Abstract

In this article we apply a recently established transference principle in order to obtain the boundedness of certain functional calculi for semigroup generators. In particular, it is proved that if $-A$ generates a $C_0$-semigroup on a Hilbert space, then for each $\tau > 0$ the operator $A$ has a bounded calculus for the closed ideal of bounded holomorphic functions on a (sufficiently large) right half-plane that satisfy $f(z) = O(e^{-\tau \text{Re}(z)})$ as $|z| \to \infty$. The bound of this calculus grows at most logarithmically as $\tau \downarrow 0$. As a consequence, $f(A)$ is a bounded operator for each holomorphic function $f$ (on a right half-plane) with polynomial decay at $\infty$. Then we show that each semigroup generator has a so-called (strong) $m$-bounded calculus for all $m \in \mathbb{N}$, and that this property characterizes semigroup generators. Similar results are obtained if the underlying Banach space is a UMD space. Upon restriction to so-called $\gamma$-bounded semigroups, the Hilbert space results actually hold in general Banach spaces.

Keywords: Functional calculus, Transference, Operator semigroup, Fourier multiplier, $\gamma$-boundedness

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1. Introduction

Roughly speaking, a functional calculus for a (possibly unbounded) operator $A$ on a Banach space $X$ is a “method” of associating a closed operator $f(A)$ to each $f = f(z)$ taken from a set of functions (defined on some subset of the complex plane) in such a way that formulae valid for the functions turn into valid formulae for the operators upon replacing the independent variable $z$ by $A$. A common way to establish such a calculus is to start with an algebra
of “good” functions $f$ where a definition of $f(A)$ as a bounded operator is more or less straightforward, and then extend this “primary” or “elementary calculus” by means of multiplicative “regularization” (see [7, Chapter 1] and [3]). It is then natural to ask which of the so constructed closed operators $f(A)$ are actually 
*bounded*, a question particularly relevant in applications, e.g., to evolution equations, see for instance [1, 11].

The latter question links functional calculus theory to the theory of vector-valued singular integrals, best seen in the theory of sectorial (or strip-type) operators with a bounded $H^\infty$-calculus, see for instance [13]. It appears there that in order to obtain nontrivial results the underlying Banach space must allow for singular integrals to converge, i.e., be a UMD space (or better, a Hilbert space). Furthermore, even if the Banach space is a Hilbert space, it turns out that simple resolvent estimates are not enough for the boundedness of an $H^\infty$-calculus [7, Section 9.1].

However, some of the central positive results in that theory — McIntosh’s theorem [15], the Boyadzhiev–deLaubenfels theorem [4] and the Hieber–Prüss theorem [10] — show that the presence of a $C_0$-group of operators does warrant the boundedness of certain $H^\infty$-calculi. In [8] the underlying structure of these results was brought to light, namely a *transference principle*, a factorization of the operators $f(A)$ in terms of vector-valued Fourier multiplier operators. Finally, in [9] it was shown that $C_0$-semigroups also allow for such transference principles.

In the present paper, we develop this approach further. We apply the general form of the transference principle for semigroups given in [9] in order to obtain bounded functional calculi for generators of $C_0$-semigroups. These results, in particular Theorems 3.3, 3.7, and 4.3, are proved for general Banach spaces. However, they make use of (subalgebras of) the analytic $L^p(\mathbb{R};X)$-Fourier multiplier algebra (see (2.1) below for a definition), and hence are useful only if the underlying Banach space has a geometry that allows for nontrivial Fourier multiplier operators. In case $X = H$ is a Hilbert space one obtains particularly nice results, which we want to summarize here. (See Section 4 for the definition of a strong $m$-bounded calculus.)

**Theorem 1.1.** Let $-A$ be the generator of a bounded $C_0$-semigroup $(T(t))_{t \in \mathbb{R}_+}$ on a Hilbert space $H$ with $M := \sup_{t \in \mathbb{R}_+} \|T(t)\|$. Then the following assertions hold.

a) For $\omega < 0$ and $f \in H^\infty(\mathbb{R}_\omega)$ one has $f(A)T(\tau) \in \mathcal{L}(H)$ with

$$\|f(A)T(\tau)\| \leq c(\tau)M^2 \|f\|_{H^\infty(\mathbb{R}_\omega)},$$

(1.1)

where $c(\tau) = O(\|\log(\tau)\|)$ as $\tau \searrow 0$, and $c(\tau) = O(1)$ as $\tau \to \infty$.

b) For $\omega < 0 < \alpha$ and $\lambda \in \mathbb{C}$ with $\text{Re} \lambda < 0$ there is $C \geq 0$ such that

$$\|f(A)(A - \lambda)^{-\alpha}\| \leq CM^2 \|f\|_{H^\infty(\mathbb{R}_\omega)}$$

(1.2)

for all $f \in H^\infty(\mathbb{R}_\omega)$. In particular, $\text{dom}(A^\alpha) \subseteq \text{dom}(f(A))$.  

2
c) A has a strong $m$-bounded $\mathcal{H}^\infty$-calculus of type 0 for each $m \in \mathbb{N}$.

(See Corollary 3.10 for a) and b) and Corollary 4.4 for c).)

When $X$ is a UMD space one can derive similar results, stated in Section 5. In Section 6 we extend the Hilbert space results to general Banach spaces by replacing the assumption of boundedness of the semigroup by its $\gamma$-boundedness, a concept strongly put forward by Kalton and Weis [12]. In particular, Theorem 1.1 holds true for $\gamma$-bounded semigroups on arbitrary Banach spaces with $M$ being the $\gamma$-bound of the semigroup.

We stress the fact that in contrast to [7], where sectorial operators and, accordingly, functional calculi on sectors, were considered, the present article deals with general semigroup generators and with functional calculi on half-planes. (See Section 2.2 below.) The abstract theory of (holomorphic) functional calculi on half-planes can be found in [3] where the notion of an $m$-bounded calculus (for operators of half-plane type) has been introduced. Our Theorem 1.1.c) is basically contained in that paper (it follows directly from [3, Cor. 6.5 and (7.1)]).

The starting point of the present work was the article [19] by Hans Zwart, in particular [19, Theorem 2.5, 2.]. There it is shown that one has an estimate (1.1) with $c(\tau) = O(t^{-1/2})$ as $\tau \searrow 0$. (The case $\alpha > 1/2$ in (1.2) is an immediate consequence; however, that case is essentially trivial, see Lemma 2.4 below.)

In [19] and its sequel paper [17] the functional calculus for a semigroup generator is constructed in a rather unconventional way using ideas from systems theory. However, a closer inspection reveals that transference (i.e., the factorization over a Fourier multiplier) is present there as well, hidden in the very construction of the functional calculus.

Notation and terminology

We write $\mathbb{N} := \{1, 2, \ldots \}$ for the natural numbers and $\mathbb{R}_+ := [0, \infty)$ for the nonnegative reals. The letters $X$ and $Y$ are used to denote Banach spaces over the complex number field. The space of bounded linear operators on $X$ is denoted by $\mathcal{L}(X)$. For a closed operator $A$ on $X$ its domain is denoted by $\text{dom}(A)$ and its range by $\text{ran}(A)$. The spectrum of $A$ is $\sigma(A)$ and the resolvent set $\rho(A) := \mathbb{C} \setminus \sigma(A)$. For all $z \in \rho(A)$ the operator $R(z, A) := (z - A)^{-1} \in \mathcal{L}(X)$ is the resolvent of $A$ at $z$.

For $p \in [1, \infty]$, $L^p(\mathbb{R}; X)$ is the Bochner space of equivalence classes of $X$-valued $p$-Lebesgue integrable functions on $\mathbb{R}$. The Hölder conjugate of $p$ is $p'$, defined by $\frac{1}{p} + \frac{1}{p'} = 1$. The norm on $L^p(\mathbb{R}; X)$ is usually denoted by $\|\cdot\|_p$.

For $\omega \in \mathbb{R}$ and $z \in \mathbb{C}$ we let $e^{\omega}(z) := e^{\omega z}$. By $M(\mathbb{R})$ (resp. $M(\mathbb{R}_+)$) we denote the space of complex-valued Borel measures on $\mathbb{R}$ (resp. $\mathbb{R}_+$) with the total variation norm, and we write $M_\omega(\mathbb{R}_+)$ for the distributions $\mu$ on $\mathbb{R}_+$ of the form $\mu(ds) = e^{\omega s} \nu(ds)$ for some $\nu \in M(\mathbb{R}_+)$. Then $M_\omega(\mathbb{R}_+)$ is a Banach algebra under convolution with the norm

$$
\|\mu\|_{M_\omega(\mathbb{R}_+)} := \|e^{-\omega}\mu\|_{M(\mathbb{R}_+)}.
$$

3
For $\mu \in M_\omega(\mathbb{R}_+)$ we let supp$(\mu)$ be the topological support of $e^{-\omega \mu}$. A function $g$ such that $e^{-\omega g} \in L^1(\mathbb{R}_+)$ is usually identified with its associated measure $\mu \in M_\omega(\mathbb{R}_+)$ given by $\mu(ds) = g(s)ds$. Functions and measures defined on $\mathbb{R}_+$ are identified with their extensions to $\mathbb{R}$ by setting them equal to zero outside $\mathbb{R}_+$.

For an open subset $\Omega \neq \emptyset$ of $\mathbb{C}$ we let $H^\infty(\Omega)$ be the space of bounded holomorphic functions on $\Omega$, a unital Banach algebra with respect to the norm

$$\|f\|_\infty := \|f\|_{H^\infty(\Omega)} := \sup_{z \in \Omega} |f(z)| \quad (f \in H^\infty(\Omega)).$$

We shall mainly consider the case where $\Omega$ is equal to a right half-plane

$$\mathbb{R}_\omega := \{ z \in \mathbb{C} \mid \text{Re}(z) > \omega \}$$

for some $\omega \in \mathbb{R}$ (we write $\mathbb{C}_+$ for $\mathbb{R}_0$).

For convenience we abbreviate the coordinate function $z \mapsto z$ simply by the letter $z$. Under this convention, $f = f(z)$ for a function $f$ defined on some domain $\Omega \subseteq \mathbb{C}$.

