SQUARE FUNCTION ESTIMATES AND FUNCTIONAL CALCULI

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To the memory of Nigel Kalton (1946–2010)

Abstract. In this paper the notion of an abstract square function (estimate) is introduced as an operator $X \rightarrow \gamma(H;Y)$, where $X, Y$ are Banach spaces, $H$ is a Hilbert space, and $\gamma(H;Y)$ is the space of $\gamma$-radonifying operators. By the seminal work of Kalton and Weis, this definition is a coherent generalisation of the classical notion of square function appearing in the theory of singular integrals. Given an abstract functional calculus $(\mathcal{E}, \mathcal{F}, \Phi)$ on a Banach space $X$, where $\mathcal{F}(O)$ is an algebra of scalar-valued functions on a set $O$, we define a square function $\Phi_\gamma(f)$ for certain $H$-valued functions $f$ on $O$. The assignment $f \mapsto \Phi_\gamma(f)$ then becomes a vectorial functional calculus, and a “square function estimate” for $f$ simply means the boundedness of $\Phi_\gamma(f)$. In this view, all results linking square function estimates with the boundedness of a certain (usually the $H^\infty$-) functional calculus simply assert that certain square function estimates imply other square function estimates. In the present paper several results of this type are proved in an abstract setting, based on the principles of subordination, integral representation, and a new boundedness concept for subsets of Hilbert spaces, the so-called $\ell_1$-frame-boundedness. These abstract results are then applied to the $H^\infty$-calculus for sectorial and strip type operators. For example, it is proved that any strip type operator with bounded scalar $H^\infty$-calculus on a strip over a Banach space with finite cotype has a bounded vectorial $H^\infty$-calculus on every larger strip.

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1. INTRODUCTION

Square functions and square function estimates are a classical topic and a central tool in harmonic analysis, in particular in the so-called Littlewood–Paley theory. Their history can be traced back to almost a century ago, see [38] for a historical account and [39, 40, 41] for the development from the 1960s on. One of the classical instances of a square function is

\[(1.1) \quad (S_\phi f)(x) := \left( \int_0^\infty |(\phi_t \ast f)(x)|^2 \frac{dt}{t} \right)^{1/2} \]

where \( \phi \in L_2(\mathbb{R}^d) \) decays reasonably fast at infinity and \( \phi_t(x) = t^{-d} \phi(x/t) \) for \( x \in \mathbb{R}^d \) and \( t > 0 \). A “square function estimate” then reads

\[(1.2) \quad \|S_\phi f\|_{L_p} = \left\| \left( \int_0^\infty |(\phi_t \ast f)(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L_p} \lesssim \|f\|_{L_p}. \]

In many situations, \( \phi \) is radial. Then its Fourier transform is radial, too, and can be written as \( \hat{\phi}(\xi) = \psi(|\xi|) \) for \( \xi \in \mathbb{R}^d \). Hence,

\[ \phi_t \ast f = \mathcal{F}^{-1}(\hat{\phi}(t\xi) \cdot \hat{f}(\xi)) = \mathcal{F}^{-1}(\hat{\psi}(|t\xi|) \cdot \hat{f}(\xi)) = \psi(t \sqrt{-\Delta}) f, \]

where we employ the functional calculus for the Laplace, or better, the Poisson operator. Hence, the abstract form of (1.2) is

\[(1.3) \quad \left\| \left( \int_0^\infty |\psi(tA)f|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L_p} \lesssim \|f\|_{L_p}, \]

where \( A := \sqrt{-\Delta} \) and taking \( \psi(z) := ze^{-z} \) we recover the classical Littlewood–Paley \( g \)-function.

From the mid 1980’s on, the theory of functional calculus for sectorial operators was developed by several people. Building on the seminal works [32] and [7] and inspired by [5] Cowling, Doust, McIntosh and Yagi in [6] established a strong link between the boundedness of the \( H^\infty \)-calculus for sectorial operators \( A \) on (closed subspaces of) \( L_p \)-spaces and square functions of the form (1.3). Kalton and Weis in an unpublished and unfortunately never completed manuscript [21] then showed how one could pass from \( L_p \)-spaces to general Banach spaces. Their manuscript...
subsequently circulated and inspired a considerable amount of research, e.g. [2, 10, 11, 12, 16, 17, 18, 19, 20, 23, 26, 29, 45, 46, 47, 15]. It is also the starting point of the present article.

The main novelty in Kalton and Weis’ approach from [21] was to employ the class of so-called $\gamma$-radonifying operators in order to define square functions. This step is motivated by two observations. On the one hand,

\[
\left( \int_0^\infty |\psi(tA)f|^2 \frac{dt}{t} \right)^{1/2} = \left( \sum_{k=1}^{\infty} ||Tf|e_k|^2 \right)^{1/2},
\]

where \((e_n)_{n\in\mathbb{N}}\) is an orthonormal basis of \(H := L_2(\mathbb{R}_+; dt/t)\) and \(Tf : H \to X := L_p(\mathbb{R})\) is the operator defined by

\[
[Tf] h := \int_0^\infty h(t)\psi(tA)f \frac{dt}{t} \quad (h \in H).
\]

(This holds true in each Banach lattice \(X\), see Section 2.7 below.) The second, more decisive step, is based on the norm equivalence

\[
\left\| \left( \sum_{k} |x_k|^2 \right)^{1/2} \right\|_X \sim \left( \mathbb{E} \left\| \sum_{k} \gamma_k \otimes x_k \right\|_X^2 \right)^{1/2},
\]

where \((\gamma_k)_k\) is an independent sequence of standard Gaussian random variables. (This equivalence holds true in any Banach lattice \(X\) of finite cotype, see Theorem 2.26.) Hence, the square function estimate (1.3) can be reformulated as

\[
(1.4) \quad \left( \mathbb{E} \left\| \sum_{k} \gamma_k \otimes |Tf|e_k \right\|_X^2 \right)^{1/2} \lesssim \|f\|_X
\]

with \(Tf\) as above. In other words, \(Tf \in \gamma(H;X)\), the space of $\gamma$-radonifying operators, and \(\|Tf\|_\gamma \lesssim \|f\|_X\) (see Section 2.7 below). In this formulation of the square function estimate the lattice structure of \(X = L_p\) does not appear any more and hence it can be used to define square function estimates over general Banach spaces \(X\).

In the present paper we follow this approach, but transcend it in two points. The first, minor, point is that we propose a definition of a general square function as any linear operator \(T : \text{dom}(T) \to \gamma(H;Y)\) where \(X,Y\) are Banach spaces and \(\text{dom}(T) \subseteq X\) is a linear subspace. A square function estimate for the square function \(T\) then just asserts its boundedness

\[
\|Tx\|_{\gamma(H;Y)} \lesssim \|x\|_X.
\]

If we admit finite dimensional Hilbert spaces \(H\) here, any (bounded) operator can be viewed as a trivial square function (estimate).

The second and more important point of our approach is that for the square functions of functional calculus type as before, we want to systematise the dependence on the function \(\psi\), e.g., in order to cover square functions associated with expressions of the form

\[
\psi(tA) \quad \text{instead of} \quad \psi(tA).
\]

(We work with the functional calculus for sectorial operators for the time being, as did Kalton and Weis). Although we do not claim that this could not be done by the conservative (Kalton-Weis) approach, we find it more natural to follow the basic ideology of functional calculus, i.e., to replace working with operators by working with functions. This leads to a re-reading of the operator \(Tf\) from above as

\[
(1.5) \quad [Tf] h = \int_0^\infty h(t)\psi(tA)f \frac{dt}{t} = \left( \int_0^\infty h(t)\psi(tz) \frac{dt}{t} \right)(A)f.
\]
(For good functions $\psi$ this is possible by the definition of $\psi(tA)$ as a Cauchy integral and Fubini’s theorem.) The function $\psi(t, z) = \psi(tz)$ can now be viewed as a function of two parameters, but since only integration with respect to $t$ is performed here, it should better be viewed as a mapping

$$\Psi : S_\omega \to H = L^2(\mathbb{R}_+; \mathbb{R}), \quad \Psi(z) = (t \mapsto \psi(t, z))$$

The square function $T$ from above then appears as resulting from inserting $A$ into the $H$-valued $H^\infty$-function $\Psi$, by defining

$$\Psi(A)x := h \mapsto (z \mapsto \langle h, \Psi(z) \rangle)(A)x : H \to X$$

In this way, the problem of a square function estimate turns into the problem of the boundedness of the operator $\Psi(A)$. As such, it is recognised as just another instance of the central problem of functional calculus, namely whether applying an unbounded functional calculus to a certain function leads to a bounded operator or not. The only difference now is that the functions we are considering have to be $H$-valued, and one should think of a functional calculus as a module rather than an algebra homomorphism.

In this view, the classical line of research on the connection of square function estimates and the boundedness of a certain (usually the $H^\infty$-) functional calculus changes its face, and the so far differently looking theorems become just instances of one type: namely how certain square function estimates imply other square function estimates.

In the present paper we have analysed theorems of this type and could reduce them to three basic principles. The first one is subordination, by which we mean that the square functions are connected via a bounded operator between the underlying Hilbert spaces. The second is an abstract version of how square function estimates can be proved via integral representation theorems. Here a deep result from the theory of $\gamma$-radonifying operators plays a central role. The third one is based on a new (natural, but still rather enigmatic) boundedness concept for subsets of Hilbert spaces, the so-called $\ell_1$-frame-boundedness. Basically, it asserts that if the $H$-valued function $\Psi$ as above has $\ell_1$-frame-bounded range, then the associated square function $\Psi(A)$ is bounded (Theorem 4.11).

The abstract results are then applied to operators of strip type (and hence to sectorial operators via the exp/log-correspondence). One of the main results here, Theorem 6.1, states that a densely defined operator (on a Banach space with finite cotype) with a bounded scalar $H^\infty$-calculus on a strip has a bounded vectorial $H^\infty$-calculus on each larger strip. (This result has been obtained independently of us by Le Merdy in [27].) Our proof is based on a simple representation formula for holomorphic functions on strips, see Section 6.1.

Subsequently we consider several other integral representation formulae for analytic functions on strips and interpret their application in the light of our theory (Sections 6.2–6.6).

By performing the twist in (1.5), our theory of square function estimates becomes just a natural part of functional calculus theory. The technicalities involving integrals over vector-valued functions (like $t \mapsto \psi(tA)f$ as above) are avoided. As a consequence (and maybe surprisingly) the concepts of $\gamma$- or $R$-boundedness do not appear in the present paper. We shall devote a future work to the detailed study of the role these concepts play for square function estimates, and how it can be incorporated into the general theory we develop here.

**Notation and Terminology**

Banach spaces are denoted by $X, Y, Z$ and understood to be complex unless otherwise noted. For a closed linear operator $A$ on a complex Banach space $X$ we denote by $\text{dom}(A), \text{ran}(A), \ker(A), \sigma(A)$ and $\varrho(A)$ the domain, the range, the kernel, the
spectrum and the resolvent set of A, respectively. The norm-closure of the range is written as \( \mathfrak{M}(A) \). The space of bounded linear operators on \( X \) is denoted by \( \mathcal{L}(X) \). For two possibly unbounded linear operators \( A, B \) on \( X \) their product \( AB \) is defined on its natural domain \( \text{dom}(AB) := \{ x \in \text{dom}(B) \mid Bx \in \text{dom}(A) \} \). An inclusion \( A \subseteq B \) denotes inclusion of graphs, i.e., it means that \( B \) extends \( A \).

The inner product of two elements \( f, g \) of a Hilbert space \( H \) is generically written as \( (f \mid g) \) or \( (f \mid g)_H \). The duality between a Banach space \( X \) and its dual space \( X' \) is denoted by \( \langle \cdot, \cdot \rangle \) or \( \langle \cdot, \cdot \rangle_{X,X'} \). We usually do not identify a Hilbert space \( H \) with its dual space \( H' \), except in the case that \( H \) is given concretely as \( H = L^2(\Omega) \), in which case we identify \( H' \) with \( H \) via the canonical duality.

For an open subset \( O \subseteq \mathbb{C} \) of the complex plane we let \( H^\infty(O) \) be the algebra of bounded holomorphic functions on \( O \) with norm \( \|f\|_{H^\infty} := \sup \{ |f(z)| \mid z \in O \} \).

Unless explicitly noted otherwise, the real line \( \mathbb{R} \) carries Lebesgue measure \( dt \), and the set \( (0, \infty) \) of positive reals carries the measure \( dt' \). We abbreviate \( L^*_p(0, \infty) := L_p((0, \infty); dt') \quad (0 < p \leq \infty) \).

The Fourier transform of a function \( f \in L_1(\mathbb{R}) \) is
\[
\mathcal{F}(f)(t) = \hat{f}(t) = \int_{\mathbb{R}} f(s)e^{ist} \, ds \quad (t \in \mathbb{R}).
\]
The inverse Fourier transform is then given by the formula
\[
(\mathcal{F}^{-1}g)(s) = g^\wedge(s) = \frac{1}{2\pi} \int_{\mathbb{R}} g(t)e^{ist} \, dt \quad (s \in \mathbb{R})
\]
for \( g \in L_1(\mathbb{R}) \).

### 2. \( \gamma \)-Summing and \( \gamma \)-Radonifying Operators

In this chapter we review and develop the theory of \( \gamma \)-summing and \( \gamma \)-radonifying operators to an extent that serves our purposes. At many places we shall simply refer to the excellent recent article [14] of van Neerven that contains also historical remarks and an extensive bibliography on the topic. We include proofs either for convenience or when we deviate from or go beyond van Neerven’s work.

Essentially, all presented results in this chapter are from or inspired by the unpublished and actually never completed preprint [21] by Kalton and Weis. Our contribution consists mostly in presenting the results with full and concise proofs, and we give full credits to these authors for the results themselves.

However, we want to stress the fact that whereas the two mentioned works deal exclusively with real Banach spaces, we develop the theory for complex spaces. The reason is that we are interested in functional calculus questions, where contour integrals are ubiquitous. For the theory we need the notion of a complex standard Gaussian random variable, by which we mean a random variable \( \gamma \) of the form
\[
\gamma = \gamma_r + i\gamma_i
\]
where \( \gamma_r \) and \( \gamma_i \) are independent standard real Gaussians. Basically, the whole theory for real spaces carries over to complex spaces when real Gaussians are replaced by complex ones.

#### 2.1. Definition and the Ideal Property

Let \( H \) be a Hilbert space and \( X \) a Banach space over the scalar field \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \). A linear operator \( T : H \to X \) is called \( \gamma \)-summing if
\[
\|T\|_{\gamma} := \sup_F \mathbb{E} \left( \left\| \sum_{e \in F} \gamma_e \otimes Te \right\|_X^2 \right)^{\frac{1}{2}} < \infty,
\]
where the supremum is taken over all finite orthonormal systems \( F \subseteq H \) and \((\gamma_e)_{e \in F}\) is an independent collection of \( \mathbb{K} \)-valued standard Gaussian random variables on some probability space. We let

\[
\gamma_\infty(H;X) := \{ T : H \rightarrow X \mid T \text{ is } \gamma\text{-summing} \}
\]

the space of \( \gamma \)-summing operators of \( H \) into \( X \). It is clear that each \( \gamma \)-summing operator is bounded with \( \| T \| \leq \| T \|_\gamma \).

**Remark 2.1 (Real vs. Complex).** In the case \( \mathbb{K} = \mathbb{C} \) we can view the complex spaces \( H, X \) as real spaces, and we shall indicate this by writing \( H_r, X_r \). Then \( H_r \) is a real Hilbert space with respect to the scalar product \( \langle f | g \rangle_r := \text{Re} \langle f | g \rangle \).

For \( \mathbb{C} \)-linear \( T : H \rightarrow X \) we now have two definitions of \( \| T \|_\gamma \), one using \( \langle \cdot, \cdot \rangle_r \)-orthonormal systems (called \( \mathbb{R} \)-ons’s for short) and real Gaussians, and the other using \( \mathbb{C} \)-orthonormal systems and complex Gaussians. We claim that both definitions lead to the same quantity. In particular, one has

\[
\gamma_\infty(H;X) = \gamma_\infty(H_r;X_r) \cap \mathcal{L}(H;X).
\]

In order to see this we note first that if \( \{ e_1, \ldots, e_d \} \) is a \( \mathbb{C} \)-orthonormal system, then \( \{ e_1, \ldots, e_d, ie_1, \ldots, ie_d \} \) is an \( \mathbb{R} \)-ons. Hence, if \( \tilde{\gamma}_j = \gamma_j + i\gamma_j \) are independent complex standard Gaussians,

\[
E \left\| \sum_j \gamma_j T e_j \right\|^2 = E \left\| \sum_j \gamma_j T(e_j) + \gamma_j T(ie_j) \right\|^2 \leq \| T \|_{\gamma, \mathbb{R}}^2
\]

with the obvious meaning of \( \| T \|_{\gamma, \mathbb{R}} \). This yields \( \| T \|_{\gamma, \mathbb{C}} \leq \| T \|_{\gamma, \mathbb{R}} \). On the other hand, let \( \{ f_1, \ldots, f_n \} \) be an \( \mathbb{R} \)-ons and let \( \gamma_1, \ldots, \gamma_n \) be real standard Gaussians. Pick a \( \mathbb{C} \)-ons \( \{ e_1, \ldots, e_n \} \) such that \( f_k \in \text{span}_\mathbb{C}\{ e_1, \ldots, e_n \} \) for each \( k \). Then we can find \( \lambda_{kj} = a_{kj} + ib_{kj} \) such that

\[
f_k = \sum_j (a_{kj} + ib_{kj})e_j = \sum_j a_{kj}e_j + b_{kj}(ie_j) \quad (1 \leq k \leq n).
\]

Define the real matrices \( A := (a_{kj})_{k,j} \), \( B := (b_{kj})_{k,j} \) and \( C := [AB] \), as well as \( g_j := e_j \) for \( 1 \leq j \leq n \) and \( g_j := ie_j \) for \( n < j \leq 2n \). Then, by the contraction principle (Theorem A.1),

\[
E \left\| \sum_{k=1}^n \gamma_k T f_k \right\|^2 = E \left\| \sum_{k=1}^n \gamma_k a_{kj} T f_k + b_{kj} T(ie_k) \right\|^2
\]

\[
= E \left\| \sum_{k=1}^n \sum_{j=1}^{2n} \gamma_k c_{kj} T g_j \right\|^2 \leq \| C \|^2 E \left\| \sum_{j=1}^{2n} \gamma_j T g_j \right\|^2
\]

\[
= \| C \|^2 E \left\| \sum_{j=1}^{2n} (\gamma_j + i\gamma_{n+j}) T e_j \right\|^2 \leq \| C \|^2 \| T \|_{\gamma, \mathbb{C}}^2.
\]

But \( c_{kj} = \langle f_k, g_j \rangle_r \), and hence \( \| C \| \leq 1 \). This yields \( \| T \|_{\gamma, \mathbb{R}} \leq \| T \|_{\gamma, \mathbb{C}} \) and concludes the proof of the claim.

The following approximation property is \[14\text{ Prop. 3.18}].

**Lemma 2.2 (\( \gamma \)-Fatou).** Let \( (T_n)_{n \geq 1} \) be a bounded sequence in \( \gamma_\infty(H;X) \) such that \( T_n \rightharpoonup T \in \mathcal{L}(H;X) \) in the weak operator topology. Then \( T \in \gamma_\infty(H;X) \) and

\[
\| T \|_\gamma \leq \liminf_{n \rightarrow \infty} \| T_n \|_\gamma.
\]

It is easy to see that \( \gamma_\infty(H;X) \) contains all finite rank operators. The closure in \( \gamma_\infty(H;X) \) of the space of finite rank operators is denoted by \( \gamma(H;X) \), and its elements \( T \in \gamma(H;X) \) are called \( \gamma \)-radonifying.

The following property is one of the cornerstones of the theory.
Theorem 2.3 (Ideal Property). Let $Y$ be another Banach space and $K$ another Hilbert space, let $L : X \to Y$ and $R : K \to H$ be bounded linear operators, and let $T \in \gamma_\infty(H;X)$. Then

$$LTR \in \gamma_\infty(K;Y) \quad \text{and} \quad \|LTR\|_\gamma \leq \|L\|_{\mathcal{L}(X,Y)} \|T\|_\gamma \|R\|_{\mathcal{L}(K;H)}.$$  

If $T \in \gamma(H;X)$, then $LTR \in \gamma(K;Y)$.

Proof. One can handle the left-hand and the right-hand side separately, the first being straightforward. For the latter, pick a finite orthonormal system $\{e_1, \ldots, e_n\}$ within $K$. Then find an orthonormal system $\{f_1, \ldots, f_m\}$ with

$$\text{span}\{Re_1, \ldots, Re_n\} = \text{span}\{f_1, \ldots, f_m\}.$$  

Consequently $Re_k = \sum_{j=1}^m a_{kj} f_j$ for some scalar $(n \times m)$-matrix $A = (a_{kj})_{kj}$. Then, by Theorem A.1 below,

$$\mathbb{E} \left\| \sum_{k=1}^n \gamma_k TR_e \right\|^2 = \mathbb{E} \left\| \sum_{k=1}^n \gamma_k T \sum_{j=1}^m a_{kj} f_j \right\|^2$$

$$= \mathbb{E} \left\| \sum_{k=1}^n \sum_{j=1}^m \gamma_k a_{kj} T f_j \right\|^2 \leq \|A\|^2 \mathbb{E} \left\| \sum_{j=1}^m \gamma_j T f_j \right\|^2 \leq \|A\|^2 \|T\|^2_\gamma.$$  

Since $\|A\|_{2 \to 2} \leq \|R\|_{K \to H}$, the claim is proved. \qed

See [44] Theorem 6.2 for a slightly different proof. Based on the ideal property, we can show that in the case $K = \mathbb{C}$ a difference between the complex and real approach to $\gamma(H;X)$ is only virtual.

Remark 2.4 (Real vs Complex, again). Let $H, X$ be complex spaces. We claim that

$$\gamma(H;X) = \{ T \in \gamma(H;X_r) \mid T \text{ is } \mathbb{C}-\text{linear} \} = \gamma(H_r;X_r) \cap \mathcal{L}(H;X).$$

The inclusion “$\subseteq$” is trivial, so suppose that $T : H \to X$ is $\mathbb{C}$-linear and in $\gamma(H_r;X_r)$. Then there is a sequence $T_n$ of real-linear finite rank operators such that $\|T_n - T\|_\gamma \to 0$. Define $S_n x := \frac{1}{2}(T_n x - iT_n(ix))$. Then each $S_n$ is a $\mathbb{C}$-linear finite rank operator $\|S_n - T\|_\gamma \to 0$. To prove this we note that the operator $M : x \mapsto ix$ is a linear isometry on $H_r$ commuting with $T$, whence

$$2 \|S_n - T\|_\gamma \leq \|T_n - T\|_\gamma + \|M^{-1}T_n M - T\|_\gamma \leq \|T_n - T\|_\gamma \to 0$$

by the ideal property. It follows that $T \in \gamma(H;X)$, as claimed.

One might ask whether $\gamma_\infty(H;X)$ can differ from $\gamma(H;X)$. An example from Linde and Pietsch, reproduced in [44] Exa. 4.4, shows that this indeed happens if $X = c_0$. On the other hand, by a theorem of Hoffman-Jerzensen and Kwapień, if $X$ does not contain $c_0$ then $\gamma(H;X) = \gamma_\infty(H;X)$, see [44] Theorem 4.3. Although this result was obtained for real spaces only, Remark 2.4 shows that it continues to hold in the complex case.

For later reference, we quote the following approximation results from [44] Corollaries 6.4 and 6.5. Their proofs are straightforward from the ideal property.

Theorem 2.5 (Approximation). Let $H, K$ be Hilbert and $X, Y$ be Banach spaces, and let $T \in \gamma_\infty(H;X)$. Then the following assertions hold:

a) If $(L_\alpha)_\alpha \subseteq \mathcal{L}(X;Y)$ is a uniformly bounded net that converges strongly to $L \in \mathcal{L}(X;Y)$, then $L_\alpha T \to LT$ in $\gamma_\infty(H;Y)$.

b) If $(R_\alpha^*)_\alpha \subseteq \mathcal{L}(H;K)$ is a uniformly bounded net that converges strongly to $R^* \in \mathcal{L}(H;K)$, then $TR_\alpha \to TR$ in $\gamma_\infty(K;X)$.

