Marcinkiewicz multipliers and multi-parameter structure on Heisenberg groups

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Abstract

The classical Marcinkiewicz multiplier theorem, say on $\mathbb{R}^2$, can be formulated as follows: Suppose m is a function on $\mathbb{R}^2$ satisfying conditions of the form

$$|(\xi\partial_\xi)^\alpha(\eta\partial_\eta)^\beta m(\xi,\eta)| \le C_{\alpha,\beta}. \tag{0.1}$$

Then the "function" $m(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y})$ of the commuting self-adjoint operators $i\frac{\partial}{\partial x}$ and $i\frac{\partial}{\partial y}$ is a bounded operator on the Lebesgue space $L^p(\mathbb{R}^2)$, for $1 < p < \infty$. Notice that the conditions (0.1) on the multiplier m are invariant under independent scalings in the variables x and y.

On the Heisenberg group $\mathbb{H}_n$, there are again two natural, commuting left-invariant differential operators, the sub- or Kohn-Laplacian $\mathcal{L}$ and the central derivative U, which are of fundamental importance. For instance, on the 3-dimensional Heisenberg group $\mathbb{H}_1$, $\mathcal{L}$ is of the form $\mathcal{L} = -(X^2+Y^2)$, where X and Y are real vector fields satisfying the commutation relations $[X,Y] = U$, $[X,U] = [Y,U] = 0$. $\mathcal{L}$ is "the" prototype of a subelliptic sum of squares operator in the sense of Hörmander. It arizes in various contexts, for instance in the study of the $\overline{\partial}$- Neumann problem.

In the course, I shall discuss a multiplier theorem of Marcinkiewicz type for functions $m(\mathcal{L}, \frac{1}{i}U)$ of the operators $\mathcal{L}$ and $\frac{1}{i}U$, which had been obtained in joint work with F. Ricci and E.M. Stein.

One problem arizing in this setting is the fact that the relevant scalings on the Heisenberg group are no longer automorphism. Moreover, whereas for analogous problems for elliptic Laplacians $\mathcal{L}$ in the classical setting the underlying Riemannian geometry and the spectral theory of $\mathcal{L}$ can be linked by means of suitable pseudodifferential or Fourierintegral operators, such technics are not yet available for subelliptic operators, whose underlying sub-Riemannian geometry is considerably more complicated.

Our approach is therefore based on the non-commutative group Fourier transform on the Heisenberg group.

The necessary background on the Heisenberg group and other tools like the theory of singular integrals will be briefly outlined in the lectures.

1 A primer of Calderón-Zygmund theory

The following results can be found Stein's book [23].

Let $X \neq \emptyset$ be a set. A **quasi-distance** ρ on X is a mapping $\rho : X \times X \to \mathbb{R}_{\geq 0}$, such that, for some constant $\kappa \geq 1$:

(i) $\rho(x, y) = 0 \iff x = y$;

(ii) $\rho(x, y) = \rho(y, x)$;

(iii) $\rho(x, z) \leq \kappa(\rho(x, y) + \rho(y, z))$

for every $x, y, z \in X$. $B_r(x) := \{y \in X : \rho(x, y) < r\}$ will denote the **ball of radius** $r > 0$ centered at $x \in X$. A **space of homogeneous type** is a locally compact Hausdorff space X, endowed with a quasi-distance ρ and a regular Borel measure $\mu \geq 0$ such that :

(i) the balls $B_r(x)$ form a neighborhood base $\forall x \in X$;

(ii) the measure μ is **doubling**, i.e. $\exists\, \gamma > 0$ such that

$$0 < \mu(B_{2r}(x)) \leq \gamma\, \mu(B_r(x)) < \infty \quad \forall x \in X, r > 0.$$

Examples 1.1. (a) Euclidean space $\mathbb{R}^d$, with Euclidean distance $\rho(x, y) = (x - y)$ und Lebesgue measure $d\mu = dx$.

(b) **2-step nilpotent Lie groups**:
Given $m, l \in \mathbb{N} \setminus \{0\}$, and skew-symmetric matrices $J_1, \dots, J_l$ on $\mathbb{R}^m$, consider $G = \mathbb{R}^m \times \mathbb{R}^l$, endowed with the group law

$$(z, u) \cdot (w, v) := (z + w, u + v + \tfrac{1}{2}\, {}^t z J w), \quad (z, u), (w, v) \in \mathbb{R}^m \times \mathbb{R}^l,$$

where

$${}^t z J w := ({}^t z J_1 w, \dots, {}^t z J_l w).$$

Then G is a nilpotent Lie group of step ≤ 2 (every connected, simply connected 2-step nilpotent Lie group is indeed isomorphic to such a group). Moreover, in these coordinates, we may identify the Lie algebra $\mathfrak{g}$ of G with $\mathbb{R}^m \times \mathbb{R}^l$, and the exponential mapping exp with the identity mapping.
One can define **dilations** $\delta_r, r > 0$, on G by

$$\delta_r(z, u) := (rz, r^2 u);$$

these are obviously automorphisms of G.
A **homogeneous norm** on G is given by

$$|(z, u)| := (|z|^4 + |u|^2)^{1/4},$$

i.e. it satisfies

$$\begin{array}{rcll} |\delta_r g| & = & r|g| & \forall r>0, g\in G, \\ |gh| & \leq & \kappa(|g|+|h|) & \forall g,h\in G, \end{array}$$

for some constant $\kappa \geq 1$. This implies that

$$\rho(g,h) := |g^{-1}h|$$

defines a left-invariant quasi-distance ρ on G. Lebesgue measure $dzdu$ is a bi-invariant Haar measure on G, and, since

$$B_r(g) = \delta_r B_1(\delta_r^{-1} g)$$

and

$$\int_G f\circ\delta_r(z,u)\,dzdu = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} f(rz,r^2u)\,dzdu = r^{-Q}\int f(z,u)\,dzdu,$$

where

$$Q := m+2l$$

is the **homogeneous dimension** of G, we see that

$$|B_r(g)| = r^Q|B_1(g)|\,,$$

which implies that the measure $dzdu$ is doubling.

Put $J_u := \sum_{j=1}^{l} u_j J_j$, if $u=(u_1,\ldots,u_l)\in\mathbb{R}^l$.
G is called a **Heisenberg-type group**, if

$$J_u^2 = -|u|^2 I \quad \forall u\in\mathbb{R}^l.$$

The case $l=1$ corresponds to **Heisenberg groups**. Then $m=2n$ must be even, and we may assume that J is the matrix of the canonical symplectic form on $\mathbb{R}^{2n}$, i.e.

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

I.e., the Heisenberg group $\mathbb{H}_n$ is $\mathbb{R}^{2n}\times\mathbb{R}$, with product

$$(x,y,u)\cdot(x',y',u') = (x+x', y+y', u+u'+\tfrac{1}{2}(x\cdot y'-x'\cdot y))\,,$$

if $(x,y),(x',y')\in\mathbb{R}^{2n}=\mathbb{R}^n\times\mathbb{R}^n, u,u'\in\mathbb{R}$.

We then also write $z_j := (x_j,y_j),\ j=1,\ldots,n$.

If X is a space of homogeneous type, for any locally integrable function f on X one defines the **Hardy-Littlewood maximal operator** M by

$$Mf(x) := \sup_B \frac{1}{\mu(B)}\int_B |f(y)|\,d\mu(y), \quad x\in X\,,$$

where the supremum is taken over all balls $B=B_r(y)$ containing x.

Theorem 1.2. *M is a sublinear operator, which is bounded on $L^p(X) = L^p(X, d\mu)$ for $1 < p \leq \infty$, i.e.*

$$||Mf||_p \leq C_p ||f||_p \quad \forall f \in L^p(X) .$$

Moreover, M is of **weak type (1,1)***, i.e. $\exists\, C_1 \geq 0$:*

$$\mu(\{x \in X : Mf(x) > s\}) \leq \frac{C_1||f||_1}{s} \quad \forall s > 0\,, \forall f \in L^1(X) .$$

Let

$$T : L^2(X) \to L^2(X)$$

be a bounded linear operator, and assume there is a measurable kernel function

$$K : X \times X \to \mathbb{C}$$

such that, "away from the diagonal", T is an integral operator

$$Tf(x) = \int_X K(x, y) f(y)\, d\mu(y) \tag{1.2}$$

with kernel K. Since K may have singularities along the diagonal $\{x = y\}$, this means more precisely that (1.2) holds for almost every (a.e) $x \in X \backslash B$, whenever $f \in L^2(X)$ is supported in a ball B. K is then called the **kernel associated** to T.

K is called a **Calderón-Zygmund kernel** and the operator T a **Calderón-Zygmund operator (CZO)** if $\exists\, C_1, C_2 > 0$ such that

$$\int_{\rho(x,y_0) > C_1 \rho(y,y_0)} |K(x, y) - K(x, y_0)|\, d\mu(x) \leq C_2 \tag{1.3}$$

holds for every $y, y_0 \in X$, and if the same condition holds, with the rôles of the variables x and y interchanged.

Examples 1.3. (a) The prototype of a CZO is the Hilbert transform

$$Hf(x) = \text{p.v.} \int_{\mathbb{R}} \frac{1}{x - y} f(y)\, dy\,, \quad x \in \mathbb{R}\,,$$

on $\mathbb{R}$, with associated kernel $K(x, y) = \dfrac{1}{x - y}$.

