The Nielsen method for groups acting on trees

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Abstract

We use Nielsen methods to study generating sets of subgroups of groups that act on simplicial trees and give several applications. In particular we exhibit an explicit bound of the complexity of acylindrical splittings of a f.g. group in terms of its rank and apply this bound to JSJ-decompositions of 3-manifold groups and torsion-free hyperbolic groups. We further show that no analogue of Grushko’s Theorem can hold for amalgamated products.

Introduction

Kurosh’s subgroup theorem [16] describes the structure of subgroups of free products. Together with Grushko’s theorem [8] this gives the following:

Theorem 1 (Grushko/Kurosh) Let \( G = \star_{i=1}^{n} G_i \) be a free product and \( U \triangleleft G \) a subgroup generated by a finite set \( M \subset G \). Then \( M \) is Nielsen equivalent to a set \( \tilde{M} = S_1 \cup \cdots \cup S_p \cup H \) such that

\[
U = \langle S_1 \rangle \ast \cdots \ast \langle S_p \rangle \ast F(H)
\]

where \( \langle S_i \rangle \), the subgroup generated by \( S_i \), is conjugate to a subgroup of one of the \( G_j \) and \( F(H) \) is the free group in \( H \).

One possible way of proving Theorem 1 is to apply the Nielsen method for free products (with amalgamation) as developed by Zieschang [30] and refined in [4]. This theory is a generalization of Nielsen’s ideas for free groups [19]. It is combinatorial and makes extensive use of normal forms in amalgamated products.

In this article we study finitely generated subgroups of a groups that are fundamental groups of a graphs of groups and obtain a result that generalizes Theorem 1. We will therefore develop the Nielsen method for groups acting on trees. Note that by the Bass-Serre theory this is equivalent to developing the Nielsen method for fundamental groups of graphs of groups on the basis of normal forms. For details the reader is referred to [27]. We need to introduce some notations in order to formulate the main result.
Let $G$ be a group and
\[ G \times T \to T, \quad (g, x) \mapsto g x \]
be a simplicial action of $G$ on a simplicial tree $T$ such that $G$ acts without inversion. In the following we will only mention the action $G \times T \to T$ if there is the possibility of ambiguity. The following definitions are meant to be with respect to a fixed action $G \times T \to T$.

For a subgroup $U$ of $G$ we define $T_U$ to be the minimal subtree of $T$ that contains the minimal subtree of $T$ that is invariant under the action of $U$ and all points $x \in T$ such that $ux = x$ for some $u \in U - 1$. It is clear that $T_U$ is $U$-invariant, i.e. that $uT_U = T_U$ for all $u \in U$ and that $T_{gUg^{-1}} = gT_U$. In this paper subgroups $U$ that have a global fixed point, i.e. for which there exists a point $y \in T$ such that $Uy = y$ play and important role and we call them \textit{special}. For such subgroup $U$ the minimal $U$-invariant subtree consists of the single point $y$ and $T_U$ is the union of all points $x \in T$ such that $ux = x$ for some $u \in U - 1$.

We say that $M \subset G$ is \textit{partitioned set with partition} $(S_1, \ldots, S_p, H)$ if $M = S_1 \cup \cdots \cup S_p \cup H$ and $H$ is finite. It is clear that the set $M$ is implicit in its partition, we will therefore sometimes omit the set $M$ and just mention the partition.

The following concept of equivalence is similar to the equivalence defined in [11] or [4]. Note that any two partitioned sets that are equivalent in the following sense are Nielsen equivalent but not necessarily versa.

Two sets $M, \tilde{M} \subset G$ with partitions $(S_1, \ldots, S_p, H)$ and $(\tilde{S}_1, \ldots, \tilde{S}_p, \tilde{H})$ are called \textit{equivalent} (we write $M \sim \tilde{M}$) if there exists a sequence
\[ M = M^1, \ldots, M^k = \tilde{M} \]
of partitioned sets $M^i$ with partition $(S^i_1, \ldots, S^i_p, H^i)$ for $1 \leq i \leq k$ such that for $1 \leq i \leq k - 1$ either

1. there exists $j \in \{1, \ldots, p\}$ such that $S^i_j = S^i_j$ for $l \neq j$, $H^{i+1} = H^i$ and $S^i_j = gS^i_j g^{-1}$ for some $g \in \langle M^i - S^i_j \rangle$ or

2. $S^i_l = S^i_l$ for $1 \leq l \leq p$ and $H^{i+1} = (H^i - \{h\}) \cup \{h'\}$ where $h \in H^i$ and $h' = g_1 h g_2$ for some $g_1, g_2 \in \langle M^i - \{h\} \rangle$.

It is clear that equivalent sets generate the same subgroup. We are now able to state the main result. Note that in [13] some of this language has been developed in a special case and a two-generator version of Theorem 2 has been proved.

For each set $S_i$ of a partition $(S_1, \ldots, S_p, H)$ we denote the set $T_{\langle S_i \rangle}$ by $\tilde{T}_i$. For the partition $(\tilde{S}_1, \ldots, \tilde{S}_p, \tilde{H})$ the we will denote $T_{\langle \tilde{S}_i \rangle}$ by $\tilde{T}_i$.

\textbf{Theorem 2} Let $M$ be a set with partition $(S_1, \ldots, S_p, H)$. Then either
\[ \langle M \rangle = \langle S_1 \rangle \ast \cdots \ast \langle S_p \rangle \ast F(H) \]
or $(S_1, \ldots, S_p, H) \sim (\tilde{S}_1, \ldots, \tilde{S}_p, \tilde{H})$ such that one of the following holds:

1. $\tilde{T}_i \cap \tilde{T}_j \neq \emptyset$ for some $i, j \in \{1, \ldots, p\}$ with $i \neq j$.

2. $\tilde{h} \tilde{T}_i \cap \tilde{T}_i \neq \emptyset$ for some $i \in \{1, \ldots, p\}$ and $\tilde{h} \in \tilde{H}$.

3. There exists $\tilde{h} \in \tilde{H}$ such that $\tilde{h}$ acts with a fixed point.
An important case of Theorem 2 is the case when all of the subgroups \(\langle S_i \rangle\) associated to the partition \((S_1, \ldots, S_p, H)\) are special. We call such a partitioned set a marked set with marking \((S_1, \ldots, S_p, H)\). It is clear that a partitioned set that is equivalent to a marked set also is marked set. An immediate consequence of Theorem 2 is then the following Corollary which will be the main tool in all our applications.

**Corollary 3** Let \(M\) be a set with marking \((S_1, \ldots, S_p, H)\). Then either

\[
\langle M \rangle = \langle S_1 \rangle \ast \cdots \ast \langle S_p \rangle \ast F(H)
\]

or \((S_1, \ldots, S_p, H) \sim (\tilde{S}_1, \ldots, \tilde{S}_p, \tilde{H})\) such that one of the following holds:

1. \(\tilde{T}_i \cap \tilde{T}_j \neq \emptyset\) for some \(i, j \in \{1, \ldots, p\}\) with \(i \neq j\).
2. \(h\tilde{T}_i \cap \tilde{T}_i \neq \emptyset\) for some \(i \in \{1, \ldots, p\}\) and \(h \in \tilde{H}\).
3. There exists \(\tilde{h} \in \tilde{H}\) such that \(\tilde{h}\) acts with a fixed point.

It turns out that Corollary 3 is particularly useful under the assumption that the trees \(T_i\) have small diameter. This is clearly always the case if the action of \(G\) on \(T\) is \(k\)-acylindrical in the sense of [26], i.e. that no non-trivial element of \(G\) fixes a segment of length greater than \(k\).

Sela [26] proves that for any finitely generated freely indecomposable group \(G\) and integer \(k\) there exists a constant \(c(k, G)\) such that any \(k\)-acylindrical minimal action of \(G\) on a simplicial tree \(T\) has at most \(c(k, G)\) \(G\)-equivalence classes of vertices, i.e. that \(G\backslash T\) has at most \(c(k, G)\) vertices. The original proof is difficult and makes extensive use of deep results on group actions on \(\mathbb{R}\)-trees due to Rips and Bestvina-Feighn [2]. We give a simple proof of Sela’s result using Theorem 2 and further show that \(c(k, G)\) can be bounded in terms of the rank of \(G\) and \(k\). For the case of finitely presented groups an explicit bound has been given by Delzant [5] in terms of the presentation and \(k\).

**Theorem 4** Let \(G\) be a non-cyclic freely indecomposable f.g. group and \(G \times T \to T\) be a minimal \(k\)-acylindrical action. Then \(G\backslash T\) has at most \(1 + 2k(\text{rank } G - 1)\) vertices.

The condition that \(G\) is non-cyclic is needed since \(\mathbb{Z}\) allows \(k\)-acylindrical free actions with arbitrarily many vertices in \(G\backslash T\), where \(T\) is the real line and \(G\) acts by translation. Imposing reducedness as done in [26] would eliminate this exception.

It turns out that the bound given in Theorem 4 is not the best possible and we give a better one in the case that the action is 1-acylindrical or what we will call almost 1-acylindrical. This bound can then be applied to show that the rank of the fundamental group of a 3-manifold is bounded from below by the number of non-Seifert pieces of the JSJ-decomposition, i.e. the canonical torus decomposition as introduced by Jaco and Shalen in [9] and Johannson in [10].

**Theorem 5** Let \(M^3\) be an orientable closed 3-manifold and let \(N\) be the number of non-Seifert pieces of its JSJ-decomposition. Then \(\text{rank } (\pi_1(M^3)) \geq N + 1\).

Since the rank of the fundamental group of a 3-manifold \(M^3\) is always less or equal than the Heegaard genus of \(M^3\), we obtain the following result from [23] as an immediate Corollary to Theorem 5.

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Corollary 6 (Scharlemann, Schultens) Let $M^3$ be an orientable closed 3-manifold. Assume that $M^3$ is of HEGGARD genus $g$ and that $N$ is the number of non-Seifert pieces of its JSJ-decomposition. Then $g \geq N + 1$.

Similar arguments apply to JSJ-splittings of torsion-free word-hyperbolic groups that were introduced in [24] and generalized to finitely generated groups in [21]. Since the action on the Bass-Serre tree associated to the JSJ-splitting turns out to be almost $1$-acylindrical, we obtain the following, which has been shown in [12] for hyperbolic groups of rank $2$.

**Theorem 7** Let $G$ be a non-cyclic torsion-free freely indecomposable word-hyperbolic group. Then the JSJ-splitting of $G$ has at most rank $G - 1$ vertices.

In the final section we will construct examples of amalgamated products that show that no non-trivial analogue of Grushko’s Theorem can hold, i.e. we show that for an amalgamated product $G = A \ast_C B$ no non-trivial bound for rank $G$ can be given in terms of rank $A$, rank $B$ and rank $C$. We show:

**Theorem 8** For any integer $n$ there exists an amalgamated product $G = A \ast_C B$ such that rank $A \geq n$, rank $B \geq n$ and rank $G = \text{rank } C = 2$.

It should be noted that the techniques discussed in this article can in many cases help to decide the rank problem for fundamental groups of graphs of groups if there is sufficient control over the vertex groups and the edge groups. In [28] a refined combinatorial version of the present theory has been developed for amalgamated products and used to determine the rank of some classes of NEC-groups.

## 1 The Nielsen method

Let $F_n$ be the free group of rank $n$. Nielsen describes in [18] and [19] an algorithm which carries a given set $M \subset F_n$ over into a set $M' \subset F_n$ such that $M$ and $M'$ generate the same subgroup and that every $m' \in M'$ contains a letter which does not cancel in any freely reduced product in elements of $M'$. This clearly implies that $\langle M \rangle$ is free in $M'$. Zieschang develops in [30] (see [4] for a refinement) a similar theory for free products with amalgamation and describes the obstructions that make it impossible to obtain an equivalent of Nielsen’s result. An analogous theory has been developed for HNN-extensions in [20]. All these theories have in common that a preorder that has a lexicographical component is defined on a group $G$ and that a generating set $M$ of a subgroup is replaced by a generating set $\hat{M}$ that is (Nielsen) equivalent to $M$ and minimal in its equivalence class with respect to the defined preorder.

In this section we develop the Nielsen theory for groups acting on trees. Though many of the tools developed are merely geometric versions of generalizations of ideas from [30] it should be noted that the present theory is able to control in some way the obstructions of [30], that essentially consist in one of the trees $\hat{T}_i$ of Theorem 2 being non-trivial.

