On the Fundamental Principles of Unbounded Functional Calculi

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Dedicated to Wolfgang Arendt and Lutz Weis on the occasion of their retirement

Abstract. In this paper, a new axiomatization for unbounded functional calculi is proposed and the associated theory is elaborated, comprising, among others, uniqueness and compatibility results and extension theorems of algebraic and topological nature. In contrast to earlier approaches, no commutativity assumptions need to be made about the underlying algebras.

In a second part, the abstract theory is illustrated in familiar situations (sectorial operators, semigroup generators). New topological extension theorems are proved for the sectorial calculus and the Hille–Phillips calculus. Moreover, it is shown that the Stieltjes and the Hirsch calculus for sectorial operators are subcalculi of a (small) topological extension of the sectorial calculus.

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1. Introduction

Some instances of functional calculi are as old as modern mathematics. The very concept of a functional calculus, however, even nowadays still remains partly heuristic, with no definitive and widely accepted precise mathematical definition. One reason for this situation lies in the variety of instances such a definition has to cover. In particular those calculi are of interest, where unbounded operators are not just the starting point but the target. These calculi are called \textit{unbounded} in the following, in order to distinguish them from the \textit{bounded} calculi, which by (our) definition are those that yield only bounded operators. (In particular, we do not intend any continuity or suppose any topology when we speak of a bounded calculus here.)

To illustrate our terminology, consider the Borel calculus of a normal operator $A$ on a Hilbert space. Even if $A$ is unbounded, the calculus $f \mapsto f(A)$ yields
bounded operators as long as \( f \) is a bounded function. Hence, restricting the Borel calculus to bounded functions yields a bounded calculus in our terminology. In contrast to that, even if \( A \) is bounded, there are Borel functions \( f \) such that \( f(A) \) is unbounded. Hence, the full Borel calculus is an unbounded functional calculus.

Now, there is little controversy about what a bounded functional calculus should be, namely an algebra homomorphism \( \Phi : \mathcal{F} \to \mathcal{L}(X) \) where \( \mathcal{F} \) is an algebra and \( X \) is a Banach space. Of course, there are obvious variations possible concerning, e.g., the properties of \( \mathcal{F} \) (unital? commutative? a function algebra?) or the space \( X \) (just locally convex?) or topological requirements for \( \mathcal{F} \) and \( \Phi \). All these, however, are somehow unproblematic because the whole situation lies still within the terminological realm of classical representation theory. In contrast, unbounded functional calculi lie beyond that realm, as the set of unbounded operators on a Banach space is not an algebra any more. There is simply no classical terminological framework to cover unbounded calculi.

Surprisingly, despite the ever-growing importance of the functional calculus for sectorial operators since its inception by McIntosh [18] in 1986 and subsequently of related calculi on strips, parabolas and other regions, there have been only a few authors (most notable: deLaubenfels [4]) showing a strong will to operate with a reasonably abstract definition of a functional calculus.

The first attempt, to the best of our knowledge, to not just give an axiomatic definition of an unbounded functional calculus but also to develop the associated abstract theory is due to the author of the present article. It was published in [8] and then incorporated in and made widely known through the book [9]. However, although not without merit and already quite abstract, it has shortcomings. Namely, the definition of a functional calculus given there is intimately tied to a construction, algebraic extension by regularization. At the time, when the book was written, this was natural: algebraic extension is a central tool, an elegant and easily manageable way of reducing an unbounded calculus to its bounded part. Nevertheless, from an advanced point of view it should be obvious that a definition based on a specific construction cannot be regarded the definite answer to the axiomatization problem.

A second shortcoming of the framework from [9] is that only commutative algebras \( \mathcal{F} \) were allowed. Admittedly, the field of applications up to now almost exclusively involves commutative algebras (algebras of scalar functions). But in the future, genuinely non-commutative situations like functional calculi arising from Lie group representations will become more and more important. Hence, there is a desire to have a setting that does not rely on commutativity.

Finally, a third shortcoming of the approach from [9] is that topological ways of extending a functional calculus are disregarded. (This is of course not a failure of the axiomatic framing itself, but of its theoretical elaboration.) Actually, the need for such extensions had been formulated already in [8] and there was also a somewhat halfhearted attempt to provide them, but that did not find much resonance.
With the present article, we are making a new attempt to find an adequate axiomatization of the notion of an (unbounded) functional calculus and to develop its theory, avoiding the named shortcomings of our former one from [9].

The paper is divided into two parts. The first part is devoted to the elaboration of the theory. As such, it is quite abstract and sometimes technical (due to the lack of commutativity assumptions). The second illustrates the theory in some familiar situations, but with a stress on formerly unknown aspects, mainly regarding topological extensions. A reader chiefly interested in the second part may safely skip the technical sections of the first part for the time being and only revert to them when necessary. In the following we give a short synopsis of the two parts.

The first part starts with the axioms of a calculus and their immediate consequences (Section 2) and then proceeds with the introduction of basic auxiliary notions like determination, algebraic cores (Section 3), and anchor sets (Section 4). The main theoretical problem here consists in providing criteria ensuring that an anchor set is actually determining. Whereas this is almost trivially true in a commutative situation (Theorem 4.1), some work is necessary to find such criteria without commutativity (Theorem 4.2). This dichotomy perpetuates also the subsequent Section 5, where the problems of uniqueness and compatibility are addressed.

In Section 6 we discuss the algebraic extension. Surprisingly, no commutativity hypothesis whatsoever is needed to make algebraic extension work (Theorems 6.1 and 6.4). However, compatibility of successive extensions cannot be guaranteed without special assumptions (Theorem 6.7).

In Section 7 we briefly touch upon approximate identities. This had been missed out in [9], a first abstract result was given by Clark [3]. In Section 8 we introduce the concept of a dual calculus. In Section 9 we discuss topological extensions. This concludes the first part.

In the second part, we illustrate the theory with some familiar examples: sectorial operators (Section 10), semigroup generators (Section 13) and normal operators (Section 14). We provide new topological extension theorems for the sectorial calculus (Section 11) and the Hille–Phillips calculus (Section 13.2). In the sectorial case, we show how this topological extension covers calculi defined in the literature like the Stieltjes calculus and the Hirsch calculus (Section 12). For generators of bounded semigroups we show that the Hille–Phillips calculus is included in a certain topological extension of the sectorial calculus (Section 13.3). In the Section 14 on normal operators on Hilbert space we report on a consistent functional calculus approach to the spectral theorem. Details, however, are deferred to a separate paper [12].
Notation and Terminology
We use the letters \( X, Y, \ldots \) generically to denote Banach spaces. By default and unless otherwise stated, the scalar field is \( K = \mathbb{C} \). The space of bounded linear operators from \( X \) to \( Y \) is denoted by \( \mathcal{L}(X; Y) \), and \( \mathcal{L}(X) \) if \( X = Y \).

A subset \( D \subseteq \mathcal{L}(X) \) is called point-separating if
\[
\bigcap_{D \in D} \ker(D) = \{0\}.
\]

A (closed) linear relation in \( X \) is any (closed) linear subspace \( A \subseteq X \oplus X \). Linear relations are called multi-valued operators in [9, Appendix A], and we use freely the definitions and results from that reference. In particular, we say that a bounded operator \( T \) commutes with a linear relation \( A \) if \( TA \subseteq AT \), which is equivalent to
\[
(x, y) \in A \Rightarrow (Tx, Ty) \in A.
\]

If \( E \) is a (multiplicative) semigroup, we denote its center by
\[
Z(E) = \{ d \in E \mid \forall e \in E : de = ed \}.
\]

A proto-calculus \( \Phi : F \rightarrow \mathcal{C}(X) \) is called a calculus if the following fourth axiom is satisfied:
\[
\text{bdd}(F, \Phi) \text{ of } \Phi\text{-bounded elements is determining } \Phi \text{ on } F.
\]

2. Axioms for Functional Calculi
Let \( F \) be an algebra with a unit element \( 1 \) and let \( X \) be a Banach space. A mapping
\[
\Phi : F \rightarrow \mathcal{C}(X)
\]
from \( F \) to the set of closed operators on \( X \) is called a proto-calculus (or: \( F \)-proto-calculus) on \( X \) if the following axioms are satisfied (\( f, g \in F, \lambda \in \mathbb{C} \)):

\[
\begin{align*}
\text{(FC1)} & \quad \Phi(1) = I. \\
\text{(FC2)} & \quad \lambda \Phi(f) \subseteq \Phi(\lambda f) \quad \text{and} \quad \Phi(f) + \Phi(g) \subseteq \Phi(f + g). \\
\text{(FC3)} & \quad \Phi(f)\Phi(g) \subseteq \Phi(fg) \quad \text{and} \\
& \quad \text{dom}(\Phi(f)\Phi(g)) = \text{dom}(\Phi(g)) \cap \text{dom}(\Phi(fg)).
\end{align*}
\]

A proto-calculus \( \Phi : F \rightarrow \mathcal{C}(X) \) is called a calculus if the following fourth axiom is satisfied:

\[
\text{(FC4)} \quad \text{The set bdd}(F, \Phi) \text{ of } \Phi\text{-bounded elements is determining } \Phi \text{ on } F.
\]
Here, an element $f \in \mathcal{F}$ is called $\Phi$-bounded if $\Phi(f) \in \mathcal{L}(X)$, and the set of $\Phi$-bounded elements is

$$
\text{bdd}(\Phi) := \text{bdd}(\mathcal{F}, \Phi) := \{f \in \mathcal{F} \mid \Phi(f) \in \mathcal{L}(X)\} = \Phi^{-1}(\mathcal{L}(X)).
$$

The terminology and meaning of Axiom (FC4) shall be explained in Section 3 below. For the time being we only suppose that $\Phi : \mathcal{F} \to \mathcal{C}(X)$ is a proto-calculus. The following theorem summarizes its basic properties.

**Theorem 2.1.** Let $\Phi : \mathcal{F} \to \mathcal{C}(X)$ be a proto-calculus on a Banach space $X$. Then the following assertions hold ($f, g \in \mathcal{F}, \lambda \in \mathbb{C}$):

a) If $\lambda \neq 0$ or $\Phi(f) \in \mathcal{L}(X)$ then $\Phi(\lambda f) = \lambda \Phi(f)$.

b) If $\Phi(g) \in \mathcal{L}(X)$ then

$$
\Phi(f) + \Phi(g) = \Phi(f + g) \quad \text{and} \quad \Phi(f)\Phi(g) = \Phi(fg).
$$

c) If $fg = 1$ then $\Phi(g)$ is injective and $\Phi(g)^{-1} \subseteq \Phi(f)$. If, in addition, $fg = gf$, then $\Phi(g)^{-1} = \Phi(f)$.

d) The set $\text{bdd}(\mathcal{F}, \Phi)$ of $\Phi$-bounded elements is a unital subalgebra of $\mathcal{F}$ and

$$
\Phi : \text{bdd}(\mathcal{F}, \Phi) \to \mathcal{L}(X)
$$

is an algebra homomorphism.

**Proof.** a) One has

$$
\Phi(f) = \Phi(\lambda^{-1} \lambda f) \supseteq \lambda^{-1} \Phi(\lambda f) \supseteq \lambda^{-1} \lambda \Phi(f) = \Phi(f).
$$

Hence, all inclusions are equalities, and the assertion follows.

b) By Axiom (FC2) and a)

$$
\Phi(f) = \Phi(f + g - g) \supseteq \Phi(f + g) + \Phi(-g) = \Phi(f + g) - \Phi(g)
$$

$$
\supseteq \Phi(f) + \Phi(g) - \Phi(g) = \Phi(f).
$$

Hence, all inclusions are equalities and the first assertion in b) follows. For the second, note that by Axiom (FC3) $\Phi(f)\Phi(g) \subseteq \Phi(fg)$ with

$$
\text{dom}(\Phi(f)\Phi(g)) = \text{dom}(\Phi(g)) \cap \text{dom}(\Phi(fg)) = \text{dom}(\Phi(fg)),
$$

hence we are done.

c) By (FC3), if $fg = 1$ then $\Phi(f)\Phi(g) \subseteq \Phi(fg) = \Phi(1) = 1$. Hence, $\Phi(g)$ is injective and $\Phi(f) \supseteq \Phi(g)^{-1}$. If $fg = gf$, by symmetry $\Phi(f)$ is injective too, and $\Phi(g) \supseteq \Phi(f)^{-1}$. This yields $\Phi(f) = \Phi(g)^{-1}$ as desired.

d) follows directly from b).
3. Determination

We shall write
\[ [f]_E := \{ e \in E \mid ef \in E \} \]
whenever \( F \) is any multiplicative semigroup, \( E \subseteq F \) and \( f \in F \). In our context, \( F \) shall always be an algebra.

Given a proto-calculus \( \Phi : F \to C(X) \) and \( f \in F \) the set of its \( \Phi \)-regularizers is
\[
\text{reg}(f, \Phi) := [f]_{ \text{bd}(F, \Phi) } = \{ e \in F \mid e, ef \in \text{bd}(F, \Phi) \}.
\]
By Theorem 2.1, \( \text{reg}(f, \Phi) \) is a left ideal in \( \text{bd}(F, \Phi) \). The elements of the set
\[
\text{reg}(\Phi) := \bigcap_{f \in F} \text{reg}(f, \Phi)
\]
are called \textbf{universal regularizers}. Of course, it may happen that \( e = 0 \) is the only universal regularizer.

Remark 3.1. The definition of a regularizer here differs from and is more general than the one given in [9]. It has been argued in [10] (eventually published as [11]) that such a relaxation of terminology is useful.

Let \( f \in F \). A subset \( M \subseteq \text{bd}(F, \Phi) \) is said to determine \( \Phi(f) \) if
\[
\Phi(f)x = y \iff \forall e \in M \cap \text{reg}(f, \Phi) : \Phi(ef)x = \Phi(e)y \quad (3.2)
\]
for all \( x, y \in X \). And \( M \) is said to strongly determine \( \Phi(f) \) if the set \( [f]_M \) determines \( \Phi(f) \), i.e. if
\[
\Phi(f)x = y \iff \forall e \in [f]_M : \Phi(ef)x = \Phi(e)y \quad (3.3)
\]
for all \( x, y \in X \).

Remarks 3.2. 1) Although very useful, the terminology “\( M \) determines \( \Phi(f) \)” is to be used with caution: there might be \( g \in F \) with \( f \neq g \) and \( \Phi(g) = \Phi(f) \) and such that \( \Phi(g) \) is not determined by \( M \) in the above sense. (In other words, the expression “\( \Phi(f) \)” has to be interpreted symbolically here, and not as a mathematical object.) With this caveat in mind, there should be little danger of confusion. However, if we want to be completely accurate, we shall use the alternative formulation “\( M \) determines \( \Phi \) at \( f \)”.

2) We observe that in both equivalences (3.2) and (3.3) only the implication “\( \Rightarrow \)” is relevant, as the implication “\( \Rightarrow \)” simply means \( \Phi(e)\Phi(f) \subseteq \Phi(ef) \) for \( e \in M \cap \text{reg}(f, \Phi) \) or \( e \in [f]_M \), respectively; and this follows from Axiom (FC3).

An immediate consequence of this observation is that a strongly determining set is determining.
A set $E \subseteq \text{bdd}(F, \Phi)$ is said to be \textbf{determining} $\Phi$ on $F$ if it determines $\Phi(f)$ for each $f \in F$. (If $F$ is understood, reference to it is often dropped, and one simply speaks of $\Phi$-determining sets.) Note that by Remark 3.2, any superset of a determining set is again determining.

A subset $E$ of $\text{bdd}(F, \Phi)$ is called an \textbf{algebraic core} for $\Phi$ on $F$ if it strongly determines $\Phi(f)$ for each $f \in F$. In this terminology, Axiom (FC4) simply requires $\text{bdd}(F, \Phi)$ to be an algebraic core for $\Phi$ on $F$. Again, by Remark 3.2 any superset of an algebraic core is also an algebraic core.