Note that if $T \in \gamma(H;X)$ the operators $LT$ and $TR$ are again $\gamma$-radionifying, by the ideal property.
2.2. Fourier series and nuclear operators. For \( g \in H \) we let \( \overline{g} := (\cdot | g) \in H' \), i.e.,

\[
H \to H', \quad g \mapsto \overline{g} = (\cdot | g)
\]

is the canonical (conjugate-linear) bijection of \( H \) onto its dual. The definition

\[
(\overline{g} | \overline{h})_{H'} := (h | g)\quad (g, h \in H)
\]

(2.1)

turns \( H' \) canonically into a Hilbert space, and a short computation yields \( \overline{g} = g \) under the canonical identification \( H = H'' \). Moreover, (2.1) becomes

\[
(\overline{g} | \overline{h})_{H'} = (g | x)_{H'} \quad (x, y \in H').
\]

(2.2)

If \( H = L_2(\Omega) = L_2(\Omega; \mathbb{K}) \) for some measure space \((\Omega, \Sigma, \mu)\), we can identify \( H' = L_2(\Omega) \) via the duality

\[
H \times H \to \mathbb{K}, \quad (h, g) \mapsto \langle h, g \rangle := \int_\Omega h \cdot g \, d\mu \quad (h, g \in L_2(\Omega)).
\]

(2.3)

Under this identification, the conjugate \( \overline{g} \) of \( g \in H \) as defined above coincides with the usual complex conjugate of \( g \) as a function on \( \Omega \).

Every finite rank operator \( T : H \to X \) has the form

\[
T = \sum_{j=1}^n \overline{g}_j \otimes x_j,
\]

(2.4)

and one can view \( \gamma(H; X) \) as a completion of the algebraic tensor product \( H' \otimes X \) with respect to the \( \gamma \)-norm.

Note that if \( e_1, \ldots, e_n \) is an orthonormal system in \( H \), then \( \overline{e}_1, \ldots, \overline{e}_n \) is an orthonormal system in \( H' \), dual to \( \{e_1, \ldots, e_n\} \) in the sense that

\[
\langle e_j, \overline{e}_k \rangle = \langle e_j, \overline{e}_k \rangle_{H', H'} = \delta_{jk} \quad (j, k = 1, \ldots, n).
\]

The following shows that a “Gaussian sum” in a Banach space \( X \) can be regarded as a \( \gamma \)-norm of a finite rank operator.

**Lemma 2.6.** Let \( g_1, \ldots, g_m \in H \) be an orthonormal system in \( H \) and \( x_1, \ldots, x_m \in X \). Then

\[
\left\| \sum_{j=1}^m \overline{g}_j \otimes x_j \right\|^2_\gamma = \mathbb{E} \left\| \sum_{j=1}^n \gamma_j x_j \right\|^2_X.
\]

**Proof.** Let \( e_1, \ldots, e_n \) be any finite orthonormal system in \( H \) and let \( T \) be defined by (2.4). Then

\[
\mathbb{E} \left\| \sum_{k=1}^n \gamma_k T e_k \right\| = \mathbb{E} \left\| \sum_{k=1}^n \gamma_k \sum_{j=1}^m (e_k | g_j) x_j \right\| \leq \mathbb{E} \left\| \sum_{j=1}^n \gamma_j x_j \right\|^2
\]

by Theorem A.1, since the scalar matrix \( A := ((e_k | g_j))_{k,j} \) satisfies \( \|A\| \leq 1 \). On the other hand, if we take \( n = m \) and \( e_k := g_k \), then we obtain equality. \( \square \)

Let \( \{e_\alpha\}_{\alpha \in A} \) be an orthonormal basis of \( H \). For a finite set \( F \subseteq A \), let

\[
P_F := \sum_{\alpha \in F} \overline{e}_\alpha \otimes e_\alpha
\]

be the orthogonal projection onto \( \text{span}\{e_\alpha \mid \alpha \in F\} \). The net \( (P_F)_F \) is uniformly bounded and converges strongly to the identity on \( H \). Hence, the following is a consequence of Theorem \( 2.5 \) [part b)]

**Corollary 2.7** (Fourier Series). If \( T \in \gamma(H; X) \) and \( \{e_\alpha\}_\alpha \) is any orthonormal basis of \( H \), then

\[
\sum_{\alpha} \overline{e}_\alpha \otimes T e_\alpha = T
\]

in the norm of \( \gamma(H; X) \).
It follows from Lemma 2.6 that
\[ \Vert V \otimes x \Vert_\gamma = \Vert g \Vert_H \Vert x \Vert_X = \Vert V \Vert_T \Vert x \Vert_X \]
for every \( g \in H, x \in X \), i.e., the \( \gamma \)-norm is a cross-norm. The following is an immediate consequence. (Recall that \( T \) is a nuclear operator if \( T = \sum n g_n \otimes x_n \) for some \( g_n \in H, x_n \in X \) with \( \sum n \Vert g_n \Vert_H \Vert x_n \Vert_X < \infty \).)

**Corollary 2.8.** A nuclear operator \( T : H \to X \) is \( \gamma \)-radonifying and \( \Vert T \Vert_\gamma \leq \Vert T \Vert_{\text{nu}} \).

The following application turns out to be quite useful.

**Lemma 2.9.** Let \( H, X \) as before, and let \( (\Omega, \Sigma, \mu) \) be a measure space. Suppose that \( f : \Omega \to H \) and \( g : \Omega \to X \) are (strongly) \( \mu \)-measurable and
\[ \int_{\Omega} \Vert f(t) \Vert_H \Vert g(t) \Vert_X \mu(dt) < \infty. \]
Then \( \Sigma \otimes g \in L_1(\Omega; \gamma(H; X)) \), and \( T := \int_{\Omega} \Sigma \otimes g \, d\mu \in \gamma(H; X) \) satisfies
\[ Th = \int_{\Omega} (h \cdot f(t)) g(t) \mu(dt) \quad (h \in H) \]
and
\[ \Vert T \Vert_\gamma \leq \int_{\Omega} \Vert f(t) \Vert_H \Vert g(t) \Vert_X \mu(dt). \]

### 2.3. Trace duality

We follow [4, 21] and identify the dual of \( \gamma(H; X) \) with a subspace of \( \mathcal{L}(H'; X') \) via trace duality. For a finite rank operator \( U : H \to H \) given by
\[ U := \sum_{j=1}^n g_j' \otimes h_j \]
for certain \( g_1', \ldots, g_n' \in H' \) and \( h_1, \ldots, h_n \in H \), its trace is
\[ \text{tr}(U) = \sum_{j=1}^n \langle h_j, g_j' \rangle. \]

This is independent of the representation of \( U \), see [3] p. 125]. Now, for \( V \in \mathcal{L}(H'; X') \) we define
\[ \Vert V \Vert_{\gamma'} := \sup \left\{ \left| \text{tr}(V'U) \right| : U \in \mathcal{L}(H; X), \Vert U \Vert_\gamma \leq 1, \dim \text{ran}(U) < \infty \right\}, \]
where we regard \( V'U : H \to X \subseteq X'' \to H'' = H \), and let
\[ \gamma'(H'; X') := \{ V \in \mathcal{L}(H'; X') : \Vert V \Vert_{\gamma'} < \infty \}. \]

By a short computation, if \( U \in \mathcal{L}(H; X) \) has the representation \( U = \sum_{j=1}^n g_j' \otimes x_j \) and \( V \in \mathcal{L}(H'; X') \), then
\[ (2.5) \quad \text{tr}(V'U) = \sum_{j=1}^n \langle x_j, Vg_j' \rangle. \]

**Lemma 2.10** (\( \gamma' \)-Fatou). Let \( (V_n)_n \) be a bounded sequence in \( \gamma'(H'; X') \) and let \( V : H' \to X' \) be such that \( \langle x, V_n h' \rangle \to \langle x, Vh' \rangle \) for all \( x \in X \) and \( h' \in H' \). Then \( V \in \gamma'(H'; X') \) and
\[ \Vert V \Vert_{\gamma'} \leq \liminf_{n \to \infty} \Vert V_n \Vert_{\gamma'}. \]

**Proof.** It follows from [2.5] that \( \text{tr}(V_n'U) \to \text{tr}(V'U) \) for every \( U : H \to X \) of finite rank. The claim follows. \( \square \)

We now turn to an alternative description of the \( \gamma' \)-norm. To this end we note the following auxiliary result. We let Nu\((H)\) denote the class of nuclear operators on \( H \), also called operators of trace class, with its natural norm \( \Vert \cdot \Vert_{\text{nu}} \).

**Lemma 2.11.** If \( T \in \text{Nu}(H) \) then \( \text{tr}(TA) = \Vert T \Vert_{\text{nu}} \) for some \( A \in \mathcal{L}(H) \), \( \Vert A \Vert \leq 1 \).
Theorem 2.14. \( J \) \( \subseteq \mathbb{N} \) is finite, \( J = \mathbb{N} \) the \( s_j \) are regular except for \( T = \sum_{j \in J} j e_j \). Define \( A := \sum_{j \in J} j e_j \). Then the ideal property of \( \sum_{j \in J} j e_j \) converges strongly. Then \( \|A\| \leq 1 \) and \( TA = \sum_{j \in J} j e_j \). Hence
\[
\text{tr}(TA) = \sum_{j \in J} s_j (f_j | A^\ast e_j) = \sum_{j \in J} s_j = \|T\|_{\text{nu}}. 
\]
\( \square \)

As a consequence we arrive at the following characterisation of the \( \gamma' \)-norm.

Corollary 2.12. Let \( V \in \mathcal{L}(H'; X') \). Then
\[
\|V\|_{\gamma'} = \sup \left\{ \|V'U\|_{\text{nu}} \mid U \in \mathcal{L}(H; X), \|U\|_{\gamma} \leq 1, \dim \text{ran}(U) < \infty \right\}. 
\]

Proof. Let \( U : H \to X \) be of finite rank with \( \|U\|_{\gamma} \leq 1 \). Then \( \|\text{tr}(V'U)\| \leq \|V'U\|_{\text{nu}} \).

On the other hand, applying Lemma 2.11 to \( T := V'U \) we find \( A \in \mathcal{L}(H) \) with \( \|A\| \leq 1 \) and
\[
\|V'U\|_{\text{nu}} = \text{tr}(V'UA) \leq \|V\|_{\gamma'}, \|UA\|_{\gamma} \leq \|V\|_{\gamma'}, \|UA\|_{\gamma} \leq \|V\|_{\gamma'}
\]
by the ideal property. \( \square \)

As a consequence of Corollary 2.12 we obtain the ideal property of \( \gamma'(H'; X') \).

Corollary 2.13 (Ideal Property). Let \( R : H \to K \) and \( L : Y \to X \) be bounded operators, and \( V \in \gamma'(H'; X') \). Then \( L'V'R' \in \gamma'(K'; Y') \) with
\[
\|L'V'R'\|_{\gamma'} \leq \|L\| \|V\|_{\gamma'} \|R\|.
\]

Proof. Let \( U : K \to Y \) be of finite rank. Then
\[
\|(L'V'R')U\|_{\text{nu}} = \|RV'(L'U)\|_{\text{nu}} \leq \|R\| \|V'(LU)\|_{\text{nu}} \leq \|R\| \|V'\|_{\gamma'} \|LU\|_{\gamma} \leq \|R\| \|V'\|_{\gamma'} \|U\|_{\gamma}
\]
by the ideal property of \( \text{Nu}(K) \) and \( \gamma(K; Y) \). \( \square \)

With the following results we extend \cite{21}, Prop. 5.1 and 5.2.

Theorem 2.14. a) If \( U \in \gamma(H; X) \) and \( V \in \gamma'(H'; X') \), then \( V'U \in \text{Nu}(H) \) with \( \|V'U\|_{\text{nu}} \leq \|V\|_{\gamma'} \|U\|_{\gamma} \). Moreover, the mapping
\[
\gamma'(H'; X') \to \mathcal{L}\left( \gamma(H; X); \text{Nu}(H) \right), \quad V \mapsto (U \mapsto V'U)
\]
is isometric.

b) The bilinear mapping (“trace duality”)
\[
\gamma(H; X) \times \gamma'(H'; X') \to \mathbb{C}, \quad (U, V) \mapsto \langle U, V \rangle := \text{tr}(V'U)
\]
establishes an isometric isomorphism \( \gamma(H; X)' \cong \gamma'(H'; X') \).

c) Let \( (e_\alpha)_{\alpha} \) be an orthonormal basis of \( H \). Then
\[
\langle U, V \rangle = \text{tr}(V'U) = \sum_{\alpha} \langle U e_\alpha, V e_\alpha \rangle_{X', X'}
\]
for every \( U \in \gamma(H; X) \) and \( V \in \gamma'(H'; X') \).

d) If \( V \in \gamma(H'; X') \) then \( V \in \gamma'(H'; X') \), with \( \|V\|_{\gamma'} \leq \|V\|_{\gamma} \).
Proof. a) follows from Corollary 2.12 and approximation of a general \( U \in \gamma(H; X) \) by finite rank operators.

b) By a) the trace duality is well defined, and it reproduces the norm on \( \gamma'(H'; X') \) by construction. For surjectivity, let \( \Lambda : \gamma(H; X) \to \mathbb{C} \) be a bounded functional and define

\[
V : H' \to X', \quad (Vh')(x) := \Lambda(h' \otimes x).
\]

A short computation reveals that \( \text{tr}(V'U) = \Lambda(U) \) for every rank-one operator \( U = h' \otimes x \). Hence \( \text{tr}(V'U) = \Lambda(U) \) even for every finite rank-operator \( U : H \to X \).

By Corollary 2.7, c) By Corollary 2.7, \( U = \sum_{\alpha} U_{\alpha} \otimes e_{\alpha} \) and the convergence is in \( \| \cdot \| \). Hence

\[
(U, V) = \sum_{\alpha} \langle e_{\alpha} \otimes Ue_{\alpha}, V \rangle = \sum_{\alpha} \langle Ue_{\alpha}, V e_{\alpha} \rangle_{X, X'},
\]

by (2.5).

4) is proved as in [44, Theorem 10.9].

Remark 2.15. It is shown in [44, Sec. 10] that equality \( \gamma(H'; X') = \gamma'(H'; X') \) holds if \( X \) is \( K \)-convex. By a result of Pisier, a space \( X \) is \( K \)-convex if and only if it has nontrivial type. See [44, Sec. 10] for more about \( K \)-convexity in this context.

2.4. Spaces of finite cotype. A \text{Rademacher} variable is a \( \pm 1 \)-valued Bernoulli-(\( \frac{1}{2}, \frac{1}{2} \)) random variable. A \text{complex Rademacher} variable is a random variable of the form

\[
r = r_1 + i r_2
\]

where \( r_1, r_2 \) are independent real Rademachers on the same probability space. Unless otherwise stated, our Rademacher variables are understood to be complex.

By [44, Prop. 2.6] (see also [8, Lemma 12.11])

\[
\mathbb{E} \left\| \sum_{j=1}^{n} r_j x_j \right\|_X^q \leq \left( \frac{1}{2} \right)^q \mathbb{E} \left\| \sum_{j=1}^{n} \gamma_j x_j \right\|_X^q,
\]

whenever \( 1 \leq q < \infty, n \in \mathbb{N}, x_1, \ldots, x_n \in X, r_1, \ldots, r_n \) are complex Rademachers and \( \gamma_1, \ldots, \gamma_n \) are complex Gaussians. (Our reference uses real random variables, but the complex case follows by a straightforward argument, yielding the same constant.)

A converse estimate does not hold in general unless the Banach space has finite cotype. Recall that a Banach space \( X \) has \text{type} \( p \in [1, 2] \) if there exists a constant \( t_p(X) \geq 0 \) such that for all finite sequences \( (x_n)_{n=1}^{\infty} \) in \( X \),

\[
\left\| \sum_{n} r_n x_n \right\|_{L_2(\Omega, X)} \leq t_p(X) \left\| (x_n)_{n} \right\|_{\ell_p(X)} ,
\]

and \( X \) has \text{cotype} \( q \in [2, \infty] \) if for some constant \( c_q(X) \geq 0 \),

\[
\left\| (x_n)_{n} \right\|_{\ell_q(E)} \leq c_q(X) \left\| \sum_{n} r_n x_n \right\|_{L_2(\Omega, E)} .
\]

We refer to [8, Chapter 11] for definitions, properties and references on the notions of type and cotype of a Banach space. (Using real in place of complex Rademachers may alter the values of \( t_p(X) \) and \( c_q(X) \) by universal factors, but does not make a qualitative difference.)

Each Banach spaces has cotype \( \infty \) and type \( 1 \); therefore, \( X \) is said to have \text{nontrivial type} if it has type \( p \) for some \( p > 1 \), and it said to have \text{finite cotype} if it has cotype \( q \) for some \( q < \infty \). Each Banach space of nontrivial type has finite cotype, but the converse is false.

It is important for us that if \( X \) has finite cotype, then a converse to (2.6) holds. Namely, we have the following deep result from [8, Theorem 12.27].
Theorem 2.16. Let $2 \leq q < \infty$. Then there is a universal constant $m_q > 0$ such that
\[
\mathbb{E} \left\| \sum_{j=1}^{n} \gamma_j x_j \right\|^2 \leq m_q^2 c_q(X)^2 \mathbb{E} \left\| \sum_{j=1}^{n} r_j x_j \right\|^2
\]
whenever $X$ is a Banach space of cotype $q$ and $x_1, \ldots, x_n \in X$.

The following observation is needed in the proof of Theorem 4.9 below.

Lemma 2.17. A Banach space $X$ has the same type and cotype as $\gamma(H;X)$.

Proof. We show the result only for the case of cotype. For the type case the arguments are similar. Suppose first that $X$ has cotype $q < \infty$, and let $(U_k)_k$ be a finite sequence in $\gamma(H;X)$. Fix an orthonormal basis $(e_\alpha)_\alpha$ of $H$. Then $U_k = \sum_\alpha e_\alpha \otimes U_k e_\alpha$ for each $k$ by Corollary 2.7. Hence, with $F$ denoting finite subsets of the index set of the orthonormal basis,
\[
\sum_k \|U_k\|_\gamma^q = \sum_k \lim F \left\| \sum_{\alpha \in F} e_\alpha \otimes U_k e_\alpha \right\|_\gamma^q = \lim F \sum_k \left\| \sum_{\alpha \in F} e_\alpha \otimes U_k e_\alpha \right\|_\gamma^q
\]
\[
\leq \sup F \left\| \sum_{\alpha \in F} E' \right\| \left\| \sum_{\alpha \in F} \gamma_\alpha U_k e_\alpha \right\|_X^q = \sup F \left\| \sum_{\alpha \in F} \gamma_\alpha U_k e_\alpha \right\|_X^q
\]
\[
\leq \sup F \left( \sum_{\alpha \in F} \gamma_\alpha U_k e_\alpha \right)^q \left( \sum_{\alpha \in F} \gamma_\alpha U_k e_\alpha \right)^q
\]
\[
= \sup F \left( \sum_{\alpha \in F} \gamma_\alpha U_k e_\alpha \right)^q \left( \sum_{\alpha \in F} \gamma_\alpha U_k e_\alpha \right)^q
\]
\[
\leq c_q(X)^{q} m^q \left( \sum_{\alpha \in F} \gamma_\alpha U_k \right)^q \left( \sum_{\alpha \in F} \gamma_\alpha U_k \right)^q
\]
\[
\leq c_q(X)^{q} m^q \left( \sum_{\alpha \in F} \gamma_\alpha U_k \right)^q \left( \sum_{\alpha \in F} \gamma_\alpha U_k \right)^q
\]
\[
where m := \sqrt{q} \text{ and the non-mentioned constants come from the Khinchine–Kahane inequalities. It follows that}
\]
\[
\|U_k\|_{\ell_q(\gamma(H;X))} \lesssim c_q(X)^{2} m^q \left( \sum_k r_k U_k \right)_{L_2(\Omega;\gamma(H;X))}
\]
and this shows that $c_q(\gamma(H;X)) \lesssim c_q(X)^2 m_q$.

For the converse suppose that $\gamma(H;X)$ has cotype $q < \infty$. Let $(x_k)_k$ be a finite sequence in $X$ and let $e \in H$ be a unit vector. Abbreviate $E := \gamma(H;X)$ and $U_k := e \otimes x_k$. Then
\[
\left( \sum_k \|x_k\|_X^q \right)^{1/q} = \left( \sum_k \|U_k\|_E^q \right)^{1/q} \leq c_q(E) \left( \sum_k r_k U_k \right)_{L_2(\Omega;E)}.
\]
Moreover,
\[
\left( \sum_k r_k U_k \right)_{L_2(\Omega;E)}^2 = \mathbb{E} \left( \sum_k r_k e \otimes x_k \right)^2 = \mathbb{E} \left( \sum_k r_k x_k \right)^2 = \mathbb{E} \left( \sum_k r_k x_k \right)^2,
\]
whence it follows that $c_q(X) \leq c_q(E)$. \qed

The next result shows the significance of spaces of finite cotype for the theory of $\gamma$-radonifying operators.

Theorem 2.18. Let $X$ be a Banach space of finite cotype $q < \infty$. There is a constant $c = c(q,c_q(X))$ such that the following holds: Whenever $K$ is a compact Hausdorff space, $H$ is a Hilbert space and $T \in \mathcal{L}(H;X)$ is an operator that factorises.
as \( T = UV \) over \( C(K) \), i.e.,

\[
\begin{array}{ccc}
H & \xrightarrow{T} & X \\
\downarrow V & & \downarrow U \\
C(K) & & \\
\end{array}
\]

then \( T \in \gamma(H; K) \) and \( \|T\|_{\gamma(H; X)} \leq c \|U\| \|V\| \).

**Proof.** Let \( X \) be of cotype \( 2 \leq q < \infty \) and fix \( q < p < \infty \). By [8] Theorem 11.14 the operator \( U \) is \( p \)-absolutely summing, and one has \( \pi_p(U) \leq c \|U\| \), where \( c \) depends on \( p \) and \( c_q(X) \). By the ideal property for \( p \)-absolutely summing operators, \( T \) is \( p \)-absolutely summing with \( \pi_p(T) \leq \pi_p(U) \|V\| \). Now, a theorem of Linde and Pietsch [14] 12.1 [30] yields that \( T \in \gamma(H; X) \) with \( \|T\|_\gamma \leq \max\{K^\gamma_p, K^\gamma_{p,2}\} \pi_p(T) \). Here \( K^\gamma_{p,2} \) and \( K^\gamma_{p,2} \) are the constants in the Khinchine–Kahane inequalities for Gaussians, see [14] Prop.2.7. By taking the infimum over \( p \) we remove the dependence of the constant on \( p \). \( \square \)

2.5. The space \( \gamma(\Omega; X) \). We now consider the case that \( H = L_2(\Omega) \) for some measure space \( (\Omega, \Sigma, \mu) \). For a \( \mu \)-measurable function \( f : \Omega \to X \) we define

\[
\Sigma_f := \{ A \in \Sigma \mid 1_A f \in L_2(\Omega; X) \}, \quad D_f := \{ 1_A g : A \in \Sigma_f, g \in L_2(\Omega) \}
\]

and the operator

\[
U_f : D_f \longrightarrow X, \quad U_f(h) := \int_\Omega h f \, d\mu.
\]

We have collected some general facts about this construction in Appendix [3]. There it is shown that \( D_f \) is dense in \( L_2(\Omega) \), and that \( U_f \) extends to a bounded operator (denoted also by \( U_f \)) on the whole of \( L_2(\Omega) \) if and only if \( f \in P_2(\Omega; X) \), the space of weakly \( L_2 \)-functions. In this case, for any \( h \in L_2(\Omega) \) the value \( U_f(h) \) is the *Pettis integral* of \( hf \), i.e., it satisfies

\[
(U_f(h), x') = \int_\Omega h(x' \circ f) \, d\mu.
\]

Now we let

\[
\gamma(\Omega; X) := \{ f \in P_2(\Omega) : U_f \in \gamma(L_2(\Omega); X) \}
\]

and define \( \gamma_\infty(\Omega; X) \) similarly. We abbreviate \( \|f\|_\gamma := \|U_f\|_\gamma \) for \( f \in \gamma_\infty(\Omega; X) \).

There is a \( \gamma \)-analogue of Lemma [3.6] In fact, it follows directly from that result and Lemma [2.2]

**Lemma 2.19** (\( \gamma \)-Fatou II). If \( (f_n)_{n \in \mathbb{N}} \) is a bounded sequence in \( \gamma_\infty(\Omega; X) \) with \( f_n \to f \) almost everywhere, then \( f \in \gamma_\infty(\Omega; X) \), \( U_{f_n} \to U_f \) strongly, and

\[
\|f\|_\gamma \leq \liminf_{n \to \infty} \|f_n\|_\gamma.
\]

The spaces \( \gamma(\Omega; X) \) and \( P_2(\Omega) \) are not complete in general. This is different with

\[
\gamma_2(\Omega; X) := L_2(\Omega; X) \cap \gamma(\Omega; X),
\]

which is a Banach space with respect to the norm \( \|f\|_\gamma := \|f\|_{L_2} + \|U_f\|_\gamma \). Recall that

\[
\text{span}\{1_A \otimes x : A \in \Sigma, \mu(A) < \infty, x \in X \}
\]

is called the space of (\( X \)-valued) *step functions*.