(b) More generally, we shall be interested in CZO's T on 2-step nilpotent Lie groups G which are left-invariant, i.e.,

$$T(\lambda_g f) = \lambda_g(Tf) \quad \forall g \in G\,,$$

where $\lambda_g f(h) = f(g^{-1}h)$ denotes the left-translation of f by $g \in G$.
For any left-invariant, bounded linear operator $T : L^2(G) \to L^2(G)$, by Schwartz' kernel theorem, one finds a unique tempered distribution $k \in \mathcal{S}'(G)$ such that

$$Tf = f \star k\,, \quad \forall f \in \mathcal{S}(G) . \tag{1.4}$$

Here, $\star$ denotes the convolution product, which, e.g. for functions $f_1, f_2 \in L^1(G)$, is given by

$$f_1 \star f_2(g) := \int_G f_1(h) f_2(h^{-1}g)\, dh.$$

The convolution (1.4) has to be interpreted in the sense of distributions.

k will be called the **kernel associated** to T. The corresponding kernel K in (1.2) is then given by

$$K(x,y) = k(y^{-1}x), \quad x, y \in G\,,$$

and T is a CZO if and only if

$$\int_{|x|>C_1|y|} |k(y^{-1}x) - k(x)|\, dx \le C_2 \quad \forall y \in G\,. \tag{1.5}$$

(c) Examples on $\mathbb{R}^d$ are:

The **Riesz-kernels** $R_j(x) = c_d \frac{x_j}{|x|^{d+1}}$, $j = 1, \ldots, d$, which are homogeneous of critical degree $-d$.

The **Riesz-potentials** $J_\alpha(x) = c_\alpha |x|^{i\alpha - d}$, $\alpha \in \mathbb{R} \setminus \{0\}$.

(d) An example on $\mathbb{H}_n$ is the **Cauchy-Szegö kernel**

$$K(z,u) = c_n(u + i|z|^2)^{-n-1},$$

which is homogeneous of critical degree $-Q$ with respect to the δ_r's. This kernel plays an important rôle in higher dimensional complex analysis.

Theorem 1.4. *Every CZO is bounded on $L^p(X)$, for $1 < p < \infty$, and of weak-type (1,1).*

2 Representation theory of the Heisenberg group

For the results in this and the next chapter, see, e.g., [23], [6], [24], [25] and the original papers by D. Geller [9], [8], [10].

The group Fourier transform on the Heisenberg group $\mathbb{H}_n$ is defined in terms of the Schrödinger representations, i.e., the irreducible unitary representations of infinite dimension:

For every $\lambda \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, the **Schrödinger representation** π_λ, acting on $L^2(\mathbb{R}^n)$, is given by

$$[\pi_\lambda(x,y,u)\phi](\xi) := e^{i\lambda(u + y\cdot\xi + \frac{1}{2}x\cdot y)}\phi(\xi + x), \quad \phi \in L^2(\mathbb{R}^n)\,.$$

One checks that $\pi_\lambda : \mathbb{H}_n \to \mathcal{U}(L^2(\mathbb{R}^n))$ is a strongly continuous homomorphism from $\mathbb{H}_n$ into the group $\mathcal{U}(L^2(\mathbb{R}^n))$ of unitary operators on the representation space $L^2(\mathbb{R}^n)$.

The **Fourier transform** of a function $f \in L^1(\mathbb{H}_n)$ is the operator valued mapping $\hat{f} : \mathbb{R}^\times \to \mathcal{B}(L^2(\mathbb{R}^n))$, given (in the strong operator sense) by

$$\hat{f}(\lambda) := \int_{\mathbb{H}_n} f(z,u)\pi_\lambda(z,u)\, dzdu, \quad \lambda \in \mathbb{R}^\times .$$

One also writes $\pi_\lambda(f)$ instead of $\hat{f}(\lambda)$.
Then one checks that

$$\widehat{f_1 \star f_2}(\lambda) = \widehat{f_1}(\lambda) \circ \widehat{f_2}(\lambda) \quad \forall f_1, f_2 \in L^1(\mathbb{H}_n) .$$

For sufficiently "nice" functions, such as Schwartz functions, one then has the following **Fourier inversion formula**:

$$f(z,u) = (2\pi)^{-n-1} \int_{\mathbb{R}^\times} \operatorname{tr}\Big(\pi_\lambda(z,u)^* \hat{f}(\lambda)\Big)\, |\lambda|^n \, d\lambda . \tag{2.1}$$

Here, $\operatorname{tr}(T)$ denotes the trace of the operator T.
Applying (2.1) to $f^* \star f$, where

$$f^*(g) = \overline{f(g^{-1})}, \quad g \in G ,$$

one obtains **Plancherel's formula**:
If $f \in L^1 \cap L^2(\mathbb{H}_n)$, then

$$||f||_2^2 = (2\pi)^{-n-1} \int_{\mathbb{R}^\times} \operatorname{tr}\Big(\hat{f}(\lambda)^* \hat{f}(\lambda)\Big)\, |\lambda|^n \, d\lambda . \tag{2.2}$$

Consider now the following vector fields on $\mathbb{H}_n$:

$$X_j := \partial_{x_j} - \tfrac{y_j}{2}\partial_u, \qquad Y_j := \partial_{y_j} + \tfrac{x_j}{2}\partial_u, \quad j = 1, \dots, n,$$
$$U := \partial_u \ .$$

One checks that these are left-invariant, i.e., if X is any of these fields, then

$$X(\lambda_g f) = \lambda_g(Xf), \quad \forall g \in \mathbb{H}_n,\ f \in C^\infty(\mathbb{H}_n) .$$

Moreover, they span the tangent space at the identity element 0 (hence at every point of $\mathbb{H}_n$), and thus form a basis of the **Lie algebra** $\mathfrak{h}_n$ of $\mathbb{H}_n$. The commutation relations among these vector fields are the **Heisenberg relations**

$$[X_j, Y_k] = \delta_{jk} U ,$$

U being central.
For $j = 1, \dots, n$, we denote by

$$\mathcal{L}_j := -(X_j^2 + Y_j^2)$$

the corresponding **partial sub-Laplacian**, and by

$$\mathcal{L} := \mathcal{L}_1 + \cdots + \mathcal{L}_n$$

the **sub-Laplacian** on $\mathbb{H}_n$. Unlike the **full Laplacian**

$$\Delta := \mathcal{L} - U^2 = -\left(\sum_{j=1}^{n}(X_j^2 + Y_j^2) + U^2\right),$$

the sub-Laplacian is non-elliptic.
However, by Hörmander's sum of squares theorem, $\mathcal{L}$ is hypoelliptic. $\mathcal{L}$ is also essentially self-adjoint on $\mathcal{D}(\mathbb{H}_n) \subset L^2(\mathbb{H}_n)$, and we shall denote its closure again by $\mathcal{L}$.
$\mathcal{L}$ is **homogeneous of degree 2**, i.e.

$$\mathcal{L}(f \circ \delta_r) = r^2(\mathcal{L}f) \circ \delta_r \quad \forall f \in C^\infty(\mathbb{H}_n), r > 0\,,$$

whereas Δ is not homogeneous.
This is one of the reasons why $\mathcal{L}$, and not Δ, plays a similar fundamental rôle in the analysis on the Heisenberg group as the Laplacian does on Euclidean space.
Denote by $d\pi_\lambda$ the derived representation of the Lie algebra $\mathfrak{h}_n$, i.e.,

$$[d\pi_\lambda(X)\phi](\xi) := \frac{d}{dt}\Big|_{t=0}[\pi_\lambda(\exp tX)\phi](\xi), \quad \phi \in C_0^\infty(\mathbb{R}^n)\,.$$

From (2.1), one gets

$$\begin{aligned} d\pi_\lambda(X_j) &= \partial_{\xi_j}\,, \\ d\pi_\lambda(Y_j) &= i\lambda\xi_j\,, \\ d\pi_\lambda(U) &= i\lambda\,, \end{aligned}$$

hence

$$d\pi_\lambda(\mathcal{L}_j) = -\partial_{\xi_j}^2 + \lambda^2\xi_j^2 =: H_{\lambda,j}, \quad j = 1, \ldots, n. \tag{2.3}$$

$H_{\lambda,j}$ is a re-scaled **Hermite operator**, acting on the j-th coordinate only.
The joint eigenfunctions of $d\pi_\lambda(\mathcal{L}_1), \ldots, d\pi_\lambda(\mathcal{L}_n)$ are therefore given by

$$\Phi_{\alpha,\lambda}(\xi) := |\lambda|^{n/4}\prod_{j=1}^{n} h_{\alpha_j}(|\lambda|^{1/2}\xi_j)\,,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $h_k(t)$ is the L^2-normalized Hermite function, given by

$$h_k(t) := (2^k\sqrt{\pi}k!)^{-1/2}H_k(t)e^{-t^2/2}\,.$$

Here, $H_k(t)$ denotes the Hermite polynomial of degree k, i.e.

$$H_k(t) := (-1)^k e^{t^2}\frac{d^k}{dt^k}(e^{-t^2})\,.$$

Then

$$d\pi_\lambda(\mathcal{L}_j)\Phi_{\alpha,\lambda} = |\lambda|(2\alpha_j + 1)\Phi_{\alpha,\lambda}, \quad j = 1, \ldots, n\,, \tag{2.4}$$

hence

$$d\pi_\lambda(\mathcal{L})\Phi_{\alpha,\lambda} = |\lambda|(2|\alpha| + n)\Phi_{\alpha,\lambda} \quad \forall \alpha \in \mathbb{N}^n. \tag{2.5}$$

I.e., $d\pi_\lambda(\mathcal{L})$ diagonalizes with respect to the orthonormal basis $\{\Phi_{\alpha,\lambda}\}_{\alpha\in\mathbb{N}^n}$ of $L^2(\mathbb{R}^n)$, with eigenvalues $|\lambda|(2|\alpha| + n)$, $\alpha \in \mathbb{N}^n$.