The Fourier transform of an $X$-valued tempered distribution $\Phi$ on $\mathbb{R}$ is denoted by $\mathcal{F}\Phi$. For instance, if $\mu \in M(\mathbb{R})$ then $\mathcal{F}\mu \in L^\infty(\mathbb{R})$ is given by

$$\mathcal{F}\mu(\xi) := \int_{\mathbb{R}} e^{-i\xi s} \mu(ds) \quad (\xi \in \mathbb{R}).$$

For $\omega \in \mathbb{R}$ and $\mu \in M_\omega(\mathbb{R}_+)$ we let $\hat{\mu} \in H^\infty(\mathbb{R}_\omega) \cap C(\mathbb{R}_\omega)$,

$$\hat{\mu}(z) := \int_0^\infty e^{-zs} \mu(ds) \quad (z \in \mathbb{R}_\omega),$$

be the Laplace-Stieltjes transform of $\mu$.

### 2. Fourier multipliers and functional calculus

We briefly discuss some of the concepts that will be used in what follows.

#### 2.1. Fourier multipliers

We shall need results from Fourier analysis as collected in [7, Appendix E]. Fix a Banach space $X$ and let $m \in L^\infty(\mathbb{R}; L^p(\mathbb{R}; X))$ and $p \in [1, \infty]$. Then $m$ is a bounded $L^p(\mathbb{R}; X)$-Fourier multiplier if there exists $C \geq 0$ such that

$$T_m(\varphi) := \mathcal{F}^{-1}(m \cdot \mathcal{F}\varphi) \in L^p(\mathbb{R}; X) \quad \text{and} \quad \|T_m(\varphi)\|_p \leq C \|\varphi\|_p$$

for each $X$-valued Schwartz function $\varphi$. In this case the mapping $T_m$ extends uniquely to a bounded operator on $L^p(\mathbb{R}; X)$ if $p < \infty$ and on $C_0(\mathbb{R}; X)$ if $p = \infty$. We let $\|m\|_{M_p(X)}$ be the norm of the operator $T_m$ and let $M_p(X)$ be the unital Banach algebra of all bounded $L^p(\mathbb{R}; X)$-Fourier multipliers, endowed with the norm $\|\cdot\|_{M_p(X)}$. 


For \( \omega \in \mathbb{R} \) and \( p \in [1, \infty] \) we let
\[
\text{AM}_p^X(R_\omega) := \{ f \in H^\infty(R_\omega) \mid f(\omega + i \cdot) \in \mathcal{M}_p(X) \} \tag{2.1}
\]
be the \textit{analytic} \( L^p(\mathbb{R}; X) \)-\textit{Fourier multiplier algebra} on \( R_\omega \), endowed the norm
\[
\|f\|_{\text{AM}_p^X} := \|f\|_{\text{AM}_p^X(R_\omega)} := \|f(\omega + i \cdot)\|_{\mathcal{M}_p(X)}.
\]
Here \( f(\omega + i \cdot) \in L^\infty(\mathbb{R}) \) denotes the \textit{trace} of the holomorphic function \( f \) on the boundary \( \partial R_\omega = \omega + i \mathbb{R} \). By classical Hardy space theory,
\[
f(\omega + is) := \lim_{\omega' \searrow \omega} f(\omega' + is) \tag{2.2}
\]
exists for almost all \( s \in \mathbb{R} \), with \( \|f(\omega + i \cdot)\|_{L^\infty(\mathbb{R})} = \|f\|_{H^\infty(R_\omega)} \).

**Remark 2.1** (Important!). To simplify notation we sometimes omit the reference to the Banach space \( X \) and write \( \text{AM}_p(R_\omega) \) instead of \( \text{AM}_p^X(R_\omega) \) whenever it is convenient.

The space \( \text{AM}_p^X(R_\omega) \) is a unital Banach algebra, contractively embedded in \( H^\infty(R_\omega) \), and \( \text{AM}_p^X(R_\omega) = \text{AM}_p^X(R_\omega) \) is contractively embedded in \( \text{AM}_p(R_\omega) \) for all \( p \in (1, \infty) \), cf. [7, p. 347].

For our main results we need two lemmas about the analytic multiplier algebra.

**Lemma 2.2.** For every Banach space \( X \), all \( \omega \in \mathbb{R} \) and \( p \in [1, \infty] \),
\[
\text{AM}_p^X(R_\omega) = \left\{ f \in H^\infty(R_\omega) \mid \sup_{\omega' > \omega} \|f(\omega' + i \cdot)\|_{\mathcal{M}_p(X)} < \infty \right\}
\]
with \( \|f\|_{\text{AM}_p^X(R_\omega)} = \sup_{\omega' > \omega} \|f(\omega' + i \cdot)\|_{\mathcal{M}_p(X)} \) for all \( f \in \text{AM}_p^X(R_\omega) \).

**Proof.** Let \( \omega \in \mathbb{R} \), \( p \in [1, \infty] \) and \( f \in \text{AM}_p(R_\omega) \). For all \( \omega' > \omega \) and \( s \in \mathbb{R} \),
\[
f(\omega' + is) = \frac{\omega' - \omega}{\pi} \int_{\mathbb{R}} \frac{f(\omega - ir)}{(s - r)^2 + (\omega' - \omega)^2} \, dr
\]
by [16, Theorem 5.18]. The right-hand side is the convolution of \( f(\omega - i \cdot) \) and the Poisson kernel \( P_{\omega' - \omega}(r) := \frac{\omega' - \omega}{(r + (\omega' - \omega))^2} \). Since \( \|P_{\omega' - \omega}\|_{L^1(\mathbb{R})} = 1 \),
\[
\|f(\omega' + i \cdot)\|_{\mathcal{M}_p(X)} \leq \|f(\omega - i \cdot)\|_{\mathcal{M}_p(X)} = \|f\|_{\text{AM}_p^X(R_\omega)}.
\]
The converse follows from (2.2) and [7, Lemma E.4.1]. \( \square \)

For \( \mu \in M(\mathbb{R}) \) and \( p \in [1, \infty] \) we let \( L_\mu \in \mathcal{L}(L^p(\mathbb{R}; X)) \),
\[
L_\mu(f) := \mu * f \quad (f \in L^p(\mathbb{R}; X)), \tag{2.3}
\]
be the convolution operator associated to \( \mu \).
Lemma 2.3. For each $\omega \in \mathbb{R}$ the Laplace transform induces an isometric algebra isomorphism from $M_\omega(\mathbb{R}_+)$ onto $AM_1^\omega(\mathbb{R}_+) = AM_1^\omega(\mathbb{R}_+)$. Moreover,

$$\|\hat{\mu}\|_{AM_1^\omega(\mathbb{R}_+)} = \|L_{e^{-\omega \cdot}}\|_{L^p(X)}$$

for all $\mu \in M_\omega(\mathbb{R}_+)$, $p \in [1, \infty]$.

Proof. The mappings $\mu \mapsto e^{-\omega \cdot} \mu$ and $f \mapsto f(\cdot + \omega)$ are isometric algebra isomorphisms $M_\omega(\mathbb{R}_+) \to M(\mathbb{R}_+)$ and $AM_p(\mathbb{R}_+) \to AM_p(\mathbb{C}_+)$, respectively. Hence it suffices to let $\omega = 0$. The Fourier transform induces an isometric isomorphism $M(\mathbb{R})$ onto $M_1(\mathbb{R})$ [7, p.347, 8]. If $\mu \in M(\mathbb{R}_+)$ and $f = \mu \in H^\infty(\mathbb{C}_+)$ then $f(i) = F\mu \in M_1(\mathbb{X})$ with $\|f(i)\|_{M_1(\mathbb{X})} = \|\mu\|_{M(\mathbb{R}_+)}$. Moreover, for $p \in [1, \infty]$,

$$\|f(i)\|_{M_1(\mathbb{X})} = \sup_{\|g\|_p \leq 1} \|F^{-1}(f(i) \cdot)\|_p = \sup_{\|g\|_p \leq 1} \|\mu \cdot g\|_p = \|L_\mu\|_{L^p(\mathbb{X})}$$

If $f \in AM_1(\mathbb{C}_+)$ then $f(i) = F\mu$ for some $\mu \in M(\mathbb{R})$. An application of Liouville’s theorem shows that $\text{supp}(\mu) \subseteq \mathbb{R}_+$, hence $f = \hat{\mu}$. $\square$

2.2. Functional calculus

We assume that the reader is familiar with the basic notions and results of the theory of $C_0$-semigroups as developed, e.g., in [5], and just recall some facts which will be needed in this article.

Each $C_0$-semigroup $T = (T(t))_{t \in \mathbb{R}_+}$ on a Banach space $X$ has type $(M, \omega)$ for some $M \geq 1$ and $\omega \in \mathbb{R}$, which means that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. The generator of $T$ is the unique closed operator $-A$ such that

$$(\lambda + A)^{-1} x = \int_0^\infty e^{-\lambda t}T(t)x \, dt \quad (x \in X)$$

for $\text{Re}(\lambda)$ large. The Hille-Phillips (functional) calculus for $A$ is defined as follows. Fix $M \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $T$ has type $(M, -\omega_0)$. For $\mu \in M_{\omega_0}(\mathbb{R}_+)$ define $T_\mu \in \mathcal{L}(X)$ by

$$T_\mu x := \int_0^\infty T(t)x \mu(dt) \quad (x \in X).$$

(2.4)

For $f = \hat{\mu} \in AM(\mathbb{R}_{\omega_0})$ set $f(A) := T_\mu$. (This is allowed by the injectivity of the Laplace transform, see Lemma 2.3.) The mapping $f \mapsto f(A)$ is an algebra homomorphism. In a second step the definition of $f(A)$ is extended to a larger class of functions via regularization, i.e.,

$$f(A) := e(A)^{-1}(ef)(A)$$

if there exists $e \in AM(\mathbb{R}_{\omega_0})$ such that $e(A)$ is injective and $ef \in AM(\mathbb{R}_{\omega_0})$. Then $f(A)$ is a closed and (in general) unbounded operator on $X$ and the definition of $f(A)$ is independent of the choice of regularizer $e$. The following lemma shows in particular that for $\omega < \omega_0$ the operator $f(A)$ is defined for all $f \in H^\infty(\mathbb{R}_0)$ by virtue of the regularizer $e(z) = (z - \lambda)^{-1}$, where $\text{Re}(\lambda) < \omega$. 

6
Lemma 2.4. Let $\alpha > \frac{1}{2}$, $\lambda \in \mathbb{C}$ and $\omega, \omega_0 \in \mathbb{R}$ with $\text{Re}(\lambda) < \omega < \omega_0$. Then

$$f(z)(z - \lambda)^{-\alpha} \in \text{AM}_{\omega}(\mathbb{R})$$

for all $f \in H^\infty(\mathbb{R})$.