**Lemma 2.20.** The space of step functions is dense in \( \gamma_2(\Omega; X) \), i.e., whenever \( f \in L_2(\Omega; X) \) such that \( U_f \in \gamma(L_2(\Omega); X) \), then there is a sequence \( (f_n)_{n} \) of \( X \)-valued step functions such that \( \|f_n - f\|_2 \to 0 \) and \( \|U_{f_n} - U_f\|_\gamma \to 0 \).
Proof. Approximate $f$ in $L_2$ by $f_n := \mathbb{E}(f|\Sigma_n)$ where $\Sigma_n$ is a finite sub-$\sigma$-algebra of $\Sigma$. (Note that $f$ is essentially measurable with respect to a countably generated sub-$\sigma$-algebra of $\Sigma$.) It follows from Theorem 2.3 that $\|U_{f_n} - U_f\|_\gamma \to 0$. □

For a general $f \in \gamma(\Omega; X)$ we still have the following approximation method.

Lemma 2.21. Let $f \in \gamma(\Omega; X)$ and let $(A_n)_n \subseteq \Sigma_f$ with $1_{A_n} \not\rightarrow 1$ almost everywhere on $\{f \neq 0\}$. Then $f1_{A_n} \in \gamma_2(\Omega; X)$, $\|f1_{A_n}\|_\gamma \leq \|f\|_\gamma$ and $\|f - f1_{A_n}\|_\gamma \to 0$.

Note that a sequence $(A_n)_n$ as considered in the lemma exists by Lemma B.1.

Theorem 2.22. For a $\mu$-measurable function $f : \Omega \to X$ the following assertions are equivalent:

(i) $f \in \gamma(\Omega; X)$.

(ii) There is a $||\cdot||_\gamma$-Cauchy sequence $(f_n)_n$ of $X$-valued step functions with $f_n \to f$ almost everywhere.

Moreover, $\|f_n - f\|_\gamma \to 0$ for each such sequence as in (ii).

Proof. (ii) $\Rightarrow$ (i): By the $\gamma$-Fatou Lemma 2.19 $f \in \gamma(\Omega; X)$ and $U_{f_n} \to U_f$ strongly. Since $\gamma(L_2(\Omega); X)$ is complete, there is $T \in \gamma(L_2(\Omega); X)$ such that $\|U_{f_n} - T\|_\gamma \to 0$. This implies that $U_f = T$, whence $f \in \gamma(\Omega; X)$ and $\|f_n - f\|_\gamma \to 0$.

(i) $\Rightarrow$ (ii): By Lemma B.1 and Lemma 2.21 we can find $A_n \in \Sigma_f$, $\mu(A_n) < \infty$, $\|f - f1_{A_n}\|_\gamma \leq 1/n$ and $1_{A_n} \not\rightarrow 1_{\{f \neq 0\}}$ outside a null set $M$, say. Now let $n \in \mathbb{N}$ be fixed. Then by Lemma 2.20 we can approximate $f1_{A_n}$ in the norm $||\cdot||_{L_2} + ||\cdot||_\gamma$ by a sequence of step functions. Without loss of generality we may suppose that these step functions vanish on $A_n^c$. Passing to a subsequence we may suppose in addition that the convergence is even pointwise almost everywhere. By a variant of Egoroff’s theorem, the convergence is almost uniform, i.e., there is a step function $f_n$ such that $\{f_n \neq 0\} \subseteq A_n$ and $\|f1_{A_n} - f_n\|_\gamma < 1/n$, and there is a set $B_n \subseteq A_n$ with $\mu(A_n \setminus B_n) \leq 2^{-n}$ and $\|f_n(x) - f(x)\|_X \leq 1/n$ for $x \in B_n$.

By construction $f_n$ is a step function and $\|f_n - f\|_\gamma \leq 2/n \to 0$. To show that $f_n \to f$ almost everywhere, we form the set $N := \bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} A_n \setminus B_n$, which is a null set. Let $x \notin N \cup M$. The there is $k \in \mathbb{N}$ such that $x \in B_n \setminus A_n$ for all $n \geq k$. But for large $n$ either $f(x) = f_n(x)$ or we have $x \in A_n$, and hence $x \in B_n$. But that means that $\|f_n(x) - f(x)\|_X \leq 1/n$ for large $n \in \mathbb{N}$.

2.6. The space $\gamma'(\Omega; X')$. Again, let $H = L_2(\Omega)$, $(\Omega, \Sigma, \mu)$ any measure space. We identify $H = H'$ via the duality (2.3). We let

$$P_2'(\Omega; X') := \{g : \Omega \to X' \mid \langle x, g(\cdot) \rangle \in L_2(\Omega) \text{ for every } x \in X\}.$$ 

The closed graph theorem shows that if $g \in P_2'(\Omega; X')$ then there is $C \geq 0$ such that

$$\left(\int_{\Omega} |\langle x, g(\omega) \rangle|^2 \mu(d\omega)\right)^{\frac{1}{2}} = \|\langle x, g(\cdot) \rangle\|_{L_2(\Omega)} \leq C \|x\| \quad (x \in X).$$

Hence, the mapping

$$V_g : L_2(\Omega) \to X', \quad (V_g h)(x) := \int_{\Omega} h(\omega) \langle x, g(\omega) \rangle \mu(d\omega)$$

is a well defined bounded operator with norm

$$\|V_g\| = \sup_{\|h\|_{L_2} \leq 1} \|V_g h\|_{X'} = \sup_{\|x\| \leq 1} \|\langle x, g(\cdot) \rangle\|_{L_2} \cdot =: \|g\|_{P_2}$$

The following is a dual analogue of Lemma B.6.
Lemma 2.23 (P₂-Fatou). Let \((g_n)_{n\in\mathbb{N}}\) be a bounded sequence in \(P'_2(\Omega; X')\) with \((x, g_n(\cdot)) \to (x, g(\cdot))\) almost everywhere for every \(x \in X\), then \(f \in P'_2(\Omega; X')\), \(\|f_n\|_{P'_2} \leq \lim\inf_{n \to \infty} \|f_n\|_{P'_2}\) and \(V_{g_n} \to V_g\) in the weak* operator topology.

Proof. For \(x \in X\) and \(h \in L_2(\Omega)\) the usual Fatou lemma states that
\[
\int |h(t) \langle x, g(t) \rangle| \, \mu(dt) \leq \lim\inf_{n \to \infty} \int h(t) |\langle x, g_n(t) \rangle| \, \mu(dt)
\]
\[
\leq \lim\inf_{n \to \infty} \|h\|_{L_2} \|x\| \|g_n\|_{P'_2}.
\]
Hence \((x, g(\cdot)) \in L_2(\Omega)\) for every \(x \in X\), i.e., \(g \in P'_2(\Omega; X')\). Similar to the proof of Lemma 2.6 it follows that \((x, g_n(\cdot)) \to (x, g(\cdot))\) weakly in \(L_2\) for each \(x \in X\). But this is just the same as to say that \(V_{g_n} \to V_g\) in the weak* operator topology. \(\square\)

We define
\[
\gamma'(\Omega; X') := \{g \in P'_2(\Omega; X') \mid V_g \in \gamma'(L_2(\Omega); X')\}
\]
and write \(\|g\|_{\gamma'} := \|V_g\|_{\gamma'}\). The following result, based on [21, Corollary 5.5], yields a convenient way to use the trace duality.

Theorem 2.24. Let \(f \in \gamma(\Omega; X)\) and \(g \in \gamma'(\Omega; X')\). Then \(\langle f(\cdot), g(\cdot) \rangle \in L_1(\Omega)\) and
\[
\int _\Omega |\langle f(\cdot), g(\cdot) \rangle| \, \mu = \|f\|_\gamma \|g\|_{\gamma'}.
\]
Moreover,
\[
\langle U_f, V_g \rangle = \text{tr}(V'_g U_f) = \int _\Omega \langle f(\cdot), g(\cdot) \rangle \, d\mu.
\]

Proof. By Theorem 2.22 it suffices to prove the claim for \(f \in L_2(\Omega) \otimes X\), say \(f = \sum_{j=1}^n f_j \otimes x_j\). Then, by (2.5),
\[
\text{tr}(V'_g U_f) = \sum_{j=1}^n \langle x_j, V'_g f_j \rangle = \sum_{j=1}^n \int _\Omega \langle x_j, g(\cdot) \rangle f_j \, d\mu = \int _\Omega \langle f(\cdot), g(\cdot) \rangle \, d\mu.
\]
For the remaining statement, find \(m \in L_\infty(\Omega)\) with \(|m| \leq 1\) and
\[
\int _\Omega |\langle f, g \rangle| \, d\mu = \int _\Omega m \langle f, g \rangle \, d\mu = |\text{tr}(V'_g U_{mf})| = |\text{tr}(V'_g U_{mf})| \leq \|g\|_{\gamma'} \|mf\|_\gamma \leq \|g\|_{\gamma'} \|f\|_\gamma
\]
by what has been already shown and the ideal property. \(\square\)

2.7. Banach lattices. In this section we derive an alternative description of the \(\gamma\)-norms on Banach lattices. This will make the name “square function” plausible, and will help us relating our abstract square functions to classical ones, see the Introduction and Section 3 below.

Let \(E\) be a complex Banach lattice (we refer to [31, 33, 37] for background, but actually we shall not need so much of it). If one adapts the theory developed in [3] pp.326-329] to the setting of complex Banach lattices, one obtains the following. Whenever \(u_1, \ldots, u_n \in E\) then
\[
(\sum_{j=1}^n |u_j|^2)^{1/2} := \sup \left\{ \left( \sum_{j=1}^n \alpha_j u_j \right) \mid \alpha \in \ell_2^n, \|\alpha\|_2 \leq 1 \right\}
\]
extists in \(E\). The notation is inspired by the formula for scalars, and is coherent with usual pointwise notation in Banach function spaces such as spaces \(L_p(\Omega)\). That is to say, if \(E = L_p(\Omega)\) for some measure space \((\Omega, \Sigma, \mu), 1 \leq p \leq \infty\) and \(u_1, \ldots, u_n \in E\), then
\[
(\sum_{j=1}^n |u_j|^2)^{1/2}(\omega) = \left( \sum_{j=1}^n |u_j(\omega)|^2 \right)^{1/2}
\]
for \( \mu \)-almost every \( \omega \in \Omega \). (This follows since in computing the supremum in (2.7) one can restrict to a countable subset.)

Now let \((\Omega, \Sigma, \mu)\) be any measure space, and let \( f \in P_2(\Omega; E) \). In complete analogy to (2.7) we shall write

\[
\left( \int_{\Omega} |f(\omega)|^2 \mu(d\omega) \right)^{\frac{1}{2}} := \sup \left\{ \left| \int_{\Omega} g f \, d\mu \right| \mid g \in L_2(\Omega), \|g\|_2 \leq 1 \right\}
\]

if this supremum exists in \( E \). Our intention is to prove the following.

**Theorem 2.25.** Let \( E \) be a Banach lattice of finite cotype, let \((\Omega, \Sigma, \mu)\) be a measure space, and let \( f \in P_2(\Omega; E) \). Then the following assertions are equivalent:

(i) \( f \in \gamma(\Omega; E) \).

(ii) \( \left( \int_{\Omega} |f(\omega)|^2 \mu(d\omega) \right)^{\frac{1}{2}} \) exists in \( E \).

In this case

\[
\left\| \left( \int_{\Omega} |f(\omega)|^2 \mu(d\omega) \right)^{\frac{1}{2}} \right\|_X \approx \|f\|_{\gamma(\Omega; E)}.
\]

The proof requires several steps, and is based on the following deep theorem.

**Theorem 2.26.** If \( E \) is a Banach lattice of finite cotype, then

\[
\left( \mathbb{E} \left\| \sum_j \gamma_j u_j \right\|_E^2 \right)^{\frac{1}{2}} \approx \left\| \left( \sum_j |u_j|^2 \right)^{\frac{1}{2}} \right\|
\]

for all finite sequences \( u_1, \ldots, u_n \in E \).

**Proof.** In [8, 16.18] one can find the analogous statement for (real) Rademachers and real Banach spaces. The extension to complex spaces is straightforward. The equivalence with Gaussians in place of Rademachers follows from Theorem 2.16. \( \square \)

We remark that in the case \( E = L_p(\Omega) \) for \( 1 \leq p < \infty \) the proof of Theorem 2.26 is a straightforward application of the Khinchine–Kahane inequalities and Parseval’s identity.

**Proof of Theorem 2.25**

1) We restrict to the case that \( H := L_2(\Omega) \) is separable, the proof in the general case being a straightforward adaptation. Fix an orthonormal basis \((e_n)_n\) of \( H \) and let \( u_n := \int_{\Omega} f e_n \). If \( g \in L_2(\Omega) \) then

\[
g = \sum_n (g | e_n) e_n
\]

in \( L_2(\Omega) \), whence

\[
\int_{\Omega} fg = \sum_n (g | e_n) \int_{\Omega} f e_n = \sum_n (g | e_n) u_n
\]

in \( E \). It follows that

\[
\left( \sum_{j=1}^n |v_j + w_j|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j=1}^n |v_j|^2 \right)^{\frac{1}{2}} + \left( \sum_{j=1}^n |w_j|^2 \right)^{\frac{1}{2}}
\]

for any \( v_1, \ldots, v_n, w_1, \ldots, w_n \in E \). From this it follows that

\[
\left( \sum_{j=1}^m |u_j|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j=1}^{m+n} |u_j|^2 \right)^{\frac{1}{2}}
\]
if $n < m$. Writing $v_n := \left( \sum_{j=1}^{n} |u_j|^2 \right)^{1/2}$ we hence obtain
\[
\|v_m - v_n\| \leq \left\| \left( \sum_{j=1}^{m} |u_j|^2 \right)^{1/2} \right\| \approx \left( E \left\| \sum_{j=n+1}^{m} \gamma_j u_j \right\|_E \right)^{1/2} = \left\| \sum_{j=n+1}^{m} \overline{\gamma_j} \otimes u_j \right\|_{\gamma(H;E)}.
\]

3) Now suppose that (i) holds, i.e., $U_f \in \gamma(H;E)$. Then $U_f = \sum_j \overline{\gamma_j} \otimes u_j$ in the norm of $\gamma(H;E)$. By our considerations in 2) we conclude that $(v_n)_n$ is a Cauchy sequence and hence has a limit $v := \lim_{n \to \infty} v_n$ in $E$. It is clear that $(v_n)_n$ is increasing, which implies that $v = \sup_n v_n$. Then by (2.8) (ii) follows.

4) Conversely, suppose that (ii) holds and let $F \subseteq \mathbb{N}$ be any finite subset. Then
\[
\left( \sum_{j \in F} |u_j|^2 \right)^{1/2} \leq v := \left( \sum_{j=1}^{\infty} |u_j|^2 \right)^{1/2},
\]
which exists by hypothesis and 1). But then
\[
\left( E \left\| \sum_{j \in F} \gamma_j u_j \right\|_E \right)^{1/2} \approx \left\| \left( \sum_{j \in F} |u_j|^2 \right)^{1/2} \right\| \leq \|v\|.
\]
It follows that $U_f \in \gamma(H;E) = \gamma(H;E)$ since $E$ has finite cotype. 

Let us specialise $E = L_p(\Omega')$, $1 \leq p < \infty$ for some measure space $(\Omega', \Sigma', \mu')$.

**Corollary 2.27.** Let $(\Omega, \Sigma, \mu)$ and $(\Omega', \Sigma', \mu')$ be measure spaces, $p \in [1, \infty)$ and let $f : \Omega \times \Omega' \to \mathbb{C}$ be measurable. Then the following assertions are equivalent.

(i) $(\omega \mapsto f(\omega, \cdot)) \in \gamma(\Omega; L_p(\Omega'))$

(ii) $\left\| \left( \int_{\Omega} |f(\omega, x)|^2 \mu(\omega) \right)^{1/2} \right\|_{L_p(\Omega', \mu'(dx))} < \infty$.

If $1 < p < \infty$ then the dual space $L_{p'}(\Omega)$ has nontrivial type, whence dual square functions on $L_p$ coincide with square functions on $L_{p'}$.

### 3. Abstract square function estimates

Building on the theory of $\gamma$-radonifying operators developed in the previous chapter, we now come to a central definition.

**Definition 3.1.** Let $X, Y$ be Banach spaces. Then an (abstract) $(X, Y)$-square function is any operator
\[
Q : \text{dom}(Q) \to \gamma(H; Y), \quad \text{dom}(Q) \subseteq X
\]
for some Hilbert space $H$. A dual $(X, Y)$-square function is any operator
\[
Q^d : \text{dom}(Q^d) \to \gamma(H; Y') \cong \gamma'(H'; Y'), \quad \text{dom}(Q^d) \subseteq X'
\]
for some Hilbert space $H$.

A square function estimate or a quadratic estimate for the $(X,Y)$-square function $Q$ is any inequality of the form
\[
\|Qx\|_\gamma \leq C \|x\| \quad \text{for all } x \in \text{dom}(Q)
\]
for some constant $C \geq 0$. If $Q$ is densely defined, such a square function estimate holds true if and only if $Q$ extends to a bounded operator $Q : X \to \gamma(H; Y)$. Note that a closed and densely defined square function satisfies a square function estimate if and only if it is fully defined.

Similarly, an estimate of the form
\[
\|Q^d x'\|_\gamma \leq C \|x'\| \quad (x' \in \text{dom}(Q^d))
\]
is called a dual square function (quadratic) estimate. The usual examples of dual square functions are not densely, but only weakly*-densely defined, and hence in
general a dual square function estimate does not lead to a bounded operator $X' \to \gamma'(H';Y')$.

The following is a standard way to arrive at $(X,Y)$-square functions. Suppose that $A : \text{dom}(A) \to \mathcal{L}(H;Y)$ is an operator with $\text{dom}(A) \subseteq X$. Then we can take its part in $\gamma(H;X)$

$$A_\gamma : \text{dom}(A_\gamma) \to \gamma(H;Y)$$

with $\text{dom}(A_\gamma) = \{ x \in \text{dom}(A) \mid Ax \in \gamma(H;Y) \}$ and $A_\gamma x := Ax$. It is easy to see that $A_\gamma$ is a closed square function if $A$ is closed. (Obviously, a similar construction is possible to obtain dual square functions.)

The square function $Q : \text{dom}(Q) \to \gamma(H;Y)$ is called subordinate to the square function $R : \text{dom}(R) \to \gamma(K;Y)$, in symbols: $Q \preceq R$, if $\text{dom}(Q) \subseteq \text{dom}(R)$ and there is a bounded operator $T : H \to K$ such that

$$Qx = Rx \circ T \quad \text{for all } x \in \text{dom}(R).$$

The square functions are called strongly equivalent, in symbols $Q \approx R$, if $Q \preceq R$ and $R \preceq Q$. Note that if $Q \preceq R$ then, by the ideal property, there is a constant $c \geq 0$ such that

$$\|Qx\|_\gamma \leq c \|Rx\|_\gamma \quad \text{for all } x \in \text{dom}(R).$$

Analogously, a dual square function $Q^d : \text{dom}(Q^d) \to \gamma'(H';Y')$ is subordinate to a dual square function $Q'^d : \text{dom}(Q'^d) \to \gamma'(H';Y')$ if $\text{dom}(Q^d) \subseteq \text{dom}(Q'^d)$ and there is a bounded operator $T : H' \to K'$ such that

$$Q'^d x' = Q^d x' \circ T \quad \text{for all } x' \in \text{dom}(Q'^d).$$

It is evident that any (dual) square function subordinate to a bounded (dual) square function is itself bounded. Subordination is a (trivial) way to generate new square function estimates from known ones.

In the following we shall describe how one can associate square functions with a functional calculus in a natural way. To this end we first have to review some basic functional calculus theory.

### 3.1. A functional calculus round-up

Let $O$ be a nonempty set, $\mathcal{F}$ a unital algebra of scalar-valued functions on $O$, $\mathcal{E} \subseteq \mathcal{F}$ a subalgebra of $\mathcal{F}$ and $\Phi : \mathcal{E} \to \mathcal{L}(X)$ an algebra homomorphism, where $X$ is a Banach space. Then the triple $(\mathcal{E}, \mathcal{F}, \Phi)$ is an abstract functional calculus in the sense of [13 Chapter 1]. The mapping $\Phi : \mathcal{E} \to \mathcal{L}(X)$ is called the elementary calculus. A function $f \in \mathcal{F}$ is regularisable, i.e., there is $e \in \mathcal{E}$ (called a regulariser) such that $ef \in \mathcal{E}$ and $\Phi(e)$ is injective. In this case one can define

$$\Phi(f) := (\Phi(e))^{-1}\Phi(ef)$$

with natural domain. This definition is independent of the regulariser and consistent with the elementary calculus. One can show [13 Section 1.2.1] that the set $\mathcal{F} = \mathcal{F}_r$ of regularisable elements is a unital subalgebra of $\mathcal{F}$, so we may suppose without loss of generality that $\mathcal{F} = \mathcal{F}_r$ in the following.

In our context the most interesting case is $\mathcal{F} = \mathcal{H}^\infty(O)$, the algebra of bounded holomorphic functions on an open set $O \subseteq \mathbb{C}$. In this case, if there is $C \geq 0$ such that $\Phi(f) \in \mathcal{L}(X)$ and

$$\|\Phi(f)\| \leq C \|f\|_{\mathcal{H}^\infty} \quad \text{for all } f \in \mathcal{H}^\infty(O),$$

then we speak of $\Phi$ as a bounded $\mathcal{H}^\infty$-calculus on $O$.

**Remark 3.2.** If $O \subseteq \mathbb{C}$ is open, then by Liouville’s theorem the algebra $\mathcal{H}^\infty(\Omega)$ is only interesting if $\emptyset \neq O \neq \mathbb{C}$. We shall tacitly assume this when talking about $\mathcal{H}^\infty(O)$-functional calculus.
Now suppose that a functional calculus $\Phi : \mathcal{E} \to \mathcal{L}(X)$ is given with $\mathcal{E} \subseteq H^\infty(O)$, and suppose that $\mathbb{C} \setminus O$ has nonempty interior $U$, say. For each $\lambda \in U$ the function $r_\lambda(z) := (\lambda - z)^{-1}$ is holomorphic and bounded on $O$. If we suppose in addition that $R_A := \Phi(r_\lambda) \in \mathcal{L}(X)$, then this yields a pseudo-resolvent on $U$. Hence by [13, Prop. A.2.4] there is unique operator $A$ with $R(\lambda, A) = R_\lambda$ for all $\lambda \in U$. (This operator is single-valued if and only if one/each $R_\lambda$ is injective.) It is common to call $\Phi$ a functional calculus for $A$ and write $f(A) := \Phi(f)$ for $f \in \mathcal{F}$.

We suppose that the reader is familiar with the functional calculus for sectorial/strip type operators as developed in [13]. For the convenience of the reader, we have included a brief description of the construction in Appendix C.

3.2. **Square functions associated with a functional calculus.** In this section we shall associate square functions with a given functional calculus. As a motivating example we use the sectorial calculus (see section 5.2 below).

Given a sectorial operator $A$ of angle $\omega_0$ on a Banach space $X$ and a function $\psi \in H^\infty_0(S_\omega)$ with $\omega \in (\omega_0, \pi)$ one considers — for fixed $x \in X$ — the vector-valued function

$$(0, \infty) \to X, \quad t \mapsto \psi(tA)x.$$ 

Following Kalton and Weis [21] one should interpret this function as an operator $T_\psi x : L^*_2(0, \infty) \to X$

via (Pettis) integration, cf. Appendix B and Section 5.2. Abbreviating $H := L^*_2(0, \infty)$ one looks at estimates of the form

$$\|\psi(tA)x\|_{\gamma((0, \infty), X)} = \|T_\psi x\|_{\gamma(H; X)} \lesssim \|x\|,$$

then called a square function estimate. For $x \in \text{dom}(A) \cap \text{ran}(A)$ one can employ the definition of the functional calculus by Cauchy integrals to obtain

$$T_\psi x = \int_0^\infty h(t)\psi(tA)x \frac{dt}{t} = \int_0^\infty h(t)\frac{1}{2\pi i} \int_\gamma \psi(tz)R(z, A)x \, dz \, \frac{dt}{t}$$

$$= \frac{1}{2\pi i} \int_\gamma \left( \int_0^\infty h(t)\psi(tz) \, \frac{dt}{t} \right) R(z, A)x \, dz$$

$$= \left( \int_0^\infty h(t)\psi(tz) \, \frac{dt}{t} \right)(A)x.$$

The last step indicates an important change in perspective. The function of two variables $(t, z) \mapsto \psi(tz)$ may as well be viewed as an $L^*_2(0, \infty)$-valued $H^\infty$-function

$$\Psi : S_\omega \to H, \quad \Psi(z)(t) := \psi(tz).$$

Then

$$z \mapsto \int_0^\infty h(t)\psi(tz) \, \frac{dt}{t} = \langle h, \Psi(z) \rangle$$

is a scalar $H^\infty$-function, into which $A$ can be inserted by the functional calculus. Finally, this operator can be applied to $x \in \text{dom}(A) \cap \text{ran}(A)$. But then for fixed such $x$ this yields an operator $H \to X$, and one can ask whether this operator is $\gamma$-radonifying. (In this special case it is, see Section 5.2 below.)