3 Functions of the sub-Laplacian

Since $\mathcal{L}$ is a self-adjoint, non-negative operator on the Hilbert space $L^2(\mathbb{H}_n)$, it admits a spectral resolution

$$\mathcal{L} = \int_0^\infty s\, dE_s\,.$$

For every Borel measurable function $m : [0,\infty[\to \mathbb{C}$, we may thus define the operator

$$m(\mathcal{L}) := \int_0^\infty m(s)\, dE_s\,.$$

If m is bounded, then $m(\mathcal{L})$ is a bounded linear operator on $L^2(\mathbb{H}_n)$, which is left-invariant. Consequently, by (1.4), there exists a unique convolution kernel $k_m \in \mathcal{S}'(\mathbb{H}_n)$ such that

$$m(\mathcal{L})f = f \star k_m, \quad \forall f \in \mathcal{S}(\mathbb{H}_n)\,.$$

We shall sometimes also write

$$k_m = m(\mathcal{L})\delta_0\,.$$

We intend to deduce information on the kernel k_m, for suitable multipliers m. Notice that, since $\mathcal{L}$ is homogeneous of degree 2,

$$k_{m(r\,\cdot\,)} = (k_m)_{\sqrt{r}}, \quad \forall r > 0\,, \tag{3.1}$$

if we define the L^1-norm preserving scaling of f by

$$f_r(g) := r^{-Q} f \circ \delta_{1/r}(g), \quad r > 0.$$

Now, it is known that spectral resolution of $\mathcal{L}$ is compatible with unitary representation theory (see, e.g., [17]), i.e.,

$$\widehat{k_m}(\lambda) = \pi_\lambda(k_m) = m(d\pi_\lambda(\mathcal{L})) \quad \forall \lambda \in \mathbb{R}^\times\,, \tag{3.2}$$

for "nice" multipliers m. Thus, by (2.5),

$$\widehat{k_m}(\lambda)\Phi_{\alpha,\lambda} = m(|\lambda|(2|\alpha| + n))\Phi_{\alpha,\lambda}\,. \tag{3.3}$$

This implies

$$\operatorname{tr}\left(\widehat{k_m}(\lambda)^* \widehat{k_m}(\lambda)\right) = \sum_{\alpha\in\mathbb{N}^n} \left|m(|\lambda|(2|\alpha| + n))\right|^2,$$

hence, by Plancherel's identity,

$$||k_m||_2^2 = (2\pi)^{-n-1} \int_{\mathbb{R}^\times} \sum_{\alpha\in\mathbb{N}^n} \left|m(|\lambda|(2|\alpha| + n))\right|^2 |\lambda|^n\, d\lambda\,. \tag{3.4}$$

Interchanging summation and integration, and changing variables in the integral, one easily obtains the following **Plancherel's identity for spectral multipliers**:

$$||k_m||_2^2 = c_n \int_0^\infty |m(s)|^2 s^n\, ds, \tag{3.5}$$

for a suitable constant c_n.

Moreover, (3.3) implies

$$\operatorname{tr}\left(\pi_\lambda(z,u)^*\widehat{k_m}(\lambda)\right) = \sum_\alpha \left\langle \widehat{k_m}(\lambda)\Phi_{\alpha,\lambda}, \pi_\lambda(z,u)\Phi_{\alpha,\lambda}\right\rangle$$
$$= \sum_{\alpha\in\mathbb{N}} m(|\lambda|(2|\alpha|+n))\overline{\chi_{\alpha,\lambda}(z,u)}\,,$$

where

$$\chi_{\alpha,\lambda}(z,u) := \langle \pi_\lambda(z,u)\Phi_{\alpha,\lambda},\, \Phi_{\alpha,\lambda}\rangle\,.$$

Explicit computations show that

$$\chi_{\alpha,\lambda}(z,u) = e^{i\lambda u}\prod_{j=1}^{n} L_{\alpha_j}\left(\tfrac{|\lambda|}{2}|z_j|^2\right) e^{-\frac{|\lambda|}{4}|z_j|^2}\,, \tag{3.6}$$

where L_k denotes the **Laguerre polynomial** of degree k, i.e.,

$$L_k(t) := \frac{1}{k!}e^t\frac{d^k}{dt^k}(t^k e^{-t})\,.$$

By Fourier inversion (2.1), we thus obtain

$$k_m(z,u) = (2\pi)^{-n-1}\int_{\mathbb{R}^\times}\sum_\alpha m(|\lambda|(2|\alpha|+n))\overline{\chi_{\alpha,\lambda}(z,u)}|\lambda|^n\,d\lambda\,, \tag{3.7}$$

for nice multipliers, such as $m \in C_0(\mathbb{R}^\times)$. Notice that the function $f = k_m$ is **polyradial** in the following sense: $f(z,u)$ depends only on the central variable u and the moduli $|z_1|,\dots,|z_n|$ of the z_j's.

Remark 3.1. Denote by $L^1_r(\mathbb{H}_n)$ the subspace of $L^1(\mathbb{H}_n)$ consisting of all polyradial L^1-functions. It is known that $L^1_r(\mathbb{H}_n)$ is a commutative subalgebra of the convolution algebra $L^1(\mathbb{H}_n)$ (this is related to the theory of Gelfand pairs). Moreover, the $\chi_{\alpha,\lambda}$ are nothing but the bounded characters of $L^1_r(\mathbb{H}_n)$ which contribute to the Plancherel measure. And, if

$$\hat f(\lambda,\alpha) := \int_{\mathbb{H}_n} f(z,u)\chi_{\alpha,\lambda}(z,u)\,dzdu, \quad (\alpha,\lambda)\in\mathbb{N}^n\times\mathbb{R}^\times\,,$$

denotes the Gelfand transform of $f \in L^1_r(\mathbb{H}_n)$, then, in correspondence with (3.3), (3.7), one has the following Gelfand-inversion and Plancherel identity:

$$f(z,u) = (2\pi)^{-n-1}\int_{\mathbb{R}^\times}\sum_\alpha \hat f(\lambda,\alpha)\overline{\chi_{\alpha,\lambda}(z,u)}|\lambda|^n\,d\lambda\,, \tag{3.8}$$

$$||f||_2^2 = (2\pi)^{-n-1}\int_{\mathbb{R}^\times}\sum_\alpha |\hat f(\lambda,\alpha)|^2|\lambda|^n\,d\lambda\,, \tag{3.9}$$

for suitable polyradial functions f.

We are looking for sufficient conditions on m such that $k_m \in L^1$.

Let

$$\omega^a(z,u) = (1+|(z,u)|)^a\,, \quad a \in \mathbb{R}\,.$$

Notice that

$$\omega^a \in L^1 \Longleftrightarrow a < -Q\,.$$

Thus, by Hölder's inequality, for any $\varepsilon \geq 0$,

$$\begin{aligned}
&\int_{\mathbb{H}_n} |k_m(g)\omega^\varepsilon(g)|\,dg \\
&\leq \; ||k_m\omega^{\varepsilon+a}||_2 \left(\int_{\mathbb{H}_n} \omega^{-2a}(z,u)\,dzdu\right)^{1/2} \\
&\lesssim \; ||k_m\omega^{\varepsilon+a}||_2\,, \qquad (3.10)
\end{aligned}$$

if $a > Q/2$.
Suppose we can prove estimates of the form

$$\int_{|g|\leq 1} |k_m(g)|^2\,dg \leq A_0, \qquad (3.11)$$

$$\int_{2^{j-1}<|g|\leq 2^j} |k_m(g)|^2\,dg \;\leq\; A_j\,, \quad j = 1,2,3,\ldots\,. \qquad (3.12)$$

Then

$$\begin{aligned}
||k_m\omega^{\varepsilon+a}||_2^2 \;&=\; \int_{|g|\leq 1} |k_m(g)|^2(1+|g|)^{2(\varepsilon+a)}\,dg \\
&+\; \sum_{j=1}^{\infty}\int_{2^{j-1}<|g|\leq 2^j} |k_m(g)|^2(1+|g|)^{2(\varepsilon+a)}\,dg \\
&\lesssim\; A_0 + \sum_{j=1}^{\infty} A_j 2^{2(\varepsilon+a)_j},
\end{aligned}$$

hence

$$\int_{\mathbb{H}_n} |k_m(g)\omega^\varepsilon(g)|\,dg \;\lesssim\; \left(\sum_{j=0}^{\infty} A_j 2^{2(\varepsilon+a)j}\right)^{1/2} \qquad (3.13)$$

whenever $a > Q/2$.

Proposition 3.2. *Let m be supported in $[1/2,4]$, say, and let $m \in L^2_\alpha(\mathbb{R})$, $\alpha \geq 0$. Then*

$$\int_{|g|\geq R} |k_m(g)|^2\,dg \;\lesssim\; R^{-2\alpha}||m||^2_{L^2_\alpha} \quad \forall R \gg 1. \qquad (3.14)$$

Consequently, by (3.13),

$$\int_{\mathbb{H}_n} |k_m(g)|(1+|g|)^\varepsilon\,dg \;\lesssim\; ||m||_{L^2_\alpha}\,, \qquad (3.15)$$

whenever $\alpha > \varepsilon + Q/2$.

Proof. There are basically two known methods to obtain estimates like (3.14). The first exploits Gaussian type decay of the heat kernels associated to $\mathcal{L}$ (as established, e.g., in the work of A. Hulanicki, B. Davies, Gaffney and many others). The second, based on an idea of M. Taylor, makes use of the finite propagation speed of waves on $\mathbb{H}_n$, and will be sketched in what follows (see, e.g., [21], and [7], [1], [14] for the first approach).