\textbf{Example 3.3 (Bounded (Proto-)Calculi).} A proto-calculus $\Phi : F \to C(X)$ with $F = \text{bdd}(F, \Phi)$ is nothing else than a unital algebra representation $\Phi : F \to \mathcal{L}(X)$. By unitality, Axiom (FC4) is automatically satisfied in this situation. So each unital representation by bounded operators is a calculus.

\section{Anchor Sets}

Let $E$ be a set and let $\Phi : E \to \mathcal{L}(X)$ be any mapping. An element $e \in E$ is called an \textbf{anchor element} if $\Phi(e)$ is injective. More generally, a subset $M \subseteq E$ is called an \textbf{anchor set} if $M \neq \emptyset$ and the set $\{\Phi(e) \mid e \in M\}$ is point-separating, i.e., if

$$\bigcap_{e \in M} \ker(\Phi(e)) = \{0\}.$$

If $F$ is a semigroup, then we say that $f \in F$ is \textbf{anchored} in $E$ (with respect to $\Phi$) if the set $[f]_E$ is an anchor set. And we call $F$ \textbf{anchored} in $E$ if each $f \in F$ is anchored in $E$. If we want to stress the dependence on $\Phi$, we shall speak of $\Phi$-anchor elements/sets and of elements/sets being $\Phi$-anchored in $E$.

Suppose that $\Phi : F \to \mathcal{C}(X)$ is a proto-calculus and $f \in F$. Any set $M \subseteq \text{bdd}(\Phi, F)$ which determines $\Phi(f)$ must be an anchor set, just because $\Phi(f)$ is an operator and not just a linear relation. The converse question, i.e., whether a particular anchor set does actually determine $\Phi(f)$ is the subject of the present section. We start with a simple case.

\textbf{Theorem 4.1.} Let $\Phi : F \to \mathcal{C}(X)$ be a calculus and let $f \in F$. If $F$ is commutative and $E \subseteq \text{reg}(f, \Phi)$ is an anchor set, then $E$ determines $\Phi(f)$.

\textbf{Proof.} Let $x, y \in X$ such that $\Phi(e)f)x = \Phi(e)y$ for all $e \in E$. Then for all $e \in E$ and $g \in \text{reg}(\Phi, F)$ we have

$$\Phi(e)\Phi(gf)x = \Phi(egf)x = \Phi(g(ef)x = \Phi(g)\Phi(e)f)x = \Phi(g)\Phi(e)y = \Phi(g)e)y = \Phi(eg)y = \Phi(e)\Phi(g)y.$$

Since $E$ is an anchor set, it follows that $\Phi(gf)x = \Phi(g)y$ for all $g \in \text{reg}(\Phi, F)$. Since $\Phi$ is a calculus, this implies $\Phi(f)x = y$. \hfill \square

Without commutativity, things become much more complicated. It actually may come as a surprise that the following result is true in general.
Theorem 4.2. Let $\Phi : F \to \mathcal{C}(X)$ be a calculus, and let $E \subseteq \text{bdd}(F, \Phi)$ be such that $\text{bdd}(F, \Phi)$ is anchored in $E$. Then $E$ determines $\Phi$ on $F$. If, in addition, $E$ is multiplicative, then $E$ is an algebraic core for $\Phi$.

For the proof of Theorem 4.2 we need some auxiliary results about determining sets and anchor sets. This is the subject of the following lemma.

Lemma 4.3. Let $\Phi : F \to \mathcal{C}(X)$ be a proto-calculus, let $f \in F$, let $M \subseteq \text{bdd}(F, \Phi)$ and $B \subseteq F$. Then the following assertions hold:

a) If $M$ determines $\Phi(f)$, then it is an anchor set. The converse holds if $f \in \text{bdd}(F, \Phi)$.

b) If $M$ determines $\Phi(f)$ (is an anchor set) and $M \subseteq N \subseteq \text{bdd}(F, \Phi)$ then $N$ determines $\Phi(f)$ (is an anchor set).

c) If $M$ determines $\Phi(f)$ (is an anchor set) and $N_g \subseteq \text{bdd}(F, \Phi)$ is an anchor set for each $g \in M$, then also $N := \bigcup_{g \in M} N_g$ is determines $\Phi(f)$ (is an anchor set).

d) Suppose that $BM = \{ bg \mid b \in B, g \in M \} \subseteq \text{bdd}(\Phi, F)$. If $B \cdot M$ is an anchor set, then so is $M$. If $M \subseteq \text{reg}(f, \Phi)$ and $BM$ determines $\Phi(f)$, then so does $M$.

e) If $T \in \mathcal{L}(X)$ commutes with all operators $\Phi(e)$ and $\Phi(ef)$ for $e \in M$, and $M$ determines $\Phi(f)$, then $T$ commutes with $\Phi(f)$.

Proof. a) The first assertion follows from (3.2) and the fact that $\Phi(f)$ is an operator and not just a linear relation. For the second suppose that $\Phi(f) \in \mathcal{L}(X)$ and $M$ is an anchor set. Let $x, y \in X$ such that $\Phi(ef)x = \Phi(e)y$ for all $e \in M$. Then, since $\Phi(f)$ is bounded,

$$\Phi(e)y = \Phi(ef)x = \Phi(e)\Phi(f)x.$$ 

Since $M$ is an anchor set, it follows that $\Phi(f)x = y$.

b) is trivial (and has already been mentioned above).

c) Suppose first that $M$ determines $\Phi(f)$ and that $\Phi(egf)x = \Phi(eg)y$ for all $g \in M$ and all $e \in N_g$ such that $eg \in \text{reg}(f, \Phi)$. Fix $g \in M \cap \text{reg}(f, \Phi)$. Then for each $e \in N_g$ we have $eg \in \text{reg}(f, \Phi)$ and hence

$$\Phi(e)\Phi(gf)x = \Phi(ef)x = \Phi(eg)y = \Phi(e)\Phi(g)y.$$ 

Since $N_g$ is an anchor set, it follows that $\Phi(gf)x = \Phi(g)y$. Since $M$ determines $\Phi(f)$, it follows that $\Phi(f)x = y$ as claimed.

Similary (and even more easily) one proves that $N$ is an anchor set if $M$ is one.

d) Suppose that $B \cdot M$ is an anchor set. For each $b \in B$ and $f \in M$ we have $\Phi(bf) = \Phi(b)\Phi(f)$ and hence $\ker(\Phi(f)) \subseteq \ker(\Phi(bf))$. It follows readily that $M$ is an anchor set.
Now suppose that $\mathcal{M} \subseteq \text{reg}(f, \Phi)$ and that $\mathcal{B}\mathcal{M}$ determines $\Phi(f)$. Let $x, y \in X$ such that $\Phi(gf)x = \Phi(g)y$ for all $g \in \mathcal{M}$. Then $\Phi(bgf)x = \Phi(b)\Phi(gf)x = \Phi(b)\Phi(g)y = \Phi(bg)y$ for all $b \in \mathcal{B}$ and $g \in \mathcal{M}$. Hence, by hypothesis, $\Phi(x) = y$.

e) Suppose that $\Phi(f)x = y$. Then, for each $e \in \mathcal{M}$,

$$\Phi(ef)Tx = T\Phi(efx) = T\Phi(ex) = \Phi(ex)y.$$

Since $\mathcal{M}$ determines $\Phi(f)$, $\Phi(f)Tx = Ty$ as claimed. □

Proposition 4.4. Let $\Phi : \mathcal{F} \to \mathcal{C}(X)$ be a proto-calculus, let $f \in \mathcal{F}$, and let $\mathcal{M}, \mathcal{E} \subseteq \text{bdd}(\Phi, \mathcal{F})$ such that $\mathcal{M}$ determines $\Phi(f)$. Suppose that one of the following conditions holds:

1) $\mathcal{M}$ is anchored in $\mathcal{E}$.

2) For each $g \in \mathcal{M}$ there is an anchor set $\mathcal{N}_g$ such that $\mathcal{N}_g \subseteq \mathcal{B}\mathcal{E}$, where $\mathcal{B} := \{ h \in \mathcal{F} \mid h\mathcal{E} \subseteq \text{bdd}(\Phi, \mathcal{F}) \}$.

Then $\mathcal{E}$ determines $\Phi(f)$.

Proof. 1) For each $g \in \mathcal{M}' := \mathcal{M} \cap \text{reg}(f, \Phi)$ let $\mathcal{N}_g := \mathcal{B}\mathcal{E}$. By hypothesis, this is an anchor set. Hence, by Lemma 4.3(c), $\mathcal{N} := \bigcup_{g \in \mathcal{M}'} \mathcal{N}_g$ determines $\Phi(f)$. As $\mathcal{N} \subseteq \mathcal{E} \cap \text{reg}(f, \Phi)$, also $\mathcal{E}$ determines $\Phi(f)$.

2) By hypothesis, $\mathcal{E}$ is an anchor set and $\mathcal{M}$ determines $\Phi(f)$. Hence, by Lemma 4.3(c), $\mathcal{E}\mathcal{M} = \bigcup_{g \in \mathcal{M}} \mathcal{E}\mathcal{M}g$ determines $\Phi(f)$. Since $\mathcal{E}\mathcal{M} \subseteq \mathcal{B}\mathcal{E}$, also the latter set determines $\Phi(f)$. Lemma 4.3(d) then implies that $\mathcal{E}$ determines $\Phi(f)$. □

Remark 4.5. Theorem 4.1 is a corollary of Proposition 4.4 (Apply part 2) with $\mathcal{N}_g = \mathcal{E}$ for each $g \in \mathcal{M} := \text{reg}(\Phi, \mathcal{F})$.)

Actually, we realize that one may replace the overall commutativity assumption on $\mathcal{F}$ by a weaker condition, e.g.: $[f]_\mathcal{E} \cap \{ e \in \mathcal{E} \mid e\mathcal{M} \subseteq \mathcal{M}e \}$ is an anchor set.

Finally, we are going to prove Theorem 4.2.

Proof of Theorem 4.2. Write $\mathcal{M} := \text{bdd}(\Phi, \mathcal{F})$ and fix $f \in \mathcal{F}$. Then $\mathcal{M}$ determines $\Phi(f)$, since $\Phi$ is a calculus, and $\mathcal{M}$ is anchored in $\mathcal{E}$, by assumption. Hence, by Proposition 4.4(a), $\mathcal{E}$ determines $\Phi(f)$.

For the second assertion, suppose in addition that $\mathcal{E}$ is multiplicative. Fix $g \in \text{reg}(f, \Phi)$. Then, as above, $[g]_\mathcal{E}$ is an anchor set. Also, for each $e \in [g]_\mathcal{E}$ one has $egf \in \mathcal{M}$ and hence $[egf]_\mathcal{E}$ is also an anchor set. It follows that $\mathcal{N}_g := \bigcup_{e \in [g]_\mathcal{E}} \mathcal{N}_g e$ is an anchor set. By Lemma 4.3(c), the set

$$\mathcal{N} := \bigcup_{g \in \text{reg}(f, \Phi)} \mathcal{N}_g g$$

determines $\Phi(f)$.
determines $\Phi(f)$. But $E$ is multiplicative, and therefore

$$N = \bigcup_{g \in \text{reg}(f, \Phi)} N_g \subseteq \bigcup_{g \in \text{reg}(f, \Phi)} \bigcup_{e \in [g]E} [egf]E \subseteq [f]E.$$ 

Hence, also $[f]E$ determines $\Phi(f)$. \hfill $\square$

5. Uniqueness, Restriction, Compatibility

Let $\Phi_1, \Phi_2 : F \to \mathcal{C}(X)$ be calculi, let $f \in F$ and suppose that $M \subseteq F$ determines both calculi $\Phi_1$ and $\Phi_2$ at $f$ (cf. Remark 3.2). Suppose further that $\Phi_1$ and $\Phi_2$ agree on $M$. Can one conclude that $\Phi_1(f) = \Phi_2(f)$?

A moment’s reflection reveals that the answer might be “no” in general. The reason is that we do not know whether $\Phi_1$ and $\Phi_2$ coincide on products $ef$ for $e \in M$. This is the original motivation for introducing the concept of strong determination.

Lemma 5.1. Suppose that $\Phi_1, \Phi_2 : F \to \mathcal{C}(X)$ are two proto-calculi and $E \subseteq F$ such that $\Phi_1|_E = \Phi_2|_E$. If $E$ is an algebraic core for $\Phi_2$ then

$$\Phi_1(f) \subseteq \Phi_2(f) \quad \text{for each } f \in F.$$ 

In particular, if $E$ is an algebraic core for both calculi, then $\Phi_1 = \Phi_2$.

Proof. Let $f \in F$. And suppose that $x, y \in X$ are such that $\Phi_1(f)x = y$. Then for every $e \in [f]E$ we have

$$\Phi_1(ef)x = \Phi_1(e)y.$$ 

This is the same as $\Phi_2(ef)x = \Phi_2(e)y$, as $\Phi_1$ and $\Phi_2$ agree on $E$. By hypothesis, $[f]E$ determines $\Phi_2(f)$, so it follows that $\Phi_2(f)x = y$. This shows $\Phi_1(f) \subseteq \Phi_2(f)$. \hfill $\square$

Combining Lemma 5.1 with Theorem 4.2 we arrive at the following uniqueness statement.

Theorem 5.2 (Uniqueness). Let $\Phi_1, \Phi_2 : F \to \mathcal{C}(X)$ be calculi. Suppose that there is $E \subseteq F$ with the following properties:

1) $\Phi_1(e) = \Phi_2(e) \in \mathcal{C}(X)$ for all $e \in E$.

2) $F$ is anchored in $E$ (with respect to one/both calculi).

Then $\Phi_1 = \Phi_2$.

Proof. By passing to

$$E' := \bigcup_{n \geq 1} E^n = \{ e_1 \cdots e_n \mid n \in \mathbb{N}, e_1, \ldots, e_n \in E \},$$

the multiplicative semigroup generated by $E$ in $F$, we may suppose that $E$ is multiplicative. Then Theorem 4.2 yields that $E$ is an algebraic core for both calculi. Hence, Lemma 5.1 yields $\Phi_1 = \Phi_2$. \hfill $\square$
5.1. Pull-Back and Restriction of a Calculus

Suppose that $F$ is a unital algebra and $\Phi : F \to \mathcal{C}(X)$ is a proto-calculus. Further, let $G$ be a unital algebra and $\eta : G \to F$ a unital algebra homomorphism. Then the mapping

$$\eta^* \Phi : G \to \mathcal{C}(X), \quad (\eta^* \Phi)(g) := \Phi(\eta(g))$$

is called the pull-back of $\Phi$ along $\eta$. It is easy to see that, in general, $\eta^* \Phi$ is a proto-calculus as well. A special case occurs if $G$ is a subalgebra of $F$ and $\eta$ is the inclusion mapping. Then $\eta^* \Phi = \Phi|_G$ is just the restriction of $\Phi$ to $G$.

Lemma 5.3. Let $F$ be a unital algebra and $\Phi : F \to \mathcal{C}(X)$ a proto-calculus. Furthermore, let $G$ be a unital algebra, $\eta : G \to F$ a unital homomorphism, $E \subseteq \text{bdd}(\eta^* \Phi, G)$, and $g \in G$. Then the following assertions hold.

a) $\text{bdd}(\eta^* \Phi, G) = \eta^{-1}(\text{bdd}(\Phi, F))$,

b) $\eta(\text{bdd}(\eta^* \Phi, G)) = \text{bdd}(\Phi, F) \cap \eta(G) = \text{bdd}(\Phi|_G, \eta(G))$.

c) $\eta([g]_E) \subseteq [\eta(g)]_{\eta(E)}$.

d) $E$ determines $\eta^* \Phi$ at $g$ if and only if $\eta(E)$ determines $\Phi$ at $\eta(g)$.

e) If $E$ is an algebraic core for $\eta^* \Phi$ then $\eta(E)$ is an algebraic core for $\Phi|_G$.

Proof. a), b), and c) follow directly from the definition of $\eta^* \Phi$. d) This follows since for $x, y \in X$ and $e \in E$

$$\Phi(\eta(e)\eta(g))x = \Phi(\eta(e))y \iff (\eta^* \Phi)(eg)x = (\eta^* \Phi)(eg)y.$$

e) This follows from c) and d). □

We have already remarked that $\eta^* \Phi$ is a proto-calculus, whenever $\Phi$ is. The following example shows that, even if $\Phi$ is a calculus, $\eta^* \Phi$ need not be one.