For this whole section we fix a group $G$ and an action $G \times T \to T$ such that $G$ acts without inversion.
1.1 Definition of the (pre)orders $\leq_s$, $\leq_M$ and $\preceq_M$

Let $M \subset G$ be a set with partition $(S_1, \ldots, S_p, H)$. If $p = 0$ choose an arbitrary vertex $x$ of $T$, if $p \geq 1$ choose a vertex $x \in T_1$. Let now

$$S(x) := \{[x, y] \mid y \text{ a vertex of } T\}$$

be the set of all segments of type $[x, y] \subset T$ where $y$ is a vertex of $T$. By $l([x, y])$ we denote the combinatorial length of $[x, y]$, i.e. the number of edges in $[x, y]$. By $[x, y(i)]$ (with $i \leq l([x, y])$) we denote the prefix of $[x, y]$ of length $i$, i.e. $[x, y(i)] \subset [x, y]$ and $l([x, y(i)]) = i$.

We now choose a well-ordering $\leq$ on $S(x)$ that has the following properties:

1. $[x, y] \leq [x, z]$ implies that $l([x, y]) \leq l([x, z])$
2. $[x, y] \leq [x, z]$ and $l([x, y]) = l([x, z])$ implies that $[x, y(i)] \leq [x, z(i)]$ for $1 \leq i \leq l([x, y])$.

Such a well-ordering is frequently called a ShortLex order and can be easily constructed inductively by choosing for each vertex a well ordering on the set of all edges emanating from this vertex. We will write $[x, y] < [x, z]$ if $[x, y] \leq [x, z]$ and $[x, y] \neq [x, z]$.

We define $TT$ to be the smallest subtree of $T$ containing the trees $T_i$ for $1 \leq i \leq p$ if $p \geq 1$ and define $TT$ to be the subtree containing the single vertex $x$ if $p = 0$. Note that in any case $x \in TT$ since $x \in T_1$ if $p \geq 1$. We define a new length function $l_M$ and a new total order $\leq_M$ on $S(x)$ which depends on the tree $TT$ and therefore on $M$.

We define

$$l_M([x, y]) := l([x, y] - TT).$$

Note that $l_M([x, y]) = 0$ iff $[x, y] \subset TT$, i.e. if $[x, y] - TT = \emptyset$. This means that $l_M$ counts the edges of a segment $[x, y]$ that lie outside $TT$. For the case $TT = x$ this implies that $l_m = l$.

We will now use $l_M$ to define the total order $\leq_M$ on $S(x)$. We will say that

$$[x, y] \leq_M [x, z]$$

if

1. $l_M([x, y]) < l_M([x, z])$ or
2. $l_M([x, y]) = l_M([x, z])$ and $[x, y] \leq [x, z]$.

It is clear that $\leq_M$ is also a well-ordering. Again $\leq_M$ coincides with $\leq$ if $TT = x$. As for the ordering $\leq$ we will write $[x, y] < [x, z]$ if $[x, y] \leq_M [x, z]$ and $[x, y] \neq [x, z]$. We will now define the length function $L_M$ on $G$ that will later be used to define the total preorder $\preceq_M$ on $G$. We define

$$L_M(g) := l([x, gx] - (TT \cup gTT)).$$

This means that $L_M$ counts the number of edges of $[x, gx]$ that do not lie in $TT \cup gTT$ which implies in particular that $L_M(g) = 0$ iff $TT \cap gTT \neq \emptyset$. It
is clear that $L_M(g) = L_M(g^{-1})$ since $g$ maps $[g^{-1}x, z]$ to $[x, gx]$ and maps the segment $[g^{-1}x, z] - (g^{-1}TT \cup TT)$ onto the segment $[x, gx] - (TT \cup gTT)$. We will call the edges of $[x, gx] - (TT \cup gTT)$ the essential edges of $g$.

We define two maps

$$u : G \to S(x) \text{ and } v : G \to S(x).$$

1. If $L_M(g) = 2k \geq 2$ or $L_M(g) = 2k + 1 \geq 3$ we define $u(g)$ to be the segment $[x, u_g] \subset [x, gx]$ with $l_M([x, u_g]) = k$ and $v(g)$ to be the segment $[x, v_x] \subset [x, g^{-1}x]$ with $l_M([x, v_x]) = k$.

2. If $L_M(g) = 1$ then we define $u(g) = [x, u_g]$ to be the segment $TT \cap [x, gx]$ and $v(g) = [x, v_x]$ to be the segment $TT \cap [x, g^{-1}x]$.

3. If $L_M(g) = 0$ we define $u(g)$ and $v(g)$ to be the degenerate segment $[x, z] = \{x\}$.

This clearly implies that $u(g) = v(g^{-1})$ and $v(g) = u(g^{-1})$. The segments $u(g)$ and $v(g)$ correspond to the front and the back half of an element in the terminology of [4]. The following Lemma uses $u$ and $v$ to give a characterization of elements with non-trivial $L_M$-length that act with a fixed point, an observation that is trivial from a normal form point of view.

**Lemma 9** Let $g \in G$ with $L_M(g) \geq 1$. Then $g$ acts with a fixed point iff $L_M(g) = 2m$ and $u(g) = v(g)$.

**Proof** Suppose that $g$ acts with a fixed point $y$. Since $g$ maps $[g^{-1}x, y]$ to $[x, y]$ and $[x, y]$ to $[gx, y]$ it follows that $l([y, g^{-1}x]) = l([y, x]) = l([y, gx])$. This is only possible if the midpoints of $[g^{-1}x, x]$ and $[x, gx]$ coincide, we denote this point by $z$. It is clear that $g$ maps the midpoint of $[g^{-1}x, z]$ to the midpoint of $[x, gx]$ it follows that $z$ is fixed under the action of $g$ and $z$ must be a vertex of $T$ since $G$ acts without inversion. Since $g$ maps $[x, z] - TT$ to $[gx, z] - gTT$ it follows that $[x, gx] - (TT \cup gTT)$ contains in even number of edges, i.e. $L_M(g) = 2m$. It is further clear that $[x, z] = u(g) = u(g^{-1}) = v(g)$ as desired.

Suppose now that $L_M(g) = 2m$ and $u(g) = v(g)$. Since $L_M(g)$ is even we get that $gv_2 = u_2$ and therefore $gv_2 = u_2$, i.e. $g$ acts with fixed point $u_2 = v_2$. $\square$

For elements $g \in G$ that act without fixed point we single out one edge of $[x, gx]$ and call it the stable edge of $g$:

Suppose first that $g \in G$ with $L_M(g) = 2m \geq 2$. It follows that $[x, gx] = u(g) \cup gv(g) = [x, u_g] \cup [gx, gv_2]$ and $u(g) \cap gv(g) = u_2 = gv_2$ (see Figure 1) and that $u(g) \neq v(g)$ by Lemma 9. If $u(g) > v(g)$ we call the edge of $u(g)$ that contains $u_2 = gv_2$ the stable edge $e(g)$ of $g$. If $u(g) < v(g)$, we define $e(g)$ to be the edge of $gv(g) = [gx, gv_2] = [gx, u_2]$ that contains $u_2 = gv_2$.

Suppose now that $g \in G$ with $L_M(g) = 2m + 1 \geq 1$. Since $u(g)$ contains only $m$ edges that do not lie in $TT$ and $v(g)$ contains only $m$ edges that do not lie in $gTT$, we clearly get that $[x, gx] - (u(g) \cup gv(g))$ consists of the edge $[u_2, gv_2]$ (see Figure 2). In this case well will define the stable edge $e(g)$ to be this edge.

The definition of the essential edge shows immediately that $g$ maps the essential edge of $g^{-1}$ onto the essential edge of $g$, i.e. that $ge(g^{-1}) = e(g)$.
We now define the preorder $\succeq_M$ on the set of pairs of elements $\{g, g^{-1}\}$ where $g \in G$. Let $\{g, g^{-1}\}$ and $\{h, h^{-1}\}$ be two such pairs. Possibly after replacing $g$ with $g^{-1}$ and $h$ with $h^{-1}$ we can assume that $u(g) \leq_M v(g)$ and $u(h) \leq_M v(h)$. We will say that

$$\{g, g^{-1}\} \succeq_M \{h, h^{-1}\}$$

if

1. $L_M(g) < L_M(h)$ or
2. $L_M(g) = L_M(h)$ and $u(g) <_M u(h)$ or
3. $L_M(g) = L_M(h)$ and $u(g) = u(h)$ and $v(g) \leq_M v(h)$.

This preorder induces a preorder on $G$, we say that $g \preceq_M h$ if $\{g, g^{-1}\} \succeq_M \{h, h^{-1}\}$. We will say $g \prec_M h$ if $g \preceq_M h$ and $h \not\succeq_M g$. Note that there is no infinite, with respect to $\succeq_M$ strictly descending, sequence of elements of $G$ because $\succeq_M$ is a well-ordering.

### 1.2 The minimality assumption

The minimality assumption consists of two parts; firstly we want the distance between the trees $T_i$ (and consequently the size of $TT$) to be as small as possible and secondly we want the elements of $H$ to be minimal with respect to the ordering $\succeq_M$.

For a subtree $Y$ of $T$ we define $u(Y)$ to be segment joining $x$ and $Y$, i.e. $u(Y) = [x, q]$ where $q = [x, q] \cap Y$. This implies in particular that $[x, q]$ is degenerate, i.e. consists of the single point $x = q$, if $x \in Y$.

Let $M \subseteq G$ be a set with partition $(S_1, \ldots, S_p, H)$. By $c(M)$ we denote the number of edges of $TT - (T_1 \cup \cdots \cup T_p)$. We say that $M$ is minimal if
1. There is no partitioned set $\bar{M}$ such that $M \sim \bar{M}$ and $c(\bar{M}) < c(M)$.
2. $h \leq_M g_1 h g_2$ for all $h \in H$ and $g_1, g_2 \in \langle M - \{h\} \rangle$.

In the next subsection we prove Theorem 2 for minimal partitioned sets; the following simple Lemma guarantees that we can restrict ourselves to this case.

**Lemma 10** Every partitioned set is equivalent to a minimal marked set.

**Proof** Let $M$ be a marked set with partition $(S_1, \ldots, S_p, H)$. Since $c(M)$ only takes integer values, there must be a partitioned set $\bar{M}$ that is equivalent to $M$ and such that the first condition holds, we assume it already holds for $M$.

If $M$ does not fulfill the second condition of minimality then there exist $h \in H$ and $g_1, g_2 \in \langle M - \{h\} \rangle$ such that $g_1 h g_2 \prec_M h$ and we replace $H$ by $(H - \{h\}) \cup \{g_1 h g_2\}$, the new partitioned set is clearly equivalent to the original one. This does clearly not change any of the $T_i$ and therefore also does not change the preorder $\preceq_M$. Since $H$ has only finitely many elements and since there is no infinite with respect to $\preceq_M$ strictly descending sequence of elements it follows that finitely many of these replacements yield an equivalent partitioned set for which the second minimality condition holds. \qed

### 1.3 The proof of Theorem 2

In order to prove Theorem 2 it clearly suffices to show that any minimal partitioned set $M$ with partition $(S_1, \ldots, S_p, H)$ generates the subgroups $\langle M \rangle = \langle S_1 \rangle \ast \cdots \ast \langle S_p \rangle \ast F(H)$ unless one of the situations (1)-(3) of Theorem 2 occurs. The following simple Lemma turns out to be crucial.

**Lemma 11** Let $M$ be a minimal partitioned set with partition $(S_1, \ldots, S_p, H)$ and assume that none of the situations (1)-(3) of Theorem 2 occurs. Then

1. $g TT \cap TT = T_i = g T_i$ for all $g \in \langle S_i \rangle - 1$.
2. $h TT \cap TT = \emptyset$, i.e. $L_M(h) \geq 1$, for all $h \in H$.

**Proof** Assume first that there exists a $g \in \langle S_i \rangle - 1$ such that $g TT \cap TT \not\subset T_i$. Since $T_i = g T_i$ and since $T_i$ has trivial intersection with all $T_j$ with $j \neq i$ this implies that there exist two edges $e$ and $f$ of $TT = (T_i \cup \cdots \cup T_p)$ that both have one vertex in common with $T_i$ such that $ge = f$.