Consider the multiplier $s \mapsto \cos(t\sqrt{s})$, $t \in \mathbb{R}$, and denote by

$$W_t := \cos(t\sqrt{\mathcal{L}})\delta_0$$

the associated convolution kernel. Then, $u(t,(z,u)) := W_t(z,u)$ satisfies, in the distributional sense, the following Cauchy problem for the wave equation on $\mathbb{H}_n$ associated to $\mathcal{L}$:

$$\begin{aligned} (\partial_t^2 + \mathcal{L})u &= 0\,, \\ u(0,\cdot) &= \delta_0\,, \\ \partial_t u(0,\cdot) &= 0\,. \end{aligned}$$

It has been shown by R. Melrose [15] that the wave u has finite propagation speed (a direct proof for the case of a sub-Laplacian on a Lie group will be given in the Appendix. In particular, this implies that there is some $\kappa > 0$ such that $\operatorname{supp} W_1 \subset \overline{B}_\kappa(0)$. Scaling, using (3.1), this yields

$$\operatorname{supp} W_t \subset \overline{B}_{\kappa t}(0)\,, \quad \forall t > 0\,. \tag{3.16}$$

(see also transparency!)

Put now

$$\psi(s) := m(s^2), \quad s \in \mathbb{R}?\,,$$

and observe that

$$||\psi||_{L^2_\alpha} \approx ||m||_{L^2_\alpha}\,,$$

since $\operatorname{supp} m \subset [1/2, 4]$. For $\alpha > 1/2$, by Fourier inversion, since ψ is even,

$$\psi(s) = \frac{1}{\pi}\int_0^\infty \hat{\psi}(t)\cos(ts)\,dt\,,$$

hence (by Fubini),

$$m(\mathcal{L}) = \psi(\sqrt{\mathcal{L}}) = \frac{1}{\pi}\int_0^\infty \hat{\psi}(t)\cos(t\sqrt{\mathcal{L}})\,dt\,,$$

i.e.,

$$k_m = \frac{1}{\pi}\int_0^\infty \hat{\psi}(t) W_t\,dt\,. \tag{3.17}$$

Fix $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(t) = 1$, if $|t| \leq 1/2\kappa$, $\chi(t) = 0$, if $|t| \geq 1/\kappa$, χ even. For given $R \gg 1$, define ψ_1, ψ_2 by

$$\begin{aligned} \widehat{\psi_1}(t) &:= \chi\left(\frac{t}{R}\right)\hat{\psi}(t), \\ \widehat{\psi_2}(t) &:= (1-\chi)\left(\frac{t}{R}\right)\hat{\psi}(t)\,, \end{aligned}$$

and put

$$k_m^{(i)} := \frac{1}{\pi}\int_0^\infty \widehat{\psi}_i(t) W_t\,dt\,, \quad i = 1, 2\,.$$

Then

$$k_m = k_m^{(1)} + k_m^{(2)}\,.$$

Moreover, by (3.16),

$$\operatorname{supp} k_m^{(1)} \subset \overline{B}_R(0)\,,$$

so that

$$k_m = k_m^{(2)} \quad \text{outside} \quad \overline{B}_R(0)\,.$$

And, "computing backwards",

$$k_m^{(2)} = k_{\tilde{m}}\,,$$

where $\tilde{m}(s) := \psi_2(\sqrt{s})$. Thus, by Plancherel's identity (3.5),

$$\begin{aligned} \int_{|g|\geq R} |k_m(g)|^2\,dg &\leq ||k_m^{(2)}||_2^2 \\ &= c_n \int_0^\infty |\tilde{m}(s)|^2 s^n\,ds \\ &= 2c_n \int_0^\infty |\psi_2(s)|^2 s^{2n+1}\,ds\,. \qquad (3.18) \end{aligned}$$

But, if $\varphi \in \mathcal{S}(\mathbb{R})$ is such that $\hat{\varphi} = \chi$, then

$$\psi_2 = \psi - \psi \star \varphi_{1/R}\,,$$

where $\varphi_{1/R}(s) = R\varphi(Rs)$ is "essentially" supported in $[-2/R, 2/R]$, say. Thus, ψ_2 is essentially supported in $[1/4, 8]$, so that

$$\begin{aligned} &\int_0^\infty |\psi_2(s)|^2 s^{2n+1} ds \approx ||\psi_2||_2^2 = \frac{1}{2\pi}||\widehat{\psi_2}||_2^2 \\ \leq\ &\int_{|t|\geq R/\kappa} |\hat{\psi}(t)|^2\,dt \\ \lesssim\ &R^{-2\alpha}\int_{|t|\geq R} |\hat{\psi}(t)(1+t^2)^{\alpha/2}|^2\,dt \\ \leq\ &R^{-2\alpha}||\psi||_{L_\alpha^2}^2 \end{aligned}$$

Q.E.D.

Since $\operatorname{supp} m \subset [1/2, 4]$ in Proposition 3.2, we can write $m(s) = e^{-s} m_1(s)$, where m_1 again satisfies the hypothesis of Proposition 3.2. Thus

$$k_m = k_{m_1} * h_1 \,, \tag{3.19}$$

where $h_t = e^{-t\mathcal{L}}\delta_0$ denotes the heat kernel with time parameter $t > 0$. Since $\mathcal{L}$ is hypoelliptic, $h_t \in C^\infty$. Moreover, h_t has Gaussian decay (see, e.g., [26]). From (3.19) we find that $k_m \in C^\infty(\mathbb{H}_n)$, and that, for every left-invariant PDO D on $\mathbb{H}_n$, Dk_m satisfies (3.15) as well (compare, e.g., the proof of Lemma 6.36 in [7]), i.e.,

$$\int_{\mathbb{H}_n} |Dk_m(g)|(1+|g|)^\varepsilon \, dg \;\lesssim\; ||m||_{L^2_\alpha}, \tag{3.20}$$

if $\alpha > \varepsilon + Q/2$.

Corollary 3.3. *If $m \in C_0^\infty(\mathbb{R})$ and $\operatorname{supp} m \subset [1/2, 4]$, then $k_m \in \mathcal{S}(\mathbb{H}_n)$.*

Remarks 3.4. (i) An inspection of the proofs reveals that the support condition $\operatorname{supp} m \subset [1/2, 4]$ in Proposition 3.2 and Corollary 3.3 is not really necessary - it suffices that m has compact support. However, the constant in the estimates (3.14) and (3.20) will depend on r, if m is supported in $[0, r]$.

(ii) It is in the first inequality of (3.18) where one looses half a fractional derivative on the required regularity of the multiplier m (compare with the optimal result described in Remark 5.2 (i)). The reason why this happens on the Heisenberg group, but not on Euclidean space, is that the wave propagator W_t on $\mathbb{H}_n$, unlike the one associated to the Laplacian on Euclidean space, has singularities not only on the boundary of the "Carnot-Carathéodory" sphere of radius t (see the Appendix), but also in the interior, so that $\int_{|g|\le R} |k_m^{(2)}(g)|^2 \, dg$ is not small compared to $\int_{|g|\ge R} |k_m^{(2)}(g)|^2 \, dg$.

(iii) Corollary 3.3 is a special case of a more general result on homogeneous Lie groups due to A. Hulanicki [13]

4 Littlewood-Paley Theory

Fix $\phi \in C_0^\infty(\mathbb{R})$, supported in $[1/2, 2] \cup [-2, -1/2]$, ϕ even, real, such that

$$\sum_{j\in\mathbb{Z}} \phi^2(2^{-j}s) = 1 \quad \forall s \neq 0. \tag{4.1}$$

Put $\phi_j(s) := \phi(2^{-j}s)$, and

$$\varphi_j := \phi_j(\mathcal{L})\delta_0 = \phi(2^{-j}\mathcal{L})\delta_0, \quad j \in \mathbb{Z}\,.$$

Notice that, by (3.1),

$$\varphi_j(g) = 2^{jQ/2}\varphi_0(\delta_{2^{j/2}}g), \quad g \in \mathbb{H}_n\,, \tag{4.2}$$

and that

$$\sum_j \phi^2(2^{-j}\mathcal{L}) = I \quad \text{on } L^2(\mathbb{H}_n)\,,$$

in the strong operator topology.

Define the **Littlewood-Paley g-function** of $f \in \mathcal{S}'(\mathbb{H}_n)$ by

$$g_{\mathcal{L}}(f)(z,u) := \left(\sum_j |f * \varphi_j(z,u)|^2\right)^{1/2}, \quad (z,u) \in \mathbb{H}_n\,.$$

Proposition 4.1. *For every $p \in]1,\infty[$ there exists a $C_p > 0$, such that*

$$C_p^{-1}||f||_p \le ||g_{\mathcal{L}}(f)||_p \le C_p||f||_p\,. \tag{4.3}$$

Proof. By Corollary 3.3, we know that $\varphi_j \in \mathcal{S}(\mathbb{H}_n)$, and that φ_j results from φ_0 by dyadic scaling (see (4.2)). In view of (4.1), the proof therefore follows from standard arguments in Calderón-Zygmund theory (see, e.g., [23], [7]). One possible proof is sketched below.