Example 5.4. Let $A$ be an unbounded operator with non-empty resolvent set $\rho(A)$. Let $F$ be the algebra of all rational functions with poles in $\rho(A)$ and let $\Phi$ be the natural calculus (as described, e.g., in [9, Appendix A.6]). Let $G$ be the algebra of polynomial functions and $\eta : G \to F$ the inclusion mapping. Then $\eta^* \Phi = \Phi|_G$ is simply the restriction of $\Phi$ to $G$. And this is not a calculus, as the only functions in $G$ that yield bounded operators are the constant ones.

We say that $\eta$ is $\Phi$-regular if $\eta^* \Phi$ is a calculus. And a subalgebra $G$ of $F$ is called $\Phi$-regular, if the restriction of $\Phi$ to $G$ is a calculus, i.e., if the inclusion mapping is $\Phi$-regular.

Corollary 5.5. In the situation of Lemma 5.3 the following statements are equivalent:

(i) $\eta$ is a $\Phi$-regular mapping, i.e., $\eta^* \Phi$ is a calculus.

(ii) $\eta(G)$ is a $\Phi$-regular subalgebra of $F$, i.e., $\Phi|_{\eta(G)}$ is a calculus.

Proof. This follows from a) and d) of Lemma 5.3 with $E = \text{bdd}(\eta^* \Phi)$. □
It seems that, in general, one cannot say much more. However, here is an interesting special case, when one can simplify assumptions.

**Theorem 5.6.** Let \( F \) be a commutative unital algebra and \( \Phi : F \to \mathcal{C}(X) \) a calculus. Let \( \mathcal{G} \) be a unital subalgebra of \( F \) such that \( \text{reg}(g, \Phi) \cap \mathcal{G} \) is an anchor set for each \( g \in \mathcal{G} \). Then \( \mathcal{G} \) is \( \Phi \)-regular, i.e., \( \Phi|_{\mathcal{G}} \) is a calculus.

**Proof.** This follows immediately from Theorem 4.1. \( \Box \)

In view of Lemma 5.3 we obtain the following consequence.

**Corollary 5.7.** Let \( F \) be a commutative unital algebra and \( \Phi : F \to \mathcal{C}(X) \) a calculus. Furthermore, let \( \mathcal{G} \) be a unital algebra and \( \eta : \mathcal{G} \to F \) a unital homomorphism. Then \( \eta \) is \( \Phi \)-regular if and only if for each \( g \in \mathcal{G} \) the set \( \text{reg}(\eta(g), \Phi) \cap \eta(\mathcal{G}) \) is an anchor set.

5.2. Compatibility and Composition Rules

Suppose one has set up a functional calculus \( \Phi = (f \mapsto f(A)) \) for an operator \( A \) and a second functional calculus \( \Psi = (g \mapsto g(B)) \) for an operator \( B \) which is of the form \( B = f(A) \). Then one would expect a “composition rule” of the form \( g(B) = (g \circ f)(A) \). This amounts to the identity \( \Psi = \eta^*\Phi \), where \( \eta = (g \mapsto g \circ f) \) is an algebra homomorphism that links the domain algebras of the two calculi. The following theorem, which basically is just a combination of results obtained so far, yields criteria for this composition rule to hold true.

**Theorem 5.8.** Let \( F \) and \( G \) be unital algebras and \( \eta : \mathcal{G} \to F \) a unital algebra homomorphism. Furthermore, let \( \Phi : F \to \mathcal{C}(X) \) and \( \Psi : \mathcal{G} \to \mathcal{C}(X) \) be protocalculi, and let \( \mathcal{E} \) be an algebraic core for \( \Psi \) such that \( \Phi(\eta(e)) = \Psi(e) \) (\( e \in \mathcal{E} \)).

Then the following statements are equivalent:

(i) \( \Phi \circ \eta = \Psi \).

(ii) \( \eta(\mathcal{G}) \) is a \( \Phi \)-regular subalgebra of \( F \).

(iii) \( \eta(\mathcal{E}) \) is an algebraic core for the restriction of \( \Phi \) to \( \eta(\mathcal{G}) \).

Moreover, (i)-(iii) hold true if, e.g., \( \Phi \) is a calculus and \( F \) is commutative.

**Proof.** (i)\(\Rightarrow\)(ii): Since \( \mathcal{E} \) is an algebraic core for \( \Psi \) and, by (i), \( \Psi = \eta^*\Phi \), the set \( \eta(\mathcal{E}) \) must be an algebraic core for \( \Phi \) on \( \eta(\mathcal{G}) \), by e) of Lemma 5.3

(iii)\(\Rightarrow\)(i): If (iii) holds then (ii) follows fortiori.

(ii)\(\Rightarrow\)(i): If (ii) holds that \( \eta^*\Phi \) is a calculus. Also, by hypothesis, \( \Psi \) is a calculus. These calculi agree on \( \mathcal{E} \), and this is an anchor (since it is an algebraic core for \( \Psi \)). Hence, by the Uniqueness Theorem 5.2 \( \eta^*\Phi = \Psi \), i.e., (i).

Finally, suppose that \( \Phi \) is a calculus and \( F \) is commutative. Let \( g \in \mathcal{G} \). Then \([g]\mathcal{E} \) is a \( \Psi \)-anchor set. Hence, \( [\eta(g)]|_{\eta(\mathcal{G})} \) is a \( \Phi \)-anchor set. By Corollary 5.7, \( \eta \) is \( \Phi \)-regular, i.e., (ii). \( \Box \)
6. Algebraic Extension

From now on, we suppose that $E$, $F$ and $\Phi$ are such that

1) $F$ is a unital algebra,
2) $E$ is a (not necessarily unital) subalgebra of $F$,
3) $\Phi : E \to \mathcal{L}(X)$ is an algebra representation.

Our goal is to give conditions on $F$ such that a given representation $\Phi : E \to \mathcal{L}(X)$ can be extended to an $F$-calculus in a unique way. A glance at Theorem 4.2 leads us to hope that it might be helpful to require in addition to 1)–3) also:

4) Each $f \in F$ is anchored in $E$.

The next result tells that under these assumptions there is indeed a unique calculus on $F$ extending $\Phi$.

**Theorem 6.1 (Extension Theorem).** Let $F$ be a unital algebra and $E \subseteq F$ a subalgebra. Furthermore, let $X$ be a Banach space and let $\Phi : E \to \mathcal{L}(X)$ be an algebra homomorphism such that $F$ is anchored in $E$. Then there is a unique calculus $\hat{\Phi} : F \to \mathcal{C}(X)$ such that $\hat{\Phi}|_E = \Phi$.

**Proof.** Uniqueness follows directly from Theorem 5.2. Moreover, since $E$ is multiplicative, for each $f \in F$ the set $[f]_E$ must determine $\hat{\Phi}(f)$. Hence, for the existence proof we have no choice as to define

$$
\hat{\Phi}(f)x = y \iff \forall e \in [f]_E : \Phi(e)x = \Phi(e)y
$$

(6.1) for any $x, y \in X$ and $f \in F$. Note that since $[f]_E$ is an anchor set, $\hat{\Phi}(f) \in \mathcal{C}(X)$. It remains to show that $\hat{\Phi}$ extends $\Phi$ and satisfies (FC1)–(FC4).

(FC1): By hypothesis, $[1]_E = E$ is an anchor set. Hence, $x = y$ is equivalent to

$$
\Phi(e)x = \Phi(e)y \quad \text{for all } e \in E,
$$

which, by definition (6.1) is equivalent to $\hat{\Phi}(1)x = y$.

Next, let us show that $\hat{\Phi}$ extends $\Phi$. To that end, let $f \in E$. Then $[f]_E = E$, and $\hat{\Phi}(f)x = y$ is equivalent to

$$
\Phi(e)y = \Phi(e)f)x = \Phi(e)\Phi(f)x \quad \text{for all } e \in E,
$$

which is equivalent to $y = \Phi(f)x$ (since $E$ is an anchor).

(FC2): Let $\lambda \in \mathbb{C}$ and $f \in F$ and take $x, y \in X$ with $\lambda \hat{\Phi}(f)x = y$. We need to show that $\hat{\Phi}(\lambda f)x = y$. This is clear if $\lambda = 0$. If $\lambda \neq 0$ we find $\hat{\Phi}(f)x = \lambda^{-1}y$ and hence $\Phi(e)f)x = \Phi(e)(\lambda^{-1}y)$, or better

$$
\Phi(e(\lambda f))x = \Phi(e)y
$$
for every $e \in [f]_\mathcal{E} = [\lambda f]_\mathcal{E}$. And the latter statement just tells that $\hat{\Phi}(\lambda f)x = y$, as desired.

Now pick $f, g \in \mathcal{F}$ and suppose that $\hat{\Phi}(f)x = y$ and $\hat{\Phi}(g)x = z$. Take $h \in [f + g]_\mathcal{E}$ and $e \in [hf]_\mathcal{E}$. Then $eh \in [f]_\mathcal{E} \cap [g]_\mathcal{E}$ and hence

$$\Phi(efh)x = \Phi(ef)y \quad \text{and} \quad \Phi(efg)x = \Phi(ef)z.$$  

This yields

$$\Phi(e)\Phi(h(f + g))x = \Phi(efh)x + \Phi(efg)x = \Phi(ef)(y + z) = \Phi(e)\Phi(h)(y + z).$$

Since $[hf]_\mathcal{E}$ is an anchor set, it follows that

$$\Phi(h(f + g))x = \Phi(h)(z + y)$$

and since $h \in [f + g]_\mathcal{E}$ was arbitrary, we arrive at $\hat{\Phi}(f + g)x = y$.

(FC3): Let $\hat{\Phi}(g)x = y$ and $\hat{\Phi}(f)y = z$, and let $h \in [fg]_\mathcal{E}$. Then for each $e \in [hfg]_\mathcal{E}$ and $e' \in [ehf]_\mathcal{E}$ one has $e'eh \in [f]_\mathcal{E}$ and $e'ehf \in [g]_\mathcal{E}$ and hence

$$\Phi(e')\Phi(e)\Phi(hfg)x = \Phi(e'ehfg)x = \Phi(e'ehf)y = \Phi(e'eh)z = \Phi(e')\Phi(e)\Phi(h)z.$$  

Since $[ehf]_\mathcal{E}$ is anchor set, $\Phi(e)\Phi(hfg)x = \Phi(e)\Phi(h)(y + z) = \Phi(e')\Phi(e)\Phi(h)(y + z)$. All in all we conclude that $\hat{\Phi}(fg)x = z$. This proves the inclusion

$$\hat{\Phi}(f)\hat{\Phi}(g) \subseteq \hat{\Phi}(fg).$$

A corollary to that is the domain inclusion

$$\operatorname{dom}(\hat{\Phi}(f)\hat{\Phi}(g)) \subseteq \operatorname{dom}(\hat{\Phi}(f)) \cap \operatorname{dom}(\hat{\Phi}(g)).$$

For the converse, suppose that $x \in \operatorname{dom}(\hat{\Phi}(g)) \cap \operatorname{dom}(\hat{\Phi}(fg))$ and define $y, z \in X$ by

$$\hat{\Phi}(g)x = y \quad \text{and} \quad \hat{\Phi}(fg)x = z.$$  

Let $e \in [f]_\mathcal{E}$ and $e' \in [efg]_\mathcal{E}$. Then $e'ef \in [g]_\mathcal{E}$ and hence $\Phi(e'efg)x = \Phi(e'ef)y$. Also, $e'e \in [fg]_\mathcal{E}$ and hence $\Phi(e'efg)x = \Phi(e'ef)z$. It follows that

$$\Phi(e')\Phi(ef)y = \Phi(e'ef)y = \Phi(e'efg)x = \Phi(e'ef)z = \Phi(e')\Phi(e)z.$$  

Since $[efg]_\mathcal{E}$ is an anchor set, $\Phi(e)\Phi(g)y = \Phi(e)z$. Since $e \in [f]_\mathcal{E}$ was arbitrary, $\hat{\Phi}(f)y = z$, and hence $x \in \operatorname{dom}(\hat{\Phi}(f)\hat{\Phi}(g))$.

(FC4) is satisfied by construction. This concludes the proof. □
6.1. The Maximal Anchored Subalgebra

In practice, \( \mathcal{F} \) may be too large and may fail to satisfy the anchor-condition 4). In this case one might look for the maximal subalgebra of \( \mathcal{F} \) which is anchored in \( \mathcal{E} \). However, it is not obvious that such an object exists. To see that it does, let us define

\[
\langle \mathcal{E}, \mathcal{F}, \Phi \rangle := \{ f \in \mathcal{F} : \forall e \in \mathcal{E} : [ef]_\mathcal{E} \text{ is a } \Phi\text{-anchor set} \}.
\] (6.2)

**Lemma 6.2.** Let \( \mathcal{F} \) be a unital algebra, \( \mathcal{E} \subseteq \mathcal{F} \) a subalgebra and \( \Phi : \mathcal{E} \to \mathcal{L}(X) \) an algebra representation. Then the following statements are equivalent:

(i) \( \mathcal{E} \) is an anchor set.
(ii) \( 1 \) is anchored in \( \mathcal{E} \).
(iii) \( \mathcal{E} \neq \emptyset \) and \( \langle \mathcal{E}, \mathcal{F}, \Phi \rangle \neq \emptyset \).

**Proof.** Straightforward. \( \square \)

The algebra representation \( \Phi : \mathcal{E} \to \mathcal{L}(X) \) is called **non-degenerate** if (i)–(iii) from Lemma 6.2 are satisfied, otherwise **degenerate**.

**Remark 6.3.** If \( \Phi \) is degenerate then there are two possibilities: 1st case: \( 1 \notin \mathcal{E} \). Then \( \mathcal{E}' := \mathcal{E} \oplus \mathbb{C}1 \) is a unital subalgebra of \( \mathcal{F} \) and by

\[
\hat{\Phi}(f) := \Phi(e) + \lambda 1, \quad f = e + \lambda 1, \; e \in \mathcal{E}, \; \lambda \in \mathbb{C}
\]
a unital representation \( \hat{\Phi} : \mathcal{E} \oplus \mathbb{C}1 \to \mathcal{L}(X) \) is defined. This new representation is clearly non-degenerate if \( X \neq \{0\} \). 2nd case: \( 1 \in \mathcal{E} \). Then \( P := \Phi(1) \) is a projection and one can restrict the representation to \( \mathcal{L}(Y) \), where \( Y := \text{ran}(P) \).

All in all we see that degenerate representations can be neglected.

**Theorem 6.4.** Let \( \mathcal{F} \) be unital algebra, \( \mathcal{E} \subseteq \mathcal{F} \) a subalgebra and \( \Phi : \mathcal{E} \to \mathcal{L}(X) \) a non-degenerate algebra representation. Then \( \langle \mathcal{E}, \mathcal{F}, \Phi \rangle \) is a unital subalgebra of \( \mathcal{F} \) containing \( \mathcal{E} \) and anchored in \( \mathcal{E} \). Moreover, \( \langle \mathcal{E}, \mathcal{F}, \Phi \rangle \) contains each unital subalgebra of \( \mathcal{F} \) with this property.

**Proof.** For the proof we abbreviate \( \mathcal{F}' := \langle \mathcal{E}, \mathcal{F}, \Phi \rangle \).

Suppose that \( \mathcal{F}_0 \) is a unital subalgebra of \( \mathcal{F} \) that contains \( \mathcal{E} \) and is anchored in \( \mathcal{E} \). If \( f \in \mathcal{F}_0 \) and \( e \in \mathcal{E} \) then \( ef \in \mathcal{F}_0 \) again and hence \( [ef]_{\mathcal{E}} \) is an anchor set. This shows that \( \mathcal{F}_0 \subseteq \mathcal{F}' \).

As \( \Phi \) is non-degenerate, \( \mathcal{E} \subseteq \mathcal{F}' \) and \( 1 \in \mathcal{F}' \). Let \( f \in \mathcal{F}' \). Then

\[
\bigcup_{e \in \mathcal{E}} [ef]_{\mathcal{E}} e \subseteq [f]_{\mathcal{E}}.
\]

For each \( e \in \mathcal{E} \), \( [ef]_{\mathcal{E}} \) is an anchor set (since \( f \in \mathcal{F}' \)) and \( \mathcal{E} \) is an anchor set (since \( \Phi \) is non-degenerate). It follows that \( [f]_{\mathcal{E}} \) is an anchor set as well. Since \( f \in \mathcal{F}' \) was arbitrary, \( \mathcal{F}' \) is anchored in \( \mathcal{E} \).
It remains to show that \( F' \) is a subalgebra of \( F \). To this end, fix \( f, g \in F' \). Then

\[
\bigcup_{e \in [f]_E} [eg]_E \subseteq [f]_E \cap [g]_E \subseteq [f+g]_E.
\]

It follows that \([f+g]_E \) is an anchor set. Since by definition \( E \cdot F' \subseteq F' \), it follows that \([df+dg]_E \) is an anchor set for each \( d \in E \). Hence, \( f+g \in F' \).

Likewise, the inclusion

\[
\bigcup_{e \in [f]_E} [efg]_E \subseteq [fg]_E
\]

implies that \([fg]_E \) is an anchor set. Since as above one can replace here \( f \) by \( df \) for each \( d \in E \), it follows that \( fg \in F' \).

**Remark 6.5.** Let, as before, \( F \) be a unital algebra, \( E \subseteq F \) a subalgebra and \( \Phi : E \to \mathcal{L}(X) \) a non-degenerate representation. Then:

a) \( \langle E, F, \Phi \rangle \) contains each \( f \in F \) such that \( Z(E) \cap [f]_E \) is an anchor set.

b) If \( F \) is commutative, then \( \langle E, F, \Phi \rangle = \{ f \in F \mid [f]_E \text{ is an anchor set} \} \).

Indeed, a) follows from the inclusion

\[
Z(E) \cap [f]_E \subseteq [ef]_E \quad \text{for all } e \in E,
\]

which is easy to establish. And b) follows from a). This shows that our present approach generalizes the one in \([6, \text{Chapter 7}]\).

Let us summarize the results of this section by combining Theorems 6.1 and 6.4.

**Corollary 6.6.** Let \( F \) be a unital algebra, \( E \subseteq F \) a subalgebra and \( \Phi : E \to \mathcal{L}(X) \) a non-degenerate representation. Then there is a unique extension \( \hat{\Phi} \) of \( \Phi \) to a calculus on \( \langle E, F, \Phi \rangle \). Moreover, \( E \) is an algebraic core for \( \hat{\Phi} \).

Corollary 6.6 allows to extend any non-degenerate representation \( \Phi \) of a subalgebra \( E \) of a unital algebra \( F \) to the subalgebra \( \langle E, F, \Phi \rangle \) of \( E \)-anchored elements. We shall call this the **canonical extension** of \( \Phi \) within \( F \), and denote it again by \( \Phi \) (instead of \( \hat{\Phi} \) as in the corollary).

**6.2. Successive Extensions**

Very often, one performs an algebraic extension in a situation, when there is already some calculus present. The following situation is most common:

Let \( F \) be a unital subalgebra of a unital algebra \( G \), and let \( E \subseteq F \) be a subalgebra which is an algebraic core for a calculus \( \Phi : F \to \mathcal{C}(X) \). Furthermore, let \( E' \) be a subalgebra of \( G \) and \( \Psi : E' \to \mathcal{L}(X) \) a representation with

\[
E \subseteq E', \quad \Psi|_E = \Phi|_E.
\]
Then Ψ is non-degenerate, and one can perform an algebraic extension within G, yielding
\[ \mathcal{F}' := \langle \mathcal{E}', \mathcal{G}, \Psi \rangle. \]
We denote the extension again by Ψ. The following picture illustrates the situation:

\[ \begin{array}{c}
\mathcal{G} \\
\mathcal{E} \\
\mathcal{F} \\
\mathcal{F} := \langle \mathcal{E}', \mathcal{G}, \Psi \rangle \\
\mathcal{E}' \\
\end{array} \]

For a function \( f \in \mathcal{F} \) one may ask, under which conditions one has \( f \in \mathcal{F}' \) and \( \Psi(f) = \Phi(f) \). The following result gives some answer.

**Theorem 6.7.** In the situation described above, let \( f \in \mathcal{F} \). Then \( f \in \mathcal{F}' \) and \( \Phi(f) = \Psi(f) \) if any one of the following conditions is satisfied:

1) \( f \in \mathcal{F}' \) and \( \Phi'(f) \in \mathcal{L}(X) \).
2) \( f \in \mathcal{F}' \) and \( \Phi'(f) \) is densely defined and \( \Phi(f) \in \mathcal{L}(X) \).
3) For each \( e' \in \mathcal{E}' \) there is a \( \Psi \)-anchor set \( \mathcal{M}_{e'} \subseteq \mathcal{E}' \) such that \( \mathcal{M}_{e'} e' \subseteq \mathcal{D}' \cdot [f]_{\mathcal{E}} \), where
   \[ \mathcal{D}' := \{ d' \in \mathcal{F}' \mid d' \cdot \mathcal{E} \subseteq \mathcal{E}' \}. \]
4) \( \mathcal{E}' = \mathcal{E} \).
5) \( \mathcal{Z}(\mathcal{E}') \cap [f]_{\mathcal{E}} \) is an anchor set.
6) \( \mathcal{E} \subseteq \mathcal{Z}(\mathcal{E}') \).
7) \( \mathcal{E}' \) is commutative.

**Proof.** 1) and 2): If \( f \in \mathcal{F} \cap \mathcal{F}' \) then \( \Phi'(f) \subseteq \Phi(f) \) by Lemma 5.1. Then 1) is sufficient since \( \Phi(f) \) is an operator, and 2) is sufficient since \( \Phi'(f) \) is closed.
3) We prove first that \( f \in \mathcal{F}' \). Let \( e' \in \mathcal{E}' \). Take \( \mathcal{M}_{e'} \) as in the hypotheses. Then
   \[ \mathcal{M}_{e'} e' f \subseteq \mathcal{D}' [f]_{\mathcal{E}} f \subseteq \mathcal{D}' \mathcal{E} \subseteq \mathcal{E}'. \]
   It follows that \( \mathcal{M}_{e'} \subseteq [e' f]_{\mathcal{E}'} \). Since \( e' \in \mathcal{E}' \) was arbitrary, \( f \in \mathcal{F}' \). (Recall that \( \mathcal{F}' = \langle \mathcal{E}', \mathcal{G}, \Psi \rangle \) and cf. (6.2)).

\( ^1 \)Observe that since \( \mathcal{E} \) is an algebraic core for \( \Phi \), the calculus on \( \mathcal{F} \) can be considered an algebraic extension of \( \Phi|_{\mathcal{E}} \). Hence the title “Successive Extensions”.

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For the identity \( \Phi(f) = \Psi(f) \) it suffices to show that \([f]_\mathcal{E}\) determines \( \Psi(f) \). But this follows directly from Proposition 4.4 part 2), with \((\Phi, \mathcal{F})\) replaced by \((\Phi', \mathcal{F}')\) and \(\mathcal{M} := \mathcal{E}'\).

Let us now examine the cases 4)–7). In case 4), one has \( \mathcal{E} = \mathcal{E}' \) and one can take \( \mathcal{M}_\mathcal{E} = [e']_\mathcal{E} \) for \( e' \in \mathcal{E} \) in 3). In case 5) one can take \( \mathcal{M}_\mathcal{E} = \mathcal{Z}(\mathcal{E}') \cap [f]_\mathcal{E} \) independently of \( e' \in \mathcal{E}' \). Case 6) is an instance of case 5), since \([f]_\mathcal{E}\) is an anchor set by the assumption that \( \mathcal{E} \) is an algebraic core for \( \Phi \) on \( \mathcal{F} \). Finally, case 7) obviously implies case 6). □

7. Approximate Identities

Let \( \mathcal{F} \) be a commutative unital algebra, \( \Phi : \mathcal{F} \to \mathcal{C}(X) \) a proto-calculus, and \( \mathcal{E} \subseteq \text{bd}(\mathcal{F}, \Phi) \) a subset of \( \Phi \)-bounded elements. A sequence \((e_n)_n\) in \( \mathcal{E} \) is called a (weak) approximate identity in \( \mathcal{E} \) (with respect to \( \Phi \)), if \( \Phi(e_n) \to 1 \) strongly (weakly) as \( n \to \infty \).

Let \( f \in \mathcal{F} \). A (weak) approximate identity \((e_n)_n\) is said to be a (weak) approximate identity for \( f \) if

\[ \Phi(e_n)f = \Phi(f_n)e_n = \Phi(f_n)e_n = \Phi(f)e_n \]

More generally, \((e_n)_n\) is said to be a common (weak) approximate identity for all the elements of subset \( \mathcal{M} \subseteq \mathcal{F} \), if \((e_n)_n\) is a (weak) approximate identity for each \( f \in \mathcal{M} \).

Finally, we say that \( f \in \mathcal{F} \) admits a (weak) approximate identity in \( \mathcal{E} \) if there is a (weak) approximate identity for \( f \) in \( \mathcal{E} \). More generally, we say that the elements of a subset \( \mathcal{M} \subseteq \mathcal{F} \) admit a common (weak) approximate identity in \( \mathcal{E} \), if there is a common (weak) approximate identity in \( \mathcal{E} \) for them.

Note that by the uniform boundedness principle, a weak approximate identity \((e_n)_n\) is uniformly \( \Phi \)-bounded, i.e., satisfies \( \sup_{n \in \mathbb{N}} \|\Phi(e_n)\| < \infty \).

**Lemma 7.1.** Let \((e_n)_n\) be a weak approximate identity for \( f \in \mathcal{F} \) with respect to \( \Phi \) and let

\[ D := \text{span} \bigcup_{n \in \mathbb{N}} \text{ran}(\Phi(e_n)). \]

Then the following assertions hold:

a) \( \{e_n \mid n \in \mathbb{N}\} \) is an anchor set.

b) \( D \) is dense in \( X \) and \( D \subseteq \text{dom}(\Phi(f)) \). In particular, \( \Phi(f) \) is densely defined.

c) \( \text{dom}(\Phi(f)) \cap D \) is a core for \( \Phi(f) \). If \((e_n)_n\) is an approximate identity for \( f \) then \( \Phi(e_n)x \to x \) within the Banach space \( \text{dom}(\Phi(f)) \) for each \( x \in \text{dom}(\Phi(f_n)) \).

d) For all \( n \in \mathbb{N} \)

\[ \Phi(e_n)\Phi(f) = \Phi(e_n f) = \Phi(f e_n) = \Phi(f)\Phi(e_n). \]
Proof. a) is trivial and b) follows from Mazur’s theorem, as $D$ is clearly weakly dense in $X$.

c) Let $x, y \in X$ with $\Phi(f)x = y$. Then by hypothesis, for each $n \in \mathbb{N}$ we have $(\Phi(e_n)x, \Phi(e_n)y) \in \Phi(f)$, so $\Phi(e_n)x \in D \cap \text{dom}(\Phi(f))$. Since $(\Phi(e_n)x, \Phi(e_n)y) \to (x, y)$ weakly, the space $\Phi(f)|_D$—considered as a subspace of $X \oplus X$—is weakly dense in $\Phi(f)$. By Mazur’s theorem again, this space is strongly dense, hence $D$ is a core for $\Phi(f)$. If $(e_n)_n$ is even a strong approximate identity, then $\Phi(e_n)x \to x$ and $\Phi(f)\Phi(e_n)x = \Phi(e_n)y \to y$ strongly.

d) Suppose that $\Phi(fge_n) \in \mathcal{L}(X)$ for all $n \in \mathbb{N}$. Then $\Phi(f)\Phi(e_n) = \Phi(e_n f) \in \mathcal{L}(X)$, and hence $\text{ran}(\Phi(e_n)) \subseteq \text{dom}(\Phi(f))$. If follows that $D \subseteq \text{dom}(\Phi(f))$, and $\Phi(f)$ is densely defined, by b). By hypothesis,

$\Phi(e_n)\Phi(f) \subseteq \Phi(f)\Phi(e_n) = \Phi(f e_n)$.

Since the left-most operator is densely defined, we obtain

$\Phi(e_n)\Phi(f) = \Phi(f e_n)$.

On the other hand, by (FC3),

$\Phi(e_n)\Phi(f) \subseteq \Phi(e_n f)$

and the latter is a closed operator. It follows that $\Phi(e_n f) = \Phi(f e_n)$ as claimed. □

By virtue of the preceding lemma, we obtain the following result.

**Theorem 7.2.** Let $\Phi : \mathcal{F} \to \mathcal{L}(X)$ be a proto-calculus.

a) If $(e_n)_n$ is a (weak) approximate identity for $f, g \in \mathcal{F}$, then it is a (weak) approximate identity for $f + g$ and $\lambda f$ ($\lambda \in \mathbb{C}$) and one has

$\Phi(f) + \Phi(g) = \Phi(f + g)$.

b) If $(e_n)_n$ is a strong approximate identity for $f, g \in \mathcal{F}$, then $(e_n^2)_n$ is a strong approximate identity for $fg$, and one has

$\Phi(f)\Phi(g) = \Phi(fg)$.

Proof. a) Since $\Phi(e_n f) = \Phi(f e_n)$ and $\Phi(e_n g) = \Phi(g e_n)$ are bounded, so is

$\Phi(e_n (f + g)) = \Phi(e_n f + e_n g) = \Phi(e_n f) + \Phi(e_n g) = \cdots = \Phi((f + g)e_n)$.

It follows that $(e_n)_n$ is a (weak) approximation of identity for $f + g$ and, hence, that $D$ is a core for $\Phi(f + g)$. But $D \subseteq \text{dom}(\Phi(f)) \cap \text{dom}(\Phi(g))$, and so we are done.

b) Since $(e_n)_n$ is an approximate identity, it is bounded, and hence also $(e_n^2)_n$ is an approximate identity. Note that

$\Phi(fge_n^2) = \Phi(f(g e_n)e_n) = \Phi(f)\Phi(g e_n)\Phi(e_n) = \Phi(f)\Phi(e_n g)\Phi(e_n) = \Phi(f e_n g e_n) = \Phi(f e_n)\Phi(g e_n) \in \mathcal{L}(X)$,

and continuing the computation yields $\Phi(fge_n^2) = \Phi(e_n^2 fg)$. This proves the first claim. The second follows easily. □
8. The Dual Calculus

For a calculus \((F, \Phi)\) on a Banach space \(X\) one is tempted to define a “dual calculus” on \(X’\) by letting \(\Phi’(f) := \Phi(f)’\). This is premature in at least two respects. First, if \(\Phi(f)\) is not densely defined, \(\Phi(f)’\) is just a linear relation and not an operator. Secondly, even if the first problem is ruled out by appropriate minimal assumptions, it is not clear how to establish the formal properties of a calculus for \(\Phi’\).