Let us pass from concrete example to the general situation. We fix a functional calculus $(\mathcal{E}, \mathcal{F}, \Phi)$ over a set $O$ as discussed in the previous section. Again we suppose $\mathcal{E} = \mathcal{F}_r$, i.e., every function in $\mathcal{F}$ is regularisable.

For a Hilbert space $H$ and a function $f : O \to H'$ we abbreviate

$h \circ f : O \to \mathbb{C}, \quad (h \circ f)(z) := \langle h, f(z) \rangle_{H, H'} \quad (z \in O, h \in H).$

Then we define

$$\mathcal{F}(O; H') := \{ f : O \to H' \mid h \circ f \in \mathcal{F} \ \forall h \in H \}.$$
We now extend the functional calculus $\Phi$ to $\mathcal{F}(O;H')$ by setting

$$\Phi(f) : \text{dom}(\Phi(f)) \to \mathcal{L}(H;X),$$

$$\text{dom}(\Phi(f)) := \{x \in X \mid x \in \text{dom}(\Phi(h \circ f)) \text{ for all } h \in H\}$$

$$[\Phi(f)x] h := \Phi(h \circ f)x$$

This definition/notation is consistent with the original notation under the identification $\mathcal{F}(O;H') = \mathcal{F}$ in the case that $H = \mathbb{C}$ is one-dimensional.

In the next step we take the part of $\Phi(f)$ in $\gamma(H;X)$ to arrive at the square function $\Phi_\gamma(f) : \text{dom}(\Phi_\gamma(f)) \to \gamma(H;X)$,

$$\Phi_\gamma(f)x := \Phi(f)x,$$

$$\text{dom}(\Phi_\gamma(f)) = \{x \in \text{dom}(\Phi(f) \mid \Phi(f)x \in \gamma(H;X)\}.$$ We call the square function $\Phi_\gamma(f)$ bounded if $\text{dom}(\Phi_\gamma(f)) = X$ and

$$\Phi_\gamma(f) : X \to \gamma(H;X)$$

is a bounded operator. If $X$ does not contain a copy of $c_0$, then $\gamma(H;X) = \gamma_\infty(H;X)$. Hence, for $f \in \mathcal{F}(O;H')$ the associated square function $\Phi_\gamma(f)$ is bounded if and only if $\Phi(h \circ f) \in \mathcal{L}(X)$ for all $h \in H$ and there is a constant $c \geq 0$ such that

$$||\mathbb{E}||\sum_{\alpha \in F} \gamma_\alpha \Phi(e_\alpha \circ f)x||^2 \leq c ||x||^2$$

for all $x \in X$, a fixed orthonormal basis $(e_\alpha)_{\alpha \in I}$ of $H$ and all finite subsets $F \subseteq I$.

In the following lemma we collect some properties of the so-obtained square functions. Note that $\mathcal{F}(O;H')$ is an $\mathcal{F}$-module with respect to pointwise multiplication.

**Lemma 3.3.** In the situation just described, the following assertions hold for each $f \in \mathcal{F}(O;H')$:

a) The operators $\Phi(f)$ and $\Phi_\gamma(f)$ are closed.

b) If $g \in \mathcal{F}(O;H')$ then

$$\Phi_\gamma(f) + \Phi_\gamma(g) \subseteq \Phi_\gamma(f + g).$$

c) If $g \in \mathcal{F}$ then

$$\Phi_\gamma(f)\Phi(g) \subseteq \Phi_\gamma(f \cdot g)$$

with $\text{dom}(\Phi_\gamma(f)\Phi(g)) = \text{dom}(\Phi(g)) \cap \text{dom}(\Phi_\gamma(f \cdot g))$.

d) If $g \in \mathcal{F}$ then

$$\Phi(g) \circ \Phi_\gamma(f) \subseteq \Phi_\gamma(f \cdot g)$$

e) If $g \in \mathcal{F}$ such that $\Phi(g) \in \mathcal{L}(X)$, then

$$\Phi(g) \circ \Phi_\gamma(f) \subseteq \Phi_\gamma(f \cdot g) = \Phi_\gamma(f)\Phi(g)$$

In particular, $\text{dom}(\Phi_\gamma(f))$ is invariant under $\Phi(g)$.

The assertion d) means: if $x \in \text{dom}(\Phi_\gamma(f))$ and $\Phi(g)[\Phi_\gamma(f)x] \in \gamma(H;X)$, then $x \in \text{dom}(\Phi_\gamma(f \cdot g))$ and $\Phi(g)[\Phi_\gamma(f)x] = \Phi_\gamma(f \cdot g)x$.

**Proof.** The proof is left to the reader. The assertions in b) and c) follow more or less directly from the corresponding statements about the functional calculus $(\mathcal{E},\mathcal{F},\Phi)$ [13, Prop. 1.2.2]. Assertion d) is straightforward, and e) is a consequence of c) and d). (Note that by the ideal property of $\gamma(H;X)$, $\text{dom}(\Phi(g) \circ \Phi_\gamma(f)) = \text{dom}(\Phi_\gamma(f))$.)

From Lemma 3.3 we see that the mapping $f \mapsto \Phi_\gamma(f)$ behaves like a functional calculus, so we call it the *vectorial* $\mathcal{F}$-calculus. In particular, in the case that $\mathcal{F} = \mathcal{H}^\infty(O)$ for some open subset $O \subseteq \mathbb{C}$, the map

$$\Phi_\gamma : \mathcal{H}^\infty(O;H') \to \{H\text{-square functions on } X\}$$
is called a vectorial $H^\infty$-calculus on $O$. The vectorial $H^\infty$-calculus is bounded if $\Phi_\gamma(f)$ is a bounded square function for each $f \in H^\infty(O; H')$ and there is a constant $C \geq 0$ such that

$$\|\Phi_\gamma(f)x\|_\gamma \leq C \|f\|_{H^\infty(O)} \|x\| \quad (x \in X, f \in H^\infty(O; H')).$$

Clearly, if the vectorial $H^\infty$-calculus is bounded, then the underlying scalar $H^\infty$-calculus is bounded. We shall prove that, essentially, the converse holds for sectorial/strip type operators (Theorem 6.1).

Suppose again that $\mathcal{F} = H^\infty(O)$ for some open subset $O \subseteq \mathbb{C}$. We say that the scalar convergence lemma holds if the following is true: whenever $(f_n)_n$ is a sequence in $H^\infty(O)$ with $\sup_{n \in \mathbb{N}} \|f_n\|_\infty < \infty$ and $f_n \rightarrow f$ pointwise on $O$, $\Phi(f_n) \in \mathcal{L}(X)$ for all $n \in \mathbb{N}$ and $\sup_{n \in \mathbb{N}} \|\Phi(f_n)\|_\infty < \infty$, then $\Phi(f) \in \mathcal{L}(X)$ and $\Phi(f_n) \rightarrow \Phi(f)$ strongly as $n \rightarrow \infty$.

The scalar convergence lemma holds for the functional calculus of a sectorial operator with dense domain and range and for a densely defined operator of strip type, see [13 Section 5.1].

**Lemma 3.4 (Convergence lemma).** Let $(\mathcal{E}, H^\infty(O), \Phi)$ be a functional calculus on a Banach space $X$ such that the scalar convergence lemma holds. Suppose that $X$ does not contain a copy of $c_0$. Then the vectorial convergence lemma holds, i.e.: Let $(f_n)_n$ be a sequence in $H^\infty(O; H')$ satisfying

1. $\sup_{n \in \mathbb{N}} \|f_n\|_\infty < \infty$,
2. $f_n(z) \rightarrow f(z)$ weakly for all $z \in O$,
3. $\Phi_\gamma(f_n) \in \mathcal{L}(X; \gamma(H; X))$ for all $n \in \mathbb{N}$ and
4. $\sup_{n \in \mathbb{N}} \|\Phi_\gamma(f_n)\|_{\mathcal{L}(X; \gamma(H; X))} < \infty$.

Then $\Phi_\gamma(f) \in \mathcal{L}(X; \gamma(H; X))$ and $\Phi_\gamma(f_n)x \rightarrow \Phi_\gamma(f)x$ strongly in $\mathcal{L}(H; X)$ as $n \rightarrow \infty$, for each $x \in X$.

**Proof.** Fix $h \in H$. Then $\sup_n \|h \circ f_n\|_\infty \leq \|h\| \sup_n \|f_n\|_\infty < \infty$ and $h \circ f_n \rightarrow h \circ f$ pointwise on $O$. Moreover, $\Phi(h \circ f_n) \in \mathcal{L}(X)$ and

$$\|\Phi(h \circ f_n)x\|_X = \|\Phi_\gamma(f_n)x\|_X \leq \|h\| \|\Phi_\gamma(f_n)x\|_{\mathcal{L}(X; \gamma(H; X))} \leq \|h\| \|\Phi_\gamma(f_n)x\|_{\gamma(H; X)}$$

for all $n \in \mathbb{N}$. This yields $\sup_n \|\Phi(h \circ f_n)\|_{\mathcal{L}(X; \gamma(H; X))} \leq \|h\| \sup_n \|\Phi_\gamma(f_n)\|_{\mathcal{L}(X; \gamma(H; X))}$. By the scalar convergence lemma, $\Phi(h \circ f) \in \mathcal{L}(X)$, and $\Phi(h \circ f_n) \rightarrow \Phi(h \circ f)$ strongly on $X$. That is, for every $x \in X$ is $\Phi_\gamma(f_n)x \rightarrow \Phi(f)x$ strongly in $\mathcal{L}(H; X)$. By the $\gamma$-Fatou Lemma, $\Phi(f)x \in \gamma(H; X)$, and since $X$ does not contain a copy of $c_0$, $\Phi(f)x \in \gamma(H; X)$ for each $x \in X$. 

### 3.3. Dual square functions associated with a functional calculus.

Let again $(\mathcal{E}, \mathcal{F}, \Phi)$ be a proper functional calculus where $\mathcal{F}$ is an algebra of functions defined on the set $O$. As above, we suppose for simplicity that $\mathcal{F} = \mathcal{F}_r$.

For a Hilbert space $H$ and a function $f : O \rightarrow H$ we abbreviate

$$f \circ h' : O \rightarrow \mathbb{C}, \quad (f \circ h')(z) := (f(z), h')_{H,H'} \quad (z \in O, h' \in H')$$

and define

$$(3.2) \quad \mathcal{F}(O; H) := \{ f : O \rightarrow H \mid f \circ h' \in \mathcal{F} \ \forall h' \in H' \}.$$ 

For fixed $f \in \mathcal{F}(O; H)$ we then define the operator

$$\Phi^d(f) : \text{dom}(\Phi^d(f)) \rightarrow \mathcal{L}(H'; X')$$

$\text{dom}(\Phi^d(f)) := \{ x' \in X' \mid x' \in \text{dom}(\Phi(f \circ h')) \text{ for all } h' \in H' \}$

$$[\Phi^d(f)x']h' := \Phi(f \circ h')x'$$
Then we pass to the associated dual square function
\[ \Phi_\gamma(f) : \text{dom}(\Phi_\gamma(f)) \to \gamma'(H'; X'), \quad \Phi_\gamma(f)x' := \Phi^d(f)x' \]
\text{dom}(\Phi_\gamma(f)) = \{ x' \in \text{dom}(\Phi^d(f)) : \Phi^d(f)x' \in \gamma(H'; X') \} \subseteq X'.

Of course, this is only meaningful if \( \Phi(f \circ h') \) is single-valued, i.e., if \( \Phi(f \circ h') \) is densely defined for each \( h' \in H' \). We therefore make the following

**Standing assumption:** Whenever we speak of a dual square function associated with a function \( f \in F(O; H) \), we require that for each \( h' \in H' \) the operator \( \Phi(f \circ h') \) is densely defined.

The following lemma is the analogue of Lemma 3.3.

**Lemma 3.5.** In the situation just described, the following assertions hold for \( f \in F(O; H) \):

a) The operator \( \Phi_\gamma(f) \) is weak*-to-weak* closed.

b) If \( g \in F \) such that \( \Phi(g) \in L(X) \) and if \( x' \in \text{dom}(\Phi_\gamma(f \cdot g)) \), then \( \Phi(g)'x' \in \text{dom}(\Phi_\gamma(f)) \) and
\[ \Phi_\gamma(f)\Phi(g)'x' = \Phi_\gamma(f \cdot g)x'. \]

**Proof.** a) is again left to the reader. For the proof of b) we fix \( h' \in H' \) and note first that since \( \Phi(g) \) is bounded we have
\[ \Phi((f \cdot g) \circ h')' = \Phi((f \circ h')g)' \subseteq (\Phi(g) \Phi(f \circ h))' = \Phi(f \circ h')' \Phi(g)' \]
by [13] A.4.2 and 1.2.2. The claim now follows easily. \( \square \)

The following theorem yields a useful characterisation of “dual square function estimates”.

**Theorem 3.6.** Let \( (e_\alpha)_{\alpha \in I} \) be a fixed orthonormal basis of \( H \). The following assertions are equivalent for \( f \in F(O; H) \):

i) \( \Phi_\gamma(f) \) is a bounded operator \( \Phi_\gamma(f) : X' \to \gamma'(H'; X') \).

ii) The assignment
\[ T(h' \otimes x) := \Phi(f \circ h')x, \quad h' \in H', \ x \in \text{dom}(\Phi(f \circ h')) \]
extends to a bounded operator \( T : \gamma(H; X) \to X \).

iii) There is a constant \( c \geq 0 \) such that
\[ \left\| \sum_{\alpha \in F} \Phi((f | e_\alpha))x_\alpha \right\|_X^2 \leq c \mathbb{E} \left\| \sum_{\alpha \in F} \gamma_\alpha x_\alpha \right\|_F^2 \]
for all finite subsets \( F \subseteq I \) and \( x_\alpha \in \text{dom}(\Phi((f | e_\alpha))) \) for \( \alpha \in F \).

In this case \( T = \Phi_\gamma(f)'|_{\gamma(H; X)} \) is the pre-adjoint of \( \Phi_\gamma(f) \) (under the identification \( \gamma'(H'; X') \cong \gamma(H; X') \)), and \( c = \| T \| = \| \Phi_\gamma(f) \| \) can be chosen in (iii).

Furthermore, if \( g \in F \) is such that \( \Phi(g) \in L(X) \), then
\[ (3.3) \quad \Phi_\gamma(f)'(\Phi(g) \circ S) = \Phi(g)(\Phi_\gamma(f)'S) \quad \text{for all } S \in \gamma(H; X). \]

**Proof.** (i) \( \Rightarrow \) (ii): By hypothesis, \( \Phi_\gamma(f)' : \gamma(H; X)'' \to X'' \) is bounded. Fix \( x' \in X', \ h' \in H' \) and \( x \in \text{dom}(\Phi(f \circ h')) \). Then
\[ \langle \Phi_\gamma(f)'(h' \otimes x), x' \rangle_{X'', X'} = \langle h' \otimes x, \Phi_\gamma(f)x' \rangle = \text{tr} \left( \Phi_\gamma(f)x' \right) (h' \otimes x) = (x, (\Phi_\gamma(f)x)' h') = (x, \Phi(f \circ h')x') = \Phi(f \circ h')x. \]

Consequently, \( \Phi_\gamma(f)'(h' \otimes x) = \Phi(f \circ h')x = T(h' \otimes x) \in X \). Since \( \text{dom}(\Phi(f \circ h')) \) is dense in \( X \), the linear span of such elements \( h' \otimes x \) is dense in \( \gamma(H; X) \). The claim follows.

(ii) \( \Rightarrow \) (iii): This follows since
\[ T(\sum_{\alpha \in F} \overline{e_\alpha} \otimes x_\alpha) = \sum_{\alpha \in F} \overline{\Phi((f | e_\alpha))x_\alpha}. \]
(ii) ⇒ (i): It suffices to show that \( \Phi_{\gamma'}(f) = T' : X' \to \gamma(H; X)' \cong \gamma'(H'; X') \). Fix \( x' \in X' \). Then
\[
(x, (T' x')(h'))_{X,X'} = \langle h' \otimes x, T' x' \rangle_{X,X'} = \langle \Phi(f \circ h') x, x' \rangle_{X,X'} = \langle \Phi(f \circ h') x, x' \rangle_{X,X'}
\]
for all \( h' \in H' \) and \( x \in \text{dom}(\Phi(f \circ h')) \). Hence \( x' \in \text{dom}(\Phi(f \circ h')) \) and
\[
\Phi_{\gamma'}(f)x'| h' = \Phi(f \circ h')x' = (T' x')h' \quad \text{for all } h' \in H'.
\]
That is, \( \Phi_{\gamma'}(f) = T' \).

For the remaining statement let again \( h' \in H' \) and \( x \in \text{dom}(\Phi(f \circ h')) \). Then, with \( S := h' \otimes x, \)
\[
\Phi(g)(T(S)) = \Phi(g)(f \circ h')x = \Phi(f \circ h')\Phi(g)x = T(h' \otimes \Phi(g)x) = T(\Phi(g) \circ S).
\]
Since the linear span of such operators \( S \) is a dense subset of \( \gamma(H; X) \), the claim follows from the ideal property of \( \gamma(H; X) \). \( \square \)

3.4. Square functions over \( L_2 \)-spaces. Up to now we worked with a general Hilbert space \( H \). If one is in the special situation \( H = L_2(\Omega) = H' \) for some measure space \( (\Omega, \mathcal{F}, \mu) \), it is natural to consider functions of two variables \( f = f(t, z) \) in the construction of square functions.

To proceed further we shall suppose in addition that \( F = H^{\infty}(O) \) for some nonempty open set \( O \subseteq \mathbb{C} \) with \( O \neq \mathbb{C} \), and that \( \Phi = \Phi_A \) is a functional calculus for the (possibly multivalued) operator \( A \), cf. Remark 3.2 (Note that for any Hilbert space \( H \), the space \( F(St_\omega; H) \) derived from the space \( F = H^{\infty}(St_\omega) \) by (3.2) above, coincides with the space of \( H \)-valued bounded holomorphic functions.)

Lemma 3.7. Let \( O \subseteq \mathbb{C} \) be an open subset of the complex plane, let \( f : \Omega \times O \to \mathbb{C} \) be measurable and suppose in addition that

1) \( f(t, \cdot) \in H^{\infty}(O) \) for almost all \( t \in \Omega \) and

2) \( \sup_{z \in O} \int_{\Omega} |f(t, z)|^2 \, dt < \infty \).

Then \( z \to f(\cdot, z) \in H^{\infty}(O; L_2(\Omega)) \).

Proof. Let \( g \in L_2(\Omega) \). It remains to show that the function \( F(z) := \int_{\Omega} g(t)f(t, z) \, dt \) is holomorphic. To this end, let \( B \) be any open ball such that \( \overline{B} \subseteq O \). Then
\[
f(a,t) = \frac{1}{2\pi i} \int_{\partial B} \frac{F(z)dz}{z-a}
\]
for all \( a \in B \). By a standard result in complex function theory [36 Theorem 10.7], \( F \) is holomorphic. \( \square \)

For \( f \) as in the lemma we have
\[
[\Phi(f)x] h = \left( \int_{\Omega} h(t)f(t, z) \, dt \right)(A)x
\]
if \( x \in \text{dom}(\Phi(f)) \) and \( h \in H = L_2(\Omega) \). As in the example of sectorial operators and “dilation type” square functions discussed at the beginning of this section, one has
\[
[\Phi(f)x] h = \left( \int_{\Omega} h(t)f(t, z) \, dt \right)(A)x = \int_{\Omega} h(t)f(t, A)x \, dt
\]
in many situations at least for vectors \( x \) from a large subspace of \( X \). We therefore use the symbol \( f(\cdot, A)x \) or \( f(t, A)x \) as a convenient alternative notation — as a \( \text{façon de parler} \) — for the operator \( \Phi(f)x \). So, whenever expressions of the form
\[
\|f(t, A)x\|_\gamma
\]
appear, it is not implied that “\(f(t, A)x\)” has to make sense literally (i.e., \(x \in \text{dom}(f(t, A))\) for almost all \(t \in \Omega\) and \(\Phi(f)x = U_{f(t, A)x}\)) but just as a suggestive notation. It is actually one of the advantages of our approach to square functions that one does not have to worry about the vector-valued integration too much.

4. Square function estimates: New from old

In this chapter we discuss certain general principles how to generate new (dual) square function estimates from known ones. A fairly trivial instance of such a principle is given by subordination.

4.1. Subordination. Subordination for abstract square functions has been defined in the beginning of Chapter 3. Here we consider a special instance for the case of square functions associated with a functional calculus \((\mathcal{E}, \mathcal{F}, \Phi)\) over a set \(O\).

**Theorem 4.1.** Let \(K\) be another Hilbert space and \(T : K \to H\) a bounded linear operator.

a) If \(g \in \mathcal{F}(O; H')\) then \(T' \circ g \in \mathcal{F}(O; K')\), \(\text{dom}(\Phi_\gamma(T' \circ g)) \subseteq \text{dom}(\Phi_\gamma(g))\) and

\[
\Phi_\gamma(T' \circ g)x = \Phi_\gamma(g)x \circ T \quad \text{for all } x \in \text{dom}(\Phi_\gamma(f)).
\]

In particular, \(\Phi_\gamma(T' \circ g) \preceq \Phi_\gamma(g)\).

b) If \(f \in \mathcal{F}(O; K)\) then \(T \circ f \in \mathcal{F}(O; H)\), \(\text{dom}(\Phi_\gamma(T \circ f)) \subseteq \text{dom}(\Phi_\gamma(f))\) and

\[
\Phi_\gamma(T \circ f)x' = \Phi_\gamma(f)x' \circ T' \quad \text{for all } x' \in \text{dom}(\Phi_\gamma(f)).
\]

In particular, \(\Phi_\gamma(T \circ f) \preceq \Phi_\gamma(f)\).

**Proof.** This is an easy exercise. 

We shall abbreviate \(\Phi_\gamma(f) \preceq \Phi_\gamma(g)\) and \(\Phi_\gamma(f) \approx \Phi_\gamma(g)\) simply by

\[
f \preceq g \quad \text{and} \quad f \approx g,
\]

respectively, whenever it is convenient. The same abbreviation is used in the case of dual square functions. For applications of the subordination principle see Chapter 5 below.

4.2. Tensor products (and property (a)). Again we work with a functional calculus \((\mathcal{E}, \mathcal{F}, \Phi)\) on a Banach space \(X\), \(\mathcal{F}\) being an algebra of functions defined on a set \(O\). Let \(H, K\) be Hilbert spaces and \(f \in \mathcal{F}(O; H')\) and \(g \in \mathcal{F}(O; K')\). Then one can consider the function

\[
f \otimes g : O \to H' \otimes K' \subseteq (H \otimes K)' \quad \text{for } (f \otimes g)(z) := f(z) \otimes g(z),
\]

and we suppose in addition that \((f \otimes g) \in \mathcal{F}(O; (H \otimes K)')\). (This is the case, e.g., if \(\mathcal{F} = \mathcal{H}^\infty(O)\), and \(O\) some open subset of \(\mathbb{C}\).) Even more, suppose that the associated square functions

\[
\Phi_\gamma(f) : X \to \gamma(H; X) \quad \text{and} \quad \Phi_\gamma(g) : X \to \gamma(K; X)
\]

are bounded. It is then natural to ask whether or under which conditions the square function \(\Phi_\gamma(f \otimes g)\) is bounded as well. By the ideal property, composition with \(\Phi_\gamma(g)\) yields a bounded operator

\[
\Phi_\gamma(g) \circ : \gamma(H; X) \to \gamma(H; \gamma(K; X)), \quad \Phi_\gamma(g) \circ T := \Phi_\gamma(g) \circ T
\]

(the “tensor extension”). Hence

\[
\Phi_\gamma(g) \circ \Phi_\gamma(f) : X \to \gamma(H; \gamma(K; X))
\]

is bounded. With \(x \in X\), \(h \in H\) and \(k \in K\) we can compute

\[
\left[\left[\Phi_\gamma(g) \circ (\Phi_\gamma(f)x)\right]h\right]k = \left[\Phi_\gamma(g) \circ (\Phi_\gamma(f)x)\right]h k = \Phi_\gamma(g)\left[\Phi_\gamma(f)x\right]h k.
\]
Our question can be answered positively if the natural mapping $\gamma$ induces a bounded operator $X$ only if the Banach space $X$ has Pisier’s “property (\(\alpha\))”, see [35, Definition 2.1] for the original definition employing Rademacher sums, and [44, Chapter 13] for the stated equivalence. Every Hilbert space has property \(\alpha\) and each space $L_p(\Omega; X)$ with $1 \leq p < \infty$ inherits this property from $X$ [44, Chapter 13]. Let us summarise our considerations in the following lemma.

**Lemma 4.2.** Let $H, K$ be two Hilbert spaces and $X$ be a Banach space with property \(\alpha\). Suppose further that the square functions

$$\Phi_\gamma(f) : X \longrightarrow \gamma(H; X) \quad \text{and} \quad \Phi_\gamma(g) : X \longrightarrow \gamma(K; X)$$

are bounded. Then the tensor square function

$$\Phi_\gamma(f \otimes g) : X \longrightarrow \gamma(H \otimes K; X)$$

is bounded, too.

**4.3. Lower square function estimates I.** A lower square function estimate is an estimate of the form

$$\|x\|_X \leq C \|\Phi_\gamma(g)x\|_\gamma \quad (x \in \text{dom}(\Phi_\gamma(g))).$$

In certain situations one can combine a lower square function estimate, a usual square function estimate and a subordination to show the boundedness of an operator $\Phi(f)$.

**Lemma 4.3.** Let $H, K$ be Hilbert spaces and let $f \in \mathcal{F}(O; K')$ and $\tilde{g} \in \mathcal{F}(O; H')$. Suppose that the square function $\Phi_\gamma(\tilde{g}) : X \rightarrow \gamma(K; X)$ is bounded and that one has a lower square function estimate

$$\|x\|_X \leq C \|\Phi_\gamma(\tilde{g})x\|_\gamma \quad (x \in \text{dom}(\Phi_\gamma(\tilde{g})))$$

for $\Phi_\gamma(\tilde{g})$. Suppose further that the scalar-valued function $f \in \mathcal{F}$ is such that there is $T_f \in \mathcal{L}(H; K)$ with

$$f \cdot \tilde{g} = T_f' \circ \tilde{g}.$$

Then

$$\|\Phi(f)x\| \leq C \|T_f\| \|\Phi_\gamma(\tilde{g})\| \|x\| \quad (x \in \text{dom}(\Phi(f))).$$

**Proof.** By Lemma [3.3\(\text{c})],

$$\Phi_\gamma(\tilde{g})f \subseteq \Phi_\gamma(f \cdot \tilde{g}) = \Phi_\gamma(T_f \circ \tilde{g}) = [\Phi_\gamma(\tilde{g})] \circ T_f$$

Since the rightmost operator is fully defined, if $x \in \text{dom}(\Phi(f))$ then $\Phi(f)x \in \text{dom}(\Phi_\gamma(\tilde{g}))$ (still by Lemma [3.3\(\text{c})]) and hence

$$\|\Phi(f)x\|_X \leq C \|\Phi_\gamma(\tilde{g})\| \|\Phi(f)x\|_\gamma \leq C \|\Phi_\gamma(\tilde{g})\| \|\Phi_\gamma(f)x\|_\gamma \leq C \|T_f\| \|\Phi_\gamma(\tilde{g})\| \|x\|$$

as claimed. \(\square\)

**Lemma 4.3** is an abstract version of the “pushing the operator through the square function”-technique used by Kalton and Weis in [21] (see also [24, Theorem 10.9]) to show that a norm equivalence

$$\|R(\pm i\omega + \cdot, A)x\|_{\gamma_{(L^2(\mathbb{R};X)}} \sim \|x\|_X$$

for a strip type operator $A$ implies the boundedness of the $H^\infty$-calculus on a strip, see Section [5.6] below for details.
4.4. Lower square function estimates II. We now present some methods to establish lower square function estimates. These, however, require slightly stronger assumptions about the underlying functional calculus. Indeed, we shall work with a functional calculus $(\mathcal{E}, H^\infty(O), \Phi)$ admitting a function $e \in \mathcal{E}$ with the following properties:

1. $ef \in \mathcal{E}$ for all $f \in H^\infty(O)$;
2. if $(f_n)_{n \in \mathbb{N}}$ is a sequence in $H^\infty(O)$ with $\sup_{n \in \mathbb{N}} \|f_n\|_{H^\infty} < \infty$ and $f_n \to f$ pointwise, then $\Phi(e f_n) \to \Phi(e f)$ weakly on $X$;
3. $\Phi(e)$ is injective.

The standard functional calculi for strip type and sectorial operators are of this kind, see Lemma 5.1 below. Note that 1) and 3) just tell that the function $e$ is a “universal regulariser” for $H^\infty(O)$.

Remark 4.4. The following considerations are motivated by McIntosh’s approximation formula

$$x = \int_0^\infty \varphi(tA)\psi(tA)x \, dt$$

for $x \in \overline{\text{dom}(A)} \cap \overline{\text{ran}(A)}$, sectorial operators $A$ and appropriate functions $\varphi, \psi$, see [52] and [123, Sec. 5.2].

Let $f \in H^\infty(O; H)$, $g \in H^\infty(O; H')$ and

$$(f \circ g)(z) := (f(z), g(z))_{H, H'} \quad (z \in O).$$

(This notation is consistent with the notation $h' \circ f$ and $g \circ h$ introduced in Sections 5.2 and 3.3.) Then $f \circ g \in H^\infty(O)$ and we expect the formula

\[(\Phi(f \circ g)x, x')_{X, X'} = (\Phi_\gamma(g)x, \Phi_{\gamma'}(f)x')_{\gamma, \gamma'}\]

(4.1) to hold. The following result gives some conditions.

Lemma 4.5. In the described situation, if $e \in \mathcal{E}$ has the properties 1) and 2) above, and if $x = \Phi(e)y$ for some $y \in \text{dom}(\Phi_\gamma(g))$, then

$$\langle \Phi(f \circ g)x, x' \rangle = \langle \Phi_\gamma(g)x, \Phi_{\gamma'}(f)x' \rangle$$

for all $x' \in \text{dom}(\Phi_{\gamma'}(f))$.

Proof. Let us first note that, under the given conditions, $x \in \text{dom}(\Phi_\gamma(g))$. Indeed, this follows directly from Lemma 3.3(d).

For the proof of the claim we let $(e_\alpha)_{\alpha \in I}$ be an orthonormal basis of $H$ and denote

$$g_\alpha(z) := \langle e_\alpha, g(z) \rangle = \langle e_\alpha | g(z) \rangle, \quad f_\alpha(z) := \langle f(z), e_\alpha \rangle = \langle f(z) | e_\alpha \rangle$$

for $\alpha \in I$. Then by general Hilbert space theory

$$(f \circ g)(z) = \sum_\alpha f_\alpha(z) \cdot g_\alpha(z)$$

for each $z \in O$, and the partial sums are uniformly bounded. (Note that the sum is actually only over countably many $\alpha$ since $\{g(z), f(z) \mid z \in O\}$ is separable.) Hence, for $x' \in \text{dom}(\Phi_{\gamma'}(f))$ we can compute

$$\langle \Phi_\gamma(g)x, \Phi_{\gamma'}(f)x' \rangle_{\gamma, \gamma'} = \sum_\alpha \langle \Phi(g_\alpha)x, \Phi(f_\alpha)x' \rangle_{X, X'},$$

$$= \sum_\alpha \langle \Phi(e)\Phi(g_\alpha)y, \Phi(f_\alpha)x' \rangle_{X, X'} = \sum_\alpha \langle \Phi(f_\alpha)\Phi(e)\Phi(g_\alpha)y, x' \rangle_{X, X'},$$

$$= \sum_\alpha \langle \Phi(ef_\alpha g_\alpha)y, x' \rangle_{X, X'} = \langle \Phi(e(f \circ g))y, x' \rangle_{X, X'} = \langle \Phi(f \circ g)x, x' \rangle_{X, X'}.$$

Here we used c) of Theorem 2.14 and property 2) of the function $e$. \qed
Theorem 4.6. Let \( f \in H^\infty(O; H) \) and \( g \in H^\infty(O; H') \) and suppose that there is \( \varepsilon \in \mathcal{E} \) satisfying 1)–3) above. If \( \Phi_{\gamma}(f) \) is a bounded operator, then
\[
\Phi_{\gamma}(f)\Phi_{\gamma}(g) \subseteq \Phi(f \circ g).
\]
In other words, \( \text{dom}(\Phi_{\gamma}(g)) \subseteq \text{dom}(\Phi(f \circ g)) \) and
\[
\langle \Phi(f \circ g)x, x' \rangle = \langle \Phi_{\gamma}(g)x, \Phi_{\gamma}(f)x' \rangle \quad \text{for all } x \in \text{dom}(\Phi_{\gamma}(g)) \text{ and all } x' \in X'.
\]
In particular, one has the lower estimate
\[
\|\Phi(f \circ g)x\|_X \lesssim \|\Phi_{\gamma}(g)x\|_\gamma \quad \text{for all } x \in \text{dom}(\Phi_{\gamma}(g)).
\]
Proof. We let \( y := \Phi_{\gamma}(f)(\Phi_{\gamma}(g)x) \in X \) by Theorem 3.6. Take \( e \in \mathcal{E} \) satisfying 1), 2) and 3) above. Then by Lemma 4.3, for each \( x' \in X' \) we have
\[
\langle \Phi(e(f \circ g))x, x' \rangle = \langle \Phi(f \circ g)\Phi(e)x, x' \rangle = \langle \Phi_{\gamma}(g)\Phi(e)x, \Phi_{\gamma}(f)x' \rangle
\]
\[
= \langle \Phi(e)\Phi_{\gamma}(g)x, \Phi_{\gamma}(f)x' \rangle = \langle \Phi_{\gamma}(f)'(\Phi(e)\Phi_{\gamma}(g)x), x' \rangle
\]
\[
= \langle \Phi(e)\Phi_{\gamma}(f)'(\Phi_{\gamma}(g)x), x' \rangle = \langle \Phi(e)y, x' \rangle
\]
where we used Lemma 3.3(e) and 3.3. By construction of the functional calculus, \( x \in \text{dom}(\Phi(f \circ g)) \) and \( \Phi(f \circ g)x = y \). The remaining assertions follow easily. \( \square \)

Corollary 4.7. Let \( f \in H^\infty(O; H) \) and \( g \in H^\infty(O; H') \) such that \( \Phi_{\gamma}(g) \) and \( \Phi_{\gamma}(f) \) are bounded operators. Then \( \Phi(f \circ g) \) is a bounded operator and
\[
\langle \Phi(f \circ g)x, x' \rangle = \langle \Phi_{\gamma}(g)x, \Phi_{\gamma}(f)x' \rangle \quad \text{for all } x \in X \text{ and } x' \in X'.
\]
In particular, if \( f \circ g = 1 \) then one has the norm equivalence
\[
\|x\|_X \simeq \|\Phi_{\gamma}(g)x\|_\gamma \quad \text{for all } x \in X.
\]

The problem whether to a given function \( f \in H^\infty(O; H) \) there exists a function \( g \) with \( f \circ g = 1 \) is known as the Corona problem. For separable Hilbert spaces and bounded holomorphic functions on the disc such functions \( g \) always exist provided \( \inf_{z \in \mathbb{D}} \|f(z)\|_H > 0 \), see Tolokonnikov [42] and Uchiyama [43] and also [44] Appendix 3. By a conformal mapping this result extends to strips or sectors immediately.

Corollary 4.8. In addition to the standing assumptions of this section, suppose that \( O \) is a simply connected domain in \( \mathbb{C} \) and that \( \Phi_{\gamma}(f) \) is a bounded operator for all \( f \in H^\infty(O; H) \). Then there is a constant \( C \geq 0 \) with the following property: whenever \( g \in H^\infty(O; H') \) is such that \( \delta := \inf_{z \in \partial O} \|g(z)\|_H > 0 \), one has
\[
\|x\| \leq C \|x\|_{\gamma} \quad \text{for all } x \in \text{dom}(\Phi_{\gamma}(g))
\]
with \( c(\delta) \leq \delta^{-2}\ln(1 + \frac{1}{\delta})^{\frac{1}{2}}. \)

Proof. The closed graph theorem yields a constant \( C_1 \) with \( \|\Phi_{\gamma}(f)\| \leq C_1 \|f\|_\infty \) for all \( f \in H^\infty(O; H) \). And the Tolokonnikov–Uchiyama lemma yields for given \( g \) a function \( f \in H^\infty(O) \) with \( f \circ g = 1 \) and \( \|f\|_\infty \leq C_2\delta^{-2}\ln(1 + \frac{1}{\delta})^{\frac{1}{2}} \). Now the claim follows from Theorem 4.6 with \( C = C_1C_2. \) \( \square \)

4.5. Integral representations. In this section we describe a method of how to obtain new square function estimates from known ones via integral representations. We build on the previous results and hence work with a functional calculus \( (\mathcal{E}, H^\infty(O), \Phi) \) on a Banach space \( X \) under the same hypotheses as before in Section 4.3, i.e., we require the existence of a function \( e \in \mathcal{E} \) satisfying 1)–3) on page 26. As always, \( H \) is an arbitrary Hilbert space. The following is the main result.

Theorem 4.9. Let \( (\Omega, dt) \) be a measure space, define \( K := L_2(\Omega) \), and let \( f, g \in H^\infty(O; K) \) and \( m \in L_\infty(\Omega; H') \) such that
\begin{enumerate}
\item \( \Phi_{\gamma}(g) : X \to \gamma(K; X) \) is bounded and
\item \( \Phi_{\gamma}(f) : X' \to \gamma'(K; X') \) is bounded.
\end{enumerate}
Consider the function \( u \in H^\infty(O; H') \) defined by
\[
(4.2) \quad u(z) := \int_\Omega m(t) \cdot f(t, z) g(t, z) \, dt \in H' \quad (z \in O).
\]
If \( H \) has finite dimension or \( X \) has finite cotype, then the operator \( \Phi_\gamma(u) : X \to \gamma(H; X) \) is bounded, too, with
\[
\|\Phi_\gamma(u)\| \leq c \|m\|_{L^\infty(\Omega; H')} \|\Phi_\gamma(f)\| \|\Phi_\gamma(g)\|,
\]
where \( c \) depends on \( \dim(H) \) or the cotype (constant) of \( X \), respectively.

Proof. We define the bounded linear mapping
\[
S : H \to L^\infty(\Omega) \leftrightarrow L(K), \quad Sh := h \circ_H m
\]
and form
\[
g_h := (h \circ_H m) g = (Sh)g \in H^\infty(O; K)
\]
for \( h \in H \). Then, by definition,
\[
(h \circ_H u)(z) = \int_\Omega f(z)(h \circ_H m)g(z) = \int_\Omega f(z)g_h(z) = (f \circ_K g_h)(z) \quad (z \in O).
\]
For \( k \in K = L_2(\Omega) \),
\[
g_h \circ_K k = ((Sh)g) \circ_K k = g \circ_K ((Sh)k).
\]
Hence
\[
\Phi(g_h)x = (\Phi_\gamma(g) \circ (Sh)x) \in \gamma(K; X) \quad \text{for each } x \in X,
\]
by the ideal property. The operator
\[
T : H \to \gamma(K; X), \quad h \mapsto Th := \Phi(g_h)x
\]
factors through \( L^\infty(\Omega) \), and \( \gamma(K; X) \) has finite cotype (Lemma 2.17). Hence, by Theorem 2.18 \( T \in \gamma(H; \gamma(K; X)) \). By Theorem 3.6 and hypothesis 2), \( \Phi_\gamma(f)' : \gamma(K; X) \to X \) is bounded, and another application of the ideal property yields that \( h \mapsto \Phi_\gamma(f)' Th = \Phi_\gamma(f)' \Phi_\gamma(g_h)x \) is in \( \gamma(H; X) \). But
\[
\Phi_\gamma(f)' \Phi_\gamma(g_h) = \Phi(f \circ_K g_h) = \Phi(h \circ_H u)
\]
by Theorem 4.6. It follows that \( \Phi_\gamma(u) \) is bounded. To prove the norm estimate we trace back all these steps:
\[
\|\Phi_\gamma(u)x\|_\gamma = \|h \mapsto \Phi_\gamma(f)' \Phi_\gamma(g_h)x\|_\gamma \leq \|\Phi_\gamma(f)\| \|h \mapsto \Phi_\gamma(g_h)x\|_{\gamma(H; \gamma(K; X))}
\]
and
\[
\|h \mapsto \Phi_\gamma(g_h)x\|_{\gamma(H; \gamma(K; X))} = \|T\|_{\gamma(H; \gamma(K; X))} = \|h \mapsto \Phi_\gamma(g)x \circ Sh\|_{\gamma(H; \gamma(K; X))} \leq c \|\Phi_\gamma(g)\| \|x\| \|m\|_{L^\infty(\Omega; H')}.
\]
Here \( c = c(q, c_q(X)) \geq 0 \) is the constant coming from the application of the factorisation Theorem 2.18 (Note that by the proof of Lemma 2.17 \( c_q(\gamma(K; X)) \) depends on \( q, c_q(X) \) and some universal constants.)

Suppose that \( \Phi \) is a functional calculus for the operator \( A, H = L_2(\Omega') \), and
\[
u(s, z) = \int_\Omega m(s, t)f(t, z)g(t, z) \, dt.
\]
Then Theorem 4.9 says the following: if the square and dual square functions associated with \( g(\cdot, A) \) and \( f(\cdot, A) \), respectively, are bounded, then also the square function associated with \( u(\cdot, A) \) is bounded. (Note our convention from Section 3.4.) For \( H = \mathbb{C} \) this theorem is the main tool to infer bounded \( H^\infty \)-calculus from square and dual square function estimates. Examples are given in Chapter 6 below.
Remark 4.10. We do not know of a proper dual analogue of Theorem 4.9. However, under certain conditions one can use it to obtain bounded square functions for the dual functional calculus and by the inclusion $\gamma(H'; X') \subseteq \gamma'(H'; X')$ this yields a bounded dual square function for the original calculus.

4.6. Square function estimates from $\ell_1$-frame-boundedness. Whereas the main result of the previous section, Theorem 4.9, can be used to infer a bounded $H^\infty$-calculus from bounded (dual) square functions, in the present section we study the converse. Our results are based on a certain boundedness concept for subsets of a Hilbert space, the so called $\ell_1$-frame-boundedness.

Let $H$ be a complex Hilbert space. The $\ell_1$-frame-bound of a subset $M \subseteq H$ is defined as

$$|M|_1 := \inf \|L\| \sup_{x \in M} \sum_{\alpha \in I} |\langle Rx, e_\alpha \rangle|,$$

where the infimum is taken over all pairs $(L, R)$ of bounded linear operators $R : H \to \ell_2(I)$, $L : \ell_2(I) \to H$, $LR = I_H$, with $I$ being any (sufficiently large) index set. And $M$ is called $\ell_1$-frame-bounded if $|M|_1 < \infty$.

This notion is, to the best of our knowledge, new and — presumably — interesting in its own right. However, it plays only an auxiliary role here, so we decided to postpone a thorough treatment to Appendix D.

Theorem 4.11. Let $O \subseteq \mathbb{C}$ be an open set and let $\Phi : H^\infty(O) \to \mathcal{L}(X)$ be a bounded algebra homomorphism, where $X$ is a Banach space. Furthermore, let $f \in H^\infty(O; H)$ and $g \in H^\infty(O; H')$. Then the following assertions hold.

a) If $g$ has $\ell_1$-frame-bounded image in $H'$ and $X$ has cotype $q < \infty$, then $\Phi_g(g) \in \mathcal{L}(X; \gamma(H'; X))$ with

$$\|\Phi_g(g)x\|_\gamma \leq 2c_q(X)m_q \|\Phi\| \|g(O)\|_1 \cdot \|x\|_X \quad (x \in X),$$

where $c_q(X)$ is the cotype-$q$ constant of $X$ and $m_q$ is the $q$-th absolute moment of the normal distribution.

b) If $f$ has $\ell_1$-frame-bounded image in $H$, then $\Phi_{\gamma'}(f) \in \mathcal{L}(X'; \gamma'(H'; X'))$ with

$$\|\Phi_{\gamma'}(f)x'\|_{\gamma'} \leq \sqrt{\frac{2}{\pi}} \|\Phi\| \|f(O)\|_1 \cdot \|x'\|_{X'} \quad (x' \in X').$$

Proof. We fix an index set $I$ and bounded operators $R : H' \to \ell_2(I)$, $L : \ell_2(I) \to H'$ with $LR = I_{H'}$. Let $(r_\alpha)_{\alpha \in I}$ be independent complex Rademacher variables. Then by Theorem 2.16 for $F \subseteq I$ finite we have

$$\mathbb{E} \left\| \sum_{\alpha \in F} \gamma_\alpha \gamma_\alpha^* \gamma_\alpha \right\|_{\gamma}^2 \leq c_q(X)^2 m_q^2 \mathbb{E} \left\| \sum_{\alpha \in F} \gamma_\alpha \gamma_\alpha^* \gamma_\alpha \right\|_{\gamma}^2 \leq (c_q(X)m_q \|\Phi\| \|x\|)^2 \sup_{z \in O} \left( \sum_{\alpha \in F} \gamma_\alpha \gamma_\alpha^* \gamma_\alpha \right) \leq \left(2c_q(X)m_q \|\Phi\| \|x\| \right)^2 \left( \sup_{z \in O} \left( \sum_{\alpha \in F} \gamma_\alpha \gamma_\alpha^* \gamma_\alpha \right) \right)^2.$$

Consequently, by the ideal property,

$$\|\Phi(g)x\|_\gamma = \|\Phi(g)x \circ R'L'\|_\gamma \leq \|L\| \|\Phi(g)x \circ R'\|_\gamma \leq 2c_q(X)m_q \|\Phi\| \|x\| \|L\| \sup_{z \in O} \left( \sum_{\alpha \in I} \gamma_\alpha \gamma_\alpha^* \gamma_\alpha \right).$$
Taking the infimum over all pairs \((L, R)\) by (4.4) we obtain \(\Phi(g)x \in \gamma_\infty(H; X)\) and
\[
\|\Phi(g)x\|_\gamma \leq 2c_{\gamma}(X)m_q\|g(O)\|_1 \|x\|.
\]

Finally, since \(X\) has finite cotype it cannot contain a copy of \(c_0\) and hence by the
Hoffmann-Jørgensen–Kwapień theorems, \(\Phi(g)x \in \gamma(H; X)\).

For the proof of b) let us abbreviate \(V := \Phi^d(f)x' : H' \to X'\). We fix again an
index set \(I\) and operators \(R : H \to \ell_2(I), L : \ell_2(I) \to H\) with \(LR = I_H\), and a
family \((r_\alpha)_{\alpha \in F}\) of independent (complex) Rademachers.

Let \(F \subseteq I\) be finite, \((x_\alpha)_{\alpha \in F} \subseteq X\) and \(U := \sum_{\alpha \in F} e_\alpha \otimes x_\alpha \in \ell_2(I) \otimes X\). Then
\[
|\text{tr}(VR^\prime U)| \leq \frac{1}{2} \|x'\| \left|\sum_{\alpha \in F} \langle x_\alpha, VR^\prime e_\alpha \rangle\right| = \left|\sum_{\alpha \in F} \langle x_\alpha, \Phi(f \circ R^\prime e_\alpha)x' \rangle\right|
\]
\[
\leq \frac{1}{2} \|x'\| \left|\sum_{\alpha \in F} \langle R(f \circ e_\alpha)x, x' \rangle\sum_{\beta \in F} r_\beta x_\beta\right|
\]
\[
\leq \frac{1}{2} \|x'\| \left|\sum_{\alpha \in F} \langle R(f \circ e_\alpha)x, x' \rangle\right| \left|\sum_{\beta \in F} r_\beta x_\beta\right|_X
\]
\[
\leq \frac{1}{\sqrt{2}} \|\Phi\| \|x'\| \left|\sum_{\alpha \in O} \sqrt{\sum_{\alpha \in F} \langle Rf(z), e_\alpha \rangle}\right| \left|\sum_{\beta \in F} r_\beta x_\beta\right|_X
\]
\[
\leq \frac{1}{\sqrt{2}} \|\Phi\| \|x'\| \left|\sum_{\alpha \in O} \sqrt{\sum_{\alpha \in F} \langle Rf(z), e_\alpha \rangle}\right| \left|\sum_{\beta \in F} r_\beta x_\beta\right|_X
\]
\[
\leq \frac{1}{\sqrt{2}} \|\Phi\| \|x'\| \left|\sum_{\alpha \in O} \langle Rf(z), e_\alpha \rangle\right| \|U\|_\gamma.
\]

Thus, we can write
\[
|\text{tr}(VR^\prime U)| \leq \frac{1}{2} \|x'\| \|\Phi\| \|x'\| \left|\sum_{\alpha \in O} \langle Rf(z), e_\alpha \rangle\right| \|U\|_\gamma.
\]

We conclude the proof of b).