Sketch of the main idea:
For $p = 2$,

$$\begin{aligned} ||g_{\mathcal{L}}(f)||_2^2 = \sum_j ||\phi_j(\mathcal{L})f||_2^2 \;&=\; \sum_j \langle \phi_j^2(\mathcal{L})f, f\rangle \\ = \left\langle \left(\sum_j \phi_j^2\right)(\mathcal{L})f, f \right\rangle \;&=\; \langle f, f\rangle = ||f||_2^2\,, \end{aligned}$$

so that (4.3) holds with $C_2 = 1$.
Denote by l_2 the Hilbert space $l_2(\mathbb{Z})$, which we identify with the space $B := \mathcal{B}(\mathbb{C}, l_2)$ of all bounded linear operators from $\mathbb{C}$ to l_2, in the obvious way. Consider then the l_2-valued, i.e. B-valued, kernel function.

$$\mathbf{k}(z,u) := \{\varphi_j(z,u)\}_{j\in\mathbb{Z}}\,, \quad (z,u) \in \mathbb{H}_n\,,$$

on $\mathbb{H}_n$. It is an easy exercise to check that this kernel satisfies the CZ-condition (1.5), if one replaces the absolute value in (1.5) by the l_2-norm. But, Theorem 1.4 remains valid also for B-valued $CZO's$, so that

$$||f * \mathbf{k}||_{L^p(\mathbb{H}_n, l_2)} \le C_p||f||_{L^p(\mathbb{H}_n)}, \quad 1 < p < \infty\,.$$

And, clearly

$$||f * \mathbf{k}||_{L^p(\mathbb{H}_n, l_2)} = ||g_{\mathcal{L}}(f)||_{L^p(\mathbb{H}_n)} ,$$

which gives the second inequality in (4.3).
The first inequality then follows by a standard duality argument. Q.E.D.

In a similarly way, we can define a g-function g_U associated to U, by putting

$$\psi_l := \phi_l \left(\frac{1}{i} U \right) \delta_0 ,$$

and then

$$g_U(f)(z,u) := \left(\sum_l |f * \psi_l(z,u)|^2 \right)^{1/2} .$$

Again

$$||g_U(f)||_p \sim ||f||_p , \quad \text{if } 1 < p < \infty .$$

Notice that

$$\psi_l(z,u) = (\delta_0 \otimes \gamma_l)(z,u) ,$$

where $\widehat{\gamma_l}(\lambda) = \phi_l(\lambda), \lambda \in \mathbb{R}$.
Combining $g_{\mathcal{L}}$ and g_U, we define

$$g_1(f)(z,u) := \left(\sum_{j,l} |f * (\varphi_j * \psi_l)(z,u)|^2 \right)^{1/2} .$$

Then, again

$$(4.4) \qquad ||g_1(f)||_p \sim ||f||_p , \quad \text{for } 1 < p < \infty$$

(see [19] for details). Observe that g_1 involves scaling with respect to the automorphisms δ_r, as well as an independent scaling in the central variable U, i.e. a **multi-parameter** scaling!

5 Joint spectral multipliers for $\mathcal{L}$ and $\frac{1}{i}U$

The two self-adjoint operators $\mathcal{L}$ and $\frac{1}{i}U$ commute, so that we can form joint functions $m(\mathcal{L}, \frac{1}{i}U)$, in the sense of joint spectral theory of these operators on the Hilbert space $L^2(\mathbb{H}_n)$. Since

$$d\pi_\lambda \left(\frac{1}{i} U \right) = \lambda I \quad \text{and} \quad d\pi_\lambda(\mathcal{L})\Phi_{\alpha,\lambda} = |\lambda|(2|\alpha| + n)\Phi_{\alpha,\lambda} ,$$

arguing as in Section 3, the convolution kernel

$$k_m := m(\mathcal{L}, \frac{1}{i}U)\delta_0$$

is explicitly given by

$$k_m(z,u) = (2\pi)^{-n-1} \int_{\mathbb{R}^\times} \sum_\alpha m(|\lambda|(2|\alpha|+n),\lambda)\overline{\chi_{\alpha,\lambda}(z,u)}|\lambda|^n\, d\lambda \tag{5.1}$$

(compare (3.7)).
Consider next the operator $|U|^{-1}\mathcal{L}$, defined spectrally by

$$\pi_\lambda(|U|^{-1}\mathcal{L})\Phi_{\alpha,\lambda} = \frac{|\lambda|(2|\alpha|+n)}{|\lambda|}\Phi_{\alpha,\lambda} = (2|\alpha|+n)\Phi_{\alpha,\lambda}\,.$$

It turns out to be favourable to study joint functions of $|U|^{-1}\mathcal{L}$ and $\frac{1}{i}U$ first, since, for these two operators, a separation of the Fourier variables λ and α takes place. Accordingly, we define another g-function as follows: Put

$$\Phi_j = \phi_j(|U|^{-1}\mathcal{L})\delta_0, \quad j\in\mathbb{Z}\,,$$

and let

$$g_2(f)(z,u) := \left(\sum_{j,l}\left|f*(\Phi_j*\psi_l)(z,u)\right|^2\right)^{1/2}.$$

Notice that

$$\Phi_j*\psi_l = (\phi_j\otimes\phi_l)(|U|^{-1}\mathcal{L},\frac{1}{i}U)\delta_0\,.$$

One can then show (see [19]) that, again,

$$||g_2(f)||_p \sim ||f||_p\,,\quad 1<p<\infty\,. \tag{5.2}$$

We shall make use of (5.2) in order to prove the following multiplier theorem.

Fix a non-trivial bump function $\eta_0\in C_0^\infty(\mathbb{R})$ supported in $\mathbb{R}_+ :=]0,\infty[$, and put

$$\eta_1(s) := \eta_0(s)+\eta_0(-s)\,,\quad s\in\mathbb{R}\,,$$

and

$$\chi(\tau,\lambda) := \eta_0(\tau)\eta_1(\lambda).$$

For every continuous function h on $\mathbb{R}_+\times\mathbb{R}$ put

$$h^r(\tau,\lambda) := h(r_0\tau, r_1\lambda)\,,$$

if $r=(r_0,r_1)\in\mathbb{R}_+^2$.
We define mixed Sobolev spaces $L^2_{\alpha,\beta} = L^2_{\alpha,\beta}(\mathbb{R}\times\mathbb{R})$ by

$$\begin{aligned}||f||_{L^2_{\alpha,\beta}} &:= ||(1+|t|)^\alpha(1+|t|+|u|)^\beta\hat{f}(t,u)||_{L^2}\\ &= c\,||(1+|\partial_\tau|)^\alpha(1+|\partial_\tau|+|\partial_\lambda|)^\beta f||_{L^2}\,.\end{aligned}$$

Then our main theorem is as follows:

Theorem 5.1. *Let $h \in C(\mathbb{R}_+ \times \mathbb{R})$ be such that*

$$||h||_{L^2_{\alpha,\beta,sloc}} := \sup_r ||h^r \chi||_{L^2_{\alpha,\beta}} < \infty$$

for some $\alpha > n, \beta > 1/2$. Then the operators $h(|U|^{-1}\mathcal{L}, \frac{1}{i}U)$ and $h(\mathcal{L}, \frac{1}{i}U)$ are bounded on $L^p(\mathbb{H}_n)$ for $1 < p < \infty$, with norms controlled by $||h||_{L^2_{\alpha,\beta,sloc}}$.

Remarks 5.2. (i) If h depends only on the first variable τ, i.e., if $h(\tau, \lambda) = m(\tau)$, we obtain, as a corollary, the following multiplier theorem of Mihlin-Hörmander type:

Suppose $m \in C(\mathbb{R}_+)$ satisfies

$$||m||_{L^2_{\alpha,sloc}} := \sup_{r>0} ||m(r\cdot)\eta_0||_{L^2_\alpha} < \infty$$

for some $\alpha > d/2$, where

$$d := 2n + 1$$

is the **Euclidean dimension** of $\mathbb{H}_n$. Then $m(\mathcal{L})$ is bounded on $L^p(\mathbb{H}_n)$ for $1 < p < \infty$.

This result had first been proved in [16], where it had also been shown that the condition $\alpha > d/2$ is sharp (for an alternative proof, see also [12]). In these articles it had in fact been proved that $m(\mathcal{L})$ is a Calderòn-Zygmund operator, so that it is also of weak-type (1,1). Notice that this improves on Proposition 3.2, where $\alpha > Q/2$ was required.

(ii) Under the same hypothesis, $||m||_{L^2_{\alpha,sloc}} < \infty$ for some $\alpha > d/2$, one also obtains that $m(\Delta)$ is bounded on $L^p(\mathbb{H}_n)$, for $1 < p < \infty$, where Δ is the full Laplacian on $\mathbb{H}_n$. Recently, extensions of the latter result to functions of the Hodge-Laplacian, acting on differential forms on $\mathbb{H}_n$, have been proved by Müller-Peloso-Ricci.

(iii) Theorem 5.1 is a special case of a multiplier theorem for Heisenberg type groups (see [19]; also [18]), where in general the homogeneous dimension is much bigger than the Euclidean dimension. Variants of these results, for joint functions of the partial sub-Laplacian, have been studied by A. Veneruso [27].

On the proof of Theorem 5.1.

Observe first that

$$h(\mathcal{L}, \frac{1}{i}U) = \tilde{h}(|U|^{-1}\mathcal{L}, \frac{1}{i}U)\,,$$

if we put

$$\tilde{h}(\tau, \lambda) = h(|\lambda|\tau, \lambda)\,.$$

And, one can prove that (check for derivatives of entire order!)

$$||\tilde{h}||_{L^2_{\alpha,\beta,sloc}} \sim ||h||_{L^2_{\alpha,\beta,sloc}}\,.$$

We may therefore restrict ourselves to estimating operators $h(|U|^{-1}\mathcal{L}, \frac{1}{i}U)$.
In order to defray the proof from certain technical complications which arise if $n \geq 2$, let us assume that $n = 1$.

