DIMENSIONS OF LIMIT SETS OF KLEINIAN GROUPS

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Abstract. In this paper we give a brief survey of results on dimension gaps for limit sets of geometrically infinite Kleinian groups. We concentrate on an important notion from geometric group theory, amenability, as a criterion for the existence of such gaps.

1. Dimensions and invariants in conformal dynamics

One goal often encountered in mathematics is to prove equality of independently defined invariants for large classes of a certain mathematical object. An instance of this in conformal dynamics has been the attempt to show that the critical exponent of the Poincaré series associated to a conformal dynamical system , e.g. a Kleinian group or a rational map on the Riemann sphere, coincides with the Hausdorff dimension of the corresponding limit set or Julia set, respectively. Since here we will be dealing mainly with *Kleinian groups*, i.e. discrete, torsion-free subgroups of the group of orientation preserving isometries of $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, $n \in \mathbb{N}$, let us first explain briefly what these invariants are. The *limit set* $L(G)$ of a Kleinian group G is the set of accumulation points of some and thus any orbit Gx of G, $x \in \mathbb{H}^{n+1}$, and is a subset of the boundary $\partial\mathbb{H}^{n+1}$ of $\mathbb{H}^{n+1}$ due to the discreteness of G. The *Poincaré series* of G is defined as a Dirichlet series

$$P(x, y, s) := \sum_{g\in G} e^{-s\, d(x,gy)},$$

where d is the hyperbolic metric, $x, y \in \mathbb{H}$ are arbitrary but fixed and are often chosen to be 0 when working in the Poincaré unit ball model of hyperbolic geometry, and s is a real parameter. The abscissa of convergence of the series reduces for real values of s to the *critical exponent* of G, which can be expressed in two different ways:

$$\begin{aligned} \delta(G) &:= \inf\{s > 0 : P(x, y, s) < \infty\} \\ &= \limsup_{R\to\infty} \frac{\log \#(B(x, R) \cap Gy)}{R}\,. \end{aligned}$$

By a theorem of Roblin [41], the upper limit above is in fact a limit. G is called *of convergence type* if $P(x, y, \delta(G)) < \infty$, and *of divergence type* otherwise. It is not too difficult to see that for *non-elementary* Kleinian groups, i.e. groups which are not generated by only one isometry, the critical exponent is positive and less or equal to n, the dimension of the boundary of hyperbolic space. For more details on Kleinian groups and hyperbolic geometry we refer to [4], [30], [38], [31], [28] and [33].

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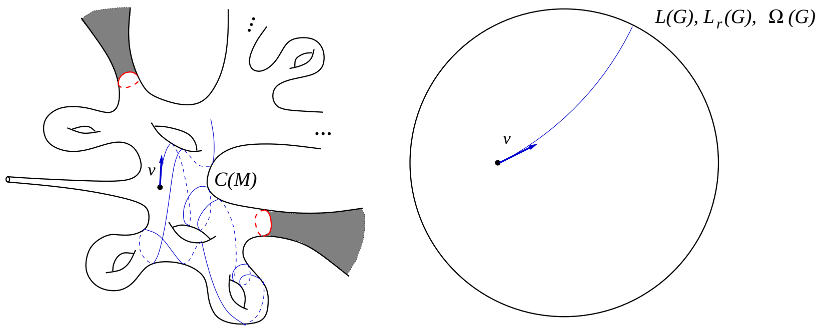

FIGURE 1. Types of dynamics in a hyperbolic manifold.

While some early results of Patterson's [34] (see also [37]) suggested that equality should hold in fairly general circumstances, it was also Patterson [35], [36] (see also [3] and [20]) who gave the first examples of Kleinian groups G for which $\delta(G)$ is strictly less than the Hausdorff dimension $\dim_H(L(G))$ of the limit set. These are groups *of the first kind*, i.e. so that $L(G)$ is the whole boundary of $\mathbb{H}^{n+1}$, which obviously have limit sets of full dimension n.