5. Examples

In this chapter we discuss several applications of the constructions and results of
the previous chapters. Our main protagonist is the functional calculus for strip
type operators as sketched in Appendix C. Via the exp / log-correspondence these
results all have sectorial versions.

Remark 4.12. It is notable that part b) of the theorem holds without any geometric
assumption on the Banach space. On the other hand, the appearance of finite
cotype in the formulation of a) is natural as one needs to estimate a Gaussian
sum in terms of a Rademacher sum. This raises the question why we do not work
with Rademacher sums exclusively right from the start. The reason is that “\(R\)
radonifying” operators do not have as nice properties as the \(\gamma\)-radonifying ones, in
particular the right ideal property fails. Since this property was also needed in the
proof of a), nothing would be gained by working with “\(R\)-radonifying” operators.
5.1. **Strip type operators.** Suppose that $A$ is an operator of strip type $\omega_0 \geq 0$ on a Banach space $X$, and let $\omega > \omega_0$. Then we can consider the functional calculus $(\mathcal{E}(\text{St}_\omega), H^\infty(\text{St}_\omega), \Phi)$ for $A$ as defined in Appendix C.

**Lemma 5.1.** In the described situation the following assertions hold:

a) Let $g \in H^\infty(\text{St}_\omega; H')$ and let $\Phi_\gamma(g)$ be the associated square function

$$\Phi_\gamma(g) : \text{dom}(\Phi_\gamma(g)) \to \gamma(H; X), \quad \Phi_\gamma(g)x = \left( h \mapsto (h \circ g)(A)x \right).$$

Then $\text{dom}(\Phi_\gamma(g))$ contains $\text{ran}(e(A))$ for each $e \in \mathcal{E}(\text{St}_\omega)$.

b) Suppose that $A$ is densely defined. Then each operator $f(A)$, $f \in H^\infty(\text{St}_\omega)$, is densely defined and dual square functions are well defined.

Let $f \in H^\infty(\text{St}_\omega; H)$ and let $\Phi_\gamma(f)$ be the associated square function

$$\Phi_\gamma(f) : \text{dom}(\Phi_\gamma(f)) \to \gamma(H; X), \quad \Phi_\gamma(f)x' = \left( h' \mapsto (f \circ h')(A)x' \right).$$

Then $\text{dom}(\Phi_\gamma(f))$ contains $\text{ran}(e(A)')$ for each $e \in \mathcal{E}(\text{St}_\omega)$.

**Proof.**

a) Let $e \in \mathcal{E}(\text{St}_\omega)$, let $x \in X$ and $h \in H$. Then

$$[\Phi(f)e(A)]x = (h \circ f)(A)x = ((h \circ f)e)(A)x = \frac{1}{2\pi i} \int_{\partial \text{St}_\omega} \langle h, f(z) \rangle e(z) R(z, A)x \, dz$$

with $\omega' \in (\omega_0, \omega)$. This shows that $e(A)x \in \text{dom}(\Phi(f))$ and that

$$\Phi(f)e(A)x = \frac{1}{2\pi i} \int_{\partial \text{St}_\omega} f(z) \otimes e(z) R(z, A)x \, dz$$

is nuclear, whence in $\gamma(H; X)$ (Lemma 2.9). The proof of [b] is similar. \qed

As a consequence we obtain that this functional calculus satisfies the requirements of Section 4.3. Indeed, every $e \in \mathcal{E}(\text{St}_\omega)$ satisfies 1) and 2) from page 26.

In the following, we shall discuss several instances of square functions for strip type operators. We recall our standing assumption that whenever we speak of dual square functions the dual calculus is supposed to be well defined, cf. Section 3.3.

We begin with the square functions “of shift type”.

**Example 5.2** (Shift type square functions). Let $\omega' > \omega$ and $\psi \in \mathcal{E}(\text{St}_\omega)$, and define $g(t, z) := \psi(t + z)$ for $t \in \mathbb{R}$ and $z \in \text{St}_\omega$. Then $g : \mathbb{R} \times \text{St}_\omega \to \mathbb{C}$ satisfies the hypotheses of Lemma 3.7. This gives rise to the (dual) square function

$$[\Phi_\gamma(g)x] = \left( \int_{\mathbb{R}} h(t) \psi(t + z) \, dt \right)(A)x \quad (h \in L^2(\mathbb{R}), x \in \text{dom}(\Phi_\gamma(g)))$$

$$[\Phi_\gamma(g)x'] = \left( \int_{\mathbb{R}} h(t) \psi(t + z) \, dt \right)(A)'x' \quad (h \in L^2(\mathbb{R}), x' \in \text{dom}(\Phi_\gamma(g))).$$

For $x \in \text{ran}(e(A))$, $e \in \mathcal{E}(\text{St}_\omega)$, Lemma 5.1 and a simple Fubini argument show that $\Phi_\gamma(g)x$ is integration against the vector-valued function $t \mapsto \psi(t + A)x$. If $X$ is itself a Hilbert space, one has

$$\|\Phi_\gamma(f)x\|_\gamma = \left( \int_{\mathbb{R}} \left\| \psi(t + A)x \right\|^2 \, dt \right)^{1/2},$$

and hence a square function estimate takes the form $\int_{\mathbb{R}} \left\| \psi(t + A)x \right\|^2 \, dt \leq C \|x\|^2$.

Similar remarks hold true for the dual square function.

**Claim:** The function $g$ considered as a mapping $\text{St}_\omega \to L^2(\mathbb{R})$ has $\ell_1$-frame-bounded range.
Proof. We first note that 
\[ \sup_{z \in St_\omega} \int_{\mathbb{R}} |\psi(t + z)| \, dt < \infty \]
by Lemma C.2.a. Moreover, \( \psi', \psi'' \in \mathcal{E}(St_\alpha) \) for each \( \alpha \in (\omega, \omega') \) by Lemma C.2.d. As before, this implies that
\[ \sup_{z \in St_\omega} \|\psi(\cdot + z)\|_{W^2_1(\mathbb{R})} = \sup_{z \in St_\omega} \int_{\mathbb{R}} |\psi(t + z)| + |\psi'(t + z)| + |\psi''(t + z)| \, dt < \infty. \]
Now by Lemma D.6 the claim is proved. \( \square \)

Corollary 5.3. Suppose that the \( H^\infty(St_\omega) \)-calculus for \( A \) is bounded and \( \psi \in \mathcal{E}(St_\omega') \) for some \( \omega' > \omega \). Then the dual square function associated with \( \psi(t + z) \) is bounded. If \( X \) has finite cotype, then also the square function associated with \( \psi(t + z) \) is bounded.

Proof. Simply combine Theorem 4.11 with Example 5.2. \( \square \)

The Fourier transform is an isomorphism on \( L^2(\mathbb{R}) \). Hence (dual) square functions related by the Fourier transform are strongly equivalent.

Example 5.4 (Weighted group orbits). Let as before \( \omega' > \omega \) and \( \psi \in \mathcal{E}(St_{\omega'}) \). Taking the inverse Fourier transform with respect to the variable \( t \) in the \( L^2(\mathbb{R}) \)-valued function \( \psi(t + z) \) yields the function
\[ \psi^\vee(s) e^{-isz} = \mathcal{F}_{t}^{-1}(\psi(t + z))(s). \]
Hence, the (dual) square functions associated with \( \psi(t + z) \) and \( \psi^\vee(s) e^{-isz} \) are strongly equivalent. In particular,
\[ \text{if } \psi(z) = \frac{\pi/\omega}{\cosh((\pi/2\omega)z)} \text{ then } \psi^\vee(s) = \frac{1}{\cosh \omega s} \]
(see Remark 6.4 below), whence
\[ \frac{\pi/\omega}{\cosh((\pi/2\omega)(t + z))} \approx \frac{e^{-isz}}{\cosh(\omega s)}. \]
Hence, by the results of Example 5.2 the latter function also has \( \ell_1 \)-frame-bounded range in \( L^2(\mathbb{R}) \).

Employing the subordination principle repeatedly one obtains
\[ \frac{\pi/\omega}{\cosh((\pi/2\omega)(t + z))} \approx \frac{e^{-isz}}{\cosh(\omega s)} \approx e^{-\omega|s|} e^{-isz} \approx (1_{\mathbb{R}_+}(s) e^{-\omega s} e^{-isz}, 1_{\mathbb{R}_-}(s) e^{-\omega s} e^{isz}) \approx (\pm i\omega + t - z)^{-1}. \]
Therefore we may, informally, write
\[ \|\cosh(\omega s)^{-1} e^{-isz} A_x\|_{\gamma} \sim \|R(\pm i\omega + t, A)x\|_{\gamma}. \]
Such square functions were considered in \cite{21} Theorem 6.2, see Section 6.6 below.

5.2. Sectorial operators. The results of the previous section have their natural analogues for sectorial operators via the \( \exp/\log \)-correspondence. Of course, one has to use the Hilbert space \( L^2_s(0, \infty) \) and the “shift type” square functions become “dilation type” square functions of the form \( \psi(tz) \). The analogue of the Fourier transform is the Mellin transform, and the “weighted group orbits” square functions are of the form \( \psi(s) z^{-is} \), i.e., the group of imaginary powers emerges here.
5.3. **Ritt operators.** A bounded operator on a Banach space is a Ritt operator if

\[ \sum_{k \geq 1} k \| T^{k-1}(I - T) \| < \infty. \]

The semigroup \( \{ T^n \mid n \geq 0 \} \) is the discrete analogue of an analytic semigroup, see [27]. The spectrum of a Ritt operator is contained in a Stolz domain and one has a natural functional calculus there, see [1] [22] [23] [27]. In the recent article [27], LeMerdy considers square functions associated with the \( \ell_2 \)-valued \( H^\infty \)-mappings

\[ f_m(k, z) := k^{m-\frac{1}{2}} z^{-k-1} (1 - z)^m \quad (k \in \mathbb{N}) \]

which are the discrete analogues of the \( L_2^\ast(0, \infty) \)-valued mappings

\[ g_m(t, z) := (tz)^m e^{-tz} \]

of dilation type. To some extent, the theory of bounded \( H^\infty \)-calculus and square function estimates on Stolz domains is equivalent to the strip or the sector case, by conformal equivalence of the underlying complex domains.

6. **Applications**

In this chapter we present several applications of the integral representation Theorem 4.9. In each case one starts from very specific bounded square and dual square functions and concludes the boundedness of an \( H^\infty \)-calculus or even, in the case that the Banach space has finite cotype, the boundedness of a vectorial \( H^\infty \)-calculus. However, one usually has to pay a price in the form that the domain set for the holomorphic functions represented by the integral formula has to be larger than the domain set used for the square functions.

6.1. **Cauchy–Gauß representation.** Our first instance uses the variant of the usual Cauchy integral formula with an additional Gaussian factor.

Let \( 0 < \omega < \omega' \), and let \( \Gamma := \partial St_\omega \) with arc length (=Lebesgue) measure. Then it is simple complex analysis to show that

\[ u(z) = \frac{1}{2\pi i} \int_\Gamma u(w) e^{-\frac{(w-z)^2}{w-\bar{z}}} \, dw \quad (|\text{Im } z| < \omega) \]

whenever \( u \in H^\infty(St_\omega; H) \), cf. Formula (C.2). To interpret it in the light of Theorem 4.9 we define

\[ m(w) := u(w), \quad f(w, z) := e^{-\frac{1}{4} \frac{(w-z)^2}{w-\bar{z}}}, \quad g(w, z) := e^{-\frac{1}{4} \frac{(w-z)^2}{w-\bar{z}}} \quad (w \in \Gamma, z \in St_\omega) \]

and \( K := L_2(\Gamma) \). Then \( f, g \in H^\infty(St_\alpha; K) \) for each \( \alpha \in (0, \omega) \). Consequently, if for an operator \( A \) of strip type \( \omega_0 < \omega < \omega' \) on a Banach space \( X \) the square and dual square functions associated with \( f \) and \( g \), respectively, are bounded, then \( A \) has a bounded \( H^\infty(St_\omega) \)-calculus. And if \( X \) has finite cotype, then \( A \) has a bounded vectorial \( H^\infty(St_\omega) \)-calculus.

Actually, one can say more here. Theorem 4.9 yields a constant \( C \geq 0 \) such that

\[ \| f(A) \|_\gamma \leq C \| f \|_{H^\infty(St_\omega)} \quad \text{for all } f \in \bigcup_{\omega' > \omega} H^\infty(St_\omega) \]

If the operator \( A \) is densely defined, then by the scalar/vectorial convergence lemma one obtains a bounded (vectorial) \( H^\infty(St_\omega) \)-calculus.

Combining these results with Theorem 4.11 or rather with Corollary 5.3 we arrive at the following central result.

**Theorem 6.1.** Let \( \alpha > 0 \) and let \( \Phi : H^\infty(St_\alpha) \to \mathcal{L}(X) \) be a bounded \( H^\infty \)-calculus over the strip \( St_\alpha \) on a Banach space \( X \) of finite cotype. Further, let \( \beta > \alpha \) and \( H \) be an arbitrary Hilbert space. Then, for each \( u \in H^\infty(St_\beta; H') \) the square function
Lemma 6.3. \( \Phi_{\gamma}(u) : X \to \gamma(H;X) \) is a bounded operator and there is a constant \( C \geq 0 \) such that
\[
\| \Phi_{\gamma}(u)x \|_{\gamma} \leq C \| u \|_{\mathcal{H}^{\infty}(St_\omega)} \| x \|_{X} \quad \text{for all} \quad u \in \mathcal{H}^{\infty}(St_\omega;H'), \, x \in X.
\]

Proof. Fix \( \omega \in (\alpha, \beta) \). Then, as in Example 5.2,
\[
\sup_{z \in St_\omega} \| g(z) \|_{W^2_1(\Gamma)} + \| f(z) \|_{W^2_1(\Gamma)} < \infty
\]
and hence \( g,f : St_\omega \to K \) have \( \ell_1 \)-frame-bounded range. By Theorem 4.11, the associated square and dual square functions are bounded. As explained above, the claim now follows from Theorem 4.9. \( \square \)

Clearly, Theorem 6.1 has a straightforward analogue for sectorial operators. Note that the vectorial calculus in Theorem 6.1 “lives” on a slightly larger strip. Consequently, in the sectorial version one needs to enlarge the sector.

Remark 6.2. While we were working on the present manuscript, Christian Le Merdy independently found the equivalent result of Theorem 6.1 for sectorial operators [27, Theorem 6.3]. (His “quadratic” \( \mathcal{H}^{\infty} \)-calculus is essentially what we call a “bounded vectorial” \( \mathcal{H}^{\infty} \)-calculus.) Le Merdy’s proof, which rests implicitly on an \( \ell_1 \)-frame-boundedness argument, is based on the Franks–McIntosh decomposition, to be treated below in Section 6.5.

6.2. Poisson representation. Our next example uses a variant of the Poisson formula for the strip.

Lemma 6.3. Let \( 0 < \omega < \omega' \) and \( u \in \mathcal{H}^{\infty}(St_{\omega'};H) \). Then
\[
(6.1) \quad u(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\pi/2}{\cosh(\pi/2(z+s))} (u(i\omega - s) + u(-i\omega - s)) \, ds
\]
where \( |\text{Im} \, z| < \omega \).

Proof. Fix \( 0 < \alpha \leq \pi/2 \omega \), then for \( |\text{Im} \, z| < \omega \) the function
\[
f(w) = \frac{\alpha(z-w)}{\sinh(\alpha(w-z))} u(w)
\]
is analytic in a strip larger than \( St_\omega \). (Note that \( w = z \) is a removable singularity.) Hence, by Cauchy’s integral formula,
\[
u(z) = f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{u(w)}{\sinh(\alpha(w-z))} \, dw
\]
where \( \Gamma := \partial St_\omega \) with the natural orientation. Now write out the parametrisation, specialise \( \alpha = \pi/2 \omega \) and use that \( \sinh(a \pm i\pi/2) = \pm i \cosh(a) \). \( \square \)

Remark 6.4. Specialising \( u(z) = e^{izx} \) and \( z = 0 \) in (6.1) one obtains again the formula
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\pi/2}{\cosh(\pi/2s)} e^{ist} \, ds = \frac{1}{\cosh(\omega t)} \quad (t \in \mathbb{R})
\]
used in Example 5.4.

In order to apply Theorem 4.9, we need to factorise the integral kernel in (6.1). A possibility is
\[
(6.2) \quad \frac{\pi/2}{\cosh(\pi/2z)} = \frac{\alpha \cosh(\pi/2s)}{\omega \cosh(\pi/2s)} \cdot \frac{\pi/2}{\cosh(\pi/2z)} = f(z) \cdot g(z),
\]
for \( \alpha > \omega \). With \( m_u(s) := (u(i\omega - s) + u(-i\omega - s)) \), Formula (6.1) then becomes
\[
(6.3) \quad u(z) = \frac{1}{2\pi} \int_{\mathbb{R}} m_u(s) f(z-s) \cdot g(z-s) \, ds.
\]
This is an instance of (4.2), whence Theorem 4.9 can be applied. The function \( f \) still looks a little unwieldy, but turns out to be strongly equivalent to \( g \), since
\[
\frac{\alpha \cosh(\frac{\gamma}{2\alpha} (z+s))}{\omega \cosh(\frac{\gamma}{2\omega} (z+s))} \approx \frac{2\alpha}{\pi} \cos \left( \frac{\pi\omega}{2\alpha} \right) \cosh(\omega t) e^{-it\omega} \approx \frac{e^{-it\omega}}{\cosh(\omega t)}.
\]
Here, the first equivalence comes from taking the inverse Fourier transform, and the second holds by multiplying by \( L \)-functions.

**Theorem 6.5.** Let \( A \) be a densely defined operator of strip type \( \omega_0 \geq 0 \) on a Banach space \( X \) (of finite cotype). Let \( \omega > \omega_0 \) and suppose that the square and dual square functions associated with the weighted group orbit \( e^{-it\omega}/\cosh(\omega t) \) are bounded, i.e.,
\[
\| e^{-itA} f \|_{\gamma} \lesssim \| f \| \quad \text{and} \quad \| e^{-itA^*} f' \|_{\gamma'} \lesssim \| f' \|.
\]

Then \( A \) has a bounded (vectorial) \( H^\infty \)-calculus on \( St_\omega \).

**Proof.** We apply the preceding remarks to obtain
\[
\| \Phi_\gamma(f) \|_{\gamma} \lesssim \| f \|_{H^\infty(St_\omega)}
\]
for \( f \in \bigcup_{\omega'>\omega} H^\infty(St_{\omega'}) \). The remaining step to a full vectorial \( H^\infty(St_\omega) \)-calculus is made via the vectorial convergence lemma (Lemma 3.4).

**Remark 6.6.** The factorisation [6.2] has been used in [14] to prove the transference principle for groups. A close inspection reveals that Formula (6.3) is — after taking a Fourier transform — just the transference identity in disguise. Using the arguments in the proof of [14] Theorem 3.2 leads to an alternative proof of Lemma 6.3, see the following section.

### 6.3. CDMcY-representation.

A variant of the Poisson type representation in the previous section was used by Cowling, Doust, McIntosh and Yagi in their influential paper [6]. To motivate it we first sketch an

**Alternative proof of Lemma 6.3.** Suppose first that \( f = \hat{g} \) is the Fourier transform of a function \( g \) on \( \mathbb{R} \) with \( \int_{\mathbb{R}} \cosh(\omega t) |g(t)| \, dt < \infty \). We abbreviate \( g_\omega(t) := \cosh(\omega t)g(t) \). Then
\[
f(z) = \int_{\mathbb{R}} e^{-iz\omega} g(t) \, dt = \int_{\mathbb{R}} \frac{e^{-it\omega}}{\cosh(\omega t)} \cosh(\omega t) g(t) \, dt = \int_{\mathbb{R}} \frac{e^{-it\omega}}{\cosh(\omega t)} g_\omega(t) \, dt
\]
and
\[
\mathcal{F}^{-1}(g_\omega)(s) = \frac{1}{2\pi} \int_{\mathbb{R}} g(t) \cosh(\omega t) e^{i\omega s} \, dt = \frac{1}{4\pi} \int_{\mathbb{R}} g(t) \left( e^{-i(\omega-s)t} + e^{-i(-\omega-s)t} \right) \, dt
\]
\[
= \frac{1}{4\pi} \left( f(\omega+s) + f(-\omega-s) \right).
\]
Hence, (6.1) is valid for such functions \( f \), and the general case is proved by approximation.

The idea behind the CDMcY-representation is to sneak in an additional factor in the previous argument and compute formally\(^1\)
\[
f(z) = \int_{\mathbb{R}} e^{-iz\omega} g(t) \, dt = \int_{\mathbb{R}} \psi(\omega)(t) e^{-it\omega} \frac{g_\omega(t)}{\psi'(t) \cosh(\omega t)} \, dt
\]
\[
= \int_{\mathbb{R}} \psi(z+s) \mathcal{F}^{-1} \left[ \frac{g_\omega(t)}{\psi'(t) \cosh(\omega t)} \right] (s) \, ds
\]
\(^1\)In order to keep our own presentation consistent, we deviate inessentially from [6] in that we use inverse Fourier transforms in place of Fourier transforms, and work on strips in place of sectors.
\[ \int_{\mathbb{R}} \psi(z+s) \left[ F^{-1}\left( \frac{1}{\psi'(t) \cosh(\omega t)} \right) \ast F^{-1}(g_\omega) \right] (s) \, ds. \]

To make this work, the authors require that

\[
(6.4) \quad \frac{1}{\psi'(t) \cosh(\nu t)} \in L_\infty(\mathbb{R}) \quad \text{for some } \nu < \omega.
\]

In order to obtain an \( L_\infty \)-bound on

\[ m_f(t) := F^{-1}\left( \frac{1}{\psi'(t) \cosh(\omega t)} \right) \ast F^{-1}(g_\omega) \]

in terms of the \( H_\infty \)-norm of \( f \) it remains to ensure that the first factor in the convolution is in \( L_1(\mathbb{R}) \). Hence, by the well known Carlson–Bernstein criterion and under the hypothesis \((6.4)\), it suffices to have

\[
\frac{(\psi')' \cosh(\nu t)}{\psi' \cosh(\omega t)} \in L_2(\mathbb{R}).
\]

Under the additional assumption (made in [6]) that \( \psi(z) = \varphi(e^z) \), and \( \varphi \in H_0^\infty \) on a sector, this is the case, see [6, p. 67].

**Remark 6.7.** The authors of [6] used this representation to infer bounded \( H_\infty \)-calculus from “weak quadratic estimates” of the form

\[ \int_{\mathbb{R}} |\langle \psi(t+A)x, x' \rangle| \, dt \lesssim \|x\| \|x'\|. \]

This notion is not covered so far in our approach (which avoids computing with \( X \)-valued functions). However, when it comes to square function estimates, it is not clear whether there is really a surplus compared with Theorem 6.5. The reason is that requirement \((6.4)\) implies that

\[ \frac{e^{-itz}}{\cosh(\nu t)} \lesssim e^{-itz} \psi'(t) \approx \psi(z+s) \]

and hence the boundedness of the shift-type square function associated with \( \psi \) implies the boundedness of the “weighted group orbit”-square functions considered in Theorem 6.5. (Even more, the CDMcY-choice of \( \psi \) implies also that \( \psi'/\cosh(\omega') \in L_\infty(\mathbb{R}) \) for some \( \omega' \) and hence square function estimates for \( \psi \) are basically equivalent with square function estimates for weighted group orbits.)