We first re-write the condition in Theorem 5.1. Re-scaling in the variables τ and λ, one finds that

$$||h^r\chi||^2_{L^2_{\alpha,\beta}} \tag{5.3}$$
$$= \frac{1}{r_0 r_1}\int \left|(1+|r_0\partial_\tau|)^\alpha(1+|r_0\partial_\tau|+|r_1\partial_\lambda|)^\beta(h\eta_r)(\tau,\lambda)\right|^2 d\tau d\lambda\,,$$

if we put

$$\eta_r(\tau,\lambda) := \eta_0\left(\frac{\tau}{r_0}\right)\eta_1\left(\frac{\lambda}{r_1}\right).$$

Since $n = 1$, we shall write $\alpha = k \in \mathbb{N}$ for the spectral parameter α.
Notice that the joint spectrum of $|U|^{-1}\mathcal{L}$ and $\frac{1}{i}U$ contains $(2\mathbb{N}+1)\times\mathbb{R}^\times \subset \mathbb{Z}\times\mathbb{R}$.
We define the **first order difference operator** Δ in the $\mathbb{Z}$-variable, acting on a function $f : \mathbb{Z}\times\mathbb{R}\to\mathbb{C}$, by

$$\Delta f(k,\lambda) := f(k,\lambda) - f(k-1,\lambda)\,.$$

If, on $\mathbb{T}\times\mathbb{R}$,

$$\hat{f}(t,u) := \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} f(k,\lambda)e^{-i(kt+\lambda u)}\,d\lambda, \qquad t\in[0,2\pi[,\lambda\in\mathbb{R},$$

denotes the Fourier transform of f, we have

$$\begin{aligned}\widehat{\Delta f}(t,u) &= (1-e^{-it})\hat{f}(t,u)\,,\\ \widehat{\partial_\lambda f}(t,u) &= iu\hat{f}(t,u)\,.\end{aligned}$$

We therefore define fractional powers $|\Delta|^\alpha$, $|\partial_\lambda|^\alpha$, $\alpha\in\mathbb{C}$, by

$$\begin{aligned}(\widehat{|\Delta|^\alpha f})(t,u) &:= |1-e^{-it}|^\alpha\hat{f}(t,u)\,,\\ (\widehat{|\partial_\lambda|^\alpha f})(t,u) &:= |u|^\alpha\hat{f}(t,u).\end{aligned}$$

We then actually prove the following, sharper version of Theorem 5.1.

Theorem 5.3. *Let $m = m(2k+1,\lambda)$ be a continuous function defined on the joint spectrum $(2\mathbb{N}+1)\times\mathbb{R}$ of $|U|^{-1}\mathcal{L}$ and $\frac{1}{i}U$. Extend m by putting*

$$\tilde{m}(k,\lambda) := \begin{cases} m(2k+1,\lambda) & , \text{ if } k\geq 0\,,\\ 0 & , \text{ if } k<0\end{cases}$$

as a function $\tilde{m}$ on $\mathbb{Z}\times\mathbb{R}$. With η_0,η_1 as before, assume that

$$||m||^2_{\alpha,\beta,sloc}$$
$$:= \sup_{r_j>0}\frac{1}{r_0r_1}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left|(1+|r_0\Delta|)^\alpha(1+|r_0\Delta|+|r_1\partial_\lambda|)^\beta(\tilde{m}\eta_r)(k,\lambda)\right|^2 d\lambda<\infty$$

for some $\alpha>1, \beta>1/2$. Then the operator $m(|U|^{-1}\mathcal{L}, \frac{1}{i}U)$ is bounded on $L^p(\mathbb{H}_1)$ for $1<p<\infty$, with norm controlled by $||m||_{\alpha,\beta,sloc}$.

In view of (5.3), it is plausible that this implies Theorem 5.1.

Proof. One can show that the norms $||m||_{\alpha,\beta,sloc}$ are equivalent for different choices of η_0.
Let us now choose $\eta_0 \geq 0$ in $C_0^\infty(\mathbb{R})$ supported in $[1/4, 4]$ such that the associated even function η_1 is identical 1 on $\operatorname{supp}\phi$ (ϕ as in Section 4). Define $\eta_j(s) := \eta_1(2^{-j}s)$, so that

$$\phi_j = \phi_j \eta_j \,,$$

and put $m_{j,l} := m(\eta_j \otimes \eta_l)$,

$$M_{j,l} := m_{j,l}(|U|^{-1}\mathcal{L}, \frac{1}{i}U)\delta_0 \,.$$

Observe that $M_{j,l} = 0$ if $j \leq 1$, since $\operatorname{spec}(|U|^{-1}\mathcal{L}) \subset 2\mathbb{N}+1$.
Similarly as in the proof of Proposition 3.2, we need size estimates on the kernels $M_{j,l}$. The crucial result will be

Lemma 5.4. *For $j, l \in \mathbb{Z}, j \geq 1,$ we have*

$$\begin{aligned} &\int_{\mathbb{H}_1} \left|(1 + 2^{j+l}|z|^2)(1 + 2^l|u|)M_{j,l}(z,u)\right|^2 dzdu \\ &\leq C2^{j+2l}\frac{1}{2^j 2^l}\sum_{k\in\mathbb{Z}}\int \left|(1+2^j|\Delta|)^2(1+2^j|\Delta|+2^l|\partial_\lambda|)\widetilde{m_{j,l}}(k,\lambda)\right|^2 d\lambda \,. \end{aligned} \tag{5.4}$$

Proof. Since $|U|^{-1}\mathcal{L}$ is homogeneous of degree 0 and U is homogeneous of degree 2 with respect to $\{\delta_r\}_{r>0}$, by re-scaling it suffices to consider the case $l = 0$. Fix $j \geq 1$. We concentrate on the leading term in (5.4), which is

$$J := \int_{\mathbb{H}_1} \left|2^j|z|^2 u\, M_{j,0}(z,u)\right|^2 dzdu \,.$$

We want to show that

$$J \lesssim \sum_k \int \left|(2^{2j}|\Delta|^2)(2^j|\Delta| + |\partial_\lambda|)\widetilde{m_{j,0}}(k,\lambda)\right|^2 d\lambda.$$

Now, from classical properties of Laguerre polynomials, one knows that if F is a polyradial function on $\mathbb{H}_1$, with Gelfand transform $\hat{F}(\lambda, k)$, then $|z|^2 F(z,u)$ and $iuF(z,u)$ are also polyradial, and their Gelfand transforms are given by

$$\partial_{|z|^2}\hat{F} \quad \text{and} \quad \partial_{iu}\,\hat{F},$$

where $\partial_{|z|^2}$ and ∂_{iu} are the operators, acting on functions $f : \mathbb{Z}\times\mathbb{R} \to \mathbb{C}$, given by

$$(\partial_{|z|^2}f)(k,\lambda) := \frac{2}{|\lambda|}(k\Delta - \tau_1 k\Delta)f(k,\lambda)\,, \tag{5.5}$$

$$(\partial_{iu}f)(k,\lambda) := \partial_\lambda - \frac{1}{2\lambda}(k\Delta + \tau_1 k\Delta)f(k,\lambda)\,, \tag{5.6}$$

where τ_p denotes the shift operator

$$\tau_p f(k,\lambda) := f(k+p,\lambda)\,, \quad p\in\mathbb{Z},$$

and where k means the multiplication operator $f\mapsto kf(k,\lambda)$(see, e.g., [4] and [24]; for explicit expressions for fractional powers of $\partial_{|z|^2-iu}$, see [11]).

And, by Plancherel's identity for polyradial functions,

$$J=(2\pi)^{-2}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^\times}\left|2^j\partial_{|z|^2}\partial_{iu}\tilde m_{j,0}(k,\lambda)\right|^2|\lambda|\,d\lambda\,.$$

Observe now that $\partial_{|z|^2}$ and ∂_{iu} can be re-written as

$$\begin{aligned}\partial_{|z|^2} &= -\frac{2}{|\lambda|}(\tau_1 k\Delta^2+\Delta)\,,\\ \partial_{iu} &= \partial_\lambda-\frac{1}{2\lambda}(\tau_0+\tau_1)k\Delta\,.\end{aligned}$$

But

$$k\sim 2^j \quad \text{and} \quad |\lambda|\sim 1 \text{ on } \operatorname{supp}\tilde m_{j,0}\,,$$

so that $2^j\partial_{|z|^2}$ behaves essentially like

$$2^{2j}\Delta^2\,,$$

and $|\partial_{iu}|$ is dominated by

$$|\partial_\lambda|+2^j|\Delta|\,.$$

This leads to (5.4), for $l=0$ (for details, see [19]). Q.E.D.

By Plancherel's identity, we also have

$$\int_{\mathbb{H}_1}|M_{j,l}(z,u)|^2\,dzdu \quad\lesssim\quad 2^{j+2l}\frac{1}{2^j2^l}\sum_k\int|\widetilde{m_{j,l}}(k,\lambda)|^2\,d\lambda\,.$$

Interpolating this with (5.4), we obtain, for $0\le\varepsilon\le 1/2$,

$$\int_{\mathbb{H}_1}\left|(1+2^{j+l}|z|^2)^{1/2+\varepsilon}(1+2^l|u|)^{1/2+\varepsilon}M_{j,l}(z,u)\right|^2dzdu\lesssim 2^{j+2l}||m||^2_{1+2\varepsilon,1/2+\varepsilon,sloc}\,.$$

If we define the weight $W=W_{j,l}$ by

$$W(z,u):=2^{-j-2l}(1+2^{j+l}|z|^2)^{1+2\varepsilon}(1+2^l|u|)^{1+2\varepsilon}\,,$$

then this can be re-written as

$$\int_{\mathbb{H}_1}|M_{j,l}(g)|^2W(g)\,dg\lesssim||m||^2_{1+2\varepsilon,1/2+\varepsilon,sloc}\,. \tag{5.7}$$

Notice that $1/W \in L^1(\mathbb{H}_1)\ \forall \varepsilon > 0$, and

$$||1/W||_1 \ \le\ C_\varepsilon, \quad \forall\, j, l$$

($1/W$ decays indeed better than needed for being in L^1).