Proof. After shifting we may suppose that $\omega = 0$. Set $h(z) := f(z)(z - \lambda)^{-\alpha}$ for $z \in \mathbb{C}_+$. Then $h(i\cdot) \in L^2(\mathbb{R})$ with

$$\|h(i\cdot)\|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}} \frac{|f(is)|^2}{|is - \lambda|^{2\alpha}} \, ds \leq \|f\|_{H^\infty(\mathbb{C}_+)}^2 \int_{\mathbb{R}} \frac{1}{|is - \lambda|^{2\alpha}} \, ds,$$

hence the Paley-Wiener Theorem [16, Theorem 5.28] implies that $h = \hat{g}$ for some $g \in L^2(\mathbb{R}_+)$. Then $e^{-\omega_0}g \in L^1(\mathbb{R}_+)$ and $\tilde{e}^{-\omega_0}\hat{g}(z) = h(z + \omega_0)$ for $z \in \mathbb{C}_+$. Lemma 2.3 yields $h \in \text{AM}_{\omega}(\mathbb{R}_0)$ with

$$\|h\|_{\text{AM}_{\omega}(\mathbb{R}_0)} = \|h(\cdot + \omega_0)\|_{\text{AM}_{\omega}(\mathbb{C}_+)} = \|e^{-\omega_0}g\|_{L^1(\mathbb{R}_+)}.$$

The Hille-Phillips calculus is an extension of the holomorphic functional calculus for the operators of half-plane type discussed in [3]. An operator $A$ is of half-plane type $\omega_0 \in \mathbb{R}$ if $\sigma(A) \subseteq \mathbb{R}_{\omega_0}$ with

$$\sup_{\lambda \in \mathbb{C} \setminus \mathbb{R}_{\omega}} \|R(\lambda, A)\| < \infty \quad \text{for all } \omega < \omega_0.$$

One can associate operators $f(A) \in \mathcal{L}(X)$ to certain elementary functions via Cauchy integrals and regularize as above to extend the definition to all $f \in H^\infty(\mathbb{R})$. If $-A$ generates a $C_0$-semigroup of type $(M, -\omega_0)$ then $A$ is of half-plane type $\omega_0$, and by combining [3, Proposition 2.8] and [7, Proposition 3.3.2] one sees that for $\omega < \omega_0$ and $f \in H^\infty(\mathbb{R})$ the definitions of $f(A)$ via the Hille-Phillips calculus and the half-plane calculus coincide.

For a proof of the next, fundamental, lemma see [3, Theorem 3.1].

Lemma 2.5 (Convergence Lemma). Let $A$ be a densely defined operator of half-plane type $\omega_0 \in \mathbb{R}$ on a Banach space $X$. Let $\omega < \omega_0$ and $(f_j)_{j \in J} \subseteq H^\infty(\mathbb{R})$ be a net satisfying the following conditions:

1) $\sup \{ |f_j(z)| \mid z \in \mathbb{R}_{\omega}, j \in J \} < \infty$;
2) $f_j(A) \in \mathcal{L}(X)$ for all $j \in J$ and $\sup_{j \in J} \|f_j(A)\| < \infty$;
3) $f(z) := \lim_{j \in J} f_j(z)$ exists for all $z \in \mathbb{R}_{\omega}$.

Then $f \in H^\infty(\mathbb{R}_{\omega})$, $f(A) \in \mathcal{L}(X)$, $f_j(A) \to f(A)$ strongly and

$$\|f(A)\| \leq \limsup_{j \in J} \|f_j(A)\|.$$

Let $A$ be an operator of half-plane type $\omega_0$ and $\omega < \omega_0$. For a Banach algebra $F$ of functions continuously embedded in $H^\infty(\mathbb{R})$, we say that $A$ has a bounded $F$-calculus if there exists a constant $C \geq 0$ such that $f(A) \in \mathcal{L}(X)$ with

$$\|f(A)\|_{\mathcal{L}(X)} \leq C \|f\|_F \quad \text{for all } f \in F. \quad (2.5)$$
The operator \(-A\) generates a \(C_0\)-semigroup \((T(t))_{t \in \mathbb{R}_+}\) of type \((M, \omega)\) if and only if \(-(A + \omega)\) generates the semigroup \((e^{-\omega t}T(t))_{t \in \mathbb{R}_+}\) of type \((M, 0)\). The functional calculi for \(A\) and \(A + \omega\) are linked by the simple composition rule "\(f(A + \omega) = f(\omega + z)(A)\)" [7, Theorem 2.4.1]. Henceforth we shall mainly consider bounded semigroups; all results carry over to general semigroups by shifting.

3. Functional calculus for semigroup generators

Define the function \(\eta : (0, \infty) \times (0, \infty) \times [1, \infty] \to \mathbb{R}_+\) by

\[
\eta(\alpha, t, q) := \inf \left\{ \|\psi\|_q \|\varphi\|_{q'} \mid \psi * \varphi \equiv e^{-\alpha} \text{ on } [t, \infty) \right\}.
\] (3.1)

The set on the right-hand side is not empty: choose for instance \(\psi := 1_{[0, t]}e^{-\alpha}\) and \(\varphi := \frac{1}{t}e^{-\alpha}\). By Appendix A.1,

\[\eta(\alpha, t, q) = O(\|\log(at)\|) \quad \text{as } at \to 0,\]

for \(q \in (1, \infty)\).

For the following result recall the definitions of the operators \(L_\mu\) from (2.3) and \(T_\mu\) from (2.4).

**Proposition 3.1.** Let \((T(t))_{t \in \mathbb{R}_+}\) be a \(C_0\)-semigroup of type \((M, 0)\) on a Banach space \(X\). Let \(p \in [1, \infty], \tau, \omega > 0\) and \(\mu \in M_\omega(\mathbb{R}_+)\) with \(\text{supp}(\mu) \subseteq [\tau, \infty)\). Then

\[
\|T_\mu\|_{\mathcal{L}(X)} \leq M^2\eta(\omega, \tau, p) \|L_{e^{-\omega}}\|_{\mathcal{L}(L^p(X))} \|\varphi\|_{p'}.
\] (3.2)

**Proof.** We can factorize \(T_\mu\) as \(T_\mu = P \circ L_{e^{-\omega}} \circ \iota\), where

- \(\iota : X \to L^p(\mathbb{R}; X)\) is given by

\[
\iota(x)(s) := \begin{cases} 
\psi(-s)T(-s)x & \text{if } s \leq 0 \\
0 & \text{if } s > 0
\end{cases} \quad (x \in X).
\]

- \(P : L^p(\mathbb{R}; X) \to X\) is given by

\[
Pf := \int_0^\infty \varphi(t)T(t)f(t)\, dt \quad (f \in L^p(\mathbb{R}; X)).
\]

- \(\psi \in L^p(\mathbb{R}_+)\) and \(\varphi \in L^{p'}(\mathbb{R}_+)\) are such that \(\psi * \varphi \equiv e^{-\omega}\) on \([\tau, \infty)\).

This is deduced as in the transference principle from [9, Section 2], using that \(\mu = (\psi * \varphi)e_{\omega}\mu\). Hölder’s inequality then implies

\[
\|T_\mu\| \leq M^2 \|\psi\|_p \|L_{e^{-\omega}}\|_{\mathcal{L}(L^p(X))} \|\varphi\|_{p'},
\]

and taking the infimum over all such \(\psi\) and \(\varphi\) yields (3.2). \(\square\)
Now define, for a Banach space $X$, $\omega \in \mathbb{R}$, $p \in [1, \infty]$ and $\tau > 0$, the space

$$\text{AM}^X_p(\mathbb{R}_\omega) := \left\{ f \in \text{AM}_p^X(\mathbb{R}_\omega) \mid f(z) = O(e^{-\tau \text{Re}(z)}) \text{ as } |z| \to \infty \right\},$$

endowed with the norm of $\text{AM}^X_p(\mathbb{R}_\omega)$.

**Lemma 3.2.** For every Banach space $X$, $\omega \in \mathbb{R}$, $p \in [1, \infty]$ and $\tau > 0$

$$\text{AM}^X_p(\mathbb{R}_\omega) = \text{AM}^X_p(\mathbb{R}_\omega) \cap e^{-\tau}H^\infty(\mathbb{R}_\omega) = e^{-\tau}\text{AM}^X_p(\mathbb{R}_\omega). \quad (3.3)$$

In particular, $\text{AM}^X_p(\mathbb{R}_\omega)$ is a closed ideal in $\text{AM}^X_p(\mathbb{R}_\omega)$.

**Proof.** The first equality in (3.3) is clear, and so is the inclusion $e^{-\tau}\text{AM}^X_p(\mathbb{R}_\omega) \subseteq \text{AM}^X_{p,\tau}(\mathbb{R}_\omega)$. Conversely, if $f \in \text{AM}^X_p(\mathbb{R}_\omega) \cap e^{-\tau}H^\infty(\mathbb{R}_\omega)$ then $e_\tau f \in \text{AM}^X_p(\mathbb{R}_\omega)$ since

$$\|e^{\tau(\omega+i)}f(\omega+i)\|_{\mathcal{M}_p(X)} = e^{\tau\omega}\|f(\omega+i)\|_{\mathcal{M}_p(X)}.$$

Now suppose that $(f_n)_{n \in \mathbb{N}} \subseteq \text{AM}^X_{p,\tau}(\mathbb{R}_\omega)$ converges to $f \in \text{AM}^X_p(\mathbb{R}_\omega)$. The Maximum Principle implies $\|e_\tau f_n\|_{H^\infty(\mathbb{R}_\omega)} = e^{\tau\omega}\|f_n\|_{H^\infty(\mathbb{R}_\omega)}$, hence $(e_\tau f_n)_{n \in \mathbb{N}}$ is Cauchy in $H^\infty(\mathbb{R}_\omega)$. Since it converges pointwise to $e_\tau f$, (3.3) implies $f \in \text{AM}^X_{p,\tau}(\mathbb{R}_\omega)$. \hfill \Box

We are now ready to prove the main result of this section. Note that the union of the ideals $\text{AM}^X_{p,\tau}(\mathbb{R}_\omega)$ for $\tau > 0$ is dense in $\text{AM}^X_p(\mathbb{R}_\omega)$ with respect to pointwise and bounded convergence of sequences. If there were a single constant independent of $\tau$ bounding the $\text{AM}^X_{p,\tau}(\mathbb{R}_\omega)$-calculus for all $\tau$, the Convergence Lemma would imply that $A$ has a bounded $\text{AM}^X_p(\mathbb{R}_\omega)$-calculus, but this is known to be false in general [7, Corollary 9.1.8].