Let now $J$ be the set that consists of all $j \in \{1, \ldots, p\}$ such that $T_j$ lies in the component of $TT - \hat{e}$ that contains $f$. This clearly is the component that contains $T_i$. We define $T_J$ to be the subtree spanned by the $T_j$ with $j \in J$ and $\bar{T}_J$ to be the subtree spanned by the $T_j$ with $j \in \hat{J} := \{1, \ldots, p\} - J$. It is clear that $f \subset T_J$ and that

$$TT - (T_i \cup \cdots \cup T_p) = (T_J - \bigcup_{i \in J} T_i) \cup (T_J - \bigcup_{i \in J} T_i) \cup \cup_{i \in \hat{J}} (y, z)$$

where $[y, z]$ is the segment joining $T_J$ ($y \in T_J$) and $T_J$ ($z \in T_J$). Since $e$ must lie in $[y, z]$ we clearly get that $y \in T_i$. 

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We define $\tilde{S}_j := S_j$ for $i \in J$, $e$, $\tilde{S}_j := gS_jg^{-1}$ for $j \in J$ and $\tilde{R} = H$. This yields a partitioned set $\tilde{M}$ with partition $(\tilde{S}_1, \ldots, \tilde{S}_p, \tilde{R})$ which is equivalent to $M$. It is clear that $\tilde{T}_j = T_j$ for $j \in J$ and $\tilde{T}_j = gT_j$ for $j \in \tilde{J}$. Now $\tilde{TT}$, the tree spanned by the $\tilde{T}_j$, is clearly contained in $T_j \cup gT \cup \{gy, gz\}$ since $gy \in T_i \subset T_j$ and $gz \in gT_j$. It follows that $\tilde{TT} - (\tilde{T}_1 \cup \cdots \cup \tilde{T}_p) \subset (\tilde{T}_j - \cup \tilde{T}_i) \cup (gT - \cup \tilde{T}_i) \cup g(y, z)$, i.e. that

$$\tilde{TT} - (\tilde{T}_1 \cup \cdots \cup \tilde{T}_p) \subset (\tilde{T}_j - \cup \tilde{T}_i) \cup g(T_j - \cup \tilde{T}_i) \cup g(y, z).$$

Since $f = g \epsilon$ lies in $T_j - \cup \tilde{T}_i$ and in $g(y, z)$, we get that $\tilde{TT} - (\tilde{T}_1 \cup \cdots \tilde{T}_p)$ contains less edges than $TT - (T_1 \cup \cdots T_p)$, a contradiction to the minimality of $c(M)$.

Assume now that $hTT \cap TT \neq \emptyset$ for some $h \in H$, i.e. that there exist vertices $y$ and $z$ of $TT$ such that $hy = z$. Note first that $y \neq z$ since otherwise $h$ acts with a fixed point which means that situation (3) of Theorem 2 occurs. We can further exclude that $x, y \in T_i$ for some $i \in \{1, \ldots, p\}$ since we assume that situation (2) of Theorem 2 does not occur. Since $T_i \cap T_j = \emptyset$ for $i \neq j$, this implies that there exists an edge $e$ of $TT - (T_1 \cup \cdots \cup T_p)$ such that $e \subset \{y, z\}$. We define $TT_0$ to be the component of $TT - \tilde{e}$ that contains $y$ and $TT_0$ to be the component that contains $z$. Let now $J$ be the subset of $\{1, \ldots, p\}$ that consists of all $i$ such that $T_i \subset TT_0$. As in the first case we define $\tilde{J}, T_j$ and $TT$. We define $\tilde{S}_1 := hS_1h^{-1}$ for $i \in J$, $\tilde{S}_i := S_i$ for $i \in \tilde{J}$ and $\tilde{R} := H$ which implies that $\tilde{T}_i = hT_i$ for $i \in J$ and $\tilde{T}_i = T_i$ for $i \in \tilde{J}$. It is clear that $\tilde{T}_i \subset hTT_0$ for $i \in J$ and $\tilde{T}_i \subset TT_0$ for $i \in \tilde{J}$. Since $z = hy \in TT_0 \cap hTT_0$ we deduce that $\tilde{TT} \subset TT_0 \cup hTT_0$. Arguments similar to the first case yield a contradiction to the minimality of $c(M)$ since $e$ has no counterpart in $TT$.

The next step is to study the action of a product of elements of the subgroups $\langle S_i \rangle$. The following Lemma shows in particular that $\langle S_1 \rangle * \cdots * \langle S_p \rangle$ acts non-trivially on $T$.

**Lemma 12** Let $M$ be as in Lemma 11 and let $w = g_1 \cdots g_n$ with $g_j \in \langle S_i \rangle$. Define further $\overline{g}_0 = 1$ and $\overline{g}_i := g_1 \cdots g_i$ for $1 \leq i \leq n$. Then

1. $\overline{g}_j TT \cap \overline{g}_{j+1} TT = \overline{g}_j T_{i+1} = \overline{g}_{j+1} T_{i+1}$, for $1 \leq i \leq n - 1$ and

2. $\overline{g}_i TT \cap \overline{g}_k TT = \emptyset$ for $|j - k| \geq 2$.

**Proof** The first assertion follows immediately from Lemma 11: Since $\overline{g}_{j+1} TT \cap TT = T_{j+1}$ multiplication with $\overline{g}_j$ shows that $\overline{g}_j TT \cap \overline{g}_{j+1} TT = \overline{g}_j T_{j+1}$. Let now $|j - k| \geq 2$. W.l.o.g. we can assume that $k \leq j - 2$. We will show that $(\overline{g}_j TT \cup \cdots \cup \overline{g}_{j-2} TT) \cap \overline{g}_j TT = \emptyset$. This clearly implies the second assertion since $\overline{g}_j TT \subset \overline{g}_j TT \cup \cdots \cup \overline{g}_{j-2} TT$. The proof is by induction on $j$.

Assume first that $j = 2$. We know that $\overline{g}_2 TT \cap \overline{g}_1 TT = T_{i_1} = \overline{g}_1 T_{i_1}$ and that $\overline{g}_2 TT \cap \overline{g}_3 TT = \overline{g}_1 T_{i_2}$. We know that $\overline{g}_1 T_{i_1}$ and $\overline{g}_1 T_{i_2}$ are disjoint since $T_{i_1}$ and $T_{i_2}$ are disjoint. Let $e$ be an edge of the segment joining $\overline{g}_1 T_{i_1}$ and $\overline{g}_1 T_{i_2}$.

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It is clear that $\tilde{g}_0 TT$ and $\tilde{g}_2 TT$ lie in $T - \tilde{h}$. Since they clearly lie in different components, they must be disjoint.

Assume now that the assertion holds for $j - 1$. Since $(\tilde{g}_0 TT \cup \cdots \cup \tilde{g}_{j-2} TT) \cap \tilde{g}_{j-1} TT = \emptyset$ we get that $(\tilde{g}_0 TT \cup \cdots \cup \tilde{g}_{j-2} TT) \cap \tilde{g}_{j-1} TT = \tilde{g}_{j-2} TT \cap \tilde{g}_{j-1} TT = \tilde{g}_{j-2} T_{i_j} = \tilde{g}_{j-1} T_{i_j}$. We further know that $\tilde{g}_{j-1} TT \cap \tilde{g}_{j} TT = \tilde{g}_{j} T_{i_j}$ and since $\tilde{g}_{j-1} T_{i_j}$ and $\tilde{g}_{j-1} T_{i_{j-1}}$ are disjoint, we can argue as before which completes the proof. □

The following two Lemmas are the main tools to prove Theorem 2. They show why stable edges deserve their name.

**Lemma 13** Let $M$ be as in Lemma 11 and $h, \tilde{h} \in H \cup H^{-1}$. Assume that $h \neq \tilde{h}$. Then $e(h) \notin [x, \tilde{h} x]$.

**Proof** We split the proof in the cases where $L_M(h)$ is odd or even.

Suppose that $L_M(h) = 2m + 1 \geq 1$ and that $e(h) \subset [x, \tilde{h} x]$. We first show that $e(h) \subset [x, \tilde{h} x] - (TT \cup \tilde{h} TT)$. Since $e(h)$ is an essential letter of $h$, it does not lie in $TT$. Assume that $e(h) \subset \tilde{h} TT$. This implies that at most $m$ edges lie in $[x, \tilde{h} x] - (TT \cup \tilde{h} TT)$. Left-multiplication with $h^{-1}$ shows that at most $m$ edges lie in $[x, h^{-1} \tilde{h} x] - (TT \cup h^{-1} \tilde{h} TT)$ which implies that $L_M(h^{-1} \tilde{h}) < L_M(h)$ and therefore $h^{-1} \tilde{h} \prec_M h$, a contradiction to the minimality of $M$. Thus $e(h) \subset [x, \tilde{h} x] - (TT \cup \tilde{h} TT)$ and at least $m + 1$ essential edges of $h$ and at least $m + 1$ essential edges of $\tilde{h}$ lie in $[x, h x] \cap [x, \tilde{h} x]$ (see Figure 3). If $h^{-1} = \tilde{h}$ it follows that $e(h) = e(h^{-1}) = h^{-1} e(h)$, i.e., that $h^{-1}$ (and $h$) acts with a fixed point, a contradiction. We are left with the case $h^{-1} \neq \tilde{h}$ (by assumption we have $h \neq \tilde{h}$). It follows that at most $L_M(\tilde{h}) = 1$ edges lie in $[x, h x] - (TT \cup \tilde{h} TT)$. Left multiplication with $h^{-1}$ shows that at most $L_M(\tilde{h}) = 1$ edges lie in $[x, h^{-1} \tilde{h} x] - (TT \cup h^{-1} \tilde{h} TT)$, which means that $L_M(h^{-1} \tilde{h}) < L_M(\tilde{h})$ and therefore $h^{-1} \tilde{h} \prec_M \tilde{h}$; contradiction to the minimality of $M$.

![Figure 3: $[x, h x]$ and $[x, \tilde{h} x]$ (after left-multiplication with $h^{-1}$)](image)

Suppose that $L_M(h) = 2m \geq 2$ and that $e(h) \in [x, h x] \cap [x, \tilde{h} x]$. Similar as in the first case we argue that $e(h) \subset [x, \tilde{h} x] - (TT \cup \tilde{h} TT)$. This implies that at least $m$ essential edges of $h$ and of $\tilde{h}$ lie in $[x, h x] \cap [x, \tilde{h} x]$. Suppose first that more than $m$ essential edges of $h$ lie in $[x, h x] \cap [x, \tilde{h} x]$. If $h^{-1} \neq \tilde{h}$ we argue as before that $h^{-1} \tilde{h} \prec_M \tilde{h}$. If $h^{-1} = \tilde{h}$ we get that $u(h) = u(\tilde{h}) = u(h^{-1}) = v(h)$ which implies by Lemma 9 that $h$ acts with a fixed point, a contradiction.
Suppose now that exactly $m$ essential edges of $h$ lie in $[x, hx] \cap [x, \tilde{h}x]$. This implies that $e(h) \subset u(h)$ and therefore that $v(h) \prec_M u(h)$. Note that we can assume that $L_M(\tilde{h}) \geq 2m$, otherwise we get a contradiction as before after interchanging $h$ and $\tilde{h}$. It is clear that $L_M(h^{-1}\tilde{h}) = L_M(\tilde{h})$ (see Figure 3). It is further clear that $u(\tilde{h}^{-1}) = u(h^{-1})$, i.e. $v(\tilde{h}) = v(h^{-1})$ (think of Figure 3 after left-multiplication with $\tilde{h}^{-1}$). Left multiplication with $h^{-1}$ shows that $u(h^{-1})$ differs from $u(\tilde{h})$ by having $u(h^{-1}) = v(h)$ as a prefix instead of $u(h)$. Since $v(h) \prec_M u(h)$ this implies that $u(h^{-1}) \prec_M u(\tilde{h})$ and therefore $h^{-1}\tilde{h} \prec_M \tilde{h}$, a contradiction to the minimality of $M$. \hfill \Box

**Lemma 14** Let $M$ be as in Lemma 11 and let $g \in (S_1 \cup \cdots \cup S_p) - 1$ and $h, \tilde{h} \in H \cup H^{-1}$. Then no essential edge of $h$ lies in $[x, gz] \cup [gx, ghx]$. 

**Proof** We write $g$ as the product $g = g_1 \cdots g_n$ with $g_j \in (S_i) - 1$ for $1 \leq j \leq n$ and $i_j \neq i_{j+1}$ for $1 \leq j \leq n-1$. We define $\tilde{g}_i := g_1 \cdots g_i$. The remainder of the proof is to check the two cases $n \geq 2$ and $n = 1$.

1. case: $n \geq 2$. Note that $[x, hx] \cap g_i TT \subset TT$ since otherwise there would be less than $L_M(h)$ edges in $[hx, gx] = (hTT \cup g_i T T)$ and therefore less then $L_M(h)$ edges in $[x, h^{-1}g_j x] - (TT \cup h^{-1}g_j TT)$ which implies that $L_M(h^{-1}g_j) = L_M(g_j^{-1}h) < L_M(h)$ and therefore $g_j^{-1}h \prec_M h$, a contradiction to the minimality of $M$. By Lemma 11 $[x, hx] \cap g_i TT \subset TT$ implies that $[x, hx] \cap g_i TT \subset T_i$.

As in the proof of Lemma 12 we see that $\tilde{g}_i TT \cap [x, hx] = \emptyset$ for $i \geq 2$ and that $(\tilde{g}_i TT \cup \cdots \cup \tilde{g}_n TT) \cap [x, hx] \subset \tilde{g}_i TT = TT$. Since $[x, gx] \subset \tilde{g}_i TT \cup \tilde{g}_i TT$ it follows that $[x, gx] \cap [x, hx] \subset TT$, i.e. that no essential edge of $h$ lies in $[x, gx]$. Analogously we see that $(\tilde{g}_i TT \cup \cdots \cup \tilde{g}_n TT) \cap [gx, ghx] \subset \tilde{g}_i TT = TT$. It also follows from Lemma 12 that $TT \cap g_i TT = \emptyset$. This shows that $[x, hx] \cap [gx, ghx] = \emptyset$ since otherwise $[x, hx] \cup [gx, ghx]$ would have to be connected and contain the segment joining $TT$ and $gTT$. This segment however contains edges of $(\tilde{g}_i TT \cup \cdots \cup \tilde{g}_n TT) - (TT \cup gTT)$ which we have shown to not lie in $[x, hx] \cup [gx, ghx]$; this proves the assertion.

2. case: $n = 1$. Note that in this case $g = \tilde{g}_i = g_1$. As in the first case we see that $[x, hx] \cap g_i TT \subset T_i$. It follows that $TT \cup gTT$ contains no essential edge of $h$, as in the first case this implies that $[x, gx]$ contains no essential edges of $h$. Analogously we see that $g_i^{-1}TT \cap [x, hx] \subset T_i = g_i^{-1}T_i$ and therefore $TT \cap [gx, ghx] \subset T_i$. Let now $e$ be the essential edge of $h$ such that $e \cap TT'$ contains exactly one vertex $y$. Assume now that an essential edge $f$ of $h$ lies in $[gx, ghx]$. It follows that $e \subset [gx, ghx]$. This is trivial if $e = f$ and if $e \neq f$ this follows since $[gx, ghx]$ is connected, contains $f$ and $gx$ and $f$ and $gx$ lie in different components of $TT - \{e\}$. This implies that $y \in T_i$ since $y \in TT \cap [gx, ghx]$. Since $e \notin TT \cup gTT$ it follows that $e$ is the edge of $[gx, ghx] - (TT \cup ghTT)$ that contains one vertex of $gTT$, namely $y$ (see Figure 4).

Suppose first that $L_M(h) = 1$. If $h \neq \tilde{h}^{-1}$ we get that less that $L_M(\tilde{h})$ edges lie in $[hx, ghx] - (hTT \cup ghTT)$. This implies that less than $L_M(\tilde{h})$ edges lie in $[x, h^{-1}ghx] - (TT \cup h^{-1}ghTT)$ and therefore $L_M(h^{-1}gh) < L_M(h)$ which implies that $h^{-1}gh \prec_M \tilde{h}$, a contradiction to the minimality of $M$.

If $h = \tilde{h}^{-1}$ then $e$ is clearly the only edge of $[x, hx] - (TT \cup hTT)$ and the only edge of $[gx, ghx] - (gTT \cup ghTT)$. If $h = \tilde{h}$ this implies that $g$ fixes $e$, a contradiction to $e \notin T_i$.\hfill \Box
If \( h = \tilde{h}^{-1} \) then \( gh^{-1} \) maps the single edge \( c \) of \([x, hx] - (TT \cup hTT)\) onto the single edge of \([gx, gh^{-1}x] - (gTT \cup gh^{-1}TT)\). It is clear that \( gh^{-1} \) acts on \( c \) as an inversion which contradicts our assumption on the action of \( G \) on \( T \).

Suppose now that \( L_M(h) \geq 2 \). Note first that we can assume that also \( L_M(\tilde{h}) \geq 2 \) since we otherwise get same contradiction as before after interchanging \( h \) and \( \tilde{h} \). This implies that \( u(h), v(h), u(\tilde{h}) \) and \( v(\tilde{h}) \) are segments of non-zero \( L_M\)-length. Let \( u_1 \) be the prefix of \( u(h) \) of \( L_M\)-length 1 and \( u_2 \) be the prefix of \( u(\tilde{h}) \) of \( L_M\)-length 1. This means in particular that \( c \) is the only edge of \( u_1 - TT \). Let further be \( f \) the only edge of \( u_2 - TT \). It is clear that \( gf = c \) since we have seen that \( c \) is the edge of \([gx, gh^{-1}x] - (gTT \cup gh^{-1}TT)\) that contains one vertex of \( gTT \). Since \( g \notin TT \) we know that \( g \) does not fix \( c \). This clearly implies that \( u_1 \neq u_2 \), i.e. either \( u_1 <_M u_2 \) or \( u_2 <_M u_1 \). Note that \( u(gh) \) differs from \( u(h) \) only in that \( u(gh) \) has \( u_1 \) as a prefix meanwhile \( u(h) \) has \( u_2 \) as a prefix and that \( u(g^{-1}h) \) differs from \( u(h) \) only in that \( u(g^{-1}h) \) has \( u_2 \) as a prefix meanwhile \( u(h) \) has \( u_1 \) as a prefix (see Figure 4). It is further clear that \( L_M(h) = L_M(g^{-1}h) \) and \( L_M(\tilde{h}) = L_M(g\tilde{h}) \) and that \( v(h) = v(g^{-1}h) \) and \( v(\tilde{h}) = v(g\tilde{h}) \). In the case \( u_1 <_M u_2 \) this implies that \( gh \sim_M h \), in the case \( u_2 <_M u_1 \) this means that \( g^{-1}h \sim_M h \), in either case a contradiction to the minimality of \( M \).

We now have all tools necessary to prove Theorem 2.

**Proof of Theorem 2** In view of Lemma 10 it suffices to show that \( \langle M \rangle = \langle S_1 \rangle * \cdots * \langle S_p \rangle * F(H) \) if \( M \) is a minimal set with partition \( (S_1, \ldots, S_p, H) \) for which none of the situation (1)-(3) occurs. It is clear that \( \langle M \rangle \) is a quotient of the group \( \langle S_1 \rangle * \cdots * \langle S_p \rangle * F(H) \). We have to show that the kernel of the quotient map is trivial. We do this by showing that the kernel of the action of \( \langle S_1 \rangle * \cdots * \langle S_p \rangle * F(H) \) on \( T \) is trivial, i.e. that no element \( w \in \langle S_1 \rangle * \cdots * \langle S_p \rangle * F(H) - 1 \) acts trivially on \( T \). After conjugating \( w \) appropriately in \( \langle S_1 \rangle * \cdots * \langle S_p \rangle * F(H) \) we can assume that

\[
w = h_{\epsilon_1} g_1 h_{\epsilon_2}^2 g_2 g_{n-1} h_{\epsilon_n}^2 g_n h_{n+1}^{-1},
\]

where \( h_i \in H \) and \( \epsilon_i \in \{-1, 1\} \) for \( 1 \leq i \leq n+1 \), \( g_i \in \langle S_1 \rangle * \cdots * \langle S_p \rangle \) for \( 1 \leq i \leq n \) such that \( h_{\epsilon_i} \neq h_{\epsilon_{i+1}}^{-1} \) if \( g_i = 1 \) for \( 1 \leq i \leq n \). We define \( \bar{g}_i := h_{\epsilon_i}^2 g_1 h_{\epsilon_2}^2 g_2 g_{i-1} h_{\epsilon_i}^2 g_i \) for \( 1 \leq i \leq n \) and \( \bar{g}_0 = 1 \).

We will show by induction on \( i \) that \([x, \bar{g}_i h_{i+1} x] \) contains the edge \( \bar{e}_i e(h_{i+1}^2) \). For \( i = n \) this implies that \([x, \bar{g}_n h_{n+1} x] = [x, gx] \) is a non-trivial segment and
therefore that $g$ acts non-trivially, which proves Theorem 2.

For $i = 0$ this is trivial since by Lemma 11 we know that $L_M(h_i^{e_i}) \geq 1$ and since $h$ is assumed to act without fixed point. This implies that $e(h_i^{e_i}) \subset [x, h_i^{e_i}x] = [x, gx]$ is defined.

Assume that the assertion is proven for $i$. It follows from Lemma 13 if $g_{i+1} = 1$ and from Lemma 14 otherwise that $e(h_{i+1}^{e_{i+1}}) \notin [g_{i+1}x, g_{i+1}h_{i+2}^{e_{i+2}}x]$. Left multiplication with $g_{i+1}h_{i+1}^{e_{i+1}}$ shows that $g_{i+1}h_{i+1}^{e_{i+1}}e(h_{i+1}^{e_{i+1}}) = g_{i+1}e(h_{i+1}^{e_{i+1}}) \notin [g_{i+1}x, g_{i+1}h_{i+2}^{e_{i+2}}x]$. Since we know by the induction hypothesis that $g_{i+1}e(h_{i+1}^{e_{i+1}}) \subset [x, g_ih_{i+1}^{e_{i+1}}x]$, this shows that

$$[x, g_ih_{i+1}^{e_{i+1}}x] \cap [g_{i+1}x, g_{i+1}h_{i+2}^{e_{i+2}}x] \subset [g_{i+1}x, g_{i+1}h_{i+1}^{e_{i+1}}x] \ (\ast)$$

since all of $[x, g_ih_{i+1}^{e_{i+1}}x] \cap [g_{i+1}x, g_{i+1}h_{i+2}^{e_{i+2}}x]$ must lie in the segment joining $g_{i+1}e(h_{i+1}^{e_{i+1}})$ and $g_{i+1}h_{i+1}^{e_{i+1}}x$ which clearly is a subset of $[g_{i+1}x, g_{i+1}h_{i+1}^{e_{i+1}}x]$.

Again using Lemma 13 or Lemma 14 we get that $e(h_{i+2}^{e_{i+2}}) \notin [x, g_i^{-1}x] \cup [g_{i+1}x, g_{i+1}h_{i+1}^{e_{i+1}}x]$, and left multiplication with $g_{i+1}$ shows that $g_{i+1}e(h_{i+2}^{e_{i+2}}) \notin [g_{i+1}x, g_{i+1}h_{i+2}^{e_{i+2}}x] \cup [g_{i+1}x, g_{i+1}h_{i+2}^{e_{i+2}}x]$. Since $g_{i+1}e(h_{i+2}^{e_{i+2}}) \subset [g_{i+1}x, g_{i+1}h_{i+1}^{e_{i+1}}x]$ we conclude with (\ast) that $g_{i+1}e(h_{i+2}^{e_{i+2}}) \notin [x, g_{i+1}h_{i+1}^{e_{i+1}}x]$. Together with $g_{i+1}e(h_{i+2}^{e_{i+2}}) \notin [g_{i+1}x, g_{i+1}x]$ we get $g_{i+1}e(h_{i+2}^{e_{i+2}}) \notin [x, g_{i+1}x]$, it follows that $g_{i+1}e(h_{i+2}^{e_{i+2}}) \subset [x, g_{i+1}h_{i+2}^{e_{i+2}}x]$ which finishes the proof of Theorem 2.

\textbf{Remark} A careful analysis of the proof shows that we do not only show that $\langle M \rangle = \langle S_1 \rangle \ast \cdots \ast \langle S_n \rangle \ast F(H)$ if none of the situations (1)-(3) of Theorem 2 occurs, but also that every element of $\langle M \rangle$ that acts with a fixed point is conjugate to an element of one of the subgroups $\langle S_i \rangle$.

2 Acylindricity and JSJ-splittings

In [1] B. Baumslag defines a subgroup $U$ of a group $G$ to be \textit{malnormal} if $gUg^{-1} \cap U = 1$ for all $g \in G - U$ and studies amalgamated free products $G = A \ast_C B$ where $C$ the amalgam is malnormal in $A$ and $B$ and therefore in $G$. In [14] Karrass and Solitar prove that such a group $G$ cannot be generated by two elements if $C \neq 1$. They further generalize the notion of malnormality and say that the amalgam $A$ of an amalgamated product $G = \prod_{i=1}^{n} G_i$ is \textit{k-step malnormal} if every element $g \in G$ such that $gAg^{-1} \cap A \neq 1$ is of length at most $k$ with respect to the length function derived from the normal form for amalgamated products. The case $k = 0$ recovers malnormality. Note that except in the case $k = 0$ it does not make sense to say that $A$ is $k-$step-malnormal in $G$ without referring to a specific decomposition of $G$ into an amalgamated product with amalgam $A$.

An amalgamated product $G$ with $k$-step malnormal amalgam can be rewritten as a graph of groups and it is easy to see (proof of Lemma 19) that the action of a $G$ on the Bass-Serre tree associated to this splitting is $2k + 2$-acylindrical. Sela [26] now calls a splitting of a group (as the fundamental group of a graph of groups) $k$-acylindrical if the action on the Bass-Serre tree is $k$-acylindrical. Theorem 4 therefore bounds the complexity of a $k$-acylindrical splitting of a group in terms of its rank. This can be regarded as a generalization of the result of Karrass and Solitar.
2.1 $k$-acyclindrical actions

In this subsection we will give a proof of Theorem 4. In fact we will prove the following Proposition which clearly implies Theorem 4 when we choose $M$ to be a minimal generating set of $G$ with the trivial marking $(H)$ where $H = M$.

**Proposition 15** Let $G$ be a non-cyclic freely indecomposable group, $G \times T \to T$ be a minimal $k$-acyclindrical action and $M$ be a generating set with marking $(S_1, \ldots, S_p, H)$. Then the number of vertices of $G \setminus T$ is bounded by $1 + 2k(\# H + p - 1)$.

**Proof** We define the signature of a marking $(S_1, \ldots, S_p, H)$ to be the pair $(p, \# H) \in \mathbb{N} \times \mathbb{N}$. We further define a well ordering on $\mathbb{N} \times \mathbb{N}$ by saying that $(n, m) \leq (n', m')$ if either $m < m'$ or $m = m'$ and $n < n'$.

Assume now that Proposition 15 does not hold. Among all groups $G$, actions $G \times T \to T$ and marked sets that satisfy the hypothesis of Proposition 15 and for which the conclusion of Proposition 15 does not hold choose an action $G \times T \to T$ and a marked set $M$ with marking $(S_1, \ldots, S_p, H)$ such that the signature $(p, \# H)$ of the marking is minimal.

We apply Corollary 3 to $M$. If $(M) = \langle S_1 \rangle \cdots \langle S_p \rangle \ast F(H)$ then either $p = 1$ and $\# H = 0$ or $p = 0$ and $\# H = 1$ since $(M) = G$ is assumed to be freely indecomposable. In the first case $G = \langle M \rangle = \langle S_1 \rangle$ acts with a global fixed point $x$ which implies that $T = \{x\}$ since $T$ is assumed to be minimal, i.e. $G \setminus T = \{x\}$ which proves the Proposition. In the second case $G = \langle M \rangle = F(H)$ is cyclic, a contradiction to our assumption on $G$. It follows that the marked set $M$ is equivalent to a marked set $\tilde{M}$ with marking $(\tilde{S}_1, \ldots, \tilde{S}_p, \tilde{H})$ such that one of the situations (1)-(3) of Corollary 3 occurs. It is clear that the conclusion of Proposition 15 also fails for $\tilde{M}$.

(1) Possibly after renumbering the $\tilde{S}_i$ we can assume that $\tilde{T}_1 \cap \tilde{T}_p \neq \emptyset$. Choose a vertex $z \in \tilde{T}_1 \cap \tilde{T}_p$ and vertices $x \in \tilde{T}_1$ and $y \in \tilde{T}_p$ such that $\langle \tilde{S}_1 \rangle x = z$ and $\langle \tilde{S}_p \rangle y = y$. Since the action of $G$ is $k$-acyclindrical and since there exists an element of $\langle \tilde{S}_1 \rangle$ that fixes $x$ and $z$ we get that $d(x, z) \leq k$. By the same reasoning we get that $d(y, x) \leq k$ and therefore $d(x, y) \leq 2k$. We define a new tree $T'$ obtained from $T$ by collapsing all edges that are $G$-equivalent to an edge of $[x, y]$. Since this is done $G$-equivariantly, $G$ acts on $T'$ and this action is clearly $k$-acyclindrical and minimal. We denote the quotient map $T \to T'$ by $\pi$. Note that $\langle \tilde{S}_1 \rangle$ and $\langle \tilde{S}_p \rangle$ act on $T'$ with the common fixed point $\pi(x) = \pi(y)$, i.e. $\langle \tilde{S}_1 \cup \tilde{S}_p \rangle$ is special with respect to the action $G \times T' \to T'$. It follows that we can endow $\tilde{M}$ with the marking $(\tilde{S}_1 \cup \tilde{S}_p, \tilde{S}_2, \ldots, \tilde{S}_{p-1}, \tilde{H})$. This marking however has a smaller signature and therefore Proposition 15 holds for the action $G \times T' \to T'$ and $\tilde{M}$ with marking $(\tilde{S}_1 \cup \tilde{S}_p, \tilde{S}_2, \ldots, \tilde{S}_{p-1}, \tilde{H})$. It follows that $G \setminus T'$ has at most $1 + 2k(\# \tilde{H} + (p - 1) - 1)$ vertices. Since the quotient map $\pi : T \to T'$ identifies at most $2k+1$ $G$-inequivalent vertices this implies that $G \setminus T'$ has at most $1 + 2k(\# \tilde{H} + (p - 1) - 1) + 2k = 1 + 2k(\# \tilde{H} + p - 1) = 1 + 2k(\# H + p - 1)$ vertices, i.e. the conclusion of Proposition 15 holds for the action $G \times T \to T'$ and the marked set $\tilde{M}$ with marking $(\tilde{S}_1, \ldots, \tilde{S}_p, \tilde{H})$: a contradiction.

(2) After renumbering the $\tilde{S}_i$ we can assume that $\tilde{H} \tilde{T}_1 \cap \tilde{T}_p \neq \emptyset$. Choose $x, y \in \tilde{T}_p$ such that $\tilde{H} x = y$ and $z \in \tilde{T}_p$ such that $\langle \tilde{S}_p \rangle z = z$. As in the first case we observe that $d(x, z) \leq k$ and that $d(y, z) \leq k$. We define $T'$ to be the tree obtained from
by collapsing all edges that are $G$-equivalent to edges of $[x, z] \cup [x, y]$ and let 
$\pi : T \to T'$ be the quotient map. It is clear that $\langle S_p \rangle$ and $\langle \bar{h} \rangle$ act with 
the common fixed point $\pi(x) = \pi(y) = \pi(z)$, i.e. $\langle S_p \cup \{ \bar{h} \} \rangle$ is a special subgroup 
with respect to the action $G \times T' \to T'$. It follows that we can endow $\tilde{M}$ with the 
marking $(\tilde{S}_1, \ldots, \tilde{S}_{p-1}, \tilde{S}_p \cup \{ \bar{h} \}, \tilde{H} - \{ \bar{h} \})$. The signature of the new marking 
is smaller than the signature of the marking of $\tilde{M}$, as in case (1) we conclude that $G \setminus T'$ has at most 
$1 + 2k(p + \#(\tilde{H} - \{ \bar{h} \}) - 1) = 1 + 2k(p + \#\tilde{H} - 2)$ vertices. We argue as in (1) and 
deduce that $G \setminus T$ has at most $1 + 2k(p + \#\tilde{H} - 2) + 2k = 1 + 2k(p + \#\tilde{H} - 1)$; a 
contradiction.

(3) This case is the easiest. If $\bar{h} = 1$ we drop $\bar{h}$ from $\tilde{H}$ and $\tilde{M}$. The marking 
of the new marked set has smaller signature and therefore the conclusion of Proposition 15 holds which clearly implies that it also holds for the original marked set; a contradiction. If $\bar{h} \neq 1$ then $\bar{h}$ generates a special subgroup, i.e.
we can assign to $\tilde{M}$ the new marking $(\tilde{S}_1, \ldots, \tilde{S}_p, \tilde{S}_{p+1} = \{ \bar{h} \}, \tilde{H} - \{ \bar{h} \})$. 
The new marking has lower signature, therefore Proposition 15 holds, i.e. $G \setminus T$ has at most 
$1 + 2k(p + 1 + \#(\tilde{H} - \{ \bar{h} \}) - 1) = 1 + 2k(p + \#\tilde{H} - 1)$ vertices. This implies that Proposition 15 also holds 
for the original marking of $\tilde{M}$; a contradiction. □

Remark The main tool Sela uses to prove the existence of the constants $c(k, G)$ 
mentioned in the introduction is Theorem 3.2 of [26] which states that if $G$ is a f.g.
freely indecomposable group with generating set $X = \{x_1, \ldots, x_n\}$ then there exists a 
constant $\lambda_k$ such that for every $k$-acylindrical minimal $G$-tree $T$ there exists a vertex 
v $\in T$ and an automorphism $\phi \in \text{Aut}(G)$ such that $d(v, \phi(x_i)(v)) \leq \lambda_k$ for $1 \leq i \leq n$.
The proof of Theorem 4 shows that our approach gives a constant $\alpha_k$ such that for 
every $k$-acylindrical minimal $G$-tree $T$ there exists a vertex $v \in T$ and a set $X'$ that 
is Nielsen-equivalent to $X$ such that $d(v, x'(v)) \leq \alpha_k$ for all $x' \in X'$. The advantage 
of the present approach is that $\alpha_k$ can be bounded in terms of $n$, the disadvantage 
is that we do not obtain a generating set that gets mapped onto the original one by an 
automorphism of $G$. This implies that the techniques in this paper do not give a proof 
of Sela’s result on acylindrical super accessibility, compare with Theorem 4.3 of [26].

2.2 Almost 1-acylindrical actions

Let $G$ be a group that can be written as an amalgamated product $G_1 *_{A} G_2$ 
where $A$ is malnormal in $G_1$ and $G_2$. The action of $G$ on the Bass-Serre tree 
with respect to this splitting is clearly 1-acylindrical.

Assume now that $G$ is an amalgamated product of type $\ast_{i=1}^n \text{G}_i$ where $A$ is 
malnormal in $G_i$ for $1 \leq i \leq n$. We rewrite $G$ as a graph of groups with vertices 
with labels $A$ and $G_i$ for $1 \leq i \leq n$, edges with label $A$ joining the vertices 
with labels $A$ and $G_i$ for $1 \leq i \leq n$ and the boundary monomorphisms being 
the identity on $A$. The action of $G$ on the Bass-Serre tree with respect to 
this splitting is clearly not 1-acylindrical since any element of $A$ fixes all edges 
emanating from the vertex that has $A$ as a stabilizer. It is however clear that 
the conjugates of elements of $A$ are the only elements of $G$ that fix segments of 
non-trivial length and that the action is 2-acylindrical since the malnormality of 
$A$ in $G_i$ guarantees that no non-trivial element of $G$ fixes two edges emanating 
from a vertex fixed under the action of a conjugate of $G_i$ for $1 \leq i \leq n$. This 
means that this action is morally 1-acylindrical if we forget about this artificial
new vertex. We make this more precise:

Let $G \times T \to T$ be an action. We call a vertex $v$ of $T$ that has non-trivial stabilizer and for which there exists no non-trivial element that fixes two distinct edges emanating from $v$ essential. All other vertices are called inessential. We further call an action $G \times T \to T$ almost 1-acylindrical if the stabilizer of every inessential vertex also fixes all edges emanating from this vertex and if the action is 2-acylindrical. The 2-acylindricity clearly implies that there are no two adjacent inessential vertices.

It is easy to see that a vertex $v$ is essential iff the stabilizer of any edge emanating from $v$ is malnormal in the stabilizer of $v$ and the stabilizer of two $G$-inequivalent edges emanating at $v$ are conjugacy separated in the stabilizer of $v$. (We say that $U < G$ and $V < G$ are conjugacy separated if $gUg^{-1} \cap V = 1$ for all $g \in G$.)

Let $U$ be a special subgroup of $G$. We say that $U$ is small if either $U$ stabilizes an edge of $T$ or if $T_U$ consists of a single vertex. We further call a special subgroup $U$ big if it is not small.

**Lemma 16** Let $G \times T \to T$ be almost 1-acylindrical and $U < G$ be special. Then the following hold:

(a) If $U$ is small, then it is contained in the stabilizer of every point of $T_U$.

(b) If $U$ is big then there exists an unique vertex of $T_U$ that is fixed under the action of $U$.

**Proof** (a) If $T_U$ consists of a single vertex, this is trivial. If $U$ stabilizes an edge of $T$ but not all of $T_U$, then there must be two adjacent edges $e$ and $f$ of $T_U$ such that $e$ is stabilized by $U$ and the $f$ is stabilized by a non-trivial proper subgroup of $U$ but not by $U$. This however is clearly impossible if the action of $G$ on $T$ is almost 1-acylindrical.

(b) This is clear since the existence of two such vertices would imply that $U$ also stabilizes the segment joining them and therefore also stabilizes at least one edge of $T$, which implies that $U$ is small. □

We will prove a stronger version of Theorem 4 for almost 1-acylindrical actions; the proof is a refinement of the proof of Proposition 15.

**Theorem 17** Let $G$ be a non-cyclic, freely indecomposable group and $G \times T \to T$ be a minimal almost 1-acylindrical action. Let $N$ be the number of $G$-equivalence classes of essential vertices of $T$. Then rank $G \geq N + 1$.

**Proof** As in the case of the proof of Theorem 4 we prove a slightly stronger claim. For any marked set $M$ with marking $(S_1, \ldots, S_p, H)$ we define $r$ to be the number of big subgroups $(S_i)$. We will show that if $M$ is a generating set of $G$ then $\#H + p + r \geq N + 1$, for a minimal generating set with marking $(H = M)$ this proves Theorem 17. As in the proof of Proposition 15 we assume that there is a counterexample and choose this counterexample to be minimal with respect to the signature $(p, \#H)$. If $T$ consists of a single vertex then any special subgroup is small. We can therefore argue as in the proof of Proposition 15 to exclude that $G \cong (S_1) * \cdots * (S_p) * F(H)$. As before we conclude that $M$
is equivalent to a marked set $\tilde{M}$ with marking $(\tilde{S}_1, \ldots, \tilde{S}_p, \tilde{H})$ such that one of the cases (1)-(3) of Corollary 3 occurs. It is clear that also exactly $r$ of the new special subgroups $\langle \tilde{S}_i \rangle$ are big. This implies in particular that the claim also fails for the marked set $\tilde{M}$.

(1) After renumbering the $\tilde{S}_i$ we can assume that $\tilde{T}_1 \cap \tilde{T}_p \neq \emptyset$. We distinguish the cases where one, two or none of $\langle \tilde{S}_1 \rangle$ and $\langle \tilde{S}_p \rangle$ are small.

Assume first that $\langle \tilde{S}_1 \rangle$ and $\langle \tilde{S}_p \rangle$ are small. Let $v$ be a vertex of $\tilde{T}_1 \cap \tilde{T}_p$. By Lemma 16 $\langle \tilde{S}_1 \rangle$ and $\langle \tilde{S}_p \rangle$ fix $v$, it follows that $\langle \tilde{S}_1 \cup \tilde{S}_p \rangle$ fixes $v$ and is therefore special, i.e., we can endow $\tilde{M}$ with the marking $(\tilde{S}_1 \cup \tilde{S}_p, \tilde{S}_2, \ldots, \tilde{S}_{p-1}, \tilde{H})$.

Since $\langle \tilde{S}_1 \cup \tilde{S}_p \rangle$ might be big, we get that at most $r+1$ of the special subgroups associated to the new marking are big. Since the signature of the new marking is smaller than the signature of the original marking it follows that the claim holds for the new marking, i.e. that $\#H + (p - 1) + (r + 1) \geq N + 1$. This clearly implies that $\#H + p - r \geq N + 1$, i.e. that the claim also holds for the original marking, a contradiction to the assumption.

Assume now that $\langle \tilde{S}_1 \rangle$ is small and that $\langle \tilde{S}_p \rangle$ is big, the opposite case is analogous. Let $x$ be the vertex of $\tilde{T}_p$ that is fixed under the action of $\langle \tilde{S}_p \rangle$ and $y$ be a vertex of $\tilde{T}_1 \cap \tilde{T}_p$. The segment $[x, y]$ is fixed under the action of at least one element of $\langle \tilde{S}_p \rangle$. This implies that no essential vertex lies in the interior of $[x, y]$ since the action is assumed to be almost 1-acyldindrical. Let now $T'$ be the tree obtained from collapsing all edges that are $G$-equivalent to edges of $[x, y]$ and let $\pi : T \to T'$ be the quotient map. We consider the induced action $G \times T' \to T'$ which is clearly again almost $1$-acyldindrical and minimal. It is further clear that $\pi(v)$ is essential if $v$ is essential and that every special subgroup that is small with respect to $G \times T \to T$ is small with respect to $G \times T' \to T'$. Since at most two $G$-equivalence classes of essential vertices get identified, it follows that $T'$ has at least $N - 1$ $G$-equivalence classes of essential vertices. It is clear that $\langle \tilde{S}_1 \cup \tilde{S}_p \rangle$ is special with respect to the new action, i.e., we can assign $\tilde{M}$ the marking $(\tilde{S}_1 \cup \tilde{S}_p, \tilde{S}_2, \ldots, \tilde{S}_{p-1}, \tilde{H})$. Of the $p - 1$ special subgroups associated to the marking at most $r$ are big. Since the signature of the new marking is smaller, the claim holds, i.e. $\#H + (p - 1) + r \geq N - 1 + 1 = N$. This however implies that $\#H + p + r \geq N + 1$, a contradiction.

Assume now that $\langle \tilde{S}_1 \rangle$ and $\langle \tilde{S}_p \rangle$ are big. Choose a vertex $z$ that is fixed under the action of $\langle \tilde{S}_1 \rangle$, a vertex $y$ that is fixed under the action of $\langle \tilde{S}_p \rangle$ and a vertex $z$ of $\tilde{T}_1 \cap \tilde{T}_p$. As before we see that no essential vertex lies in the interior of $[x, z]$ or $[y, z]$, it follows that at most three essential vertices lie in $[x, z] \cup [y, z]$. Let $T'$ be the tree obtained from $T$ by collapsing all edges that are $G$-equivalent to edges of $[x, z] \cup [y, z]$. Again $G$ acts of $T'$ and this action is almost 1-acyldindrical and minimal. Since at most three $G$-equivalence classes of essential vertices are identified it follows that $T'$ has at least $N - 2$ essential vertices. It further follows that $\langle \tilde{S}_1 \cup \tilde{S}_p \rangle$ is special with respect to the action $G \times T' \to T'$, i.e. we can endow $\tilde{M}$ with the marking $(\tilde{S}_1 \cup \tilde{S}_p, \tilde{S}_2, \ldots, \tilde{S}_{p-1}, \tilde{H})$ and of the special subgroups associated to this marking at most $r - 1$ are big since two have fallen together. This new marking has smaller signature, thus the claim holds for the new marking, similar arguments as before give a contradiction.

(2) After renumbering the $\tilde{S}_i$ we can assume that $\tilde{T}_1 \cap \tilde{T}_p \neq \emptyset$. We distinguish the cases where $\langle \tilde{S}_p \rangle$ is small and where $\langle \tilde{S}_p \rangle$ is big.

Assume that $\langle \tilde{S}_p \rangle$ is small. If $\tilde{T}_p$ consists of a single vertex $v$, then $\tilde{T}_1 \cap \tilde{T}_p \neq \emptyset$. If $\tilde{T}_p$ consists of more than one vertex, then we can assume that $\tilde{T}_p$ is a union of $p$ vertices of $\tilde{T}_p$. We denote the vertex of $\tilde{T}_p$ that is not fixed by $\tilde{S}_1$ by $v$. Then $\tilde{S}_1$ fixes $\tilde{T}_1 \cap \tilde{T}_p$ and $\tilde{S}_1 \cup \tilde{S}_2 \cup \ldots \cup \tilde{S}_{p-1}$ fixes $v$. This implies that $\tilde{M}$ is equivalent to a marked set $\tilde{M}'$ with marking $\langle \tilde{S}_1 \cup \tilde{S}_2 \cup \ldots \cup \tilde{S}_{p-1}, \tilde{H} \rangle$ and of the special subgroups associated to this marking at most $r - 1$ are big since two have fallen together. This new marking has smaller signature, thus the claim holds for the new marking, similar arguments as before give a contradiction.
\( \emptyset \) implies that \( \bar{h} \) fixes \( v \). It follows that \( \langle S_p \cup \{ \bar{h} \} \rangle \) is special, possibly big. This means that we can assign \( \bar{M} \) the marking \((S_1, \ldots, S_{p-1}, S_p \cup \{ \bar{h} \}, \bar{H} - \{ \bar{h} \})\). Of the special subgroup associated to the new marking at most \( r + 1 \) are big. The new marking has lower signature, arguments as before yield a contradiction.

If \( \langle S_p \rangle \) lies in the stabilizer of an edge \( e = [x, y] \) and this edge is all of \( \bar{T}_p \), the minimality and the almost 1-acylindricity of the action clearly imply that \( x \) and \( y \) are essential. Now \( \bar{h} \bar{T}_p \cap \bar{T}_p \neq \emptyset \) implies that either \( \bar{h} \) fixes either \( x \) or \( y \) or that \( \bar{h} \) maps \( x \) onto \( y \) or \( y \) onto \( x \). In the first case we argue as before, in the second we define \( T' \) to be the tree obtained from \( T \) by collapsing all edges that are \( G \)-equivalent to \( e \). Since only \( G \)-equivalent vertices get identified, it follows that \( T' \) also has \( N \) essential vertices. Now \( \langle S_p \cup \{ \bar{h} \} \rangle \) is a special, possibly big, subgroup with respect to the action \( G \times T' \to T' \). It follows that we can endow \( \bar{M} \) with the marking \((S_1, \ldots, S_{p-1}, S_p \cup \{ \bar{h} \}, \bar{H} - \{ \bar{h} \})\) and that at most \( r + 1 \) of the special subgroups associated to this marking are big. The new marking has a smaller signature. Argument as before yield a contradiction.

If \( \langle S_p \rangle \) lies in the stabilizer of an edge \( e = [x, y] \) but \( \bar{T}_p \neq \emptyset \) it follows that either \( x \) or \( y \) is inessential and therefore \( \langle S_p \rangle \) lies in the stabilizer of an inessential vertex, i.e. \( \bar{T}_p \) consists of an inessential vertex (say \( x \)) and all edges emanating from \( x \). It is clear that \( x \) is the only inessential vertex of \( \bar{T}_p \) and that therefore no other vertex of \( \bar{T}_p \) is \( G \)-equivalent to \( x \). Now \( \bar{h} \bar{T}_p \cap \bar{T}_p \neq \emptyset \) implies that there exist vertices \( y \) and \( z \) of \( \bar{T}_p \neq \emptyset \) such that \( \bar{h}y = z \). If \( y = z \), i.e. \( \bar{h} \) fixes \( y \), we argue as before. If \( y \neq z \) we conclude that \( y \neq x \) and \( z \neq x \) since otherwise exactly one of \( y \) and \( z \) would be \( G \)-equivalent to \( x \) which is clearly impossible since \( y \) and \( z \) are \( G \)-equivalent. We now define \( T' \) to be the tree obtained from \( T \) by identifying \( g[x, y] \) with \( g[x, z] \) for all \( g \in G \) and by subsequently removing the new edges if the image of \( x \) under the quotient map \( \pi : T \to T' \) is of valence 1. This will guarantee the minimality of the induced action \( G \times T' \to T' \). The same arguments as in the case before yield a contradiction.

Assume now that \( \langle S_p \rangle \) is big. Choose \( x \in \bar{T}_p \) such that \( \langle S_p \rangle x = x \) and \( y \in \bar{T}_p \cap \bar{h}\bar{T}_p \). As in case (1) we conclude that no essential vertex lies in the interior of \([x, y]\) or \([y, \bar{h}x]\). It follows that at most three essential vertices lie in \([x, \bar{h}x] \subset [x, y] \cup [y, \bar{h}x] \). It further clear that at least two of them, namely \( x \) and \( \bar{h}x \), are \( G \)-equivalent if there are three. We define \( T' \) to be the tree obtained from \( T \) by collapsing all edges that are \( G \)-equivalent to an edge of \([x, \bar{h}x]\). The new tree has at least \( N - 1 \) \( G \)-equivalence classes of essential vertices since at most two got identified. With respect to the action \( G \times T' \to T' \) we can endow \( \bar{M} \) with the marking \((S_1, \ldots, S_{p-1}, S_p \cup \{ \bar{h} \}, \bar{H} - \{ \bar{h} \})\). Of the special subgroups at most \( r \) are big. Arguments as before yield a contradiction.

(3) If \( \bar{h} = 1 \) we argue as in Proposition 15. If \( \bar{h} \neq 1 \) then \( \bar{h} \) generates a special subgroup, i.e. we can endow \( \bar{M} \) with the marking \((S_1, \ldots, S_p, \{ \bar{h} \}, \bar{H} - \{ \bar{h} \})\). The special subgroup \( \langle \bar{h} \rangle \) is small since otherwise there would exist an edge \( e = [x, y] \subset T_{\langle \bar{h} \rangle} \) such that \( \bar{h} \) fixes \( x \) but does not fix \( e \), i.e. a proper power of \( \bar{h} \) fixes \( e \). This however is impossible since the almost 1-acylindricity of the action implies that the stabilizer of the edge \([x, y]\) is malnormal in the stabilizers of \( x \). It follows that \( r \) of the special subgroups associated to the new marking are big. Since the new marking has smaller signature we can argue as before to get a contradiction.

\[ \square \]

In the following we will see that only the number of essential vertices matters.
As a consequence we obtain the following:

**Corollary 18** Let $G$ be a non-cyclic freely indecomposable group and $G \times T \to T$ be a minimal action such that there are $N$ $G$-equivalence classes of essential vertices. Then rank $G \geq N + 1$.

**Proof** In view of Theorem 17 it suffices to show that $G$ admits a minimal almost 1-acylindrical action on a tree $T'$ such that $T'$ has $N$ $G$-equivalence classes of essential vertices. We first define $\hat{T}$ to the tree obtained from $T$ by collapsing all edges that connect two inessential vertices. We obtain an action $G \times \hat{T} \to \hat{T}$ that is 2-acylindrical. Let $\pi : T \to \hat{T}$ be the canonical projection. It is clear that $\hat{T}$ has at least $N$ $G$-equivalence classes of essential vertices, namely the images of the essential vertices of $T$ under $\pi$. We now have to modify $\hat{T}$ such that the stabilizer of every inessential vertex also fixes all edges emanating from this vertex. This can be done by identifying all $G$-equivalent edges that emanate from the same inessential vertex. Since only $G$-equivalent edges get identified it follows that also only $G$-equivalent vertices get identified and therefore the new tree also has at least $N$ $G$-equivalence classes of essential vertices. It is further clear that the stabilizer of the (formerly) inessential vertices fixes all edges emanating from this vertex since otherwise there would be two $G$-equivalent edges emanating from the same inessential vertex, a contradiction since these were all identified in the process described above. If there is only one edge emanating from a (formerly) inessential vertex we remove this edge in order to make the action minimal. The new tree $T'$ clearly has the desired properties. □

### 2.3 JSJ-splittings of 3-manifolds and hyperbolic groups

In this subsection we will give proofs of Theorem 5 and Theorem 7. Both turn out to be simple applications of Corollary 18.

Jaco and Shalen [9] and Johannson [10] independently proved that closed orientable 3-manifolds can be canonically decomposed by cutting along essential embedded tori. The resulting pieces are either Seifert manifolds or are acylindrical, i.e. any properly embedded incompressible annulus is boundary parallel.

This decomposition of the manifold induces a splitting of the fundamental group of the manifold as a fundamental group of a graph of groups. In this splitting the edge groups correspond to the essential tori and are isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and the vertex groups correspond to the fundamental groups of the pieces of the decomposition.

**Proof of Theorem 5** Let $N$ be the number of acylindrical pieces. We have to show that rank $G \geq N + 1$. In view of Corollary 18 it clearly suffices to show that $G = \pi_1(M^3)$ admits a minimal action on a tree $T$ such that there are at least $N$ $G$-equivalence classes of essential vertices.

We take the decomposition of $G$ as the fundamental group of the graph of groups associated to the JSJ-decomposition. Let $v$ be a vertex of this graph that comes from an acylindrical piece. It follows directly from the definition of acylindricity that the edge groups of edges emanating from $v$ are malnormal
in the vertex group of \( v \) and that the edge groups of two such are conjugacy
separated in the vertex group.

Let now \( T \) be the Bass-Serre tree associated to this decomposition of \( G \). The
action \( G \times T \to T \) is minimal. The observation just made now guarantees that
all vertices of \( T \) that are associated to the fundamental group of an acyclic
piece of \( M^3 \) are essential. The assertion follows since there are clearly \( N \)
\( G \)-equivalence classes of such vertices.

\[ \square \]

It has turned out that large classes of groups admit canonical decompositions
that are similar to the JSJ-decomposition of 3-manifolds and their fundamental
groups. The first result is due to Sela [24]; he proves that torsion-free hyperbolic
groups admit a canonical decomposition as the fundamental group of a
graph of groups where all edge groups are cyclic and calls this the JSJ-splitting.
An alternative construction was carried out by Bowditch [3]. The theory of
JSJ-decompositions is crucial to Sela’s solution of the isomorphism problem for
torsion-free hyperbolic groups; see [25] for a solution for hyperbolic groups that
admit no cyclic splitting.

Recently the theory of JSJ-splittings has been generalized to finitely
presented groups by Rips and Sela [21]. Other ways of constructing JSJ-decom-
positions were introduced by Dunwoody and Sageev [6] and by Fujiwara and
Papasoglu [7].

The JSJ-decomposition of a torsion-free hyperbolic groups \( G \) is a presenta-
tion of \( G \) as the fundamental group of a graph of groups where all edge groups
are cyclic and of infinite index in the vertex groups. We will now prove Theorem
7; the proof is similar to the proof of Lemma 4.7 of [26].

**Proof of Theorem 7** Since \( G \) is hyperbolic we known that the normalizer \( N(U) \)
of a cyclic subgroup \( U \) is the unique maximal cyclic group containing \( U \).

Let now \( T \) be the Bass-Serre tree corresponding to the JSJ-splitting of \( G \)
and \( G \times T \to T \) be the natural action. Let \( e = [x, y] \) be an edge of \( T \) and
\( U \) be its (cyclic) stabilizer. We first show that \( U \) is maximal cyclic in the
stabilizer of \( x \) or in the stabilizer of \( y \). Assume that \( U \) is neither maximal in
\( \text{Stab}(x) \) nor in \( \text{Stab}(y) \). It follows that there exist \( g \in (\text{Stab}(x) - U) \cap N(U) \)
and \( h \in (\text{Stab}(y) - U) \cap N(U) \). It is clear that \( h \) and \( g \) do not have a common
fixed point. It follows that \( (g, h) \neq N(U) \) is not cyclic which implies that \( N(U) \)
is not cyclic, a contradiction.

We define a new tree \( \tilde{T} \) which is obtained from \( T \) by identifying two edges
\( e_1 \) and \( e_2 = ge_2 \), if they have a vertex in common, are stabilized by the same
subgroup \( U \) and if \( g \in N(U) \). This identification is \( G \)-equivariant and we obtain
therefore a new action \( G \times \tilde{T} \to \tilde{T} \). For the new action the stabilizer of every
edge \( e = [x, y] \) is maximal cyclic in \( \text{Stab}(x) \) and \( \text{Stab}(y) \). Since maximal cyclic
subgroups of torsion-free hyperbolic groups are either identical or have trivial
intersection and since every edge group of the JSJ-decomposition is hyperbolic,
the same holds for stabilizers of two distinct edges emanating at a vertex.

We further define the tree \( T' \) by first subdividing every edge of \( \tilde{T} \) and then
identifying edges \( e_1 \) and \( e_2 \) if they have a vertex in common that comes from \( \tilde{T} \)
(not the ones introduced by the subdivision) and if they have the same stabilizer.
We obtain a new action \( G \times T' \to T' \) were no elements fixes two edges emanating
from a vertex that comes from \( \tilde{T} \); in particular \( T' \) has as many \( G \)-equivalence
classes of essential vertices as the JSJ-decomposition had vertices. The Theorem now follows from Corollary 18.

3 On the rank of amalgamated products

Grushko’s theorem implies that the rank is additive under free products, i.e. that \( \text{rank } G = \sum_{i=1}^{n} \text{rank } G_i \) if \( G = \star_{i=1}^{n} G_i \). In [17] it has been asked whether there exits a satisfactory formula for amalgamated products. We will show that in general no non-trivial rank formula holds for amalgamated products.

3.1 Some examples and amalgamated products with \( k \)-step malnormal amalgam

Rosenberger remarks in [22] that a class of Fuchsian groups can be written as amalgamated products \( G = A \ast_{C} B \) such that the formula \( \text{rank } G \geq \text{rank } A + \text{rank } B - \text{rank } C \) does not hold. In [15] a family of Coxeter-groups \( G_n = A_n \ast_{C_n} B_n \) with rank \( A_n = \text{rank } B_n = n \), rank \( C_n = 1 \) and rank \( G_n = n+1 \) is described. These groups can also be written as amalgamated products with \( 2n \) factors, i.e. are examples for amalgamated products where the rank is lower than the number of factors. It turns out that this can only happen if the amalgam is not malnormal, since the rank of the amalgamated product of \( n \) factors is at least \( n+1 \) if the amalgam is malnormal. (See [14] for the case \( n = 2 \) and [29] or [28] otherwise, it also follows immediately from Corollary 18.) See also Rosenberger’s discussion alongside the example mentioned above.

We will look at the situation where \( G = \star_{i=1}^{n} G_i \) is an amalgamated product with \( k \)-step malnormal amalgam \( A \). We will prove the following:

**Lemma 19** Let \( G = \star_{i=1}^{n} G_i \) and \( A \neq 1 \). If \( A \) is \( k \)-step malnormal in \( G = \star_{i=1}^{n} G_i \) then \( \text{rank } G \geq n/(4k+4) \).

**Proof** We rewrite \( G \) as a splitting as in the beginning of subsection 2.2. We will see that the action of \( G \) on the Bass-Serre tree \( T \) with respect to this splitting is \((2k+2)\)-acylindrical. It then follows from Theorem 4 that \( n+1 \leq (\text{rank } G - 1)(4k+4) + 1 \) and therefore \( n \leq (\text{rank } G - 1)(4k+4) \) and \( n \leq \text{rank } G(4k+4) \) which implies that \( \text{rank } G \geq n/(4k+4) \) which proves the assertion.

Assume now that there exists a \( g \in G - 1 \) such that \( g \) fixes a segment \([x, y]\) of length \( 2k+3 \). Let \( z \) be the vertex of \( T \) which is fixed under the action of \( A \). Since every second vertex of \([x, y]\) is \( G \)-equivalent \( z \), there is a subsegment \([x', y']\) of \([x, y]\) of length \( 2k+2 \) such that \( x' \) and \( y' \) are \( G \)-equivalent to \( z \), after conjugation of \( g \) we can assume that \( x' = z \). It follows that there exists a \( h \in G \) such that \( hz = y' \). Now \( g \) fixes \( z \) which implies that \( g \in A \) and \( g \) fixes \( y' \) which implies that \( g \in hAh^{-1} \). It follows that \( h^{-1}gh \in A \) which contradicts the assumption that \( A \) is \( k \)-step malnormal in \( G = \star_{i=1}^{n} G_i \) since \( h \) is an element with normal form of length \( k + 1 \) (one letter for every vertex of \([z, y']\) that is not \( G \)-equivalent to \( z \)). \( \square \)
3.2 Constructing the examples

It is now not difficult to find an amalgamated product $G = G' *_A G''$ such that rank $G' \geq m$, rank $G'' \geq m$, rank $A = 2$ and rank $G = 2$ for arbitrary $m \in \mathbb{N}$. In view of Lemma 19 it suffices to construct an amalgamated product $G = \bigoplus_{i=1}^{2n} G_i$ with $n = 8m$ such that rank $G = \text{rank } A = 2$ and that $A$ is 1-step malnormal in $G' := \bigoplus_{i=1}^{n} G_{2i-1}$ and $G'' := \bigoplus_{i=1}^{n} G_{2i}$. The definition of $G'$ and $G''$ clearly implies that $G = G' *_A G''$ and Lemma 19 guarantees that rank $G' \geq m$ and rank $G'' \geq m$.