We can now perform the L^p-estimation of $T := m(|U|^{-1}\mathcal{L}, \frac{1}{i}U)$.
Observe first that, since $\phi_j = \phi_j \eta_j$, we have

$$(Tf) \star (\Phi_j \star \psi_l) = [m(\eta_j \otimes \eta_l)(\phi_j \otimes \phi_l)](|U|^{-1}\mathcal{L}, \frac{1}{i}U)f = f_{j,l} \star M_{j,l}\,,$$

if we put

$$f_{j,l} := f \star (\Phi_j \star \psi_l)\,.$$

Thus, by (5.2),

$$||Tf||_p \lesssim ||g_2(Tf)||_p = \Big|\Big| \Big(\sum_{j,l} |f_{j,l} \star M_{j,l}|^2 \Big)^{1/2} \Big|\Big|_p\,. \tag{5.8}$$

Let us assume without loss of generality that $p \ge 2$ (the case $p < 2$ then follows by duality). By Cauchy-Schwarz' inequality and (5.7), we have

$$\begin{aligned} |f_{j,l} \star M_{j,l}(g)|^2 \ &=\ \Big| \int_{\mathbb{H}_1} \frac{f_{j,l}(gh^{-1})}{W(h)^{1/2}} \cdot (M_{j,l} W^{1/2}(h))\, dh \Big|^2 \\ &\lesssim\ ||m||^2_{1+2\varepsilon, 1/2+\varepsilon, sloc} \int_{\mathbb{H}_1} \frac{|f_{j,l}(gh^{-1})|^2}{W(h)}\, dh, \quad g \in \mathbb{H}_1\,. \end{aligned} \tag{5.9}$$

Moreover, we can choose a function $\eta \in L^{(p/2)'}(\mathbb{H}_1)$, $\eta \ge 0, ||\eta||_{(p/2)'} = 1$, such that

$$\begin{aligned} \Big|\Big| \Big(\sum_{j,l} |f_{j,l} \star M_{j,l}|^2 \Big)^{1/2} \Big|\Big|_p^2 \ &=\ \Big|\Big| \sum_{j,l} |f_{j,l} \star M_{j,l}|^2 \Big|\Big|_{p/2} \\ &\le\ 2 \int_{\mathbb{H}_1} \sum_{j,l} |f_{j,l} \star M_{j,l}(g)|^2 \eta(g)\, dg\,. \end{aligned}$$

In combination with (5.8), (5.9), by Fubini's theorem we therefore get, putting $C_m := ||\tilde{m}||_{1+2\varepsilon, 1/2+\varepsilon, sloc}$,

$$\begin{aligned} ||Tf||_p^2 &\lesssim C_m^2 \int_{\mathbb{H}_1} \int_{\mathbb{H}_1} \sum_{j,l} \frac{|f_{j,l}(gh^{-1})|^2}{W(h)} \eta(g)\, dh dg \\ &=\ C_m^2 \int_{\mathbb{H}_1} \Big(\sum_{j,l} |f_{j,l}(g)|^2 \Big) \Big(\int_{\mathbb{H}_1} \frac{\eta(gh)}{W(h)}\, dh \Big)\, dg\,. \end{aligned}$$

But, $1/W(z, u)$ is "essentially" supported where $|z| \le 2^{-(j+l)/2}$, $|u| \le 2^{-l}$, so that integration against $1/W$ behaves essentially like averaging on the product domain $\{|z| \le 2^{-(j+l)/2},\ |u| \le 2^{-l}\}$.

If we define the **strong maximal operator**

$$M_s\eta(g) := \sup_{r_j>0} \frac{1}{r_0 r_1} \int_{|z|\le r_0^2} \int_{|u|\le r_1} |\eta(g\cdot(z,u))|\, dzdu\,,$$

one can indeed prove the domination

$$\int_{\mathbb{H}_1} \frac{\eta(gh)}{W(h)}\, dh \quad \lesssim \quad M_s\eta(g)\,,$$

so that

$$||Tf||_p^2 \lesssim C_m^2 \int_{\mathbb{H}_1} \left(\sum_{j,l} |f_{j,l}(g)|^2 \right) M_s\eta(g)\, dg\,.$$

However, it is known that also M_s is bounded on $L^q(\mathbb{H}_1)$, for $1 < q \le \infty$, (see, e.g. [2], or [20]). Therefore, by Hölder's inequality,

$$\begin{aligned} ||Tf||_p^2 \;&\lesssim\; C_m^2 \left(\int \left(\sum_{j,l} |f_{j,l}(g)|^2 \right)^{p/2} dg \right)^{2/p} \cdot ||M_s\eta||_{(p/2)'} \\ &\lesssim\; C_m^2\, ||g_2(f)||_p^2\, ||\eta||_{(p/2)'}\,, \end{aligned}$$

hence, by (5.2),

$$||Tf||_p \quad \lesssim \quad C_m ||f||_p\,.$$

Q.E.D.

6 Appendix: Finite propagation speed

We shall here give a proof of the finite propagation speed of waves associated to sums of squares operators on Lie groups, which will be based on a modification of the energy estimates in [5], Theorem 5.3 (alternatively, see [15]).

Let V be a connected C^∞-manifold of dimension n, and let $X_1, \ldots, X_k$ be smooth, everywhere linearly independent vector fields on V satisfying Hörmander's condition, i.e., the vector fields $X_1, \ldots, X_k$ and their iterated commutators span the tangent space to V at every point of V. An absolutely continuous path $\gamma : [0,1] \to V$ is called **horizontal** (w.r. to $X_1, \ldots, X_k$), if there exist functions $a_j(t)$ such that

$$\dot\gamma(t) = \sum_{j=1}^{k} a_j(t) X_j(\gamma(t)) \quad \text{for a.e.}\, t \in [0,1]\,.$$

Regarding $X_1, \ldots, X_k$ as an "orthonormal" system, one defines the length of γ by

$$|\gamma| := \int_0^1 \left(\sum_{j=1}^{k} a_j^2(t) \right)^{1/2} \mathrm{dt}\,.$$

For $x, y \in V$, we put

$$\rho(x,y) := \inf\{|\gamma| : \gamma \quad \text{is horizontal}, \gamma(0) = x, \gamma(1) = y\},$$

if there exists a horizontal path connecting x and y, and $\rho(x,y) := \infty$ otherwise.
Hörmander's condition ensures that $\rho(x,y) < \infty$ for every $x, y \in V$, and that ρ is a distance which induces the topology of V (see, e.g., [26]), the **so-called Carnot-Carathéodony distance**.
We now specialize V to the case of a connected Lie group G, with Lie algebra $\mathfrak{g}$ and right-invariant Haar measure dg, and assume that $X_1, \ldots, X_k$ are left-invariant vector fields on G. Identifying $X \in \mathfrak{g}$ with the left-invariant vector field

$$Xf(g) := \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} f(g \exp tX), \quad f \in C^\infty(G), g \in G,$$

we then have

$$\int_G (X\varphi)(x)\psi(x)\mathrm{dx} = -\int_G \varphi(x)(X\psi)(x)\mathrm{dx}, \tag{6.1}$$

say, for $\varphi, \psi \in C_0^\infty(G)$. Moreover, ρ is left-invariant, i.e.

$$\rho(gx, gy) = \rho(x,y) \quad \forall\, g, x, y \in G. \tag{6.2}$$

Let $\{\chi_\varepsilon\}_{\varepsilon>0} \subset C_0^\infty(G)$ be an approximation to the identity, consisting of non-negative functions with compact supports tending to the identity element as $\varepsilon \to 0$, and consider the regularization

$$\begin{aligned} \rho_\varepsilon(x,y) &:= \int_G \chi_\varepsilon(g)\rho(gx,y)\, dg \\ &= \int_G \chi_\varepsilon(gx^{-1})\rho(g,y)\, dg. \end{aligned}$$

Then $\rho_\varepsilon(x,y) \to \rho(x,y)$, uniformly on compacta as $\varepsilon \to 0$, and $\rho_\varepsilon(\cdot, y) \in C^\infty(G)$ for every $y \in G$.

Lemma 6.1. *If* $X = \sum_{j=1}^{k} \alpha_j X_j$, *with* $\alpha_j \in \mathbb{R}$, *and if we put* $|X| := \left(\sum_{j=1}^k \alpha_j^2\right)^{1/2}$, *then*

$$|X\rho_\varepsilon(\cdot, y)| \le |X| \quad \forall\, y \in G. \tag{6.3}$$

In particular,

$$\sum_{j=1}^{k} (X_j \rho_\varepsilon(\cdot, y))^2 \quad \le 1. \tag{6.4}$$

Proof. For $s \in \mathbb{R}$, $x, y \in G$ fixed, we put

$$\Delta(s) := \rho(x \exp(sX), y) - \rho(x, y)\,.$$

Since ρ is a distance, one easily sees that

$$|\Delta(s)| \leq \rho(x \exp(sX), x)\,.$$

Denote by γ the path

$$\gamma(t) := x \exp(tsX),\ t \in [0,1]\,,$$

so that

$$\dot\gamma(t) = sX(\gamma(t))\,.$$

Then γ is horizontal, connects x with $x \exp(sX)$, and

$$|\gamma| = |s|\ |X|\,.$$

Consequently, $|\Delta(s)| \leq |s|\ |X|$, i.e.

$$|\rho(x(\exp sX), y) - \rho(x, y)|\ \leq\ |s|\ |X|\,,$$

independently of x and y. This implies

$$\begin{aligned} & |\rho_\varepsilon(x \exp(tX), y) - \rho_\varepsilon(x, y)| \\ \leq\ & \int_G \chi_\varepsilon(g)|\ |\rho(gx \exp(tX), y) - \rho(gx, y)|\ dg \\ \leq\ & |s|\ |X|\,, \end{aligned}$$

for every $\varepsilon > 0$, which implies (6.3). Q.E.D.