The existence of such examples raises the natural question of what the 'dynamical meaning' of such a gap might be. The answer was given only later by the rather elegant theorem of Bishop and Jones [8] who showed that, for all non-elementary Kleinian groups G,

$$\dim_{\mathrm{H}}(L_r(G)) = \delta(G).$$

Thus, the dimension gap is in fact a gap between recurrent and transient dynamics within the convex core of the corresponding hyperbolic manifold. We now need to again briefly explain some of the notions used above. The *convex hull of the limit set* $CH(G)$ of a Kleinian group G is defined to be the closed metric convex hull of the union of all geodesics in $\mathbb{H}^{n+1}$ with both ends in the limit set $L(G)$. The group G not only acts discontinuously on $\mathbb{H}$, but also on $C(G)$, and the quotient $C(M_G) := CH(G)/G$ is called the *convex core* of the hyperbolic manifold $M_G := \mathbb{H}^{n+1}/G$. Equivalently, $C(M_G)$ is the closed metric convex hull of the union of all closed geodesics in M_G, thus being the region in M_G where closed (geodesic) orbits occur. A limit point $\xi \in L(G)$ is called *radial* if the projection to M_G of some infinite geodesic ray in $\mathbb{H}$ towards ξ returns infinitely often to some compact subset of M_G. The set of radial limit points of G is called the *radial limit set* of G and is denoted by $L_r(G)$. If we fix one point $o \in C(M_G)$ we get a very clear and suggestive 1-1 correspondence between the dynamical behaviour of geodesic rays originating at o and the type of point at infinity the lifts of these geodesic rays to $\mathbb{H}$ end in: geodesic movements staying in $C(M_G)$ correspond to limit points of G and recurrent geodesic movements (which then necessarily stay in $C(M_G)$) correspond to radial limit points of G (see also Figure 1).

The group G and the manifold M_G are called *geometrically finite* if, for some and thus all $\varepsilon > 0$, the ε-neighbourhood of $C(M_G)$ has finite hyperbolic volume, and *geometrically infinite* otherwise. From a modern perspective it is clear that the dimension gap between recurrent and transient dynamics showcased by Bishop and Jones' theorem above can only occur for geometrically infinite groups, as shown by the classic result of Beardon and Maskit [5] which states that a Kleinian group G

is geometrically finite if and only if $L(G) = L_r(G) \dot{\cup} L_p(G)$. Here, $L_p(G)$ denotes the set of *bounded parabolic fixed points* of G, i.e. those parabolic fixed points for which any invariant horosphere projects to a region whose intersection with the convex core has finite hyperbolic volume. G is called *convex cocompact* if it acts cocompactly on $CH(G)$ or, equivalently, if $C(M_G)$ is compact.

Nonetheless, it is of course important to decide when equality of these invariants occurs. For non-elementary geometrically finite Kleinian groups, it is well-known that $\delta(G) = \dim_{\mathrm{H}}(G)$ (see the book of Nicholls [33] for more details), and that $\dim_{\mathrm{H}}(G) = \dim_{\mathrm{M}}(G)$, where $\dim_{\mathrm{M}}$ denotes the *box counting* or *Minkowski dimension* (see Stratmann and Urbański [49]). For Kleinian groups acting on the 3-dimensional hyperbolic space $\mathbb{B}^3$, Bishop [6], [7] has shown $\delta(G) = \dim_{\mathrm{M}}(G)$ if the group is non-elementary and analytically finite and if its limit set has null 2-dimensional measure (see also [8]). Taking the solution of the Ahlfors measure conjecture (which follows from the Tameness Conjecture proven in [2] and [15]) into consideration, the above results imply that all finitely generated non-elementary Kleinian groups acting on $\mathbb{H}^3$ satisfy $\delta(G) = \dim_{\mathrm{M}}(G)$.

Having said this, let us return to the problem of identifying Kleininan groups with a dimension gap. It is the seminal work of R. