$-equivalent vertices have even distance (in a Bass-Serre tree of an amalgamated product) this implies that $d(p, q) \in \{ 0, 2 \}$ and that $p$ and $q$ are distance 1 from either
We only investigate the case that \(d(p,x) = d(q,x) = 1\), the other case is analogous. Choose now \(g_1\) and \(g_2\) from \(\mathrm{Stab}\ x\) such that \(g_1p = y\) and \(g_2q = y\) this implies that \(g_1hg_1^{-1} = g_2hg_2 = y\), i.e. we can choose \(S' = S \cup \{g_1, g_2, g_2hg_1^{-1}\} \subset \mathrm{Stab}\ x \cup \mathrm{Stab}\ y\). \(\square\)

We now have all the necessary tools for the proof of Proposition 6.

**Proof of Proposition 6** Let \(M\) be an arbitrary minimal generating set of \(G\) with partition \((S_1, \ldots, S_p, H)\) where every set \(S_i\) is good and suppose that there exists no good generating set of cardinality \(p(M)\). Suppose further that among all such sets \(M\) is smallest with respect to the well-ordering on the signatures. The partition cannot consist of a single good set \(S_1\) since in this case \(p(M) = \#S_1\) and \(S_1\) is a good generating set. Since the induced splitting of \(\langle M \rangle = G\) is evidently the original splitting of \(G\) as the amalgamated product \(A \ast_C B\), the induced splitting cannot contain any trivial edge groups. Theorem 5 therefore implies that after replacing \(M\) with an equivalent partitioned set which we denote again by \(M\) one of the situations (1)-(3) occurs. This new set is a new generating set which clearly has the same signature and potential; it therefore still satisfies the assumptions made before.

We now look at the situation (1)-(3) and find in any of these cases a partitioned set with smaller signature and at most the same potential that still satisfies the assumptions made, yielding a contradiction to the minimality of the originally chosen set and therefore proving Proposition 6.

(1) \(T_{S_i} \cap T_{S_j} \neq \emptyset\). We replace the two good sets \(S_i\) and \(S_j\) by a good set \(S_{ij}\) having the properties spelt out in Lemma 7. This yields a new partitioned generating set which clearly has smaller signature. Taking into account how the numbers \(p_1, p_2, p_3\) and \(\Sigma\) change (if for example \(S_i\) and \(S_j\) are tiny they get replaced by a small set, i.e. \(p_1\) decreases by two and \(p_2\) increases by 1 and \(p_3\) and \(\Sigma\) stay unchanged) it is further easy to check the potential of the new set is at most as big as the potential of the original set.

(2) \(T_{S_i} \cap hT_{S_i} \neq \emptyset\). We remove \(h\) from \(H\) and replace \(S_i\) by a set \(S'_i\) as described in Lemma 8. Since the new set \(H\) has one less elements the signature has clearly decreased in this process. As before it is easy to check that the potential has not increased.

(3) If an element \(h \in H\) acts with a fixed point we remove \(h\) from \(H\) and add a tiny good set \(S_{p+1} = \{h\}\). As before we see that the new set has smaller signature and that the potential is unchanged. \(\square\)

### 3 Is the given bound the best possible?

At a first glance the bound given in Theorem 1 does not seem to be very good. In this section however we briefly discuss why we believe that probably no significantly better bound is possible. In [2] I. Bumagina constructs examples of amalgamated products \(G = A \ast_C B\) where \(C\) is infinite cyclic and malnormal and where \(\text{rank } G = \text{rank } A = 2n\) and \(\text{rank } B = n\). We describe how to construct amalgamated products \(G = A \ast_C B\) with malnormal \(C \cong \mathbb{Z}\) such that rank \(G\) is roughly \(1/2(\text{rank } A + \text{rank } B)\). A careful analysis of the proof of Theorem 1 further shows that the replacement done when situation (3) of Lemma 8 occurs is that 1 element of the generating set gets replaced by 3 elements of the good
generating set. It is not clear though that this replacement is not too generous, i.e. whether the factor 1/3 does actually occur. It is furthermore not difficult to construct amalgamated products that have good generating sets of cardinality rank \( A + \text{rank } B - 2 \text{rank } C \) and there seems to be no immediate reason why this cannot combined with the other phenomenon. We indicate how to construct the examples: Choose a two-generated group

\[ H = \langle a_1, a_2 \rangle R \]

such that \( C = \langle a_1 \rangle \) is malnormal and infinite cyclic. Choose further malnormal infinite cyclic subgroups \( C_1 = \langle c_1 \rangle, \ldots, C_n = \langle c_n \rangle \) of \( H \) such that the subgroups \( C, C_1, \ldots, C_n \) are pairwise conjugacy separated, i.e. that \( hC_i h^{-1} \cap C_j = 1 \) for all \( i \neq j \) and \( h \in H \). It is easy to see that this group can be chosen to be the two-generated free group \( F(a_1, a_2) \).

Now define \( A \) to be the \( n \)-fold HNN-extension of \( H \) where the \( k \)-th stable letter makes \( C \) and \( C_k \) conjugate, i.e.

\[ A := \langle a_1, a_2, t_1, \ldots, t_n \rangle R, t_1 a_1 t_1^{-1} = c_1, \ldots, t_n a_1 t_n^{-1} = c_n \].

A simple normal form calculation in \( A \) shows that \( C \) is malnormal in \( A \). It is further clear that rank \( A \geq n + 1 \) since \( A \) has the free group of rank \( n \) as its proper quotient (quotient out the normal closure of \( H \)).

Now choose \( B \) to be a group of rank \( n + 1 \)

\[ \langle b, b_1, \ldots, b_n \rangle [S] \]

where \( C = \langle b \rangle \) is malnormal and infinite cyclic and \( b_i \) is in \( B \) conjugate to \( b \), i.e. \( b_i = g_i b_i g_i^{-1} \) for some \( g_i \in B \). Such groups have been constructed explicitly in [2].

We define

\[ G = A \ast_C B = A \ast_{\langle a_2 b \rangle} B \]

which is an amalgamated product with malnormal cyclic amalgam where rank \( A \) and rank \( B \) are at most \( n + 1 \). We show that \( G \) is generated by the set \( M = \{ a_1, a_2, t_i g_i^{-1}, \ldots, t_n g_n^{-1} \} \), i.e. that rank \( G \leq n + 2 \).

Since \( a_1, a_2 \in M \) and \( c_i \in \langle a_1, a_2 \rangle \) it is clear that \( c_i \in \langle M \rangle \) for \( 1 \leq i \leq n \). This implies that \( (t_i g_i^{-1})^{-1} c_i (t_i g_i^{-1}) = g_i t_i^{-1} c_i t_i g_i^{-1} = g_i a_1 g_i^{-1} = g_i b_i g_i^{-1} = b_i \in \langle M \rangle \) \( 1 \leq i \leq n \) and therefore \( B \subset \langle M \rangle \). It follows that \( g_i \) and therefore \( (t_i g_i^{-1}) g_i = t_i \in \langle M \rangle \) which implies that \( A \subset \langle M \rangle \), i.e. \( M \) is a generating set of \( G \).

References