To prove (6.4), given $x \in G$, choose a unit vector $\xi = (\xi_1, \ldots, \xi_k) \in \mathbb{R}^k$ so that

$$l := \left(\sum_{j=1}^k (X_j \rho_\varepsilon(x, y))^2\right)^{1/2} = \sum_{j=1}^k \xi_j X_j \rho_\varepsilon(x, y)\,,$$

where X is considered to act on the first variable, and put $X := \sum_{j=1}^k \xi_j X_j$. Then $|X| = 1$, and $l = X\rho_\varepsilon(x, y)$, so that (6.4) follows from (6.3).

Let $L = -\sum_{j=1}^k X_j^2$.

Theorem 6.2. *Suppose $u \in C^2(G \times [0, T])$ satisfies the wave equation*

$$\partial_t^2 u + Lu = 0\,.$$

Assume further that $u = \partial_t u = 0$ on the ball

$$B := \{(x, 0) : \rho(x, x_0) \leq t_0\}$$

in the hyperplane $t = 0$, where $x_0 \in G$ and $0 < t_0 < T$. Then u vanishes in the conical region

$$\Gamma = \{(x, t) : 0 \leq t \leq t_0 \quad \text{and} \quad \rho(x, x_0) \leq t_0 - t\}\,.$$

Proof. Taking real and imaginary parts, we may assume that u is real. For $0 \leq t \leq t_0$ let

$$B_t = \{x \in G : \rho(x, x_0) \leq t_0 - t\}$$

be the "t-section of Γ". Given any small $\delta > 0$, we choose $\chi = \chi_\delta \in C^\infty(\mathbb{R})$ such that $\chi(s) = 1$, if $s \leq 1$, $\chi(s) = 0$, if $s \geq 1 + \delta$, and so that $\psi := -\chi' \geq 0$. Notice that then $\psi \in C_0^\infty(\mathbb{R})$, and $\operatorname{supp}\psi \subset [1, 1+\delta]$, and that $\chi\Big(\dfrac{\rho(x,x_0)}{t_0 - t}\Big)$ approximates the indicator function of the set B_t. Denote by $E(t) = E_{\delta,\varepsilon}(t)$ the "energy" of u essentially contained in B_t, given by

$$E(t) := \frac{1}{2}\int_G |\operatorname{grad}_H u(x,t)|^2 \chi\left(\frac{\rho_\varepsilon(x,x_0)}{t_0 - t}\right) dx\,,$$

where $\operatorname{grad}_H u$ denotes the "horizontal" gradient

$$\operatorname{grad}_H u := (u_t, X_1 u, \ldots, X_k u)\,.$$

Then

$$\frac{dE}{dt}(t) = \int_G [u_t u_{tt} + \sum_{j=1}^k (X_j u)(X_j u_t)]\; \chi\left(\frac{\rho_\varepsilon(x,x_0)}{t_0 - t}\right) dx \qquad (6.5)$$

$$-\frac{1}{2}\int_G |\operatorname{grad}_H u(x,t)|^2 \psi\left(\frac{\rho_\varepsilon(x,x_0)}{t_0 - t}\right)\frac{\rho_\varepsilon(x,x_0)}{(t_0-t)^2}\, dx\,.$$

Moreover

$$X_j((X_j u)u_t) = (X_j u)(X_j u_t) + (X_j^2 u)u_t\,. \qquad (6.6)$$

This implies

$$\begin{aligned} J &:= \int_G [u_t u_{tt} + \sum_{j=1}^k (X_j u)(X_j u_t)]\; \chi\left(\frac{\rho_\varepsilon(x,x_0)}{t_0 - t}\right) dx \\ &= \int_G u_t(u_{tt} + Lu)\; \chi\left(\frac{\rho_\varepsilon(x,x_0)}{t_0 - t}\right) dx \\ &+ \int_G \sum_{j=1}^k X_j((X_j u)u_t)\; \chi\left(\frac{\rho_\varepsilon(x,x_0)}{t_0 - t}\right) dx\,. \end{aligned}$$

The first term vanishes, since u satisfies the wave equation. And,

$$-X_j\left(\chi\left(\frac{\rho_\varepsilon(\cdot,x_0)}{t_0 - t}\right)\right) = \psi\left(\frac{\rho_\varepsilon(\cdot,x_0)}{t_0 - t}\right)\frac{X_j\rho_\varepsilon(\cdot,x_0)}{t_0 - t}\,,$$

so that, by (6.1),

$$J = \int_G u_t\left[\sum_{j=1}^k (X_j u) X_j \rho_\varepsilon(x,x_0)\right]\psi\left(\frac{\rho_\varepsilon(x,x_0)}{t_0 - t}\right)\frac{dx}{t_0 - t}\,.$$

Cauchy-Schwarz' inequality thus leads to

$$|J| \le \int_G |u_t| \left(\sum_{j=1}^k (X_j u)^2\right)^{1/2} \left(\sum_{j=1}^k (X_j \rho_\varepsilon(x,x_0))^2\right)^{1/2} \psi\left(\frac{\rho_\varepsilon(x,x_0)}{t_0-t}\right)\frac{dx}{t_0-t}.$$

From Lemma 6.1 we thus obtain

$$\begin{aligned} |J| &\le \int_G |u_t| \left(\sum_{j=1}^k (X_j u)^2\right)^{1/2} \psi\left(\frac{\rho_\varepsilon(x,x_0)}{t_0-t}\right)\frac{dx}{t_0-t} \\ &\le \frac{1}{2}\int_G \left(|u_t|^2 + \sum_{j=1}^k (X_j u)^2\right) \psi\left(\frac{\rho_\varepsilon(x,x_0)}{t_0-t}\right)\frac{dx}{t_0-t} \\ &\le \frac{1}{2}\int |\operatorname{grad}_H u(x,t)|^2 \psi\left(\frac{\rho_\varepsilon(x,x_0)}{t_0-t}\right)\frac{\rho_\varepsilon(x,x_0)}{t_0-t}\frac{dx}{t_0-t}, \end{aligned}$$

since $\dfrac{\rho_\varepsilon(x,x_0)}{t_0-t} \ge 1$, by the support property of ψ. Together with (6.5), we see that $\dfrac{dE}{dt} \le 0$, hence

$$E_{\delta,\varepsilon}(t) \le E_{\delta,\varepsilon}(0) \quad \forall\, t \in [0,t_0], \varepsilon > 0.$$

Letting ε tend to zero, this implies, by the dominated convergence theorem, that

$$E_\delta(t) \le E_\delta(0), \quad \forall\, t \in [0,t_0],$$

where

$$E_\delta(t) := \frac{1}{2}\int_G |\operatorname{grad}_H u(x,t)|^2\, \chi_\delta\left(\frac{\rho(x,x_0)}{t_0-t}\right) dx.$$

Letting finally δ tend to zero, we obtain, for $0 \le t \le t_0$,

$$\int_{B_t} |\operatorname{grad}_H u(x,t)|^2\, dx \quad \le \quad \int_{\{\rho(x,x_0)\le t_0\}} |\operatorname{grad}_H u(x,0)|^2\, dx. \tag{6.7}$$

Since the Cauchy data of u vanish on B, the right-hand side of (6.7) vanishes, so that $\operatorname{grad}_H u = 0$ on Γ. But, $u(x,0) = 0$, so that $u = 0$ on Γ.

Q.E.D.

Let us denote by e the identity element in G, and by

$$B_r(y) := \{x \in G : \rho(x,y) < r\}$$

the ball of radius $r > 0$ centered at $y \in G$, and by $\overline{B}_r(y) := \{x \in G : \rho(x,y) \le r\}$, $(r \ge 0)$.

Corollary 6.3. *Let $W_t := \cos(t\sqrt{L})\delta_e$ be the wave propagotar associated to L. Then*

$$\operatorname{supp} W_t \subset \overline{B}_t(e) \quad (t \ge 0). \tag{6.8}$$

Proof. Let $\varepsilon > 0$, and let $\varphi \in \mathcal{D}(\overline{B}_\varepsilon(e))$. Then $u(x,t) := \varphi * W_t(x)$ solves the wave equation $\partial_t^2 u + Lu = 0$, and $u(\cdot, 0) = \varphi$, $u_t(\cdot, 0) = 0$. Fix $t_0 > 0$, and let $x_0 \in G \backslash \overline{B}_{t_0+\varepsilon}(e)$, i.e. $\rho(x_0, e) > t_0 + \varepsilon$. Then $\overline{B}_{t_0}(x_0) \cap \overline{B}_\varepsilon(e) = \emptyset$, and so the Cauchy data of u vanish on $B := \{(x,0) : \rho(x, x_0) \le t_0\}$. From Theorem 6.2 we conclude that $u(x_0, t_0) = 0$.
This implies

$$\text{supp } \varphi * W_{t_0} \subset \overline{B}_{t_0+\varepsilon}(e)\,,$$

for every $\varphi \in \mathcal{D}(\overline{B}_\varepsilon(e))$. Applying this to a Dirac-sequence supported in $\overline{B}_\varepsilon(e)$, we obtain

$$\text{supp } W_{t_0} \subset \overline{B}_{t_0+\varepsilon}(e)\,, \quad \forall\, \varepsilon > 0\,.$$

This gives (6.8).

Q.E.D.

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