**Theorem 3.3.** For each $p \in (1, \infty)$ there exists a constant $c_p \geq 0$ such that the following holds. Let $A$ generate a $C_0$-semigroup $(T(t))_{t \in \mathbb{R}_+}$ of type $(M,0)$ on a Banach space $X$ and let $\tau, \omega > 0$. Then $f(A) \in \mathcal{L}(X)$ and

$$\|f(A)\| \leq \begin{cases} c_p M^2 \log(\tau) \|f\|_{\text{AM}^X_p} & \text{if } \omega \tau \leq \min\left(\frac{1}{\omega}, \frac{1}{\tau}\right), \\ 2M^2 e^{-\omega \tau} \|f\|_{\text{AM}^X_p} & \text{if } \omega \tau > \min\left(\frac{1}{\omega}, \frac{1}{\tau}\right) \end{cases}$$

for all $f \in \text{AM}^X_{p,\tau}(\mathbb{R}_\omega)$. In particular, $A$ has a bounded $\text{AM}^X_{p,\tau}(\mathbb{R}_\omega)$-calculus.

**Proof.** First consider $f \in \text{AM}^X_{1,\tau}(\mathbb{R}_\omega)$. Let $\delta_\tau \in M_{-\omega}(\mathbb{R}_+) = \text{the unit point mass at } \tau$. By Lemmas 3.2 and 2.3 there exists $\mu \in M_{-\omega}(\mathbb{R}_+)$ such that $f = e_{-\tau}\mu = \delta_\tau * \mu$. Since $\delta_\tau * \mu \in M_{-\omega}(\mathbb{R}_+)$ with $\text{supp}(\delta_\tau * \mu) \subseteq [\tau, \infty)$, Proposition 3.1 and Lemma 2.3 yield

$$\|f(A)\| \leq M^2 \eta(\omega, \tau, p) \|f\|_{\text{AM}^X_p}. \quad (3.4)$$
Now suppose \( f \in \text{AM}_{p,\tau}(\mathbb{R}-\omega) \) is arbitrary. For \( \epsilon > 0 \), \( k \in \mathbb{N} \) and \( z \in \mathbb{R}-\omega \) set \( g_k(z) := \frac{k}{z-\omega} \) and \( f_{k,\epsilon}(z) := f(z + \epsilon)g_k(z + \epsilon) \). Lemma 2.4 yields \( f_{k,\epsilon} \in \text{AM}_{1,\tau}(\mathbb{R}-\omega) \), hence, by what we have already shown,

\[ \|f_{k,\epsilon}(A)\| \leq M^2\eta(\omega, \tau, p) \|f_{k,\epsilon}\|_{\text{AM}_p^X} . \]

The inclusion \( \text{AM}_1(\mathbb{R}-\omega) \subseteq \text{AM}_p(\mathbb{R}-\omega) \) is contractive, so Lemma 2.3 implies that \( g_k \in \text{AM}_p(\mathbb{R}-\omega) \) with

\[ \|g_k\|_{\text{AM}_p} \leq \|g_k\|_{\text{AM}_1} = k\|e_{-k}\|_{L^1(\mathbb{R}_+)} = 1. \]

Combining this with Lemma 2.2 yields

\[ \|f_{k,\epsilon}\|_{\text{AM}_p^X} \leq \|f(\cdot + \epsilon)\|_{\text{AM}_p^X} \|g_k(\cdot + \epsilon)\|_{\text{AM}_p} \leq \|f\|_{\text{AM}_p^X}. \]

In particular, \( \sup_{k,\epsilon} \|f_{k,\epsilon}\|_{\infty} < \infty \) and \( \sup_{k,\epsilon} \|f_{k,\epsilon}(A)\| < \infty \). The Convergence Lemma 2.5 implies that \( f(A) \in \mathcal{L}(X) \) satisfies (3.4). Appendix A.1 concludes the proof.

\[ \square \]

**Remark 3.4.** Because \( \text{AM}_1(\mathbb{R}-\omega) = \text{AM}_\infty(\mathbb{R}-\omega) \) is contractively embedded in \( \text{AM}_p(\mathbb{R}-\omega) \), Theorem 3.3 also holds for \( p = 1 \) and \( p = \infty \). However, \( A \) trivially has a bounded \( \text{AM}_1 \)-calculus by Lemma 2.3 and the Hille-Phillips calculus.

Note that the exponential decay of \( |f(z)| \) is only required as the real part of \( z \) tends to infinity. If \( |f(z)| \) decays exponentially as \( |z| \to \infty \) the result is not interesting, by Lemma 2.4.

We can equivalently formulate Theorem 3.3 as a statement about composition with semigroup operators.

**Corollary 3.5.** Under the assumptions of Theorem 3.3, \( f(A)T(\tau) \in \mathcal{L}(X) \) and

\[
\|f(A)T(\tau)\| \leq \begin{cases} 
 c_p M^2 \log(\omega \tau) e^{\omega \tau} \|f\|_{\text{AM}_p^X} & \text{if } \omega \tau \leq \min\left\{ \frac{1}{p}, \frac{1}{p'} \right\}, \\
 2M^2 \|f\|_{\text{AM}_p^X} & \text{if } \omega \tau > \min\left\{ \frac{1}{p}, \frac{1}{p'} \right\}
\end{cases}
\]

for all \( f \in \text{AM}_p^X(\mathbb{R}-\omega) \).

**Proof.** Note that \( f(A)T(\tau) = (e_{-\tau}f)(A) \) and \( \|e_{-\tau}f\|_{\text{AM}_p^X} = e^{\omega \tau} \|f\|_{\text{AM}_p^X} \). \[ \square \]

**Additional results**

As a first corollary of Theorem 3.3 we obtain a sufficient condition for a semigroup generator to have a bounded \( \text{AM}_p \)-calculus.

**Corollary 3.6.** Let \(-A\) generate a bounded \( C_0\)-semigroup \( (T(t))_{t \in \mathbb{R}_+} \subseteq \mathcal{L}(X) \) with

\[ \bigcup_{\tau > 0} \text{ran}(T(\tau)) = X. \]

Then \( A \) has a bounded \( \text{AM}_p^X(\mathbb{R}_-) \)-calculus for all \( \omega < 0 \), \( p \in [1, \infty] \).
Proof. Using Corollary 3.5, note that \( f(A)T(\tau) \in \mathcal{L}(X) \) implies \( \text{ran}(T(\tau)) \subseteq \text{dom}(f(A)) \). An application of the Closed Graph Theorem (using the Convergence Lemma) yields (2.5).

**Theorem 3.7.** Let \( p \in (1, \infty) \), \( \omega > 0 \) and \( \alpha, \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) < 0 < \text{Re}(\alpha) \). There exists a constant \( C = C(p, \alpha, \lambda, \omega) \geq 0 \) such that the following holds. Let \( -A \) generate a \( C_0 \)-semigroup \( (T(t))_{t \in \mathbb{R}_+} \) of type \((M,0)\) on a Banach space \( X \). Then \( \text{dom}((A - \lambda)\alpha) \subseteq \text{dom}(f(A)) \) and

\[
\left\| f(A)(A - \lambda)^{-\alpha} \right\| \leq CM^2 \| f \|_{\text{AM}_p^X}
\]

for all \( f \in \text{AM}_p^X((\mathbb{R} - \omega)) \).

**Proof.** First note that \( -(A - \lambda) \) generates the exponentially stable semigroup \( (e^{\lambda t}T(t))_{t \in \mathbb{R}_+} \). Hence Corollary 3.3.6 in [7] allows us to write

\[
(A - \lambda)^{-\alpha} x = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha - 1} e^{\lambda t} T(t) x \, dt \quad (x \in X).
\]

Fix \( f \in \text{AM}_p((\mathbb{R} - \omega)) \) and set \( a := \frac{1}{\omega} \min \left\{ \frac{1}{p}, \frac{1}{p'} \right\} \). By Corollary 3.5,

\[
\int_0^\infty t^{\text{Re}(\alpha) - 1} e^{\text{Re}(\lambda)t} \| f(A)T(t)x \| \, dt \leq CM^2 \| f \|_{\text{AM}_p^X} \| x \| < \infty
\]

for all \( x \in X \), where

\[
C = c_p \int_0^a t^{\text{Re}(\alpha) - 1} |\log(\omega t)| e^{(\text{Re}(\lambda) + \omega)t} \, dt + 2 \int_a^\infty t^{\text{Re}(\alpha) - 1} e^{\text{Re}(\lambda)t} \, dt
\]

is independent of \( f, M, \) and \( x \). Since \( f(A) \) is a closed operator, this implies that \( (A - \lambda)^{-\alpha} \) maps into \( \text{dom}(f(A)) \) with

\[
f(A)(A - \lambda)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha - 1} e^{\lambda t} f(A)T(t) \, dt
\]

as a strong integral. 

**Remark 3.8.** Theorem 3.7 shows that for each analytic multiplier function \( f \) the domain \( \text{dom}(f(A)) \) is relatively large, it contains the real interpolation spaces \( (X, \text{dom}(A))_{\theta,q} \) and the complex interpolation spaces \([X, \text{dom}(A)]_{\theta} \) for all \( \theta \in (0,1) \) and \( q \in [1,\infty] \). This follows from [14, Proposition 1.1.4] and [7, Corollary 6.6.3] for real interpolation spaces and then from [14, Proposition 2.1.10] for the complex interpolation spaces.