**6.4. Laplace (transform) representation.** In this section we work with a sectorial operator \( A \) of angle \( \theta < \pi/2 \), i.e., \(-A\) generates a (sectorially) bounded holomorphic semigroup \((e^{-tA})_{t>0}\). Ubiquitous square functions in this context are dilation type square functions \( \psi(tz) \) with \( H_\mathbb{L}_2^\infty(0, \infty) \), in particular for the choice \( \psi = \psi_\alpha \), where

\[ \psi_\alpha(z) = z^\alpha e^{-z} \quad (\alpha > 0) \]

and \( z \) is from a sufficiently large sector. Aiming at an application of Theorem 4.9 we look for a representation

\[ u(z) = \int_0^\infty m_u(t) \psi_\alpha(tz) \psi_\beta(tz) \, dt = z^{\alpha+\beta} \int_0^\infty m_u(t) t^{\alpha+\beta-1} e^{-2tz} \, dt = \frac{1}{2^{\alpha+\beta}} z^{\alpha+\beta} \int_0^\infty m_u(t/2) t^{\alpha+\beta-1} e^{-tz} \, dt \]

with \( m_u \in L_\infty(0, \infty) \). This means that \( \frac{1}{2^{\alpha+\beta}} m_u(t/2) t^{\alpha+\beta-1} \) is the inverse Laplace
transform of $u(z)/z^{\alpha+\beta}$. Now let us suppose that $u \in H^{\infty}(S_{\omega})$ for some $\omega' > \frac{\pi}{2}$. Then one can use the complex inversion formula to compute

$$m_u(t/2)^{\alpha+\beta-1} = \frac{1}{2\pi i} \int_{\Gamma_{\omega,t}} \frac{u(z)}{z^{\alpha+\beta}} e^{zt} \, dz$$

Here, $\frac{\pi}{2} < \omega < \omega'$ and the contour $\Gamma_{\omega,t}$ is the boundary of the region $S_{\omega} \setminus \{|z| \leq t\}$. Hence, with a change of variable,

$$m(t/2) = \frac{2^{\alpha+\beta}}{2\pi i} \int_{\Gamma_{\omega,t}} \frac{u(z)t^{\alpha+\beta-1}}{z^{\alpha+\beta}} e^{zt} \, dz$$

and this yields an estimate

$$\|m_u\|_{L^\infty(0,\infty)} \lesssim \left( \int_{\Gamma_{\omega,t}} |e^{Re z}/z^{\alpha+\beta}| \, |dz| \right) \|u\|_{H^{\infty}(S_{\omega})}.$$  

Combining these considerations with Theorem 4.9, we obtain the following result.

**Theorem 6.8.** Let $A$ be a sectorial operator, with dense domain and range, of angle $\theta < \frac{\pi}{2}$ on a Banach space $X$ (of finite cotype). Let $\alpha, \beta > 0$ and suppose that the square function associated with $\varphi_\alpha(tz) = (zt)^\alpha e^{-tz}$ and the dual square function associated with $\varphi_\beta(tz) = (zt)^\beta e^{-tz}$ are bounded operators. Then $A$ has a bounded (vectorial) $H^{\infty}$-calculus on each sector $S_{\omega'}$ with $\omega' > \frac{\pi}{2}$.

**Remark 6.9.** If $\alpha + \beta > 1$, then one can choose $\omega = \frac{\pi}{2}$ in the complex inversion formula. Hence one obtains an estimate $\|m_u\|_{L^\infty} \lesssim \|u\|_{H^{\infty}(S_{\omega})}$ and then, by the convergence lemma, a bounded $H^{\infty}(S_{\omega})$-calculus.

It is an intriguing question under which conditions one can actually push the “$H^{\infty}$-angle” (that is, the angle $\omega$ such that $A$ has a bounded (vectorial) $H(St_{\omega})$-calculus) down below $\frac{\pi}{2}$. To the best of our knowledge, this requires using the concept of $R$-boundedness and the multiplier theorem for $\gamma$-spaces. Recently [28], Christian Le Merdy has shown that if $X$ has Pisier’s property (a), then boundedness of the (dual) square function associated with $\varphi_{\gamma}(tz) = (tz)^{\gamma} e^{-tz}$ already suffices. Apart from a result by Kalton and Weis involving $R$-boundedness, Le Merdy needed to “improve the exponent”, i.e., to pass from $\varphi_{\gamma}$ to $\varphi_1$ and even to $\varphi_{\omega}$. His clever argument, carried out for $X$ being an $L_p$-space, can be covered by our abstract theory.

**Lemma 6.10 (Le Merdy).** Suppose that $X$ is a Banach space with Pisier’s property (a), and let $A$ be a sectorial operator of angle $\theta < \frac{\pi}{2}$, with sectorial functional calculus $\Phi$. Suppose that for given $\alpha, \beta > 0$ the square functions $\Phi_\gamma(\varphi_\alpha)$ and $\Phi_\gamma(\varphi_\beta)$ are bounded operators. Then $\Phi_\gamma(\varphi_{\alpha+\beta})$ is bounded, too.

**Proof.** The proof relies on the tensor product square function and subordination. We abbreviate $H = L^2(\mathbb{R}_+)$. Since $X$ has property (a), Lemma 4.2 shows that the function

$$(\varphi_\alpha \otimes \varphi_\beta)(s, t, z) = s^\alpha t^\beta z^{\alpha+\beta} e^{-(t+s)z}$$

yields a bounded square function on $L^2(0, \infty) \otimes L^2(0, \infty)$. Equivalently, the function

$$(f_\alpha \otimes f_\beta)(s, t, z) = s^{\alpha-\frac{\gamma}{2}} t^{\beta-\frac{\gamma}{2}} z^{\alpha+\beta} e^{-(t+s)z}$$

yields a bounded square function on $H \otimes H$, where we have put $f_\alpha(t, z) := t^{\alpha-\frac{\gamma}{2}} z^\alpha e^{-tz}$. 


Next, observe that $T : H \to H \otimes H$ defined by $(Tf)(s,t) = (t+s)^{-\frac{1}{2}}f(t+s)$ is isometric. Indeed,
\[
\int_0^\infty \int_s^\infty \frac{|f(t+s)|^2}{t+s} \, dt \, ds = \int_0^\infty \int_s^\infty \frac{|f(t)|^2}{t} \, dt \, ds = \int_s^\infty |f(t)|^2 \left( \frac{1}{t} \int_0^t \, ds \right) \, dt.
\]
Therefore, $T^*T = \text{Id}_H$. As a consequence, $\Phi_\gamma(f) \in \mathcal{L}(X;\gamma(H;X))$ if and only if $\Phi_\gamma(T \circ f) \in \mathcal{L}(X;\gamma(H \otimes H;X))$. Now,
\[
T^*(f_\alpha \otimes f_\beta)(t,s,z) = \frac{1}{\sqrt{t}} \int_0^t f_\alpha(t-s,z)f_\beta(s,z) \, ds = c_{\alpha,\beta} \, t^{\alpha+\beta-\frac{1}{2}} z^{\alpha+\beta} e^{-tz},
\]
and this concludes the proof. \qed

Remark 6.11. Passing from $L^2_2(0,\infty)$ to $L^2_2(\mathbb{R}_+)$ and then to $L^2_2(\mathbb{R})$ via the Laplace transform, one has
\[
\varphi_{\gamma_2}(tz) = (tz)^{\frac{\gamma_2}{2}} e^{-tz} \quad \text{on} \quad L^2_2(0,\infty) \approx \frac{z^{\frac{\gamma_2}{2}}}{z+is} \quad \text{on} \quad L^2_2(\mathbb{R})
\]
These square functions — in the form $A^{\gamma_2} R(is,A)x$ and $(A')^{\gamma_2} R(is,A')x'$ — were considered by Kalton and Weis in [21, Theorem 7.2].

6.5. Franks–McIntosh representation. In [9] Franks and McIntosh prove the following result: Given $\theta \in (0,\pi)$ there exist sequences $(f_n),(g_n)$ in $H^\infty(S_\theta)$ such that
\begin{itemize}
  \item[a)] $\sup_{z \in S_\theta} \sum_n |f_n(z)| + |g_n(z)| \leq C$,
  \item[b)] Any $\phi \in H^\infty(S_\theta;X)$ decomposes as $\phi(z) = \sum_n a_n f_n(z)g_n(z)$ with coefficients $a_n \in X$ satisfying $\|a_n\| \lesssim \|\phi\|_\infty$.
\end{itemize}
The decomposition [b] is an instance of our representation formula (4.2) for $K = \ell_2$. Condition [a] tells — in our terminology — that the $\ell_2$-valued $H^\infty$-functions $F(z) = (f_n(z))_n$ and $G(z) = (g_n(z))_n$ have $\ell_1$-frame-bounded range.

In [27] Le Merdy employs this representation to prove that on a space $X$ of finite cotype each sectorial operator with a bounded $H^\infty$-calculus on a sector has bounded sectorial (“quadratic”) $H^\infty$-calculus on each larger sector, i.e., the sectorial equivalent to our Theorem 6.1 cf. Remark 6.2.

6.6. Singular Cauchy representation. All the results in this chapter so far were applications of Theorem 4.9 that is, they infer a bounded (vectorial) $H^\infty$-calculus from bounded square and dual square functions. In the present section, however, we shall treat an application of Lemma 4.3. That is, we want to infer bounded $H^\infty$-calculus from upper and lower square function estimates. We discuss an example due to Kalton and Weis [21], see also [21, Theorem 10.9].

Let $A$ be a densely defined operator of strip type $\omega_0 \geq 0$ on a Banach space $X$. We fix $\omega > \omega_0$ and let $\Gamma_\omega = \partial St_\omega = (i\omega + \mathbb{R}) \cup (-i\omega + \mathbb{R})$ with arc length measure, let $H := L^2_2(\Gamma_\omega)$ and consider the $H$-valued function $g(\lambda,z) := \frac{1}{\lambda - z}$. Under the canonical isomorphism $H \cong L^2_2(\mathbb{R}) \oplus L^2_2(\mathbb{R})$, the function $g$ is strongly equivalent with the pair $(\pm i\omega + s - z)^{-1}$ of shift-type square functions, which — as demonstrated in Section 5.2 — is again strongly equivalent with the weighted group orbit square function associated with $e^{-is\omega}/\cosh(\omega s)$. Our aim is to prove the following remarkable result of Kalton and Weis [21, Theorem 6.2].

Theorem 6.12. Let $A$ be a densely defined operator of strip type $\omega_0$ and let $\omega > \omega_0$. Suppose that
\[
\|R(\lambda,A)x\|_{\gamma(\ell_2(\Gamma_\omega);X)} \sim \|x\| \quad (x \in X).
\]
Then $A$ has a bounded $H^\infty(St_\omega)$-calculus.
Proof. Let $0 \leq \omega_0 < \alpha < \omega < \omega'$ and let $f \in H^\infty(\text{St}_{\omega'})$ and $\lambda \in \Gamma_\omega$. Let $\Gamma_{\varepsilon, \lambda} = \partial(\text{St}_\omega \cup B(\lambda, \varepsilon))$, oriented positively.

Then for $z \in \text{St}_\alpha$

$$\frac{f(w)}{(w - z)(\lambda - z)} = \frac{f(w)}{(w - \lambda)(\lambda - z)} - \frac{f(w)}{(w - \lambda)(w - z)}.$$ Integrating this with respect to $w$ over $\{w \in \Gamma_{\varepsilon, \lambda}, |w| \leq r\}$ and letting $r \to \infty$ yields

$$\frac{f(z)}{\lambda - z} = \frac{f(\lambda)}{\lambda - z} - \frac{1}{2\pi i} \int_{\Gamma_{\varepsilon, \lambda}} \frac{f(w)}{(w - \lambda)(w - z)} \, dw.$$ By the fractional Cauchy theorem, the limit as $\varepsilon \to 0$ of the integral over the half circle avoiding $\lambda \in \Gamma_\omega$ at distance $\varepsilon$ is

$$\frac{1}{2\pi i} \cdot (i\pi) \frac{f(\lambda)}{\lambda - z} = \frac{1}{2} \frac{f(\lambda)}{\lambda - z}.$$ Hence, as $\varepsilon \to 0$ we obtain

$$\frac{f(z)}{\lambda - z} = \frac{f(\lambda)}{\lambda - z} - \frac{1}{2\pi i} \int_{\Gamma_\omega} \frac{f(w)}{(w - \lambda)(w - z)} \, dw - \frac{f(\lambda)}{2(\lambda - z)}$$

$$= \frac{f(\lambda)}{2(\lambda - z)} + \frac{1}{2\pi i} \text{p.v.} \int_{\Gamma_\omega} \frac{f(w)}{(\lambda - w)(w - z)} \, dw.$$ Let $T_f : L_2(\Gamma_\omega) \to L_2(\Gamma_\omega)$ be defined by

$$(T_fh)(\lambda) := \frac{f(\lambda)}{2} h(\lambda) + \frac{1}{2\pi i} \text{p.v.} \int_{\Gamma_\omega} \frac{f(w)}{(\lambda - w)} h(w) \, dw \quad (\lambda \in \Gamma_\omega).$$ Note that $T_f$ is bounded by a constant times $\|f\|_{H^\infty(\text{St}_\omega)}$: the first summand is simply multiplication by $\frac{1}{2} f$ and the second is multiplication with $f$ composed with convolution with $1/w$.

Now, by the computations above, we have

$$f(z)g(\lambda, z) = (T_f(g(\cdot, z)))(\lambda) \quad (z \in \text{St}_\alpha, \lambda \in \Gamma_\omega).$$ Viewing $g$ as a function in $H^\infty(\text{St}_\alpha; L_2(\Gamma_\omega))$ we hence have

$$f \cdot g = T_f \circ g$$ as in the hypotheses of Lemma 4.3 (Note that we as usual identify $L_2(\Gamma_\omega) = L_2(\Gamma_\omega)'$ here.) We hence obtain a constant $C \geq 0$ independent of $\omega' > \omega$ such that

$$\|f(A)\| \leq C \|f\|_{H^\infty(\text{St}_\omega)} \quad \text{for all } f \in H^\infty(\text{St}_{\omega'}).$$ The claim now follows from the scalar convergence lemma [13 Section 5.1]. □
APPENDIX A. THE CONTRACTION PRINCIPLE FOR GAUSSIAN SUMS

The aim of this section is to give a complete and concise proof of the following fundamental result. We work over the scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

**Theorem A.1** (Contraction Principle). Let $\gamma_1, \gamma_2, \ldots$ be independent scalar standard Gaussians on some probability space, let $X$ be a Banach space, $x_1, \ldots, x_m \in X$ and let $A = (a_{kj})_{kj}$ be a scalar $n \times m$-matrix. Then

$$
\mathbb{E} \left\| \sum_{k=1}^n \sum_{j=1}^m \gamma_k a_{kj} x_j \right\|^2 \leq \|A\|^2 \mathbb{E} \left\| \sum_{j=1}^m \gamma_j x_j \right\|^2_X,
$$

where the matrix $A$ is considered as an operator $A : \ell_2^m \to \ell_2^n$.

The proof proceeds in three steps. In the first step one reduces the problem to the case that $n = m$. If $m > n$ one just extends $A$ to an $m \times m$-matrix by adding 0-rows. If $m < n$ one extends $A$ to an $n \times n$-matrix by adding 0-columns, and defines $x_j := 0$ for $m < j \leq n$.

Now, if $m = n$ we may suppose by scaling that $A$ is a contraction. Then the following lemma reduces the claim to $A$ being an isometry.

**Lemma A.2.** Every contraction on the Euclidean space $\mathbb{K}^d$ is a convex combination of at most $d$ isometries.

**Proof.** This is well known, see but the proof is given here for the convenience of the reader. We may suppose that $\|A\| = 1$. By polar decomposition $A = U |A|$ where $|A| = (A^*A)^{1/2}$, and $U$ is isometric. Hence we may assume that $A = A^*$ is positive semi definite. By the spectral theorem we may even further reduce the problem to $A$ being a diagonal matrix with entries $1 = \lambda_d \geq \cdots \geq \lambda_1 \geq 0$. (Note that 1 has to be an eigenvalue since $\|A\| = 1$.) Now we set $\lambda_0 = 0$ and write

$$
\text{diag}(\lambda_1, \ldots, \lambda_d) = \sum_{j=1}^d (\lambda_j - \lambda_{j-1})P_j
$$

where $P_j(x_1, \ldots, x_d) := (x_1, \ldots, x_j, 0, \ldots, 0)$ is projection onto the first $j$ coordinates. (So $P_d = I$.) This is convex combination of projections. But for any orthogonal projection $P$ on a Hilbert space,

$$
P = \frac{1}{2} I + \frac{1}{2} (2P - I)
$$

is a representation as a convex combination of unitaries, since $(2P - I)^*(2P - I) = (2P - I)^2 = 4P^2 - 4P + I = I$. Since in the representation above always the identity $I$ is used, we can collect terms and arrive at a convex combination of at most $d$ terms. \hfill \square

Finally, we are reduced to the case that $n = m$ and $A$ is an orthogonal/unitary matrix. Then by the rotation invariance of the $n$-dimensional, resp. $2n$-dimensional, standard Gaussian measure [5, p.239],

$$
\mathbb{E} \left\| \sum_{k=1}^n \sum_{j=1}^n \gamma_k a_{kj} x_j \right\|^2 = \mathbb{E} \left\| \sum_{k=1}^n \left( \sum_{k=1}^n a_{kj} \gamma_k \right) x_j \right\|^2 = \mathbb{E} \left\| \sum_{j=1}^m \gamma_j x_j \right\|^2.
$$

This concludes the proof of Theorem A.1.

APPENDIX B. WEAKLY SQUARE INTEGRABLE FUNCTIONS AND PETTIS INTEGRALS

Let $(\Omega, \Sigma, \mu)$ be a measure space. Recall that a function $f : \Omega \to X$ is $\mu$-measurable if there is a sequence $(f_n)_n$ of step functions (finite linear combinations of functions $1_A \otimes x$ with $\mu(A) < \infty$ and $x \in X$) such that $f_n \to f$ pointwise almost everywhere. Each $\mu$-measurable function is essentially separably valued and vanishes outside a $\sigma$-finite set.
For a $\mu$-measurable function $f : \Omega \to X$ we let

$$\Sigma_f := \{ A \in \Sigma \mid 1_A f \in L_2(\Omega; X) \}.$$ 

Then $\Sigma_f$ is closed under taking finite unions and measurable subsets. The following result shows that $\Sigma_f$ is quite rich.

**Lemma B.1.** Let $f : \Omega \to X$ be $\mu$-measurable. Then the following assertions hold:

a) Simply note that $f$ is bounded and hence $D_f$ is an ideal of $L_2(\Omega)$.

b) There is a sequence $A_n \in \Sigma_f$ such that $\mu(A_n) < \infty$ and $1_{A_n} \not\to 1_{\{f \neq 0\}}$ almost everywhere.

c) span$\{1_A \mid \mu(A) < \infty, A \in \Sigma_f\}$ is dense in $L_2(\Omega)$.

**Proof.**

[a] Simply note that $A \cap \{\omega \mid (f(\omega)) \leq n\} \not\to A$ as $n \to \infty$.

[b] This follows from a) and the fact that $\{f \neq 0\}$ is $\sigma$-finite.

[c] This follows from a) and the fact that the step functions are dense in $L_2$. \qed

We let

$$D_f := \{1_A g \mid g \in L_2(\Omega), A \in \Sigma_f\}.$$ 

Then $D_f$ is an ideal of $L_2(\Omega)$, i.e., a linear subspace of $L_2(\Omega)$ with $h \in D_f$ whenever $|h| \leq |k|$ and $k \in D_f$. Moreover, $1_A \in D_f$ for each $A \in \Sigma_f$ with finite measure and hence $D_f$ is dense in $L_2(\Omega)$. Now we define the operator

$$U_f : D_f \to X \quad U_f(h) := \int_{\Omega} hf \, d\mu.$$ 

The following lemma shows that the mapping $f \mapsto U_f$ is essentially one-to-one.

**Lemma B.2.** Let $f : \Omega \to X$ be $\mu$-measurable such that $U_f = 0$. Then $f = 0$ $\mu$-almost everywhere.

**Proof.** Let $x' \in X'$ and $A \in \Sigma_f$. Then $\int_A (x' \circ f)g = 0$ for all $g \in L_2(\Omega; X)$, i.e., $x' \circ 1_A f = 0$ almost everywhere. Since $f$ is essentially separably-valued, it follows that $1_A f = 0$ almost everywhere. But every set of finite measure differs from a set from $\Sigma_f$ by as little as we like, and hence $f = 0$ almost everywhere on each set of finite measure. Since the set $\{f \neq 0\}$ is $\sigma$-finite, the claim follows. \qed

In certain cases the operator $U_f$ extends to a bounded operator $L_2(\Omega) \to X$, which we denote also by $U_f$.

**Remark B.3.** If $f \in L_2(\Omega; X)$ then $D_f = L_2(\Omega)$, $U_f$ is bounded, and $\|U_f\| \leq \|f\|_2$. However, we stress that for a general $\mu$-measurable $f : \Omega \to X$ the space $D_f$ need not be equal to all of $L_2(\Omega)$ even if $U_f$ is bounded.

In the following we characterise those functions $f$ such that $U_f$ is bounded.

**Theorem B.4.** Let $f : \Omega \to X$ be $\mu$-measurable, and let $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma_f$ with $1_{A_n} \not\to 1$ a.e. on the set $\{f \neq 0\}$. Then the following assertions are equivalent:

(i) $U_f$ is bounded;

(ii) $f$ is "weakly $L_2$", i.e., $x' \circ f \in L_2(\Omega)$ for each $x' \in X'$;

(iii) $\sup_{n \in \mathbb{N}} \|U_{f1_{A_n}}\| < \infty$.

If (i) and (iii) hold, then

$$\sup_{n \in \mathbb{N}} \|U_{f1_{A_n}}\| = \|U_f\| = \sup_{\|x'\| \leq 1} \|x' \circ f\|_2 =: \|f\|_{L_2}$$

and $U_{f1_{A_n}} \to U_f$ strongly.
Proof. (i) ⇒ (iii) For \( g \in L_2(\Omega) \) and \( n \in \mathbb{N} \) we have
\[
\|U_{1A_n}(g)\| = \|U_f(1_{A_n}g)\| \leq \|U_f\| \|1_{A_n}g\|_2 \leq \|U_f\| \|g\|_2
\]
which implies that \( \|U_{1A_n}f\| \leq \|U_f\| \).

(iii) ⇒ (ii) For each \( n \in \mathbb{N} \),
\[
\int_{A_n} |x' \circ f|^2 = \int |x' \circ (f1_{A_n})|^2 \leq \sup_{\|h\|_2 \leq 1} \left| \int (f1_{A_n})h, x' \right|^2 \leq \sup_{\|h\|_2 \leq 1} \|x'\|^2 \|U_{1A_n}h\|^2 = \|x'\|^2 \|U_{1A_n}\|^2.
\]
Since \( A_n \setminus \{ f \neq 0 \} \) up to a null set, (ii) follows from (iii).

(ii) ⇒ (i) If \( f \) is weakly \( L_2 \), then by the closed graph theorem there must be \( c \geq 0 \) such that \( \|x' \circ f\|_2 \leq c \|x'\| \) for all \( x' \in X' \). Fix \( h = 1_A g \in D_f \), with \( A \in \Sigma_f \) and \( g \in L_2 \). Then \( (U_f(h), x') = f(x' \circ f)h \), whence by Cauchy–Schwarz
\[
|\langle U_f(h), x' \rangle| \leq \|x' \circ f\|_2 \|h\|_2 \leq c \|x'\| \|h\|_2.
\]
This yields \( \|U_f(h)\| \leq c \|h\|_2 \), and (i) follows.

Finally, suppose that (i), (iii) hold. Then (B.1) has already been shown. For the strong convergence note that \( U_{1A_n}(h) \to U_f(h) \) for each \( h \in D_f \), and this is dense in \( L_2(\Omega) \).

We let \( P_2(\Omega; X) \) be the space of strongly measurable functions \( f : \Omega \to X \) such that \( U_f \) is bounded, viz. \( f \) is weakly \( L_2 \). If \( f \in P_2(\Omega; X) \) then the extension \( U_f \) of \( U_f \) to all of \( L_2 \) can be described by a weak (= Pettis) integral. Namely for each \( x' \in X' \) the function \( x' \circ f \) is \( L_2 \) and hence
\[
\langle U_f(h), x' \rangle = \int_{\Omega} \langle h(\omega)f(\omega), x' \rangle \mu(d\omega) \quad \text{for all } h \in L_2(\Omega).
\]

It is sometimes convenient to decide the boundedness of \( U_f \) on a subset of its natural domain.

Lemma B.5. Let \( f : \Omega \to X \) be \( \mu \)-measurable, and let \( D \) be a subspace of \( D_f \), dense in \( L_2(\Omega) \) and invariant under the multiplication by characteristic functions. If \( U_f \) is bounded on \( D \) then it is bounded on \( D_f \), with the same norm.