For arbitrary $n \in \mathbb{N}$, $n \geq 2$ we construct an amalgamated product $G$ which has these properties. The amalgam is $A = \langle a_1, a_2 \rangle$, a free group of rank 2. In $A$ we will choose (free) subgroups $U_1, U_2, \ldots, U_{2n+1}$, such that $U_i$ is freely generated by $X_i = \{ x_{i,1}, \ldots, x_{i,2n+2-i} \} \subset A$ for $1 \leq i \leq 2n+1$; this implies in particular that $U_i$ is free of rank $2n + 2 - i$. We further assume that $x_{1,1} = a_1$ and $x_{2n+1,1} = a_2$, that $U_i = \langle U_i, U_i+1 \rangle$ is free in $X_i \cup X_{i+1}$ for $1 \leq i \leq 2n$ and that $U_i$ and $U_j$ are conjugacy separated in $A$ for $1 \leq i, j \leq 2n$ and $|i - j| \geq 2$. Lemma 20 will guarantee the existence of such subgroups.

We define

$$H_i := \langle \tilde{x}_{i,1}, \ldots, \tilde{x}_{i,2n+2-i}, \tilde{x}_{i+1,1}, \ldots, \tilde{x}_{i+1,2n+1-i}, y_i y_i^{-1} = \tilde{x}_{i+1,1}, \ldots, y_i \tilde{x}_{i,2n+1-i} y_i^{-1} = \tilde{x}_{i+1,2n+1-i}; (y_i \tilde{x}_{i,2n+2-i})^3 \rangle$$

for $1 \leq i \leq 2n$. Lemma 21 will show that the subgroups $\tilde{G}_i = \langle \tilde{x}_{i,1}, \ldots, \tilde{x}_{i,2n+2-i}, \tilde{x}_{i+1,1}, \ldots, \tilde{x}_{i+1,2n+1-i} \rangle < H_i$ are free in $\{ \tilde{x}_{i,1}, \ldots, \tilde{x}_{i,2n+2-i}, \tilde{x}_{i+1,1}, \ldots, \tilde{x}_{i+1,2n+1-i} \}$. The enables us to define

$$G_i := A *_{\tilde{G}_i} H_i$$

for $1 \leq i \leq 2n$, where the isomorphism $\phi_i$ is given by $\phi_i(x_{i,j}) = \tilde{x}_{i,j}$ for $1 \leq j \leq 2n + 2 - i$ and $\phi_i(x_{i+1,j}) = \tilde{x}_{i+1,j}$ for $1 \leq i \leq 2n + 1 - i$. In the following we will not distinguish between $x_{i,j}$ and $\tilde{x}_{i,j}$ and $U_i$ and $\tilde{U}_i$. We define

$$G := \bigoplus_{i=1}^{2n} G_i, \quad G' := \bigoplus_{i=1}^{n} G_{2i-1}, \quad G'' := \bigoplus_{i=1}^{n} G_{2i}.$$

It is clear that $G \cong G' *_A G''$. If $a \in A - 1, \ g \in G_i - A$ and $gag^{-1} \in A$ then the definition of $G_i$ implies that $a$ is in $A$ conjugate to an element of $\tilde{U}_i = \tilde{U}_i$. Assume now that $A$ is not 1-step malnormal in $G' := \bigoplus_{i=1}^{n} G_{2i-1}$. It follows that there exists an element $h = x_1 x_2$ with $x_i \in G_{2i-1} - A$ and $x_2 \in G_{2j-1} - A$ and $i \neq j$ such that $hah^{-1} = x_1 x_2 x_2^{-1} x_1^{-1} = a'$ for some $a \in A - 1$. It follows that also $a'' := x_2 x_2^{-1} \in A$ which implies that $a''$ is in $A$ conjugate to an element of $\tilde{U}_{2j-1}$. Since $x_{2j} x_{2j}^{-1} \in A$ we deduce that $a''$ is in $A$ also conjugate to an element $\tilde{U}_{2j-1}$, a contradiction since $\tilde{U}_{2j-1}$ and $\tilde{U}_{2j-1}$ were assumed to be conjugacy separated in $A$. It follows that $A$ is 1-step
malnormal in $G' := \bigwedge_{i=1}^{n} G_{2i-1}$. Analogously we show that $A$ is 1-step malnormal in $G'' := \bigwedge_{i=1}^{n} G_{2i}$.

We now show that $G$ is generated by the two elements $a_1$ and $y_1 \cdots y_{2n}$. Since $y_1^{-1} \cdots y_{2n}^{-1} x_1 y_1 \cdots y_{2n} = y_1^{-1} \cdots y_{2n}^{-1} x_2 y_2 \cdots y_{2n} = \cdots = y_1^{-1} x_{2n+1} y_1 = x_{2n+1} = a_2$ it follows that $(a_1, y_1 \cdots y_{2n}) = \langle a_1, a_2, y_1 \cdots y_{2n} \rangle$. We show by induction on $k$ that $B_k := \langle a_1, a_2, y_1 \cdots y_k \rangle = \langle a_1, a_2, y_1, \ldots, y_k \rangle$ for $k = 2n$ this is the assertion. For $k = 1$ this is trivial. Assume now the claim is proved for $k$. Since $x_{1,2n+1-k} \in A = \langle a_1, a_2 \rangle \subset B_k$ it follows that $y^{-1}_{k+1} x_{1,2n+1-k} y_{k+1} = y^{-1}_{k+1} x_{k+1,2n+1-k} y_{k+1} \in B_k$ and therefore also $x_{k+1,2n+1-k} y_{k+1} x_{k+1,2n+1-k} \in \langle a_1, a_2, y_1 \cdots y_{k+1} \rangle \in B_{k+1}$ because of the relation $(x_{k+1,2n+1-k} y_{k+1})^3$ in $G_{k+1}$. As $x_{2k+1,2n+1-k} \in A \subset B_{k+1}$ we also have $y_{k+1} \in \langle a_1, a_2, y_1 \cdots y_{k+1} \rangle \in B_{k+1}$ and deduce that $B_{k+1} = \langle a_1, a_2, y_1 \cdots y_{k+1}, y_{k+1} \rangle = \langle a_1, a_2, y_1 \cdots y_{k+1}, y_{k+1} \rangle = \langle B_k, y_{k+1} \rangle$. The assertion now follows immediately from the induction hypothesis.

The existence of these examples clearly proves Theorem 8. In the remainder of this subsection we state and prove the two lemmas mentioned before.

**Lemma 20** Let $A = \langle a_1, a_2 \rangle$ be the free group of rank 2 and $n > 2$. There exist subgroups $U_1, \ldots, U_{2n+1}$ of $A$ such that the following hold:

1. $U_i$ is free in $X_i = \{ x_{i,1}, \ldots, x_{i,2n+1-i} \} \subset A$ for $1 \leq i \leq 2n+1$ with $x_{i,1} = a_1$ and $x_{2n+1-i} = a_2$.

2. $\langle U_i, U_{i+1} \rangle$ is free in $X_i \cup X_{i+1}$ for $1 \leq i \leq 2n$.

3. $\langle U_i, U_{i+1} \rangle$ and $\langle U_j, U_{j+1} \rangle$ are conjugacy separated if $1 \leq i, j \leq 2n$ and $|i - j| \geq 2$.

**Proof** The proof is constructive. Let $w_1 = (a_1 a_2)^5 a_1$ and $w_2 = (a_2 a_1)^5 a_2$. Let $\phi_n$ be the endomorphism that maps $a_1$ to $a_1^n$ and $a_2$ to $a_2^n$. We define $x_{1,1} = a_1$, $x_{1,i} = a_1^j \phi_{n+1}(w_1) a_2^j$ for $2 \leq j \leq 2n+1$, $x_{2,j} = a_2^j \phi_{n+2}(w_1) a_2^{-j}$ for $1 \leq j \leq 2n$, and $x_{i,j} = a_1^j \phi_{n+1}(w_2) a_2^{-j}$ for $2 \leq i \leq 2n+1$ and $1 \leq j \leq 2n+2-i$.

It follows easily from the choice of the elements $x_{i,j}$ that the inequalities $|x^{-1} y^2| > |x|$ and $|x^{-1} y^2| > |y|$ hold for two elements $x, y \in X_i \cup X_{i+1}$, where we denote by $|g|$ the length of the freely reduced element representing $g$. Nielsen’s theory for finitely generated subgroups of free groups [19] immediately guarantees the first two claims.

Let now $u$ and $v$ be freely reduced and cyclically reduced non-trivial words in $\{ X_i \cup X_{i+1} \}$ and $\{ X_j \cup X_{j+1} \}$, respectively, with $1 \leq i < j \leq 2n$ and $|i - j| \geq 2$. Let further $\tilde{u}$ and $\tilde{v}$ be the reduced cyclic words in $\{ a_1, a_2 \}$ representing the conjugacy classes of $u$ and $v$ in $A$. Now either $i = 1$ and $u = x_{1,1}^i = a_1^i$ or $\tilde{u}$ contains a subword of the form $(\phi_{5n+3}(a_1 a_2)^4)^{\pm 1}$ or $(\phi_{5n+3+1}(a_1 a_2)^4)^{\pm 1}$. Also either $j = 2n$ and $v = x_{2n+1,1}^i = a_2^i$ or $\tilde{v}$ contains a subword of the form $(\phi_{5n+3}(a_1 a_2)^4)^{\pm 1}$ or $(\phi_{5n+3+1}(a_1 a_2)^4)^{\pm 1}$. This follows immediately from the definition of the $x_{k,l}$ and the cancellation between two such elements. Similar arguments show that $\tilde{u}$ contains no subwords of the form $(\phi_{5n+3}(a_1 a_2)^4)^{\pm 1}$ or $(\phi_{5n+3+1}(a_1 a_2)^4)^{\pm 1}$, the analogue holds for $\tilde{v}$. It follows that $\tilde{u}$ and $\tilde{v}$ are
different as cyclic words and are therefore not conjugate. This proves the last claim.

Lemma 21. Let $H = \langle x_1, \ldots, x_{n+1}, z_1, \ldots, z_n, y^2, yx_1y = z_1, \ldots, yx_ny = z_n, (yx_{n+1})^3 \rangle$. The subgroup generated by $X = \{ x_1, \ldots, x_{n+1}, z_1, \ldots, z_n \}$ is free in $X$.

Proof. We write $H$ as an amalgamated product $\langle x_1, \ldots, x_n, z_1, \ldots, z_n, y^2, yx_1y = z_1, \ldots, yx_ny = z_n \rangle \ast_{\langle y^2 \rangle} \langle x_{n+1}, y^2, (yx_{n+1})^3 \rangle$. Abelianizing shows that $y$ is non-trivial in the first factor, that $y$ is non-trivial in the second factor follows since $\langle y^2 \rangle$ is a free factor in $\langle x_{n+1}, y^2, (yx_{n+1})^3 \rangle \cong \langle \bar{x}(=yx_{n+1}), y^2, \bar{z}^3 \rangle$.

We further show that $\{ x_1, \ldots, x_n, z_1, \ldots, z_n \}$ freely generates a subgroup of the first factor and $\{ x_{n+1} \}$ freely generates a subgroup of the second factor. The assertion then follows since they both have trivial intersection with the amalgam. Since $x_{n+1} = y^{-1} \bar{x}$ is an element of length 2 in a free product it generates an infinite cyclic subgroup. After some Tietze transformations the first factor has the presentation $\langle x_1, \ldots, x_n, y^2 \rangle \cong F_n \ast \mathbb{Z}_2$ and it remains to verify that $\{ x_1, \ldots, x_n, yx_1y(=z_1), \ldots, yx_ny(=z_n) \}$ is a free generating system. This however is clear since $\{ x_1, \ldots, x_n \}$ and $\{ yx_1y, \ldots, yx_ny \}$ are both free generating systems of distinct conjugates of the same factor in a free product.

\noindent □

References