**Remark 3.9.** We can describe the range of \( f(A)(A - \lambda)^{-\alpha} \) in Theorem 3.7 more explicitly. In fact,

\[
\text{ran}(f(A)(A - \lambda)^{-\alpha}) \subseteq \text{dom} \left( (A - \lambda)^{\frac{\alpha}{\delta}} \right)
\]
for all $\text{Re}(\beta) < \text{Re}(\alpha)$. Indeed, this follows if we show that $\text{ran}(A - \lambda)^{-\alpha} \subseteq \text{dom}((A - \lambda)^\beta f(A))$, and [7, Theorem 1.3.2] implies

$$\text{dom}((A - \lambda)^\beta f(A)) = \text{dom}(f(A)) \cap \text{dom} \left( \left( z - \lambda \right)^\beta f(z)(A) \right).$$

The inclusion $\text{ran}(A - \lambda)^{-\alpha} \subseteq \text{dom}(f(A))$ follows from Theorem 3.7. Since

$$\left( z - \lambda \right)^\beta f(z)(A - \lambda)^{-\alpha} = \left( z - \lambda \right)^{\beta - \alpha} f(z)(A) = f(A)(A - \lambda)^{\beta - \alpha},$$

the same holds for the inclusion $\text{ran}(A - \lambda)^{-\alpha} \subseteq \text{dom} \left( \left( z - \lambda \right)^\beta f(z)(A) \right)$.

Semigroups on Hilbert spaces

If $X = H$ is a Hilbert space, Plancherel’s Theorem implies $\mathcal{A}M_2^H = H^\infty$ with equality of norms. Hence the theory above specializes to the following result, implying a) and b) of Theorem 1.1.

**Corollary 3.10.** Let $-A$ generate a bounded $C_0$-semigroup $(T(t))_{t \in \mathbb{R}_+}$ of type $(M,0)$ on a Hilbert space $H$. Then the following assertions hold.

a) There exists a universal constant $c \geq 0$ such that the following holds. Let $\tau, \omega > 0$. Then $f(A) \in \mathcal{L}(H)$ and

$$\|f(A)\| \leq \begin{cases} cM^2 \|\log(\omega \tau)\| \|f\|_\infty & \text{if } \omega \tau \leq \frac{1}{2}, \\ 2M^2e^{-\omega \tau} \|f\|_\infty & \text{if } \omega \tau > \frac{1}{2} \end{cases}$$

for all $f \in e^{-\tau}H^\infty(R_{-\omega})$. Moreover, $f(A)T(\tau) \in \mathcal{L}(H)$ with

$$\|f(A)T(\tau)\| \leq \begin{cases} cM^2 \|\log(\omega \tau)e^{\omega \tau}\| \|f\|_\infty & \text{if } \omega \tau \leq \frac{1}{2}, \\ 2M^2 \|f\|_\infty & \text{if } \omega \tau > \frac{1}{2} \end{cases}$$

for all $f \in H^\infty(R_{-\omega})$.

b) If

$$\bigcup_{\tau > 0} \text{ran}(T(\tau)) = H,$$

then $A$ has a bounded $H^\infty(R_\omega)$-calculus for all $\omega < 0$.

c) For $\omega < 0$ and $\alpha, \lambda \in \mathbb{C}$ with $\text{Re}(\lambda) < 0 < \text{Re}(\alpha)$ there is $C = C(\alpha, \lambda, \omega) \geq 0$ such that

$$\|f(A)(A - \lambda)^{-\alpha}\| \leq CM^2 \|f\|_\infty$$

for all $f \in H^\infty(R_\omega)$. In particular, $\text{dom}(A^\alpha) \subseteq \text{dom}(f(A))$.

Part c) shows that, even though semigroup generators on Hilbert spaces do not have a bounded $H^\infty$-calculus in general, each function $f$ that decays with polynomial rate $\alpha > 0$ at infinity yields a bounded operator $f(A)$. For $\alpha > \frac{1}{2}$ this is already covered by Lemma 2.4, but for $\alpha \in (0, \frac{1}{2}]$ it appears to be new.
Remark 3.11. Part c) of Corollary 3.10 yields a statement about stability of numerical methods. Let $-A$ generate an exponentially stable semigroup $(T(t))_{t \geq 0}$ on a Hilbert space, let $r \in H^\infty(C_+)$ be such that $\|r\|_{H^\infty(C_+)} \leq 1$, and let $\alpha, h > 0$. Then
\[
\sup \{ \|r(hA)^n x\| \mid n \in \mathbb{N}, x \in \text{dom}(A^\alpha) \} < \infty \tag{3.5}
\]
follows from c) in Corollary 3.10 after shifting the generator. Elements of the form $r^n(hA)x$ are often used in numerical methods to approximate the solution of the abstract Cauchy problem associated to $-A$ with initial value $x$, and (3.5) shows that such approximations are stable whenever $x \in \text{dom}(A^\alpha)$.

4. $m$-Bounded functional calculus

In this section we describe another transference principle for semigroups, one that provides estimates for the norms of operators of the form $f^{(m)}(A)$ for $f$ an analytic multiplier function and $f^{(m)}$ its $m$-th derivative, $m \in \mathbb{N}$. We use terminology from Section 5 of [3]. Moreover, we recall our notational simplification $AM_p(R_\omega) := AM_p^X(R_\omega)$ (Remark 2.1).

Let $\omega < \omega_0$ be real numbers. An operator $A$ of half-plane type $\omega_0 \in \mathbb{R}$ on a Banach space $X$ has an $m$-bounded $AM_p^X(R_\omega)$-calculus if there exists $C \geq 0$ such that $f^{(m)}(A) \in L(X)$ with
\[
\left\| f^{(m)}(A) \right\| \leq C \|f\|_{AM_p^X} \quad \text{for all } f \in AM_p^X(R_\omega).
\]
This is well defined since the Cauchy integral formula implies that $f^{(m)}$ is bounded on every half-plane $R_{\omega'}$ with $\omega' > \omega$.

We say that $A$ has a strong $m$-bounded $AM_p^X$-calculus of type $\omega_0$ if $A$ has an $m$-bounded $AM_p^X(R_\omega)$-calculus for every $\omega < \omega_0$ and such that for some $C \geq 0$ one has
\[
\left\| f^{(m)}(A) \right\| \leq \frac{C}{(\omega_0 - \omega)^m} \|f\|_{AM_p^X(R_\omega)} \tag{4.1}
\]
for all $f \in AM_p^X(R_\omega)$ and $\omega < \omega_0$.

Lemma 4.1. Let $A$ be an operator of half-plane type $\omega_0 \in \mathbb{R}$ on a Banach space $X$, and let $p \in [1, \infty]$ and $m \in \mathbb{N}$. If $A$ has a strong $m$-bounded $AM_p^X$-calculus of type $\omega_0$, then $A$ has a strong $n$-bounded $AM_p^X$-calculus of type $\omega_0$ for all $n > m$.

Proof. Let $\omega < \alpha < \beta < \omega_0$, $f \in AM_p(R_\omega)$ and $n \in \mathbb{N}$. Then
\[
f^{(n)}(\beta + is) = \frac{n!}{2\pi i} \int_{\mathbb{R}} \frac{f(\alpha + ir)}{(\alpha + ir - (\beta + is))^{n+1}} \, dr
\]
\[
= \frac{n!}{2\pi i} \left( f(\alpha + i \cdot) * (\alpha - \beta - i \cdot)^{-n-1} \right)(s)
\]
for all \( s \in \mathbb{R} \), by the Cauchy integral formula. Hence, using Lemma 2.2,

\[
\left\| f^{(n)}(\beta + i\cdot) \right\|_{\mathcal{M}_p(X)} \leq \frac{n!}{2\pi} \left\| (\alpha - \beta - i\cdot)^{-n-1} \right\|_{L^1([\beta])} \|f(\alpha + i\cdot)\|_{\mathcal{M}_p(X)} \\
\leq \frac{C}{(\beta - \alpha)^n} \|f\|_{\mathcal{M}_p(R_\omega)}
\]

for some \( C = C(n) \geq 0 \) independent of \( f, \beta, \alpha \) and \( \omega \). Letting \( \alpha \) tend to \( \omega \) yields

\[
\left\| f^{(n)} \right\|_{\mathcal{M}_p(R_\beta)} = \left\| f^{(n)}(\beta + i\cdot) \right\|_{\mathcal{M}_p(X)} \leq \frac{C}{(\beta - \omega)^n} \|f\|_{\mathcal{M}_p(R_\omega)}. \quad (4.2)
\]

Now let \( n > m \). Applying (4.2) with \( n - m \) in place of \( n \) shows that \( f^{(n-m)} \in \mathcal{M}_p(R_\beta) \) with

\[
\left\| f^{(n)}(A) \right\| \leq \frac{C'}{(\omega_0 - \beta)^m} \|f^{(n-m)}\|_{\mathcal{M}_p(R_\beta)} \leq \frac{CC'}{(\omega_0 - \beta)^m(\beta - \omega)^{n-m}} \|f\|_{\mathcal{M}_p(R_\omega)}.
\]

Finally, letting \( \beta = \frac{1}{2} (\omega + \omega_0) \),

\[
\left\| f^{(n)}(A) \right\| \leq \frac{C''}{(\omega_0 - \omega)^n} \|f\|_{\mathcal{M}_p(R_\omega)}
\]

for some \( C'' \geq 0 \) independent of \( f \) and \( \omega \).

For the transference principle in Proposition 3.1 it is essential that the support of \( \mu \in \mathcal{M}_\omega(\mathbb{R}_+) \) is contained in some interval \([\tau, \infty)\) with \( \tau > 0 \). In general one cannot expect to find such a transference principle for arbitrary \( \mu \), as this would allow one to prove that semigroup generators have a bounded analytic multiplier calculus. But this is known to be false in general, cf. [7, Corollary 9.1.8]. However, if we let \( t\mu \) be given by \((t\mu)(dt) := t\mu(dt)\) then we can deduce the following transference principle. We use the conventions \( 1/\infty := 0 \), \( \infty^0 := 1 \).

**Proposition 4.2.** Let \(-A\) be the generator of a \( C_0\)-semigroup \((T(t))_{t \in \mathbb{R}_+}\) of type \((M, 0)\) on a Banach space \( X \). Let \( p \in [1, \infty] \), \( \omega < 0 \) and \( \mu \in \mathcal{M}_\omega(\mathbb{R}_+) \). Then

\[
\|T_{t\mu}\| \leq \frac{M^2}{|\omega|} p^{-1/p} (p')^{-1/p'} \left\| L_{e^{-\omega\cdot}} \right\|_{\mathcal{L}(L^p(X))}.
\]

**Proof.** We can factorize \( T_{t\mu} \) as \( T_{t\mu} = P \circ L_{e^{-\omega\cdot}} \circ \iota \), where \( \iota \) and \( P \) are as in the proof of Proposition 3.1 with \( \psi, \varphi := 1_{\mathbb{R}_+} e_\omega \). This follows from the abstract transference principle in [9, Section 2], since \((\psi \ast \varphi)e^{-\omega \cdot} = t\mu\). Then

\[
\|T_{t\mu}\| \leq M^2 \|e_\omega\|_{p'} \left\| L_{e^{-\omega\cdot}} \right\|_{\mathcal{L}(L^p(X))} \|e_\omega\|_p \\
= M^2 \frac{1}{|\omega|} p^{-1/p} (p')^{-1/p'} \left\| L_{e^{-\omega\cdot}} \right\|_{\mathcal{L}(L^p(X))},
\]

by Hölder’s inequality. \( \square \)
We are now ready to prove our main result on $m$-bounded functional calculus, a generalization of Theorem 7.1 in [3] to arbitrary Banach spaces.

**Theorem 4.3.** Let $A$ be a densely defined operator of half-plane type 0 on a Banach space $X$. Then the following assertions are equivalent:

(i) $-A$ is the generator of a bounded $C_0$-semigroup on $X$.

(ii) $A$ has a strong $m$-bounded $\text{AM}_p^X$-calculus of type 0 for some/all $p \in [1, \infty]$ and some/all $m \in \mathbb{N}$.

In particular, if $-A$ generates a bounded $C_0$-semigroup then $A$ has an $m$-bounded $\text{AM}_p^X(R_\omega)$-calculus for all $\omega < 0$, $p \in [1, \infty]$ and $m \in \mathbb{N}$.

**Proof.** (i) $\Rightarrow$ (ii) By Lemma 4.1 it suffices to let $m = 1$. We proceed along the same lines as the proof of Theorem 3.3. Let $(T(t))_{t \in \mathbb{R}^+} \subseteq \mathcal{L}(X)$ be the semigroup generated by $-A$ and fix $\omega < 0$, $p \in [1, \infty]$ and $f \in \text{AM}_p(R_\omega)$.

Define the functions $f_{k,\epsilon} := f(\cdot + \epsilon)g_k(\cdot + \epsilon)$ for $k \in \mathbb{N}$ and $\epsilon > 0$, where $g_k(z) := \frac{z^k}{z^{k+\epsilon}}$ for $z \in R_\omega$. Then $f_{k,\epsilon} \in \text{AM}_p(R_\omega)$ by Lemma 2.4, and Lemma 2.3 yields $\mu_{k,\epsilon} \in \mathcal{M}_p(\mathbb{R}_+)$ with $f_{k,\epsilon} = \hat{\mu}_{k,\epsilon}$. Now

$$f'_{k,\epsilon}(z) = \lim_{h \to 0} \frac{f_{k,\epsilon}(z+h) - f_{k,\epsilon}(z)}{h} = \lim_{h \to 0} \int_0^\infty \frac{e^{-zt} - e^{-(z+h)t}}{h} \mu_{k,\epsilon}(dt) = -\int_0^\infty t e^{-zt} \mu_{k,\epsilon}(dt)$$

for $z \in R_\omega$, by the Dominated Convergence Theorem. Hence $f'_{k,\epsilon}(A) = -T_{t\mu_{k,\epsilon}}$, and Proposition 4.2 and Lemma 2.3 imply

$$\|f'_{k,\epsilon}(A)\| \leq \frac{M^2}{|\omega|} p^{-1/p}(p')^{-1/p'} \|f_{k,\epsilon}\|_{\text{AM}_p^X}.$$ 

Furthermore, $\sup_{k,\epsilon} \|f_{k,\epsilon}\|_{\text{AM}_p^X} \leq \|f\|_{\text{AM}_p^X}$. In particular, the $f_{k,\epsilon}$ are uniformly bounded. By the Cauchy integral formula, so are the derivatives $f'_{k,\epsilon}$ on every smaller half-plane. Since $f'_{k,\epsilon}(z) \to f'(z)$ for all $z \in R_\omega$ as $k \to \infty$, $\epsilon \to 0$, the Convergence Lemma yields $f'(A) \in \mathcal{L}(X)$ with

$$\|f'(A)\| \leq \frac{M^2}{|\omega|} p^{-1/p}(p')^{-1/p'} \|f\|_{\text{AM}_p^X},$$

which is (4.1) for $m = 1$.

For (ii) $\Rightarrow$ (i) assume that $A$ has a strong $m$-bounded $\text{AM}_p$-calculus of type 0 for some $p \in [1, \infty]$ and some $m \in \mathbb{N}$. Then

$$e^{-t} \in \text{AM}_p(R_\omega) \subseteq \text{AM}_p(R_\omega)$$

for all $t > 0$ and $\omega < 0$, with

$$\|e^{-t}\|_{\text{AM}_p(R_\omega)} \leq \|e^{-t}\|_{\text{AM}_p(R_\omega)} = e^{-t\omega}.$$
Now \((e^{-t})^{(m)} = (-t)^m e^{-t}\) implies
\[
|t^m e^{-tA}| \leq \frac{C}{|\omega|^m} e^{-t\omega}.
\]

Letting \(\omega := -\frac{1}{t}\) and using Lemma 2.5 in [3] yields the required statement. \(\square\)

If \(X = H\) is a Hilbert space then Plancherel’s theorem yields the following result, which is a generalization of Theorem 7.1 in [3]. It contains part c) of Theorem 1.1.

**Corollary 4.4.** Let \(A\) be a densely defined operator of half-plane type 0 on a Hilbert space \(H\). Then the following assertions are equivalent:

(i) \(-A\) is the generator of a bounded \(C_0\)-semigroup on \(H\).

(ii) \(A\) has a strong \(m\)-bounded \(H^\infty\)-calculus of type 0 for some/all \(m \in \mathbb{N}\).

In particular, if \(-A\) generates a bounded \(C_0\)-semigroup then \(A\) has an \(m\)-bounded \(H^\infty(R_\omega)\)-calculus for all \(\omega < 0\) and \(m \in \mathbb{N}\).

5. Semigroups on UMD spaces

A Banach space \(X\) is a **UMD space** if the function \(t \mapsto \text{sgn}(t)\) is a bounded \(L^2(X)\)-Fourier multiplier. For \(\omega \in \mathbb{R}\) we let
\[
H^\infty_1(R_\omega) := \{ f \in H^\infty(R_\omega) \mid (z - \omega)f'(z) \in H^\infty(R_\omega) \}
\]
be the **analytic Mikhlin algebra** on \(R_\omega\), a Banach algebra endowed with the norm
\[
\|f\|_{H^\infty_1} = \|f\|_{H^\infty_1(R_\omega)} := \sup_{z \in R_\omega} |f(z)| + |(z - \omega)f'(z)| (f \in H^\infty_1(R_\omega)).
\]

The vector-valued Mikhlin multiplier theorem [7, Theorem E.6.2] and Lemma 2.2 yield the continuous inclusion
\[
H^\infty_1(R_\omega) \hookrightarrow \text{AM}_p(X)
\]
for each \(p \in (1, \infty)\), if \(X\) is a UMD space. Combining this with Theorems 3.3 and 4.3 and Corollaries 3.5 and 3.6 proves the following theorem.

**Theorem 5.1.** Let \(-A\) generate a \(C_0\)-semigroup \((T(t))_{t \in \mathbb{R}_+}\) of type \((M, 0)\) on a UMD space \(X\). Then the following assertions hold.

a) For each \(p \in (1, \infty)\) there exists a constant \(c_p = c(p, X) \geq 0\) such that the following holds. Let \(\tau, \omega > 0\). Then \(f(A) \in \mathcal{L}(X)\) with
\[
\|f(A)\| \leq \begin{cases} 
    c_p M^2 \log(\omega \tau) & \text{if } \omega \tau \leq \min \left\{ \frac{1}{p}, \frac{1}{p'} \right\}, \\
    2c_p M^2 e^{-\omega \tau} \|f\|_{H^\infty_1} & \text{if } \omega \tau > \min \left\{ \frac{1}{p}, \frac{1}{p'} \right\}
\end{cases}
\]
for all \( f \in H_1^\infty(\mathbb{R}_- \cap e^{-\tau}H_\infty(\mathbb{R}_-)), \) and \( f(A)T(\tau) \in \mathcal{L}(X) \) with

\[
\|f(A)T(\tau)\| \leq \begin{cases} \ c_p M^2 \|\log(\omega \tau)\|e^{\omega \tau} \| f \|_{H_1^\infty} & \text{if } \omega \tau \leq \min \left\{ \frac{1}{p}, \frac{1}{p'} \right\}, \\ 2c_p M^2 \| f \|_{H_1^\infty} & \text{if } \omega \tau > \min \left\{ \frac{1}{p}, \frac{1}{p'} \right\} \end{cases}
\]

for all \( f \in H_1^\infty(\mathbb{R}_-) \).

b) If

\[ \bigcup_{\tau > 0} \text{ran}(T(\tau)) = X, \]

then \( A \) has a bounded \( H_1^\infty(\mathbb{R}_-)^\)-calculus for all \( \omega < 0 \).

c) \( A \) has a strong \( m \)-bounded \( H_1^\infty \)-calculus of type 0 for all \( m \in \mathbb{N} \).

**Remark 5.2.** Theorem 3.7 yields the domain inclusion \( \text{dom}(A^\alpha) \subseteq \text{dom}(f(A)) \) for all \( \alpha \in \mathbb{C}_+, \omega < 0 \) and \( f \in H_1^\infty(\mathbb{R}_-) \), on a UMD space \( X \). However, this inclusion in fact holds true on a general Banach space \( X \). Indeed, for \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) < 0 \), Bernstein’s Lemma [2, Proposition 8.2.3] implies \( \frac{f(z)}{(\lambda - z)} \in \text{AM}_1(\mathbb{C}_+) \), hence \( f(A)(\lambda - A)^{-\alpha} \in \mathcal{L}(X) \) and \( \text{dom}(A^\alpha) \subseteq \text{dom}(f(A)) \). An estimate

\[ \|f(A)(\lambda - A)^{-\alpha}\| \leq C \| f \|_{H_1^\infty(\mathbb{R}_-)} \]

then follows from an application of the Closed Graph Theorem and the Convergence Lemma.

**Remark 5.3.** To apply Theorem 5.1 one can use the continuous inclusion

\[ H_\infty(\mathbb{R}_- \cup (S_\phi + a)) \subseteq H_1^\infty(\mathbb{R}_-) \]

for \( \omega' > \omega, a \in \mathbb{R} \) and \( \phi \in (\pi/2, \pi] \). Here \( \mathbb{R}_- \cup (S_\phi + a) \) is the union of \( \mathbb{R}_- \) and the translated sector \( S_\phi + a \), where

\[ S_\phi := \{ z \in \mathbb{C} \mid |\text{arg}(z)| < \phi \}. \]

Indeed, to derive (5.1) it suffices to let \( a = 0 \), and using Cauchy’s integral formula as in [7, Lemma 8.2.6] yields the desired result.

### 6. \( \gamma \)-Bounded semigroups

The geometry of the underlying Banach space \( X \) played an essential role in the results of Sections 3 and 4 in the form of properties of the analytic multiplier algebra \( \text{AM}_p^X \). To wit, in order to identify non-trivial functions in \( \text{AM}_p^X \) one needs a geometric assumption on \( X \), for instance that it is a Hilbert or a UMD space. In this section we take a different approach and make additional assumptions on the semigroup instead of the underlying space. We show that if the semigroup in question is \( \gamma \)-bounded then one can recover the Hilbert space results on an arbitrary Banach space \( X \).
For this section we assume the reader to be familiar with the basics of the theory of $\gamma$-radonifying operators and $\gamma$-boundedness as collected in the survey article by van Neerven [18]. We use terminology and results from [9].

Let $H$ be a Hilbert space and $X$ a Banach space. A linear operator $T : H \to X$ is $\gamma$-summing if

$$
\|T\|_\gamma := \sup_F \left( \mathbb{E} \left\| \sum_{h \in F} \gamma_h Th \right\|_X^2 \right)^{1/2} < \infty,
$$

where the supremum is taken over all finite orthonormal systems $F \subseteq H$ and $(\gamma_h)_{h \in F}$ is an independent collection of complex-valued standard Gaussian random variables on some probability space. Endow

$$
\gamma_\infty(H;X) := \{T : H \to X \mid T \text{ is } \gamma\text{-summing}\}
$$

with the norm $\|\cdot\|_\gamma$ and let the space $\gamma(H;X)$ of all $\gamma$-radonifying operators be the closure in $\gamma_\infty(H;X)$ of the finite-rank operators $H \otimes X$.

For a measure space $(\Omega, \mu)$ let $\gamma(\Omega;X)$ (resp. $\gamma_\infty(\Omega;X)$) be the space of all weakly $L^2$-functions $f : \Omega \to X$ for which the integration operator $J_f : L^2(\Omega) \to X$,

$$
J_f(g) := \int_\Omega g \cdot f \, d\mu \quad (g \in L^2(\Omega)),
$$

is $\gamma$-radonifying ($\gamma$-summing), and endow it with the norm $\|f\|_\gamma := \|J_f\|_\gamma$.

A collection $T \subseteq \mathcal{L}(X)$ is $\gamma$-bounded if there exists a constant $C \geq 0$ such that

$$
\left( \mathbb{E} \left\| \sum_{T \in T'} \gamma_T x_T \right\|_X^2 \right)^{1/2} \leq C \left( \mathbb{E} \left\| \sum_{T \in T'} \gamma_T x_T \right\|_X^2 \right)^{1/2}
$$

for all finite subsets $T' \subseteq T$, sequences $(x_T)_{T \in T'} \subseteq X$ and independent complex-valued standard Gaussian random variables $(\gamma_T)_{T \in T'}$. The smallest such $C$ is the $\gamma$-bound of $T$ and is denoted by $\|T\|_\gamma$. Every $\gamma$-bounded collection is uniformly bounded with supremum bound less than or equal to the $\gamma$-bound, and the converse holds if $X$ is a Hilbert space.

An important result involving $\gamma$-boundedness is the multiplier theorem. We state a version that is tailored to our purposes. Given a Banach space $Y$, a function $g : \mathbb{R} \to Y$ is piecewise $W^{1,\infty}$ if $g \in W^{1,\infty}(\mathbb{R} \setminus \{a_1, \ldots, a_n\}; Y)$ for some finite set $\{a_1, \ldots, a_n\} \subseteq \mathbb{R}$.

**Theorem 6.1** (Multiplier Theorem). Let $X$ and $Y$ be Banach spaces and $T : \mathbb{R} \to \mathcal{L}(X,Y)$ a strongly measurable mapping such that

$$
T := \{T(s) \mid s \in \mathbb{R}\}
$$
is $\gamma$-bounded. Suppose furthermore that there exists a dense subset $D \subseteq X$ such that $s \mapsto T(s)x$ is piecewise $W^{1,\infty}$ for all $x \in D$. Then the multiplication operator

$$M_T : L^2(\mathbb{R}) \otimes X \to L^2(\mathbb{R}; Y) \quad M_T(f \otimes x) = f(\cdot)T(\cdot)x$$

extends uniquely to a bounded operator

$$M_T : \gamma(L^2(\mathbb{R}); X) \to \gamma(L^2(\mathbb{R}); Y)$$

with $\|M_T\| \leq J_T K_\gamma$.

**Proof.** That $M_T$ extends uniquely to a bounded operator into $\gamma(\infty(L^2(\mathbb{R}); Y))$ with $\|M_T\| \leq [T]_\gamma$ is the content of Theorem 5.2 in [18]. To see that in fact $\text{ran}(M_T) \subseteq \gamma(\mathbb{R}; Y)$ we employ a density argument. For $x \in D$ let $a_1, \ldots, a_n \in \mathbb{R}$ be such that $s \mapsto T(s)x \in W^{1,\infty}(\mathbb{R} \setminus \{a_1, \ldots, a_n\}; Y)$, and set $a_0 := -\infty$, $a_{n+1} := \infty$. Let $f \in C_c(\mathbb{R})$ be given and note that

$$\int_{a_j}^{a_{j+1}} \|f\|_{L^2(s,a_{j+1})} \|T(s)'x\| \, ds < \infty$$

for all $j \in \{1, \ldots, n\}$. Furthermore,

$$\int_{-\infty}^{a_1} \|f\|_{L^2(-\infty,s)} \|T(s)'x\| \, ds < \infty.$$

Corollary 6.3 in [9] yields $(1_{(a_j, a_{j+1})}f)(\cdot)T(\cdot)x \in \gamma(\mathbb{R}; Y)$ for all $0 \leq j \leq n$, hence $f(\cdot)T(\cdot)x \in \gamma(\mathbb{R}; Y)$. Since $C_c(\mathbb{R}) \otimes D$ is dense in $L^2(\mathbb{R}) \otimes X$, which in turn is dense in $\gamma(L^2(\mathbb{R}); X)$, the result follows.

We are now ready to prove a generalization of part a) of Corollary 3.10. Recall that

$$e_{-\tau}H^\infty(\mathbb{R}_\omega) = \left\{ f \in H^\infty(\mathbb{R}_\omega) \mid f(z) = O(e^{-\tau \text{Re}(z)}) \text{ as } |z| \to \infty \right\}$$

for $\tau > 0$, $\omega \in \mathbb{R}$.

**Theorem 6.2.** There exists a universal constant $c \geq 0$ such that the following holds. Let $-A$ generate a $\gamma$-bounded $C_0$-semigroup $(T(t))_{t \in \mathbb{R}_+}$ with $M := [T]_\gamma$ on a Banach space $X$, and let $\tau, \omega > 0$. Then $f(A) \in L(X)$ with

$$\|f(A)\| \leq \begin{cases} cM^2 \log(\omega \tau) \|f\|_\infty & \text{if } \omega \tau \leq \frac{1}{2} \\ 2M^2 e^{-\omega \tau} \|f\|_\infty & \text{if } \omega \tau > \frac{1}{2} \end{cases}$$

for all $f \in e_{-\tau}H^\infty(\mathbb{R}_{-\omega})$.

In particular, $A$ has a bounded $e_{-\tau}H^\infty(\mathbb{R}_{-\omega})$-calculus.
Proof. We only need to show that the estimate (3.2) in Proposition 3.1 can be refined to
\[ \|T \mu\| \leq M^2 \eta(\omega, \tau, 2) \|L_{e, \mu}\|_{L(\gamma(R; X))} \]  
(6.1)
for \( \mu \in M_{-\omega}(R_+) \) with \( \operatorname{supp}(\mu) \subseteq [\tau, \infty) \). Then one uses that
\[ \|L_{e, \mu}\|_{L(\gamma(R; X))} \leq \|\hat{e}_{\omega}\mu\|_{H^\infty(C_+)} = \|\hat{R}\|_{H^\infty(R_{-\omega})}, \]
by the ideal property of \( \gamma(L^2(R); X) \) [18, Theorem 6.2], and proceeds as in the proof of Theorem 3.3 to deduce the desired result.

To obtain (6.1) we factorize \( T \mu \) as \( T \mu = P \circ L_{e, \mu} \circ \iota \), where \( \iota: X \to \gamma(R; X) \)
and \( P: \gamma(R; X) \to X \) are given by
\[ \iota x(s) := \psi(-s)T(-s)x \quad (x \in X, s \in R), \]
\[ Pg := \int_0^\infty \varphi(t)T(t)g(t) \, dt \quad (g \in \gamma(R; X)), \]
for \( \psi, \varphi \in L^2(R_+) \) such that \( \psi \ast \varphi \equiv e_{-\omega} \) on \([\tau, \infty)\). This factorization follows as in Section 2 of [9] once we show that the maps \( \iota \) and \( P \) are well-defined and bounded. To this end, first note that \( s \mapsto T(-s)x \) is piecewise \( W^{1,\infty} \) for all \( x \) in the dense subset \( \operatorname{dom}(A) \subseteq X \) and that
\[ \psi(-\cdot) \otimes x \in L^2(-\infty, 0) \otimes X \subseteq \gamma(L^2(R); X). \]
Therefore Theorem 6.1 yields \( \iota x \in \gamma(R, X) \) with
\[ \|\iota x\|_\gamma = \|Jx\|_\gamma \leq M \|\psi(-\cdot) \otimes x\|_\gamma = M \|\psi\|_{L^2(R_+)} \|x\|_X. \]
As for \( P \), write
\[ Pg = \int_0^\infty \varphi(t)T(t)g(t) \, dt = J_{Tg}(\varphi) \]
and use Theorem 6.1 once again to see that \( Tg \in \gamma(R; X) \). Hence
\[ \|Pg\|_X \leq \|J_{Tg}\|_\gamma \|\varphi\|_{L^2(R_+)} \leq M \|\varphi\|_{L^2(R_+)} \|g\|_\gamma. \]
Finally, estimating the norm of \( T \mu \) through this factorization and taking the infimum over all \( \psi \) and \( \varphi \) yields (6.1).

Corollary 6.3. Corollary 3.10 generalizes to \( \gamma \)-bounded semigroups on arbitrary Banach spaces upon replacing the uniform bound \( M \) of \( T \) by \( \|T\|_\gamma \).

Theorem 4.3 can be extended in an almost identical manner to a \( \gamma \)-version.

Theorem 6.4. Let \( -A \) generate a \( \gamma \)-bounded \( C_0 \)-semigroup on a Banach space \( X \). Then \( A \) has a strong \( m \)-bounded \( H^\infty \)-calculus of type 0 for all \( m \in \mathbb{N} \).
Appendix A. Growth estimates

In this appendix we examine the function \( \eta : (0, \infty) \times (0, \infty) \times [1, \infty] \rightarrow \mathbb{R}_+ \) from (3.1):

\[
\eta(\alpha, t, q) := \inf \left\{ \|\psi\|_q \|\phi\|_{q'} \mid \psi \ast \phi \equiv e_{-\alpha} \text{ on } [t, \infty) \right\}.
\]

We will use the notation \( f \lesssim g \) for real-valued functions \( f, g : Z \rightarrow \mathbb{R} \) on some set \( Z \) to indicate that there exists a constant \( c \geq 0 \) such that \( f(z) \leq cg(z) \) for all \( z \in Z \).

**Lemma Appendix A.1.** For each \( q \in (1, \infty) \) there exist constants \( c_q, d_q \geq 0 \) such that

\[
d_q|\log(\alpha t)| \leq \eta(\alpha, t, q) \leq c_q|\log(\alpha t)| \tag{A.1}
\]

if \( \alpha t \leq \min \left\{ \frac{1}{q}, \frac{1}{q'} \right\} \). If \( \alpha t > \min \left\{ \frac{1}{q}, \frac{1}{q'} \right\} \) then

\[
e^{-\alpha t} \leq \eta(\alpha, t, q) \leq 2e^{-\alpha t}. \tag{A.2}
\]

**Proof.** First note that \( \eta(\alpha, t, q) = \eta(\alpha t, 1, q) = \eta(1, \alpha t, q) \) for all \( \alpha, t \) and \( q \).

Indeed, for \( \psi \in L^q(\mathbb{R}_+), \varphi \in L^{q'}(\mathbb{R}_+) \) with \( \psi \ast \varphi \equiv e_{-\alpha} \) on \([1, \infty)\) define

\[
\psi_t(s) := \frac{1}{e^{\pi \alpha}} \psi(s/t) \text{ and } \varphi_t(s) := \frac{1}{e^{\pi \alpha}} \varphi(s/t) \text{ for } s \geq 0.
\]

Then

\[
\psi_t \ast \varphi_t(r) = \int_{0}^{\infty} \psi \left( \frac{r-s}{t} \right) \varphi \left( \frac{s}{t} \right) \frac{ds}{t} = \psi \ast \varphi \left( \frac{t}{s} \right)
\]

for all \( r \geq 0 \), so \( \psi_t \ast \varphi_t \equiv e_{-\alpha} \) on \([t, \infty)\). Moreover,

\[
\|\psi_t\|_q^q = \int_{0}^{\infty} |\psi(\frac{s}{t})|^q \frac{ds}{t} = \int_{0}^{\infty} |\psi(s)|^q ds = \|\psi\|_q^q,
\]

and similarly \( \|\varphi_t\|_{q'}^q = \|\varphi\|_{q'}^q \). Hence \( \eta(\alpha, t, q) \leq \eta(\alpha t, 1, q) \). Considering \( \psi_{1/t} \) and \( \varphi_{1/t} \) yields \( \eta(\alpha, t, q) = \eta(\alpha t, 1, q) \). The other equality follows immediately.

Hence, to prove any of the inequalities in (A.1) or (A.2), we can assume either that \( \alpha = 1 \) or that \( t = 1 \) (but not both).

For the left-hand inequalities, we assume that \( \alpha = 1 \) and we first consider the left-hand inequality of (A.1). Let \( t < 1 \) and \( \psi \in L^q(\mathbb{R}_+), \varphi \in L^{q'}(\mathbb{R}_+) \) such that \( \psi \ast \varphi \equiv e_{-1} \) on \([t, \infty)\). Then

\[
|\log(t)| = -\log(t) = \int_{t}^{1} \frac{ds}{s} \leq e \int_{t}^{1} e^{-s} \frac{ds}{s} = e \int_{t}^{1} |\psi \ast \varphi(s)| \frac{ds}{s}
\]

\[
\leq e \int_{t}^{1} \int_{0}^{s} |\psi(s-r)| \cdot |\varphi(r)| \frac{dr}{s} \frac{ds}{s}
\]

\[
\leq e \int_{0}^{\infty} \int_{r}^{\infty} \frac{|\varphi(s-r)|}{s} ds |\varphi(r)| dr
\]

\[
= e \int_{0}^{\infty} \int_{0}^{\infty} \frac{|\psi(r)||\varphi(s)|}{s+r} ds dr \leq \frac{e\pi}{\sin(\pi/q)} \|\psi\|_q \|\varphi\|_{q'},
\]
where we used Hilbert’s absolute inequality [6, Theorem 5.10.1]. It follows that
\[ \eta(1, t, q) \geq \sin(\pi/q) e^{|\log(t)|}. \]

For the left-hand inequality of (A.2), we assume that \( \alpha = 1 \) and let \( t > 0 \) be arbitrary. Then
\[ e^{-t} = (\psi \ast \varphi)(t) \leq \int_0^t |\psi(t-s)| |\varphi(s)| \, ds \leq \|\psi\|_q \|\varphi\|_{q'}, \]
by Hölder’s inequality, hence \( e^{-t} \leq \eta(1, t, q) \).

For the right-hand inequalities in (A.1) and (A.2), we assume that \( t = 1 \) and first consider the right-hand inequality in (A.1) for \( \alpha \leq \min \left\{ \frac{1}{q}, \frac{1}{q'} \right\} \). In the proof of Lemma A.1 in [9] it is shown that
\[ (\psi_0 \ast \varphi_0)(s) = \begin{cases} s, & s \in [0, 1) \\ 1, & s \geq 1 \end{cases} \]
for
\[ \psi_0 := \sum_{j=0}^{\infty} \beta_j 1_{(j,j+1)} \quad \text{and} \quad \varphi_0 := \sum_{j=0}^{\infty} \beta'_j 1_{(j,j+1)}, \]
where \((\beta_j)\) and \((\beta'_j)\) are sequences of positive scalars such that \( \beta_j = O((1 + j)^{-1/q'}) \) and \( \beta'_j = O((1 + j)^{-1/q}) \) as \( j \to \infty \). Let \( \psi := e^{-\alpha} \psi_0 \) and \( \varphi := e^{-\alpha} \varphi_0 \). Then \( \psi \ast \varphi \equiv e^{-\alpha} \) on \([1, \infty)\) and
\[ \|\psi\|_q^q = \|e^{-\alpha} \psi_0\|_q^q = \sum_{j=0}^{\infty} \beta_j \int_j^{j+1} e^{-\alpha s} \, ds \leq \sum_{j=0}^{\infty} e^{-\alpha j} \frac{1}{1 + j} \]
\[ \leq 1 + \int_0^\infty e^{-\alpha s} \frac{1}{1 + s} \, ds = 1 + e^{\alpha} \int_0^\infty e^{-s} \frac{1}{s} \, ds. \]
The constant in the first inequality depends only on \( q \). Since \( \alpha q \leq 1 \),
\[ \|\psi\|_q^q \leq 1 + e^{\alpha q} \left( \int_0^1 \frac{e^{-s}}{s} \, ds + \int_1^\infty \frac{e^{-s}}{s} \, ds \right) \leq 1 + \frac{1}{\alpha q} \frac{1}{s} \, ds + e^{\alpha q} \int_1^\infty e^{-s} \, ds = 1 - \log(\alpha q) + e^{\alpha q} - 1 \leq \log \left( \frac{1}{\alpha} \right) + 2. \]
Moreover, \( \frac{1}{\alpha} \geq q > 1 \) hence \( \log \left( \frac{1}{\alpha} \right) \geq \log(q) > 0 \) and
\[ \log \left( \frac{1}{\alpha} \right) + 2 \leq \left( 1 + \frac{2}{\log(q)} \right) \log \left( \frac{1}{\alpha} \right). \]
Therefore
\[ \|\psi\|_q \leq \log \left( \frac{1}{\alpha} \right)^{1/q} = |\log(\alpha)|^{1/q}, \]
for a constant depending only on $q$. In a similar manner we deduce

$$\|\varphi\|_{q'} \lesssim |\log(\alpha)|^{1/q'}$$

for a constant depending only on $q'$ (and thus on $q$). This yields (A.1).

For the right-hand side of (A.2) we assume that $t = 1$ and, without loss of generality (since $\eta(\alpha, t, q) = \eta(\alpha, t, q')$), that $\alpha > \frac{1}{q}$. Let $\varphi := 1_{[0,1]} e^{\alpha(q-1)}$ and $\psi := \frac{\alpha}{e^{\alpha q} - 1} 1_{\mathbb{R}_+} e^{-\alpha}$. Then

$$\psi * \varphi(r) = \frac{\alpha q}{e^{\alpha q} - 1} \int_0^1 e^{\alpha(q-1)s} e^{-\alpha(r-s)} \, ds = e^{-\alpha r}$$

for $r \geq 1$. Hence

$$\eta(\alpha, 1, q) \leq \|\psi\|_q \|\varphi\|_{q'} = \frac{\alpha q}{e^{\alpha q} - 1} \left( \int_0^1 e^{-\alpha qs} \, ds \right)^{1/q} \left( \int_0^1 e^{\alpha(q-1)qs} \, ds \right)^{1/q'}$$

$$= \left( \frac{\alpha q}{e^{\alpha q} - 1} \right)^{\frac{q-1}{q}} = (e^{\alpha q} - 1)^{-1/q} \leq 2^{1/q} e^{-\alpha} \leq 2 e^{-\alpha},$$

where we have used the assumption $\alpha > \frac{1}{q}$ in the penultimate inequality. \hfill \Box

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