To tackle these problems, we shall take a different route and define the dual calculus by virtue of the extension procedure described in Section 6. To wit, let \(\Phi : F \to \mathcal{C}(X)\) be any proto-calculus and let \(B := \text{bdd}(F, \Phi)\) the set of \(\Phi\)-bounded elements. For \(b \in B\) we define

\[\Phi’(b) := \Phi(b)’ \in \mathcal{L}(X’).\]

As \(F\) may not be commutative, the mapping \(\Phi’\) may not be a homomorphism for the original algebra structure. To remedy this defect, we pass to the \textbf{opposite algebra} \(F^{\text{op}}\), defined on the same set \(F\) with the same linear structure but with the “opposite” multiplication

\[f \cdot_{\text{op}} g = gf \quad (g, f \in F).\]

For any subset \(M \subseteq F\) we write \(M^{\text{op}}\) when we want to consider \(M\) as endowed with this new multiplication. This applies in particular to \(B\), whence we obtain

\[[f]^{\text{op}} = \{ e \in B \mid e \cdot_{\text{op}} f \in B \} = \{ e \in B \mid fe \in B \}\]

for \(f \in F^{\text{op}}\). The mapping

\[\Phi’ : B^{\text{op}} \to \mathcal{L}(X’),\]

is a unital algebra homomorphism. We then can pass to its canonical extension to the algebra

\[F’ := \langle B^{\text{op}}, F^{\text{op}}, \Phi’ \rangle;\]

as usual, we shall denote that extension by \(\Phi’\) again. The mapping

\[\Phi’ : F’ \to \mathcal{L}(X’),\]

is called the \textbf{dual calculus} associated with \(\Phi\). By construction, it is a calculus (Theorem 6.1).

\begin{theorem}
Let \((F, \Phi)\) be a proto-calculus with dual calculus \((F’, \Phi’),\) and let \(f \in F’\). Define

\[D_f := \text{span}\{ \Phi(e)x \mid x \in X, \ e \in [f]^{\text{op}} \}.\]

Then the following assertions hold:

a) \(D\) is dense in \(X\) and \(D \subseteq \text{dom}(\Phi(f))\). In particular, \(\Phi(f)\) is densely defined.

b) \(\Phi(f)’ \subseteq \Phi’(f)\), with equality if and only if \(D_f\) is a core for \(\Phi(f)\).

c) \(\Phi(f)\) is bounded if and only if \(\Phi’(f)\) is bounded; in this case \(\Phi’(f) = \Phi(f)’\). 
\end{theorem}
Proof. a) Let \( e \in [f]_{\mathcal{B}^+} \). Then \( e, fe \in \mathcal{B} \). Hence, \( \text{ran}(\Phi(e)) \subseteq \text{dom}(\Phi(f)) \). This yields the inclusion \( D \subseteq \text{dom}(\Phi(f)) \). Since by construction, \([f]_{\mathcal{B}^+}\) is a \( \Phi' \)-anchor, one has

\[
\bigcap_{e \in [f]_{\mathcal{B}^+}} \ker(\Phi(e)') = \{0\}, \tag{8.1}
\]

By a standard application of the Hahn–Banach theorem, \( D \) is dense in \( X \).

b) Fix \( x', y' \in X \). Note the following equivalences:

\[
\Phi'(f)x' = y' \iff \forall e \in [f]_{\mathcal{B}^+} : \Phi'(e \circ_{\mathcal{B}^+} f)x' = \Phi'(e)y' \\
\iff \forall e \in [f]_{\mathcal{B}^+} : \Phi(fe)x' = \Phi(e)y' \\
\iff \forall e \in [f]_{\mathcal{B}^+} : \forall z \in X : \langle \Phi(f)\Phi(e)z, x' \rangle = \langle \Phi(e)z, y' \rangle \\
\iff (x', -y') \perp (\Phi(f) \cap (D_f \oplus X)),
\]

where we identify \( \Phi(f) \) with its graph as a subset of \( X \oplus X \). On the other hand,

\[
\Phi(f)'x' = y' \iff (x', -y') \perp \Phi(f).
\]

From this it is evident that \( \Phi(f)' \subseteq \Phi'(f) \). Furthermore, since both operators \( \Phi(f)' \) and \( \Phi'(f) \) are weakly* closed, by the Hahn–Banach theorem one has equality if and only if \( \Phi(f) \cap (D_f \oplus X) \) is dense in \( \Phi(f) \). The latter just means that \( D_f \) is a core for \( \Phi(f) \).

c) If \( \Phi(f) \) is bounded, then so is \( \Phi(f)' \), and hence by b) \( \Phi'(f) = \Phi(f)' \). Suppose, conversely, that \( \Phi'(f) \in L(X') \). Since \( \Phi'(f) \) has a closed graph for the weak* topology, it follows from Theorem A.1 that there is \( T \in L(X) \) such that \( \Phi'(f) = T' \). By b), \( \Phi(f)' \subseteq T'' \), which in turn implies that

\[
T = T'' \cap (X \oplus X) \subseteq \Phi(f)'' \cap (X \oplus X) = \Phi(f),
\]

since \( \Phi(f) \) is closed, see [9, Prop.A.4.2.d]. This implies that \( \Phi(f) = T \), so \( \Phi(f) \) is indeed bounded.

A calculus \((\mathcal{F}, \Phi)\) on a Banach space \( X \) is called dualizable if \( \mathcal{F}' = \mathcal{F}^{op} \), i.e., the dual calculus is defined on \( \mathcal{F}^{op} \). Equivalently, \((\mathcal{F}, \Phi)\) is dualizable if for each \( f \in \mathcal{F} \) and each \( b \in \text{bdd}(\mathcal{F}, \Phi) \) the space

\[
D_{fb} = \text{span}\{\Phi(e)x \mid x \in X, e, fe \in \text{bdd}(\mathcal{F}, \Phi)\}
\]

is dense in \( X \). For a dualizable calculus one has

\[
\text{bdd}(\mathcal{F}, \Phi) = \text{bdd}(\mathcal{F}', \Phi')
\]

by c) of Theorem 8.1.

If \( \Phi \) on \( \mathcal{F} \) is a non-dualizable calculus then \( \mathcal{F}^{op} \) (i.e., \( \mathcal{F}' \) with the original algebra structure) is a \( \Phi \)-regular subalgebra of \( \mathcal{F} \) (since it contains \( \text{bdd}(\mathcal{F}, \Phi) \)) and we may restrict \( \Phi \) to this algebra. In a sense, \( \mathcal{F}^{op} \) is the largest subalgebra such that the restriction of \( \Phi \) to it is a dualizable calculus.
9. Topological Extensions

The algebraic extension procedure discussed in Section 6 is based on a “primary” or “elementary” calculus $\Phi : E \to L(X)$ that can be extended. In this section we discuss the form of possible other—topological—ways of extending a primary calculus. Whereas the algebraic extension is canonical when a superalgebra is given, a topological extension depends also on the presence of a given topological structure on the superalgebra.

In the following we want to formalize the idea of a topological extension in such generality that the extant examples are covered. However, we admit that experience with topological extensions as such is scarce, so that the exposition given here is likely to be replaced by a better one some time in the future.

Let $F$ be an algebra and $\Lambda$ a set. An (algebraic) convergence structure on $F$ over $\Lambda$ is a relation $\tau \subseteq F^\Lambda \times F$ with the following properties:

1) $\tau$ is a subalgebra of $F^\Lambda \times F$.
2) For each $f \in F$ the pair $((f)_{\lambda \in \Lambda}, f)$ is in $\tau$. (Here, $(f)_{\lambda \in \Lambda}$ is the constant family.)

The convergence structure is called Hausdorff, if $\tau$ is actually an operator and not just a relation. If $\Lambda = \mathbb{N}$, we speak of a sequential convergence structure.

Given a convergence structure $\tau$, one writes $f_\lambda \xrightarrow{\tau} f$ in place of $((f_\lambda)_{\lambda \in \Lambda}, f) \in \tau$ and says that $(f_\lambda)_{\lambda \in \Lambda}$ $\tau$-converges to $f$. From 1) and 2) it follows that dom$(\tau) \subseteq F^\Lambda$ is an algebra containing all constant families. The structure $\tau$ is Hausdorff if and only if one has $f_\lambda \xrightarrow{\tau} f, f_\lambda \xrightarrow{\tau} g \Rightarrow f = g$.

From now on, we consider the following situation: $E'$ is a unital algebra, $E \subseteq E'$ is a subalgebra, and $\Phi : E \to L(X)$ is a representation; $A \subseteq L(X)$ is a subalgebra such that $\Phi(E) \subseteq A$; and $\tau = (\tau_1, \tau_2)$ is a pair of convergence structures $\tau_1$ on $E'$ and $\tau_2$ on $A$ over the same index set $\Lambda$. (The latter will be called a joint convergence structure on $(E', A)$ in the following.)

In this situation, the set $E^\tau := \{ f \in E' \mid \exists (e_\lambda)_{\lambda \in \Lambda} \text{ in } E, T \in A : e_\lambda \xrightarrow{\tau_1} f, \Phi(e_\lambda) \xrightarrow{\tau_2} T \}$ is a subalgebra of $E'$ containing $E$. Suppose in addition that $\Phi$ is closable with respect to $\tau$, which means that

$$(e_\lambda)_{\lambda \in \Lambda} \in E^A, T \in A, e_\lambda \xrightarrow{\tau_1} 0, \Phi(e_\lambda) \xrightarrow{\tau_2} T \Rightarrow T = 0. \quad (9.1)$$

Then one can define the $\tau$-extension $\Phi^\tau : E^\tau \to L(X)$ of $\Phi$ by

$$\Phi^\tau(f) := T$$
whenever \((e_\lambda)_{\lambda} \in E^\Lambda, e_\lambda \xrightarrow{\tau_1} f,\) and \(\Phi(e_\lambda) \xrightarrow{\tau_2} T.\) (Indeed, (9.1) just guarantees that \(\Phi^\tau\) is well-defined, i.e., \(\Phi^\tau(f)\) does not depend on the chosen \(\tau\)-approximating sequence \((e_\lambda)_{\lambda}.)\)

**Theorem 9.1.** The so-defined mapping \(\Phi^\tau : E^\tau \rightarrow \mathcal{L}(X)\) is an algebra homomorphism which extends \(\Phi.\)

**Proof.** Straightforward. \(\square\)

In practice, one wants to combine a topological with an algebraic extension, and that raises a compatibility issue. To explain this, let us be more specific.

Let \(E\) be an algebraic core for a calculus \(\Phi : \mathcal{F} \rightarrow \mathcal{C}(X),\) let \(G\) be a superalgebra of \(\mathcal{F}\) and let \(E'\) be a subalgebra of \(G\) containing \(E:\)

\[F \subseteq G \quad \text{and} \quad E \subseteq E' \subseteq G.\]

Suppose further that \(\tau = (\tau_1, \tau_2)\) is a joint convergence structure on \((E', A),\) where \(A\) is a subalgebra of \(\mathcal{L}(X)\) containing \(\Phi(E),\) and that \(\Phi|_E\) is closable with respect to \(\tau.\)

As above, let \(\Phi^\tau\) denote the \(\tau\)-extension of \(\Phi|_E\) to the algebra \(E^\tau \subseteq E'.\) Starting from \(E^\tau\) we can extend \(\Phi^\tau\) algebraically to

\[G^\tau := \langle E^\tau, G, \Phi^\tau \rangle,\]

and we denote this extension again by \(\Phi^\tau.\)

The question arises whether \(\Phi^\tau\) is an extension of \(\Phi.\) This problem has been already discussed in Section 6.2 in a more general context, so that Theorem 6.7 and the subsequent remarks apply. In particular, we obtain the following:

**Corollary 9.2.** In the situation described above, the following assertions hold:

a) If \(f \in F \cap G^\tau\) then \(\Phi^\tau(f) \subseteq \Phi(f),\) so that \(\Phi^\tau(f) = \Phi(f)\) if \(\Phi^\tau(f)\) is bounded. In particular, \(\Phi^\tau = \Phi\) on \(E^\tau \cap F.\)

b) If \(f \in F\) is such that \(Z(E^\tau) \cap [f]_E\) is an anchor set, then \(f \in G^\tau\) and \(\Phi^\tau(f) = \Phi(f).\) In particular, \(\Phi^\tau\) extends \(\Phi\) if \(E \subseteq Z(E^\tau).\)

If \(E\) is commutative and \(\tau_1\) is Hausdorff, then \(E^\tau\) is also commutative. Hence, in this case, 2) is applicable and it follows that \(\Phi^\tau\) extends \(\Phi.\)

**Remarks 9.3.** The idea of a topological extension in the abstract theory of functional calculus was introduced in [8], with a (however easy-to-spot) mistake in the formulation of closability [8 (5.2)]. There, with an immediate application in mind, the discussion was still informal.

In our attempt to formalize it, we here introduce the notion of a “convergence structure” which, admittedly, is *ad hoc*. We have not browsed through the literature to find a suitable and already established notion. Maybe one would prefer a richer axiomatic tableau for such a notion and is tempted to add axioms, e.g., require that \(\Lambda\) is directed and that families that coincide eventually display the same
convergence behaviour. On the other hand, only axioms 1) and 2) are needed to prove Theorem 9.1.

In practice, one may often choose \( \tau_2 \) to be operator norm or strong convergence, but other choices are possible. (See Sections 11 and 13.2 below for an interesting example. It was the latter example that led us to acknowledge that one needs flexibility of the convergence notion also on the operator side.)

**Part 2. Examples**

In this second part of the article, we want to illustrate the abstract theory with some well-known examples. However, we focus on the supposedly less well-known aspects.

**10. Sectorial Operators**

A closed operator \( A \) on a Banach space \( X \) is called **sectorial** if there is \( \omega \in [0, \pi) \) such that \( \sigma(A) \) is contained in the sector \( \mathbb{S}_\omega \) and the function \( \lambda \mapsto \lambda R(\lambda, A) \) is uniformly bounded outside every larger sector. The minimal \( \omega \) that has this property is called the **sectoriality angle** and is denoted by \( \omega_{sc}(A) \).

For \( \omega > 0 \) we let \( \mathcal{E}(\mathbb{S}_\omega) \) be the set of functions \( f \in H^\infty(\mathbb{S}_\omega) \) such that

\[
\int_{\partial \mathbb{S}_\delta} |f(z)| \frac{|dz|}{|z|} < \infty \quad \text{for all } 0 \leq \delta < \omega.
\]

If \( f \in \mathcal{E}(\mathbb{S}_\omega) \) and \( A \) is a sectorial operator on \( X \) of angle \( \omega_{sc}(A) < \omega \) then we define

\[
\Phi^\omega_A(f) := \frac{1}{2\pi i} \int_{\partial \mathbb{S}_\delta} f(z) R(z, A) \, dz
\]

(10.1)

Note that the norm condition on the resolvent of \( A \) and the integrability condition on \( f \) just match in order to render this integral absolutely convergent. It is a classical fact that

\[
\Phi^\omega_A : \mathcal{E}(\mathbb{S}_\omega) \to L(X)
\]

is an algebra homomorphism and

\[
\Phi^\omega_A \left( \frac{z}{(1 + z)^2} \right) = A(1 + A)^{-1}.
\]

Standard complex analysis arguments yield that for each \( f \in \mathcal{E}(\mathbb{S}_\omega) \) one has

\[
\lim_{z \to 0} f(z) = \lim_{z \to \infty} f(z) = 0 \quad \text{whenever } |\arg z| \leq \delta
\]

for each \( 0 < \delta < \omega \). As a consequence,

\[
\ker(A) \subseteq \ker \Phi^\omega_A(f).
\]

It follows that \( \Phi^\omega_A \) is degenerate if \( A \) is not injective.

Since we do not want to assume the injectivity of \( A \), we could follow Remark 6.3 and extend \( \Phi^\omega_A \) to the unital algebra

\[
\mathcal{E}_1(\mathbb{S}_\omega) := \mathcal{E}(\mathbb{S}_\omega) \oplus \mathbb{C} 1
\]
and make this the basis of the algebraic extension procedure from Section [6]. However, it is easily seen that the function $(1 + z)^{-1}$ is not anchored in $E_1(S_\omega)$. So, the resulting calculus would be “too small” in the sense that it would not cover some natural functions of $A$.

In order to deal with this problem, one extends $\Phi_\omega^A$ further to the algebra

$$E_e(S_\omega) := E(S_\omega) \oplus \mathbb{C}1 \oplus \mathbb{C} \frac{1}{1 + z}$$

by defining

$$\Phi_A((1 + z)^{-1}) := (1 + A)^{-1}.$$ 

It follows from properties of $\Phi_\omega^A$ on $E(S_\omega)$ that this extension is indeed an algebra homomorphism.

At this point one may perform an algebraic extension as in Theorem 6.1 within a “surrounding” algebra $F$. A natural choice for $F$ is the field $M(S_\omega)$ of all meromorphic functions on the sector $S_\omega$. The domain of the resulting calculus, which is again denoted by $\Phi_\omega^A$, is the algebra

$$\text{dom}(\Phi_\omega^A) := \langle E_e(S_\omega), M(S_\omega), \Phi_\omega^A \rangle.$$ 

Whereas the algebras $E$ and $E_e$ are described independently of $A$, the algebra $\text{dom}(\Phi_\omega^A)$ is heavily dependent on $A$, and is, as a whole, quite arcane in general.

Note that there is still a dependence of our calculus on the choice of $\omega > \omega_{se}(A)$. This can be eliminated as follows: for $\omega_1 > \omega_2 > \omega_{se}(A)$ one has a natural embedding

$$\eta : M(S_{\omega_1}) \to M(S_{\omega_2}), \quad \eta(f) := f|_{S_{\omega_2}}.$$ 

By the identity theorem of complex analysis, $\eta$ is injective. Of course we expect compatibility, i.e.,

$$\Phi_{\omega_1}^A(f|_{S_{\omega_2}}) = \Phi_{\omega_2}^A(f) \quad \text{for each } f \in \text{dom}(\Phi_{\omega_2}^A).$$

Since all involved algebras are commutative, by Theorem 5.8 this has to be verified only for functions $f \in E_e(S_{\omega_1})$, and hence effectively only for functions $f \in E(S_{\omega_1})$. For such functions it is a consequence of a path-deformation argument.

By letting $\omega$ approach $\omega_{se}(A)$ from above, we obtain a “tower” of larger and larger algebras and their “union”

$$E[S_{\omega_{se}(A)}] := \bigcup_{\omega > \omega_{se}(A)} E(S_\omega),$$

and likewise for $E_e[S_{\omega_{se}(A)}]$ and $M[S_{\omega_{se}(A)}]$. A precise definition of this union would require the notion of a meromorphic function germ on $S_{\omega_{se}(A)} \setminus \{0\}$. The resulting calculus is called the sectorial calculus for $A$ and is denoted by $\Phi_A$ here. It can be seen as an “inductive limit” of the calculi $\Phi_\omega^A$. 

On the Fundamental Principles of Unbounded Functional Calculi
11. Topological Extensions of the Sectorial Calculus

Of course the question arises whether the sectorial calculus \( \Phi_A \) covers all “natural” choices for functions \( f \) of \( A \). The answer is “no”, at least when the operator \( A \) is not injective.

To understand this, we look at functions of the form

1) \( f(z) = \int_{\mathbb{R}_+} \frac{t}{1+z} \mu(dt) \)

and

2) \( g(z) = \int_{\mathbb{R}_+} (t^2z)^n e^{-tz} \mu(dt) \),

where \( \mu \in \mathcal{M}(\mathbb{R}_+) \) is a complex Borel measure on \( \mathbb{R}_+ = [0, \infty) \). It is easy to see that by 1) a holomorphic function \( f \) on \( S_\pi \) is defined, bounded on each smaller sector. And by 2), a holomorphic function \( g \) on \( S_{\pi/2} \) is defined, bounded on each smaller sector.

Of course, in any reasonable functional calculus one would expect

\[
\Phi_A f(A) = \int_{\mathbb{R}_+} A(t+A)^{-1} \mu(dt) \quad \text{(11.1)}
\]

for each sectorial operator \( A \) and

\[
\Phi_A g(A) = \int_{\mathbb{R}_+} (t^2A)^n e^{-tA} \mu(dt) \quad \text{(11.2)}
\]

for each sectorial operator \( A \) with \( \omega_{se}(A) < \frac{\pi}{2} \). If \( A \) is injective, this is true for the sectorial calculus.

However, if \( A \) is not injective, then \( \mu \) can be chosen so that \( [f]_{E_\omega} \) is not an anchor set and, consequently, \( f \) is not contained in the domain of \( \Phi_A \). A prominent example for this situation is the function

\[
f(z) = \frac{1}{\lambda - \log z} = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2(t + A)^{-1}} dt,
\]

which plays a prominent role for Nollau’s result on operator logarithms [9, Chapter 4]. Similar remarks apply in case 2).

The mentioned “defect” of the sectorial calculus \( \Phi_A \) can be mended by passing to a suitable topological extension as described in Section 9. Actually, this has been already observed in [8], where uniform convergence was used as the underlying convergence structure.

Here, we intend to generalize the result from [8] by employing a weaker convergence structure on a larger algebra. Define

\[
H^\infty(S_\omega \cup \{0\}) := \{ f \in H^\infty(S_\omega) \mid f(0) := \lim_{z \to 0} f(z) \text{ exists} \}.
\]

We say that a sequence \( (f_n)_n \) in \( H^\infty(S_\omega \cup \{0\}) \) converges pointwise and boundedly (in short: bp-converges) on \( S_\omega \cup \{0\} \) to \( f \in H^\infty(S_\omega \cup \{0\}) \) if \( f_n(z) \to f(z) \) for each \( z \in S_\omega \cup \{0\} \) and \( \sup_n \|f_n\|_{\infty,S_\omega} < \infty \). It is obvious that bp-convergence is a Hausdorff sequential convergence structure as introduced in Section 9. We take
bp-convergence as the first component of the joint convergence structure $\tau$ we need for a topological extension.

The second component, is described as follows. For a set $\mathcal{B} \subseteq \mathcal{L}(X)$ let
\[ \mathcal{B}' := \{ S \in \mathcal{L}(X) \mid \forall B \in \mathcal{B} : SB = BS \} \]
be its **commtator** within $\mathcal{L}(X)$. Let
\[ \mathcal{A}_A := \{ (1 + A)^{-1}\} = \{ R(\lambda, A) \mid \lambda \in \varrho(A) \} \]
For $((T_n)_n, T) \in \mathcal{A}_A^\mathbb{N} \times \mathcal{A}_A$ we write
\[ T_n \overset{\tau_A}{\to} T \]
if there is a point-separating set $\mathcal{D} \subseteq \mathcal{A}_A'$ such that
\[ DT_n \to DT \quad \text{strongly, for each } D \in \mathcal{D}. \]
(Recall that $\mathcal{F}$ is point-separating if $\bigcap_{D \in \mathcal{D}} \ker(D) = \{0\}$.) Note that $\mathcal{A}_A'$ is a commutative unital subalgebra of $\mathcal{L}(X)$ closed with respect to strong convergence and containing all resolvents of $A$.

**Lemma 11.1.** The relation $\tau_A$ on $\mathcal{A}_A^\mathbb{N} \times \mathcal{A}_A$ is an algebraic Hausdorff convergence structure.

**Proof.** This follows easily from the fact that $\mathcal{A}'$ is commutative. \qed

**Remark 11.2.** We note that, in particular, one has
\[ (1 + A)^{-m}T_n \to (1 + A)^{-m}T \quad \text{strongly} \quad \implies \quad T_n \overset{\tau_A}{\to} T \]
that for any $m \in \mathbb{N}_0$. This means that $\tau_A$-convergence is weaker than strong convergence in any extrapolation norm associated with $A$.

We shall show that $\Phi_A$ on $\mathcal{E}_e(S_\omega)$ is closable with respect to the joint convergence structure
\[ \tau = ( \text{bp-convergence on } H^\infty(S_\omega \cup \{0\}) \) \] \[ , \quad \tau_A^* \text{ on } \mathcal{A}_A \]. \quad (11.3) \]
We need the following auxiliary information.

**Lemma 11.3.** Let $A$ be sectorial, let $\omega < \varrho(A) < \omega < \pi$ and let $e \in \mathcal{E}(S_\omega)$. Then
\[ \text{ran}(\Phi_A(e)) \subseteq \mathfrak{P}(A). \]

**Proof.** Let $\varphi_n := \frac{n e}{1 + nA}$. Then $\varphi_n \to 1$ pointwise and boundedly on $S_\omega$. By Lebesgue’s theorem, $\Phi_A(\varphi_n e) \to \Phi_A(e)$ in norm. But
\[ \Phi_A(\varphi_n e) = nA(1 + nA)^{-1}\Phi_A(e). \]

The claim follows. \qed

Now we can head for the main result.
Theorem 11.4. Let $A$ be a sectorial operator on a Banach space $X$, let $\omega \in (\omega_{ac}(A), \pi)$ and let $(f_n)_n$ be a sequence in $H^\infty(S_\omega \cup \{0\})$ such that $f_n \to 0$ pointwise and boundedly on $S_\omega \cup \{0\}$. Suppose that $\Phi_A(f_n)$ is defined and bounded for each $n \in \mathbb{N}$, and that $\Phi_A(f_n) \xrightarrow{\text{st}} T \in A_A$ strongly. Then $T = 0$.

Proof. For simplicity we write $\Phi$ in place of $\Phi_A$, and $E$ and $\mathcal{E}$ in place of $\mathcal{E}(S_\omega)$ and $\mathcal{E}(S_\omega)$, respectively. By passing to $f_n - f_n(0)1$ we may suppose that $f_n(0) = 0$ for each $n \in \mathbb{N}$.

By hypothesis, there is a point-separating set $\mathcal{D} \subseteq A'_A$ such that

$$\text{D} \Phi(f_n) \to DT \quad \text{strongly, for all } D \in \mathcal{D}.$$ 

We fix $D \in \mathcal{D}$ for the time being.

Now, take $e := z(1 + z)^{-2}$ and observe that $ef_n \in E$ and $\Phi(ef_n) \to 0$ in operator norm by Lebesgue’s theorem and the very definition of $\Phi$ in (10.1). On the other hand,

$$\text{D} \Phi(e f_n) = D \Phi(e) \Phi(f_n) = \Phi(e) D \Phi(f_n) \to \Phi(e) DT$$

strongly. This yields $A(1 + A)^{-2} DT = \Phi(e) DT = 0$, and hence

$$\text{ran}((1 + A)^{-2} DT) \subseteq \text{ker}(A). \quad (11.4)$$

If $\text{ker}(A) = \{0\}$ then $DT = 0$ and hence $T = 0$ since $D$ was arbitrary from $\mathcal{D}$. So suppose that $A$ is not injective and define $e_0 := (1 + z)^{-2}$.

We claim that $e_0 f_n \in \mathcal{E}$. To prove this, note that, by hypothesis, $f_n$ is anchored in $\mathcal{E}_c$. Since $A$ is not injective, $[f]_{\mathcal{E}_c}$ must contain at least one function $e_1$ with $e_1(0) \neq 0$. Write

$$e_1 = e_2 + c \mathbf{1} + \frac{d}{1 + z} = e_2 + \frac{c + d}{1 + z} + c \frac{z}{1 + z} \quad \text{for certain } c, d \in \mathbb{C} \text{ and } e_2 \in \mathcal{E}.$$ 

Now multiply by $f_n$ and $(1 + z)^{-1}$ to obtain

$$(c + d)e_0 f_n = \frac{e_1 f_n}{1 + z} - e_2 \frac{f_n}{1 + z} - c \frac{z}{(1 + z)^2} \in \mathcal{E}.$$ 

(Note that $e_1 f_n \in \mathcal{E}_c$.) But $c + d = e_1(0) \neq 0$, and hence $e_0 f_n \in \mathcal{E}$ as claimed.

Finally, apply Lemma 11.3 above with $e = e_0 f_n$ to see that

$$\text{ran}(\Phi(e_0 f_n)) \subseteq \text{ran}(A).$$

As

$$\Phi(e_0 f_n)D = D \Phi(e_0 f_n) = D \Phi(e_0) \Phi(f_n) = \Phi(e_0) D \Phi(f_n) = (1 + A)^{-2} D \Phi(f_n) \to (1 + A)^{-2} T$$

strongly, it follows that

$$\text{ran}((1 + A)^{-2} DT) \subseteq \text{ran}(A). \quad (11.5)$$

Taking (11.4) and (11.5) together yields

$$\text{ran}((1 + A)^{-2} DT) \subseteq \text{ker}(A) \cap \text{ran}(A) = \{0\},$$
since $A$ is sectorial [9 Prop. 2.1]. This means that $(1 + A)^{-2}DT = 0$, and since $D \in \mathcal{D}$ was arbitrary, it follows that $T = 0$ as desired. □

As an immediate consequence we obtain the result already announced above.

**Corollary 11.5.** The sectorial calculus $\Phi_A$ on $H^\infty(S_\omega \cup \{0\}) \cap \text{bd}(\Phi_A, \text{dom}(\Phi_A))$ is closable with respect to the joint convergence structure

( bp-convergence on $S_\omega \cup \{0\}$, $\tau_A^*$-convergence within $A$).

Based on Corollary [11.5] we apply Theorem [9.1] and obtain the **bp-extension** $\Phi_A^{bp}$ of the sectorial calculus $\Phi_A$ for $A$. Since the relevant function algebras are commutative, there is no compatibility issue, cf. Corollary [9.2].

As bp-convergence is weaker than uniform convergence, Corollary [11.5] implies in particular the result from [7, Section 5] that the sectorial calculus is closed with respect to

( uniform convergence on $S_\omega \cup \{0\}$, operator norm convergence ),

which obviously is also a joint algebraic sequential convergence structure. The respective topological extension (and also its canonical algebraic one) shall be called the **uniform extension** of the sectorial calculus and denoted by $\Phi_A^{uni}$. Obviously, we have $\text{dom}(\Phi_A^{uni}) \subseteq \text{dom}(\Phi_A^{bp})$ and $\Phi_A^{uni} = \Phi_A^{bp}$ on $\text{dom}(\Phi_A^{uni})$.

**Remark 11.6.** The bp-extension is “large” in a sense, since bp-convergence and $\tau_A^*$-convergence are relatively weak requirements. (Actually, they are the weakest we can think of at the moment.) On the other hand, the bp-extension is quite “small”. Whereas the bp-extension is interesting in order to understand what a “maximal” calculus could be for a given operator, the uniform extension is interesting in order to understand the “minimal” extension necessary to cover a given function.

12. Stieltjes Calculus and Hirsch Calculus

The defect of the sectorial calculus mentioned in the previous section has, as a matter of fact, been observed by several other people working in the field. Of course, this has not prevented people from working with operators of the form (11.1) or (11.2). The former one has actually been used already by Hirsch in [13]. It was extended algebraically by Martinez and Sanz in [15, 16] under the name of “Hirsch functional calculus”.

Dungey in [5] considers the operator $\psi(A)$ for

$$\psi(z) = \int_0^1 z^\alpha \, d\alpha = \frac{z - 1}{\log z}$$

and remarks that $\psi(A)$ is defined within the Hirsch calculus but not within the sectorial calculus.

Batty, Gomilko and Tomilov in [1] embed the Hirsch calculus (which is not a calculus in our sense because the domain set is not an algebra) into a larger
calculus (in our sense) which is an algebraic extension of a calculus for the so-called bounded Stieltjes algebra.

12.1. The Stieltjes Calculus

In [1], Batty, Gomilko and Tomilov define what they call the extended Stieltjes calculus for a sectorial operator. In this section, we introduce this calculus and show that it is contained in the uniform extension of the sectorial calculus.

According to [1, Section 4], the bounded Stieltjes algebra \( \tilde{S}_b \) consists of all functions \( f \) that have a representation

\[
f(z) = \int_{\mathbb{R}_+} \frac{\mu(ds)}{(1 + sz)^m}
\]

for some \( m \in \mathbb{N}_0 \) and some \( \mu \in M(\mathbb{R}_+) \). Each such \( f \) is obviously holomorphic on \( S_\pi \) and bounded on each smaller sector. It is less obvious, however, that \( \tilde{S}_b \) is a unital algebra, a fact which is proved in [1, Section 4.1].

**Proposition 12.1.** Suppose \( f \) is a bounded Stieltjes function with representation

\[
f(z) = \int_{\mathbb{R}_+} \frac{\mu(ds)}{(1 + sz)^m}
\]

and \( A \) is a sectorial operator on a Banach space \( X \). Then

\[
\Phi_{uni}^A(f) = \int_{\mathbb{R}_+} (1 + sA)^{-m} \mu(ds),
\]

where \( \Phi_{uni}^A \) is the uniform extension of the sectorial calculus for \( A \).

**Proof.** By subtracting \( \mu(0) \) we may suppose that \( \mu \) is supported on \((0, \infty)\). Also, we may suppose that \( m \geq 1 \). Define

\[
f_n(z) := \int_{[\frac{1}{n}, n]} \frac{\mu(ds)}{(1 + sz)^m} \quad \text{and} \quad a_n := \int_{[\frac{1}{n}, n]} 1 ds \mu.
\]

Then

\[
g_n(z) := f_n(z) - \frac{a_n}{1 + z} = \int_{[\frac{1}{n}, n]} \frac{1}{(1 + sz)^m} - \frac{1}{1 + z} \mu(ds).
\]

The function under the integral is contained in \( E \). A moment’s reflection reveals that \( g_n \in E \) also, and that one can apply Fubini’s theorem to compute \( \Phi_A(g_n) \).

This yields

\[
\Phi_A(g_n) = \int_{[\frac{1}{n}, n]} \Phi_A \left( \frac{1}{(1 + sz)^m} - \frac{1}{1 + z} \right) \mu(ds)
\]

\[
= \int_{[\frac{1}{n}, n]} (1 + sA)^{-m} - (1 + A)^{-1} \mu(ds)
\]

\[
= \int_{[\frac{1}{n}, n]} (1 + sA)^{-m} \mu(ds) - a_n (1 + A)^{-1},
\]

and hence \( f_n \in E \) with

\[
\Phi_A(f_n) = \int_{[\frac{1}{n}, n]} (1 + sA)^{-m} \mu(ds) \rightarrow \int_{\mathbb{R}_+} (1 + sA)^{-m} \mu(ds)
\]
in operator norm. Since $f_n \to f$ uniformly on $S_\omega \cup \{0\}$ for each $\omega < \pi$, the proof is complete.

Instead of operating with a topological extension, the authors of [1] use (12.2) as a definition. They then have to show independence of the representation [1, Prop. 4.3], compatibility with the holomorphic calculus [1, Lemma 4.4], and the algebra homomorphism property [1, Propositions 4.5 and 4.6]. In our approach, all these facts follow from Proposition 12.1 and general theory.

12.2. The Hirsch Calculus

Developing further the approach of Hirsch [13], Martinez and Sanz in [15] and [16] define the class $T$ of all functions $f$ that have a representation

$$f(z) = a + \int_{\mathbb{R}_+} \frac{z}{1 + zt} \nu(dt),$$

where $a \in \mathbb{C}$ and $\nu$ is a Radon measure on $\mathbb{R}_+$ satisfying

$$\int_{\mathbb{R}_+} \frac{\nu(dt)}{1 + t} < \infty.$$ 

One can easily see that $T$ is contained in the algebraically extended Stieltjes algebra. Just write

$$f(z) = a + z \int_{[0,1]} \frac{1}{1 + tz} \nu(dt) + \int_{(1,\infty)} \frac{z}{1 + zt} \nu(dt) := a + zg(z) + h(z)$$

and note that $g$ and $h$ are bounded Stieltjes functions. The latter is obvious for $g$; and for $h$ it follows from the identity

$$h(z) = \int_{(1,\infty)} \frac{z}{1 + zt} \nu(dt) = \int_{(1,\infty)} \frac{\nu(dt)}{t} - \int_{(1,\infty)} \frac{1}{1 + tz} \nu(dt). \quad (12.3)$$

We obtain

$$\Phi_{uni}^A(f) = a + A\Phi_{uni}^A(g) + \Phi_{uni}^A(h)$$

$$= a + A \int_{[0,1]} (1 + tA)^{-1} \nu(dt) + \int_{(1,\infty)} A(1 + tA)^{-1} \nu(dt)$$

by a short computation using (12.3). This coincides with how $f(A)$ is defined in [16, Def. 4.2.1]. Hence, the Hirsch calculus (which is not a functional calculus in our terms since $T$ is not an algebra) is contained in the uniform extension of the sectorial calculus.

2The reader might object that the “general theory” presented in this article is quite involved. We agree, but stress the fact that only a commutative version of this theory, which is relatively simple, is needed here.
12.3. Integrals involving Holomorphic Semigroups

If $A$ is sectorial of angle $\omega_{sc}(A) < \frac{\pi}{2}$, then $-A$ generates a holomorphic semigroup

$$T_A(\lambda) := e^{-\lambda A} := (e^{-\lambda z})(A) \quad (\lambda \in S_{\omega_{sc}(A)} - \frac{\pi}{2}),$$

see [9, Section 3.4]. If one has $\alpha = 0$ or $\Re \alpha > 0$, and one restricts $\lambda$ to a smaller sector, the function $\lambda \mapsto (\lambda A)^\alpha T_A(\lambda)$ becomes uniformly bounded. Hence, one can integrate with respect to a bounded measure. The following result shows that also these operators are covered by the uniform extension of the sectorial calculus.

**Proposition 12.2.** Let $0 \leq \varphi < \frac{\pi}{2}$, let $\mu$ be a complex Borel measure on $S_{\varphi}$ and let $\alpha = 0$ or $\Re \alpha > 0$. Then the function

$$f(z) := \int_{S_{\varphi}} (\lambda z)^\alpha e^{-\lambda z} \mu(d\lambda)$$

is holomorphic on $S_{\pi - \varphi}$ and uniformly bounded on each smaller sector. If $A$ is any sectorial operator on a Banach space with $\omega_{sc}(A) + \varphi < \frac{\pi}{2}$, then

$$\Phi_A^{uni}(f) = \int_{S_{\varphi}} (\lambda A)^\alpha e^{-\lambda A} \mu(d\lambda),$$

where $\Phi_A^{uni}$ is the uniform extension of the sectorial calculus for $A$.

We point out that Proposition 12.2 applies in particular to the case that $\varphi = 0$ and $S_{\varphi} = \mathbb{R}_+$ is just the real axis.

**Proof.** The proof follows the line of the proof of Proposition 12.1 and we only sketch it. First one subtracts a constant to reduce to the case that $\mu$ has no mass on $\{0\}$. Then one uses the approximation

$$\int_{\lambda \in S_{\varphi}, \frac{1}{n} \leq |\lambda| \leq n} \ldots \mu(d\lambda) \to \int_{S_{\varphi}} \ldots \mu(d\lambda) \quad (n \to \infty)$$

first for scalars and then for operators. This reduces the claim to establishing the identity

$$\Phi_A \left( \int_{\lambda \in S_{\varphi}, \frac{1}{n} \leq |\lambda| \leq n} (\lambda z)^\alpha e^{-\lambda z} \mu(d\lambda) \right) = \int_{\lambda \in S_{\varphi}, \frac{1}{n} \leq |\lambda| \leq n} (\lambda A)^\alpha e^{-\lambda A} \mu(d\lambda).$$

If $\Re \alpha > 0$ then this is a simple application of Fubini’s theorem. If $\alpha = 0$ then one has to write

$$e^{-\lambda z} = \frac{1}{1 + z} + \left( e^{-\lambda z} - \frac{1}{1 + z} \right)$$

and use Fubini for the second summand. \qed
13. Semigroup and Group Generators

We define a **bounded semigroup** to be uniformly bounded mapping \( T : \mathbb{R}_+ \to \mathcal{L}(X) \) which is strongly continuous on \((0, \infty)\) and satisfies the semigroup laws

\[
T(0) = I, \quad T(s + t) = T(s)T(t) \quad (t, s > 0).
\]

(This has been called a **degenerate semigroup** in [9].) For \( \mu \in M(\mathbb{R}_+) \) one can define

\[
\Psi_T(\mu) := \int_{\mathbb{R}_+} T(s) \mu(ds) \in \mathcal{L}(X)
\]

as a strong integral. The mapping

\[
\Psi_T : M(\mathbb{R}_+) \to \mathcal{L}(X)
\]

is an algebra homomorphism with respect to the convolution product. There is a unique linear relation \( B \) on \( X \), called the **generator** of \( T \), such that

\[
(\lambda - B)^{-1} = \int_0^\infty e^{-\lambda t} T(t) \, dt
\]

for one/all \( \lambda \in \mathbb{C} \) with \( \text{Re} \, \lambda > 0 \) [9, Appendix A.8].

Because of \( \Psi_T(\delta_0) = I \), the representation \( \Psi_T \) is not degenerate. However, its restriction to \( M(0, \infty) \) might be. In fact, this is the case if and only if the common kernel \( \bigcap_{t>0} \ker(T(t)) \) is not trivial, if and only if \( B \) is not operator. From now on, we confine ourselves to the non-degenerate case, i.e., we suppose that \( B \) is an operator. Instead of at \( B \) we shall be looking at

\[
A := -B
\]

it the following.

Note that the Laplace transform

\[
\mathcal{L} : M(\mathbb{R}_+) \to C_b(\mathbb{C}_+) \quad \mathcal{L} \mu(z) := \int_{\mathbb{R}_+} e^{-zs} \mu(ds) \quad (\text{Re} \, z \geq 0).
\]

is injective. Here, \( \mathbb{C}_+ := \{z \in \mathbb{C} \mid \text{Re} \, z > 0\} = \mathbb{S}_+^\circ \). Its image is the **Hille–Phillips algebra**

\[
\mathcal{L} \mathcal{M}(\mathbb{C}_+) := \{ \mathcal{L} \mu \mid \mu \in M(\mathbb{R}_+) \},
\]

a unital algebra under pointwise multiplication. The mapping

\[
\Phi_T : \mathcal{L} \mathcal{M}(\mathbb{C}_+) \to \mathcal{L}(X), \quad \Phi_T(f) = \Psi_T(\mathcal{L}^{-1} f)
\]

is called the **Hille–Phillips calculus** (HP-calculus, for short) for \( A \). One has

\[
\Phi_T((\lambda + z)^{-1}) = (\lambda + A)^{-1}
\]

for all \( \lambda \in \mathbb{C}_+ \). Finally, we extend \( \Phi_T \) algebraically within the field \( \mathcal{M}(\mathbb{C}_+) \) of meromorphic functions on \( \mathbb{C}_+ \) and call this the **extended Hille–Phillips calculus**.
13.1. The Complex Inversion Formula

The semigroup can be reconstructed from its generator by the so-called complex inversion formula. This is a standard fact from semigroup theory in the case that \( T(t) \) is strongly continuous at \( t = 0 \), i.e., if \( A \) is densely defined. However, we do not want to make this assumption here, so we need to digress a little on that topic.

**Proposition 13.1 (Complex inversion formula).** Let \(-A\) be the generator of a bounded semigroup \( T = (T(t))_{t \geq 0} \). Then the mapping \( \mathbb{R}_+ \to \mathcal{L}(X) \) \( t \mapsto T(t)(1 + A)^{-1} \)
is Lipschitz-continuous in operator norm. Moreover, for each \( \omega < 0 \)
\[
T(t)(1 + A)^{-2} = \frac{1}{2\pi i} \int_{\omega + i \mathbb{R}} \frac{e^{-tz}}{(1 + z)^2} R(z, A) \, dz \quad (t \geq 0)
\]
where the integration contour is directed top down from \( \omega + \infty \) to \( \omega - \infty \).

*Proof.* The HP-calculus turns the scalar identity
\[
e^{-tz} - e^{-sz} = -z \int_s^t e^{-rz} \, dr \quad (s, t \in \mathbb{R}_+)
\]
into the operator identity
\[
T(t) - T(s) = -A \int_s^t T(r) \, dr.
\]
Multiplying with \((1 + A)^{-1}\) from the right and estimating yields
\[
\|T(t)(1 + A)^{-1} - T(s)(1 + A)^{-1}\| \leq M \|A(1 + A)^{-1}\| |s - t| \quad (s, t \in \mathbb{R}_+).
\]
For the second claim we note first the estimate
\[
\|\lambda (1 + A)^{-1}\| \leq \frac{M}{\text{Re} \lambda} \quad (\text{Re} \lambda > 0),
\] (13.1)
where \( M := \sup_{t > 0} \|T(t)\| \). Consequently, \( A \) is an operator of strong right half-plane type 0, and hence admits a functional calculus \( \Psi \), say, on half planes as in [2]. Writing \( e^{-tA} := \Psi(e^{tz}) \) one obtains
\[
S(t) := \frac{1}{2\pi i} \int_{\omega + i \mathbb{R}} e^{-tz} R(z, A) \, dz = \Psi\left( \frac{e^{-tz}}{(1 + z)^2} \right) = e^{-tA}(1 + A)^{-2}
\]
for \( t \geq 0 \) by definition of \( \Psi \) and usual functional calculus rules. Taking Laplace transforms, by [2, Lemma 2.4] we obtain
\[
\int_0^\infty e^{-\lambda t} S(t) \, dt = \int_0^\infty e^{-\lambda t} e^{-tA}(1 + A)^{-2} \, dt = (\lambda + A)^{-1}(1 + A)^{-2}
\] (13.2)
whenever \( \text{Re} \lambda > -\omega \). Since \( \omega \) can be chosen arbitrarily close to 0, the identity (13.2) actually holds for all \( \text{Re} \lambda > 0 \). Since the Laplace transform is injective, it follows that
\[
S(t) = T(t)(1 + A)^{-2} \quad (t \geq 0)
\]
as claimed. \(\square\)
As a consequence we obtain that the commutator of the semigroup and the commutator of its generator coincide.

**Corollary 13.2.** For a bounded operator \( S \in \mathcal{L}(X) \) the following assertions are equivalent:

(i) \( S \) commutes with \((1 + A)^{-1}\)

(ii) \( S \) commutes with each \( T(t) \), \( t > 0 \).

**Proof.** The implication (ii) \( \Rightarrow \) (i) is trivial. Suppose that (i) holds. Then \( S \) commutes with \( R(\lambda, A) \) for each \( \lambda \in \varrho(A) \) [9, Prop. A.2.6]. By the complex inversion formula, \( S \) commutes with \((1 + A)^{-2} = (1 + A)^{-2}T(t)\). It follows that

\[
(1 + A)^{-2}ST(t) = S(1 + A)^{-2}T(t) = (1 + A)^{-2}T(t)S.
\]

Since \((1 + A)^{-2}\) is injective, \( T(t)S = ST(t) \). \(\square\)

13.2. A Topological Extension of the HP-Calculus

We let, as before, \(-A\) be the generator of a bounded semigroup as above. As in Section 11, we consider the algebra

\[ A_A = \{(1 + A)^{-1}\}' \]

which by Corollary 13.2 coincides with the commutant of the semigroup. Similarly to Section 11, for \((T_n, T) \in A_A \times A_A\) we write

\[ T_n \xrightarrow{\tau_n^A} T \]

if there is a point-separating subset \( D \subseteq A_A' \) such that

\[ DT_n \to DT \quad \text{in operator norm, for each } D \in D \]

Note the difference to the structure \( \tau_s^A \) considered in Section 11, where we allowed strong convergence. It is easily checked that \( \tau_n^A \) is an algebraic Hausdorff convergence structure.

**Theorem 13.3.** Let \(-A\) be the generator of a bounded semigroup \( T \) on a Banach space \( X \). Then the Hille–Phillips calculus \( \Phi_T \) is closable with respect to the joint convergence structure

(pointwise convergence on \( \mathbb{C}_+ \), \( \tau_n^A \)-convergence) \text{ on } \( \mathcal{H}^\infty(\mathbb{C}_+), A_A \).

**Proof.** Suppose that \( f_n = L\mu_n \in L\mathcal{M}(\mathbb{C}_+) \) is pointwise convergent on \( \mathbb{C}_+ \) to 0 and that \( D \subseteq A_A' \) is a point-separating subset of \( A_A \) such that

\[ D\Phi_T(f_n) \to DT \]

in operator norm. It suffices to show that \((1 + A)^{-1}DT = 0\).

To this aim, note that \( A_A' \) is a commutative unital Banach algebra. Since \( \Phi_T(f_n) \in A_A' \), we have also \( DT \in A_A' \). Hence, by Gelfand theory, it suffices to show that

\[ \chi((1 + A)^{-1}DT) = 0 \]
for each multiplicative linear functional \( \chi : A'_A \to \mathbb{C} \). Fix such a functional \( \chi \).

Since

\[
\chi((1 + A)^{-1} DT) = \chi((1 + A)^{-1}) \chi(DT)
\]

we may suppose without loss of generality that \( \alpha := \chi((1 + A)^{-1}) \neq 0 \).

Consider the function \( c : \mathbb{R}_+ \to \mathbb{C}, c(t) := \chi(T(t)) \). Then

\[
c(t + s) = c(t)c(s) \quad (t, s \geq 0)
\]

and \( c \) is bounded. Moreover,

\[
t \mapsto \alpha c(t) = \chi(T(t)(1 + A)^{-1})
\]

is continuous (by Proposition [13.1]). Since \( \alpha \neq 0 \), \( c \) is continuous. It follows from a classical theorem of Cauchy that there is \( \lambda \in \mathbb{C}_+ \) such that

\[
c(t) = e^{-\lambda t} \quad (t \geq 0).
\]

Next, we find

\[
\Phi(f_n)(1 + A)^{-1} = \int_{\mathbb{R}_+} T(t)(1 + A)^{-1} \mu_n(dt).
\]

By Proposition [13.1] the integrand is a norm-continuous function of \( t \). Hence,

\[
\chi(D\Phi_T(f_n))(1 + A)^{-1} = \chi(D) \int_{\mathbb{R}_+} \chi(T(t)(1 + A)^{-1}) \mu_n(dt)
\]

\[
= \alpha \chi(D) \int_{\mathbb{R}_+} e^{-\lambda t} \mu_n(t) = \alpha \chi(D) f_n(\lambda).
\]

Letting \( n \to \infty \) yields

\[
\chi((1 + A)^{-1} DT) = 0
\]

as desired. (It is in this last step that we need the operator norm convergence \( D\Phi_T(f_n) \to DT \).) \( \square \)

**Remark 13.4.** It is not difficult to see that the spectra of \((1 + A)^{-1}\) in \( A'_A \) and in \( \mathcal{L}(X) \) coincide. It follows from Gelfand theory that

\[
\sigma((1 + A)^{-1}) = \{ \chi((1 + A)^{-1}) \mid 0 \neq \chi \text{ is a multiplicative functional on } A'_A \}.
\]

This implies, eventually, that if \( \alpha := \chi((1 + A)^{-1}) \neq 0 \) then \( \chi(T(t)) = e^{-\lambda t} \), where \((1 + \lambda)^{-1} = \alpha \in \sigma((1 + A)^{-1})\), and hence \( \lambda \in \sigma(A) \) by the spectral mapping theorem for the resolvent. All in all we obtain that we can replace pointwise convergence on \( \mathbb{C}_+ \) by pointwise convergence on \( \sigma(A) \) in Theorem [13.3] (These arguments actually show that the failing of the spectral mapping theorem for the semigroup is precisely due to the existence of multiplicative functionals \( \chi \) on \( A'_A \) that vanish on \((1 + A)^{-1}\) but do not vanish on some \( T(t) \). However, these functionals are irrelevant in our context.)

According to Theorem [13.3] the Hille–Phillips calculus \( \Phi_T \) has a topological extension based on the joint convergence structure

(pointwise convergence on \( \mathbb{C}_+ \), \( \tau^n_A \)-convergence) on \( (\mathbb{H}^\infty(\mathbb{C}_+), A_A) \).
Let us call this the **semi-uniform extension** of the HP-calculus. We do not know whether one can replace $\tau^A_n$ by $\tau^A_s$ here in general. However, there are special cases, when it is possible.

### 13.3. Compatibility of the HP-Calculus and the Sectorial Calculus

By (13.1), the negative generator $A$ of the bounded semigroup $T$ is sectorial of angle $\omega_{\text{se}}(A) \leq \frac{\pi}{2}$. Hence, there are now two competing functional calculi for it, the Hille–Phillips calculus $\Phi_T$ and the sectorial calculus $\Phi_A$, each coming with its associated algebraic and topological extensions. Of course, we expect compatibility, so let us have a closer look.

Suppose first that $\omega_{\text{se}}(A) < \frac{\pi}{2}$. Then by Proposition [12.2], the Hille–Phillips calculus is a restriction of the uniform extension of the elementary sectorial calculus for $A$. By commutativity, compatibility is still valid for the respective algebraic extensions, that is: the extended Hille–Phillips calculus is a subcalculus of $\Phi_{A,\text{uni}}$.

Now, suppose that $\omega_{\text{se}}(A) = \frac{\pi}{2}$. It has been shown in [9, Lemma 3.3.1] that each $e \in \mathcal{E}[S_{\frac{\pi}{2}}]$ is contained in $\mathcal{L}\mathcal{M}(\mathbb{C}_+)$ with

$$\Phi_T(e) = \Phi_A(e)$$

(The actual formulation of [9, Lemma 3.3.1] yields a little less, but its proof works in the more general situation considered here.) It follows that

$$\mathcal{E}_e[S_{\frac{\pi}{2}}] \subseteq \mathcal{L}\mathcal{M}(\mathbb{C}_+) \quad \text{and} \quad \Phi_T = \Phi_A \text{ on } \mathcal{E}_e[S_{\frac{\pi}{2}}].$$

By commutativity of the algebras, the algebraic extensions of these calculi also are compatible (Theorem 6.7). That is, the (algebraically) extended Hille–Phillips calculus is an extension of the sectorial calculus for $A$. Furthermore, the uniform extension of the sectorial calculus is clearly contained in the semi-uniform extension of the Hille–Phillips calculus as described above.

**Remarks 13.5.**

1) The bounded Stieltjes algebra is actually included in the Hille–Phillips algebra. This can be seen by a direct computation. More generally, each function $f \in \mathcal{M}[S_{\frac{\pi}{2}}]$ such that $\Phi_A(f)$ is bounded for each negative generator of a bounded semigroup, is contained in $\mathcal{L}\mathcal{M}(\mathbb{C}_+)$. (Choose $T$ to be the right semigroup on $L_1(\mathbb{R}_+)$.)

2) At present, we do not know how the bp-extension of the sectorial calculus and the semi-uniform extension of the HP-calculus relate.

On the other hand, we can “reach” the HP-calculus from the sectorial calculus by employing a modification of the uniform extension. Namely, consider the joint convergence structure

$$\left( \text{uniform convergence on } \mathbb{C}_+, \text{ operator norm convergence } \right) \quad (13.3)$$

on $\mathcal{L}\mathcal{M}(\mathbb{C}_+)$ and $\mathcal{A}_A$, respectively. By compatibility and Theorem 13.3, the sectorial calculus on $\mathcal{E}_e[S_{\frac{\pi}{2}}]$ is closable with respect to that structure. The next result
show that the functions
\[ \frac{e^{-t\mathbf{z}}}{(1 + \mathbf{z})^2} \quad (t > 0) \]
are in the domain of the corresponding topological extension.

**Lemma 13.6.** Let \(-A\) be the generator of a bounded semigroup \(T\), and let \(t > 0\) and \(\omega < 0\). For any \(n \in \mathbb{N}\) the function
\[ f_n(z) := \frac{1}{2\pi i} \int_{\omega + i[-n,n]} e^{-wt} \frac{dw}{w - z} \]
is contained in \(E_e[S_{\pi/2}]\). Moreover,
\[ f_n \to \frac{e^{-t\mathbf{z}}}{(1 + \mathbf{z})^2} \quad (n \to \infty) \]
uniformly on \(\overline{\mathbb{C}}^+\) and
\[ \Phi_A(f_n) \to T(t)(1 + A)^{-2} \]
in operator norm.

**Proof.** Note that \(f_n\) is holomorphic on \(\mathbb{C} \setminus (\omega + i[-n,n])\) and hence on a sector \(S_{\varphi}\) for \(\varphi > \pi/2\). On each smaller sector we have \(f_n(z) = O(|z|^{-1})\) as \(|z| \to \infty\) and \(f_n(z) - f_n(0) = O(|z|)\) as \(|z| \to 0\). It follows that \(f_n \in E_e(S_{\varphi})\).

The remaining statements follow from the complex inversion formula. One needs the identity
\[ \Phi_A(f_n) = \frac{1}{2\pi i} \int_{\omega + i[-n,n]} e^{-wt} R(w, A) dw, \]
which is proved by standard arguments. \(\square\)

Since \(T(t)\) can be reconstructed algebraically from \(T(t)(1 + A)^{-2}\), we see that the semigroup operators are contained in the algebraic extension of the topological extension given by (13.3) of the elementary sectorial calculus.

### 14. Normal Operators

Normal operators on Hilbert spaces are known, by the spectral theorem, to have the best functional calculus one can hope for. The (Borel) functional calculus for a normal operator is heavily used in many areas of mathematics and mathematical physics. Despite this importance of the functional calculus, the spectral theorem is most frequently formulated in terms of projection-valued measures or multiplication operators, and the functional calculus itself appears merely as a derived concept.

This expositional dependence (of the functional calculus on the spectral measure) is manifest in the classical extension of the calculus from bounded to unbounded functions as described, e.g., in Rudin’s book [20]. Since the description of \(f(A)\) for unbounded \(f\) in terms of spectral measures is far from simple, working
with the unbounded part of the calculus on the basis of this exposition is rather cumbersome.

However, the situation now is different from when Rudin’s classic text was written, in at least two respects. Firstly, we now have an axiomatic notion of a functional calculus (beyond bounded operators in its range). This enables us to develop the properties of the calculus from axioms rather than from a particular construction, which makes things far more perspicuous and, eventually, far easier to handle.

Secondly, we now have an elegant tool to go from bounded to unbounded functions: the algebraic extension procedure. As a result, the unbounded part of the construction of the functional calculus for a normal operator on a Hilbert space just becomes a corollary of Theorem 6.1. Actually, all algebras in this context are commutative and there is always an anchor element, so one does not even need the full force of Theorem 6.1, but only the relatively elementary methods of [9].

In order to render these remarks less cryptic, we need of course be more specific about the theory we have in mind. Here, we shall only sketch the main features, and defer a more detailed treatment to a separate paper [12].

Let \((X, \Sigma)\) be a measurable space, i.e., \(X\) is a set and \(\Sigma\) is a \(\sigma\)-algebra of subsets of \(X\). We let

\[
\mathcal{M}(X, \Sigma) := \{ f : X \to \mathbb{C} \mid f \text{ measurable} \}.
\]

A measurable (functional) calculus on \((X, \Sigma)\) is a pair \((\Phi, H)\) where \(H\) is a Hilbert space and

\[
\Phi : \mathcal{M}(X, \Sigma) \to \mathcal{L}(H)
\]

is a mapping with the following properties \((f, g \in \mathcal{M}(X, \Sigma), \lambda \in \mathbb{C}):\)

\begin{enumerate}
  \item [MFC1] \(\Phi(1) = I\);
  \item [MFC2] \(\Phi(f) + \Phi(g) \subseteq \Phi(f + g)\) and \(\lambda \Phi(f) \subseteq \Phi(\lambda f)\);
  \item [MFC3] \(\Phi(f)\Phi(g) \subseteq \Phi(fg)\) and \(\text{dom}(\Phi(f)\Phi(g)) = \text{dom}(\Phi(g)) \cap \text{dom}(\Phi(fg))\);
  \item [MFC4] \(\Phi(f) \in \mathcal{L}(H)\) and \(\Phi(f)^* = \Phi(f^*)\) if \(f\) is bounded;
  \item [MFC5] If \(f_n \to f\) pointwise and boundedly, then \(\Phi(f_n) \to \Phi(f)\) weakly.
\end{enumerate}

Property (MFC5) is called the weak bp-continuity of the mapping \(\Phi\).

Evidently, (MFC1)–(MFC3) are just the axioms (FC1)–(FC3) of a proto-calculus. For \(f \in \mathcal{M}(X, \Sigma)\) let

\[
e := \frac{1}{1 + |f|}
\]

Then \(e\) is a bounded function and \(ef\) is also bounded. Hence, by (MFC4), \(e\) is a regularizer of \(f\). Moreover, \(\Phi(e^{-1})\) is defined, and hence \(\Phi(e^{-1}) = \Phi(e)^{-1}\) (Theorem 2.1). It follows that

\[
\Phi(f) = \Phi(e^{-1}ef) = \Phi(e)^{-1}\Phi(ef),
\]
which just means that the set \( \{e\} \) is determining for \( \Phi(f) \). This show that

\[
\mathcal{E} := \{ e \in \mathcal{M}(X, \Sigma) \mid e \text{ is bounded}\}
\]

is an algebraic core for \( \Phi \). (In particular, \( \Phi \) satisfies (FC4) and hence is a calculus.)

As a result, each measurable calculus coincides with the algebraic extension of its restriction to the bounded functions. To construct a measurable calculus, it therefore suffices to construct a calculus on the bounded measurable functions and then apply the algebraic extension procedure. And this is a far simpler method than employing spectral measures.

It is remarkable (and very practical) that only (MFC1)–(MC5) are needed to establish all the well-known properties of the Borel calculus for normal operators. For example, one can prove that the identity

\[
\Phi(f) = \Phi(f)^*
\]

holds for each \( f \in \mathcal{M}(X, \Sigma) \), and not just for bounded functions as guaranteed by (MFC4). Next, observe that for given \( f, g \) the sequence of functions

\[
e_n := \frac{n}{n + |f| + |g|}
\]

form a common approximate identity for \( f \) and \( g \). (This is actually a strong approximate identity, since strong convergence in (MFC5) holds automatically.) By Theorem 7.2 we obtain

\[
\Phi(f) + \Phi(g) = \Phi(f + g), \quad \Phi(f)\Phi(g) = \Phi(fg).
\]

One of the most important results in this abstract development of measurable calculi concerns uniqueness. We only cite a corollary of a more general theorem:

**Theorem 14.1.** Let \( X \subseteq \mathbb{C}^d \), endowed with the trace \( \sigma \)-algebra of the Borel algebra. Let \( (\Phi, H) \) and \( (\Psi, H) \) be two measurable calculi on \( X \) such that

\[
\Phi(z_j) = \Psi(z_j) \quad (j = 1, \ldots, d).
\]

Then \( \Phi = \Psi \).

This theorem implies, e.g., the composition rule

\[
(f \circ g)(A) = f(g(A))
\]

for a normal operator \( A \) on a Hilbert space \( H \), since both mappings

\[
\Phi(f) := (f \circ g)(A) \quad \text{and} \quad \Psi(f) := f(g(A))
\]

are Borel calculi on \( \mathbb{C} \) that agree for \( f = z \).

For more about the functional calculus approach to the spectral theorem we refer to [12].
Appendix A. The Closed Graph Theorem for the Weak*-Topology

In our investigations on the dual calculus, the following theorem is needed.

**Theorem A.1.** Let $X,Y$ be Banach spaces and $T \in \mathcal{L}(X', Y')$ such that $T$ has a closed graph with respect to the weak* topologies. Then $T$ is continuous with respect to the weak* topologies and hence of the form $T = S'$ for some $S \in \mathcal{L}(Y; X)$.

Theorem [A.1] is actually a special case of much more general results about operators between certain topological vector spaces. One of the earliest references for it is [17, Theorem 1]. A more explicit version for Fréchet spaces is [14, p.79], with the caveat that one really needs a closed graph (rather than just a sequentially closed graph) to make the proof work. Again less explicit, Theorem [A.1] is a special case of the results in [21]. (I am indebted to Wolfgang Ruess for these bibliographical remarks.)

However, as all these references rely on expert knowledge in the field of topological vector spaces, we include a short ad hoc proof for the convenience of the reader.

**Proof of Theorem [A.1]** Without loss of generality, we may suppose that $\|T\| \leq 1$. Let $K := \text{Ball}_{X'}[0,1]$ and $L := \text{Ball}_{Y'}[0,1]$ be the closed unit balls of $X'$ and $Y'$, respectively. Then $K, L$ are compact with respect to the weak* topologies. Clearly, $T(K) \subseteq L$, and by hypothesis, $T|_K : K \to L$ has a closed graph. Hence, $T|_K$ is continuous [19, §26, Ex.8]. This implies that for each $y \in Y$ the element $T'y$ of $X''$ is weak* continuous on $K$, so is an element of $X$ by a classical result (see [22, 1.2] for a simple proof). This means that $T'$ maps $Y$ into $X$ and with $S := T|_Y$ the claim follows. □

**References**


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