Proof. By Lemma B.1 we find a sequence \( (A_n)_n \subseteq \Sigma_f \) such that \( 1_{A_n} \to 1 \) almost everywhere on \( \{ f \neq 0 \} \). Let \( g \in D \) and \( n \in \mathbb{N} \). Then
\[
\|U_{1A_n}g\| = \|U_f(1_{A_n}g)\| \leq c \|1_{A_n}g\|_2 \leq c \|g\|_2,
\]
where \( c \) is the bound of \( U_f \) on \( D \). Since \( U_{1A_n} \) is bounded and \( D \) is dense, it follows that \( \|U_{1A_n}\| \leq c \), independent of \( n \in \mathbb{N} \). Hence, the assertion follows from Theorem B.4.

We have a Fatou-type property.

Lemma B.6 (P2–Fatou). If \( (f_n)_{n \in \mathbb{N}} \) is a bounded sequence in \( P_2(\Omega; X) \) with \( f_n \to f \) almost everywhere, then \( f \in P_2(\Omega; X) \), \( U_{f_n} \to U_f \) strongly, and
\[
\|f\|_{P_2} \leq \liminf_{n \to \infty} \|f_n\|_{P_2}.
\]

Proof. We form the set
\[
D := \{ 1_A g \mid g \in L_2(\Omega), A \in \Sigma_f, \|f - f_n\|_{L_2(A)} \to 0 \}.
\]
By Egoroff’s theorem and Lemma B.1 \( D \) is dense in \( L_2(\Omega) \). Moreover, \( D \subseteq D_f \) by construction, and \( D \) is obviously invariant under multiplication by characteristic functions.
functions. If \( h := 1_{A}g \in D \) then \( f_{n}h \to fh \) in \( L_{1} \) and hence \( U_{f_{n}}(h) = \int hf_{n} \to \int hf = U_{f}(h) \). We conclude that
\[
\|U_{f}(h)\| = \lim_{n \to \infty} \|U_{f_{n}}(h)\| \leq \lim inf_{n \to \infty} \|U_{f_{n}}\| \|h\|_{2}.
\]
By Lemma B.5 it follows that \( U_{f} \) is bounded with norm \( \|U_{f}\| \leq \lim sup_{n} \|U_{f_{n}}\| \).

The rest follows by a standard approximation argument.\( \square \)

**Appendix C. Holomorphic functional calculus on sectors and strips**

For the reader’s convenience we briefly develop the two calculi most relevant in our context, namely the calculus for sectorial and strip type operators. Although this has been done at several places in the literature, e.g. in the second author’s book [13, Chapters 2 and 4], the construction here is a little different in order to achieve perfect correspondence between the sector and the strip case.

To begin with, we fix some notation. Let \( St_{0} := \mathbb{R}, S_{0} := (0, \infty) \); further, for \( \omega > 0 \) we let
\[
St_{\omega} := \{ z \in \mathbb{C} \mid |\text{Im} z| < \omega \} \quad \text{and} \quad S_{\omega} := \{ z \in \mathbb{C} \setminus \{0\} \mid |\text{arg} z| < \omega \},
\]
where the latter is only meaningful if \( \omega \leq \pi \). In that case the transformations \( w = \log z \) and \( z = \exp(w) \) form a pair of mutually inverse holomorphic mappings from \( S_{\omega} \) to \( St_{\omega} \) and vice versa. In particular, the functional calculus theories for \( H^{\infty}(St_{\omega}) \) and \( H^{\infty}(S_{\omega}) \) are equivalent. We shall concentrate on the strip case and only briefly touch upon the sector case.

For \( \omega > 0 \) the algebra of *elementary functions* on \( St_{\omega} \) is
\[
\mathcal{E}(St_{\omega}) := \{ f \in H^{\infty}(St_{\omega}) \mid \int_{-\infty}^{\infty} |f(r + i\alpha)| dr < \infty \text{ for all } |\alpha| < \omega \}.
\]
If \( 0 < \omega \leq \pi \) we correspondingly define
\[
\mathcal{E}(S_{\omega}) := \{ f \in H^{\infty}(S_{\omega}) \mid \int_{0}^{\infty} |f(re^{i\alpha})| \frac{dr}{r} < \infty \text{ for all } |\alpha| < \omega \}.
\]
Then \( \mathcal{E}(S_{\omega}) = \{ f \circ \log \mid f \in \mathcal{E}(St_{\omega}) \} \).

**Remark C.1.** It is common in the literature to use a class of elementary functions defined via explicit growth conditions instead of integrability. In this approach, the class
\[
H_{0}^{\infty}(S_{\omega}) = \{ f \in H^{\infty}(S_{\omega}) \mid \exists s, C > 0 : |f(z)| \leq C \min(|z|^{s}, |z|^{-s}) \}
\]
features prominently. However, such growth conditions are not compatible with the \( \exp / \log \)-correspondence, whereas our definition is.

It is clear that \( f \in \mathcal{E}(St_{\omega}) \) if and only if \( f(\cdot + r) \in \mathcal{E}(St_{\omega}) \) for some/each \( r \in \mathbb{R} \). Moreover, by Cauchy’s theorem, the following formulæ hold for any elementary function \( f \in \mathcal{E}(St_{\omega}) \):

\[
(C.1) \quad f(z) = \frac{1}{2\pi i} \int_{\partial St_{\omega}} \frac{f(\zeta)}{\zeta - z} d\zeta
\]
\[
(C.2) \quad = \frac{1}{2\pi i} \int_{\partial St_{\omega}} f(\zeta) e^{-\zeta(z - z)^{2}} d\zeta \quad (z \in St_{\omega}', 0 < \omega' < \omega).
\]

Note that for \( z \in \mathbb{C} \) and \( \zeta \in St_{\omega} \)
\[
|e^{-\zeta(z - z)^{2}}| = e^{-\text{Re}(\zeta(z - z)^{2})} = e^{-(\text{Re} \zeta - \text{Re} z)^{2}} \cdot e^{(\text{Im} \zeta - \text{Im} z)^{2}} \leq e^{-(\text{Re} \zeta - \text{Re} z)^{2}} e^{(\omega + |\text{Im} z|)^{2}}.
\]

Consequently, for fixed \( z \in \mathbb{C} \) the function \( \zeta \to e^{-\zeta(z - z)^{2}} \) is an elementary function on \( St_{\omega}' \). It follows that the representation formula (C.2) actually holds for all \( f \in H^{\infty}(St_{\omega}) \).

**Lemma C.2.** Let \( 0 < \alpha < \omega \) and \( f \in \mathcal{E}(St_{\omega}) \). Then the following assertions hold:
a) \( \sup_{|s| \leq \alpha} \int_{-\infty}^{\infty} |f(r + is)| \, dr < \infty \).

b) \( f \in \mathcal{E}(\text{St}_\alpha) \cap C_0(\overline{\text{St}_\alpha}) \).

c) \( \int_{\partial \text{St}_\alpha} f(z) \, dz = 0 \).

d) \( f' \in \mathcal{E}(\text{St}_\alpha) \).

**Proof.** For the proof of a) fix \( \alpha' < \omega' < \omega \). Then for \( 0 \leq s \leq \alpha \),

\[
\int_{\partial \text{St}_s} |f(z)| \, dz \leq \frac{1}{2\pi} \int_{\partial \text{St}_s} d\zeta \int_{\partial \text{St}_s} \left| \frac{e^{-(c(s-z)^2)}}{z-\zeta} \right| \, |dz| \\
= \frac{1}{2\pi} \int_{\partial \text{St}_s} d\zeta \int_{\partial \text{St}_s} \left| \frac{e^{-(c(s-z)^2)}}{z-\zeta} \right| \, |dz| \\
\leq \frac{1}{2\pi} \|f\|_{L_1(\partial \text{St}_s)} \int_{\partial \text{St}_s} \left| \frac{e^{-(Re z)^2} e^{(\omega'+\nu)^2} \omega-\alpha}{\omega-\alpha} \right| \, |dz| = \frac{\omega' \omega}{\sqrt{\pi}(\omega'-\alpha)} \|f\|_{L_1(\partial \text{St}_s)}.
\]

b) To see that \( f(z) \rightarrow 0 \) as \( \Re z \rightarrow \infty, |\Im z| \leq \alpha \) one uses the representation formula (C.1) or (C.2) and the dominated convergence theorem.

c) By Cauchy’s formula one has \( 0 = \int_{\partial \text{St}_s} f(z) \, dz \) where \( R_n \) is the rectangle with corners at \( \pm n \pm io, n \in \mathbb{N} \). When letting \( n \rightarrow \infty \) the upper and the lower side of the rectangle approach \( \partial \text{St}_s \) and the integrals over the left and right side vanish since \( f \in C_0(\overline{\text{St}_s}) \) by b).

d) Let \( \alpha < \omega' < \omega \). Then by Cauchy’s integral formula,

\[
f'(z) = \frac{1}{2\pi i} \int_{\partial \text{St}_s} \frac{f(\zeta) \, d\zeta}{(\zeta-z)^2} \quad (|\Im z| < \omega').
\]

In particular

\[
\int_{|\Im z| = \alpha} |f(z)| \, dz \leq \left( \int_{\partial \text{St}_s} \frac{|f(\zeta)| \, |d\zeta|}{2\pi} \right) \left( \max_{\zeta = \pm i\omega'} \int_{|\Im z| = \alpha} \frac{|dz|}{(\zeta-z)^2} \right) = \infty.
\]

\( \square \)

**Operators of Strip Type.** A closed operator \( A \) on a Banach space \( X \) is called of *strip type* \( \alpha \geq 0 \), if \( \sigma(A) \subseteq \overline{\text{St}_\alpha} \) and if for all \( \beta > \alpha \), the resolvent \( R(\cdot, A) \) is uniformly bounded on \( \mathbb{C} \setminus \text{St}_\beta \). If for each \( \beta > \alpha \) we have an estimate \( \|R(\lambda, A)\| \lesssim (|\Im \lambda| - \beta)^{-1} \) on \( \mathbb{C} \setminus \text{St}_\beta \), \( A \) is called of *strong strip type* \( \alpha \).

For an operator \( A \) of strip type \( \alpha \geq 0, \omega > \alpha \) and an elementary function \( f \in \mathcal{E}(\text{St}_\omega) \) there is a natural definition of the operator \( f(A) \in \mathcal{L}(X) \) by

\[
(C.3) \quad f(A) \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{\partial \text{St}_s} f(z) R(z, A) \, dz,
\]

which is independent of \( \omega' \in (\alpha, \omega) \) by Cauchy’s theorem. The mapping \( \mathcal{E}(\text{St}_\omega) \rightarrow \mathcal{L}(X) \) given by \( f \mapsto f(A) \) is called the elementary calculus for \( A \). It is rather routine to show by virtue of the resolvent identity, the residue theorem and contour deformation arguments that this is a homomorphism of algebras, and that

\[
\left( \frac{1}{\lambda} \right) (A) = f(A) R(\lambda, A) \quad \text{and} \quad \left( \frac{1}{(X-z)(\mu-z)} \right) (A) = R(\lambda, A) R(\mu, A)
\]

for all \( \lambda, \mu \in \mathbb{C} \setminus \overline{\text{St}_\omega} \), cf. [13] Chapter 2 or [3].

An operator \( A \) of strip type \( \alpha \in [0, \omega) \) on a Banach space \( X \) has a *bounded \( H^\infty(\text{St}_\omega)\)-calculus* if there is a constant \( C \geq 0 \) such that

\[
\|f(A)\| \leq C \|f\|_{H^\infty(\text{St}_\omega)} \quad \text{for all} \ f \in \mathcal{E}(\text{St}_\omega).
\]
Lemma C.3. Let \( 0 \leq \alpha < \omega \), and let \( A \) be a densely defined operator \( A \) of strip type \( \alpha \in [0, \omega) \) on a Banach space \( X \). Then \( A \) has a bounded \( H^\infty (S_\omega) \)-calculus if and only if the elementary calculus has an extension to a bounded algebra homomorphism \( \Phi : H^\infty (S_\omega) \to \mathcal{L}(X) \). In this case, such an extension is unique and \( \| \Phi(f) \| \leq C \| f \|_\infty \) holds for every \( f \in H^\infty (S_\omega) \) if it holds for every \( f \in \mathcal{E}(S_\omega) \).

Proof. Suppose first that the bounded algebra homomorphism \( \Phi : H^\infty (S_\omega) \to \mathcal{L}(X) \) extends the elementary calculus. If \( f \in H^\infty (S_\omega) \) and \( e \in \mathcal{E}(S_\omega) \) such that \( ef \in \mathcal{E}(S_\omega) \) and \( e(A) \) is injective, then \( (ef)(A) = \Phi(ef) = \Phi(e)\Phi(f) = e(A)\Phi(f) \). Hence

\[
\Phi(f) = e(A)^{-1}(ef)(A)
\]

with the natural domain. Since each function \( e(z) = (\lambda - z)^{-2} \) with \( \text{Im} \, \lambda > \omega \) is an instance, this shows uniqueness.

Now suppose that \( \| f(A) \| \leq C \| f \|_\infty \) for each \( f \in \mathcal{E}(S_\omega) \). We consider the extension \( \Phi \) to all of \( H^\infty (S_\omega) \) by regularisation. Let \( f \in H^\infty (S_\omega) \). Then for each \( n \in \mathbb{N} \) the function \( e_n(z) := e^{-(1/n)|z|^2} \) is elementary, hence also \( fe_n \) is, and thus

\[
\| (ef_n)(A) \| \leq C \| e_n \|_\infty \| f \|_\infty \leq C \| e_n \| \| f \|_\infty \leq Ce^{-\frac{1}{n}} \| f \|_\infty .
\]

On the other hand, it is easy to see that for \( x \) in the dense subspace \( \text{dom}(A^2) \) of \( X \), one has \( x = \text{dom}(f(A)) \) and \( (ef_n)(A)x = \Phi(f)x \). Hence \( \Phi(f) \) is bounded by \( C \| f \|_\infty \), and since it is closed and densely defined, \( \Phi(f) \in \mathcal{L}(X) \) and \( \| \Phi(f) \| \leq C \| f \|_\infty \).

\[
\square
\]

Sectorial Operators. A closed operator \( A \) with dense domain and dense range on a Banach space \( X \) is called sectorial of angle \( \alpha \in [0, \pi) \), if \( \sigma(A) \subseteq S_\alpha \) and if for all \( \beta \in (\alpha, \pi) \), the mapping \( z \mapsto zR(z, A) \) is uniformly bounded on \( \mathbb{C} \setminus S_\beta \).

One can set up a functional calculus for sectorial operators on sectors analogously to the strip case. Namely, \( f(A) \) is defined for an elementary function \( f \in \mathcal{E}(S_\omega) \) by means of \( (\mathcal{C}_3) \) with \( \partial S_\omega \) replaced by \( \partial S_\omega \). For a general \( f \in H^\infty (S_\omega) \), \( f(A) \) is defined by regularisation as described above. Then the sectorial analogue of Lemma \( \mathcal{C}_3 \) holds.

It turns out \( \mathcal{E}_3 \) Prop. 3.5.2] that each sectorial operator \( A \) of angle \( \alpha \) has a logarithm \( \log(A) \), which is of (strong) strip type \( \alpha \). The functional calculi of these operators are linked via the \text{exp/log}-correspondence, i.e., \( f(\log A) = (f \circ \log)(A) \) for all \( f \in H^\infty (S_\omega) \), see \( \mathcal{E}_3 \) Theorem 4.2.4. It is not true in general that every (strong) strip type operator is the logarithm of a sectorial one \( \mathcal{E}_3 \) Example 4.4.1. However, as long as one confines oneself to operators with bounded \( H^\infty \)-calculus, the correspondence is perfect \( \mathcal{E}_3 \) Prop. 5.5.3], and hence it suffices to consider in detail only one of these cases.

Appendix D. \( \ell_1 \)-frame-bounded sets

Let \( H \) be a Hilbert space. A sequence \( (f_\alpha)_{\alpha \in I} \) in \( H \) is called a frame for \( H \) if there exist two constants \( 0 < A < B \) such that

\[
A^2 \| h \|^2_H \leq \sum \| (h, f_\alpha) \|^2 \leq B^2 \| h \|^2_H \quad \text{for all } h \in H.
\]

Equivalently, a frame is given by a pair of operators \( (L, R) \) where \( R : H \to \ell_2(I) \) and \( L : \ell_2(I) \to H \) such that \( LR = \text{Id}_H \). Indeed, in that case \( f_\alpha := R^* e_\alpha, \alpha \in I, \) is a frame, where \( (e_\alpha)_{\alpha \in I} \) is the canonical basis of \( \ell_2(I) \). (One easily obtains \( \ell_1 \) with \( A = \| L \|^{-1} \) and \( B = \| R \| \).) Conversely, if \( (f_\alpha)_{\alpha \in I} \) is a frame and \( R : H \to \ell_2(I) \) is defined by \( Rf := ((f, f_\alpha))_{\alpha \in I} \), then \( R^* R \) is a selfadjoint, positive and invertible operator, and hence \( L := (R^* R)^{-1} R^* \) satisfies \( LR = \text{Id}_H \).
Let $H$ be a Hilbert space. A subset $M$ of $H$ is called $\ell_1$-frame-bounded if there exists a frame $(f_\alpha)_{\alpha \in I}$ of $H$ such that
\[
\sup_{x \in M} \sum_{\alpha \in I} |(x | f_\alpha)| < \infty,
\]
In this case, in virtue of the above discussion, the $\ell_1$-frame-bound of a subset $M \subseteq H$ is defined as
\[
|M|_1 := \inf \|L\| \sup_{x \in M} \sum_{\alpha \in I} |(Rx | e_\alpha)|
\]
where the infimum is taken over all pairs of operators $(L, R)$ with $R : H \to \ell_2(I)$ and $L : \ell_2(I) \to H$ such that $LR = I_H$. Let $X$ be a Banach space. An operator $T : X \to H$, called $\ell_1$-frame-bounded if $T$ maps the unit ball of $X$ into an $\ell_1$-frame-bounded subset of $X$. In this case,
\[
|T|_{\ell_1} := \{|Tx | \|x\|_X \leq 1\}_1
\]
is called the $\ell_1$-frame-bound of $T$.

Remarks D.1. 1) $\ell_1$-frame-bounded sets need not be compact.
2) Let $X, Y$ be Banach spaces. If $U : X \to H$ is $\ell_1$-frame-bounded and $V : Y \to X$ is bounded, then $UV : Y \to H$ is $\ell_1$-frame-bounded and
\[
|UV|_{\ell_1} \leq |U|_{\ell_1} \|V\|.
\]
3) We point out that we do not know yet whether finite unions or simple translates of $\ell_1$-frame-bounded sets are again $\ell_1$-frame-bounded, something one would certainly expect to hold for a “good” boundedness concept. Consequently, we do not know whether the set of $\ell_1$-frame-bounded operators $X \to H$ form a vector space.

Lemma D.2. Let $H$ be any Hilbert space and $M \subseteq H$. Then the following assertions hold.

a) If $M$ is $\ell_1$-frame-bounded, then it is norm-bounded, with
\[
\sup_{x \in M} \|x\| \leq |M|_1
\]
If span$(M)$ is finite-dimensional and $M$ is norm-bounded, then it is $\ell_1$-frame-bounded.

b) If $M$ is $\ell_1$-frame-bounded and $S : H \to K$ is an isomorphism into another Hilbert space, then $S(M)$ is $\ell_1$-frame-bounded with
\[
|S(M)|_1 \leq \|S\| |M|_1
\]

c) If $M$ is $\ell_1$-frame-bounded, then $\text{absconv}(M)$ is $\ell_1$-frame-bounded.

Proof. Parts a and b are clear. For the proof of c it suffices to notice that the closed unit ball of $\ell_1(I)$ is absolutely convex and closed in $\ell_2(I)$.

Remark D.3. Every $\ell_1$-frame-bounded operator $T : X \to H$ factorises through an $\ell_1$-space, but the converse is not true in general. Indeed, let $(f_n)_{n \in \mathbb{N}}$ be a countable dense subset of the unit sphere $\{f \in \ell_2 | \|f\|_2 = 1\}$ of $\ell_2$. Let $T : \ell_1 \to \ell_2$ be the operator defined by $T(x_n)_n := \sum_n x_n f_n$. Then the image under $T$ of the unit ball of $\ell_1$ is dense in the unit ball of $\ell_2$, and hence $T$ is not $\ell_1$-frame-bounded.

For operators between Hilbert spaces, the class of $\ell_1$-frame-bounded operators coincides with the class of Hilbert–Schmidt operators.

Lemma D.4. For an operator $T : K \to H$, $K$ and $H$ Hilbert spaces, the following assertions are equivalent:

(i) $T$ is $\ell_1$-frame-bounded.
implies (iii). Finally, if $T$ is Hilbert-Schmidt, the singular value decomposition yields a representation

$$T = \sum_n \tau_n f_n \otimes e_n$$

with orthonormal systems $(e_n)_n$ and $(f_n)_n$ and scalars $\tau = (\tau_n)_n \in \ell_2(\mathbb{N})$. We extend $(e_n)_n$ in some way to an orthonormal basis $(e_{n\alpha})_{n\alpha \in I}$ of $H$. Then

$$\sum_n |\langle Tf, e_n \rangle| = \sum_n |\langle Tf, e_{n\alpha} \rangle| = \sum_n |\tau_n| |\langle f, f_n \rangle| \leq \|\tau\|_{\ell_2} \|f\|_K$$

by the Cauchy-Schwarz and the Bessel inequalities. Hence $T$ is $\ell_1$-frame-bounded with $|T|_{\ell_1} \leq \|\tau\|_{\ell_2} = \|T\|_{HS}$. \qed

Let us provide some other examples of $\ell_1$-frame-bounded sets/operators.

**Examples D.5.**

1) The *Wiener algebra* $A(\mathbb{T})$ is the set of continuous functions on $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ that have absolutely summable Fourier coefficients. Obviously, the embedding $A(\mathbb{T}) \subseteq L_2(\mathbb{T})$ is $\ell_1$-frame-bounded.

2) As a consequence of the above item, every embedding into $L_2(\mathbb{T})$ that factors through the Wiener algebra is $\ell_1$-frame-bounded. This implies, e.g., that the embedding $C^1[0,1] \subseteq L_2[0,1]$ is $\ell_1$-frame-bounded for $s > 1/2$

3) The embeddings $B_{pq}^s[0,1] \subseteq L_2[0,1]$ and $W_p^s[0,1] \subseteq L_2[0,1]$ are $\ell_1$-frame-bounded whenever $s > 1/2$.

We do not know whether the continuous analogue of Example 1) is true, namely whether the embedding

$$A(\mathbb{R}) := \{f \in L_1(\mathbb{R}) \mid \hat{f} \in L_1(\mathbb{R})\} \subseteq L_2(\mathbb{R})$$

is $\ell_1$-frame-bounded. However, we have the following.

**Lemma D.6.** The canonical embedding $W_1^2(\mathbb{R}) \hookrightarrow L_2(\mathbb{R})$ is $\ell_1$-frame-bounded.

**Proof.** Fix a function $0 \leq \eta \in C^\infty(\mathbb{R})$ with supp$(\eta) \subseteq (\pi, \pi)$ and in such a way that with $\eta_k(t) := \eta(t - k)$ for $k \in \mathbb{Z}$ one has

$$1 = \sum_{k \in \mathbb{Z}} \eta_k.$$ 

The double sequence $(f_{n,k})_{(n,k) \in \mathbb{Z}^2}$ given by $f_{n,k} := \eta_k e^{in\tau}$, forms a Gabor frame on $L_2(\mathbb{R})$. Let $g \in W_1^2(\mathbb{R})$. For $n = 0$,

$$\sum_{k \in \mathbb{Z}} \left| \int_{-\pi}^{\pi} \eta_k(s) g(s) ds \right| \leq \|g\|_{L_1}.$$ 

For $n \neq 0$, a twofold integration by parts (with vanishing boundary terms) yields

$$\int_{-\pi}^{\pi} \eta_k(s) g(s) e^{-ins} ds = -\frac{1}{n} \int_{-\pi}^{\pi} [\eta_k(s) g(s)]' e^{-ins} ds.$$
Hence

\[ \sum_{n \in \mathbb{Z}^+} \left\{ f_n, g(s) \right\} \lesssim \|g\|_{W_1^\alpha}. \]

Using interpolation techniques one can see that \( W_1^\alpha(\mathbb{R}) \subseteq L_2(\mathbb{R}) \) is \( \ell_1 \)-frame-bounded for each \( \alpha > 1 \).

References


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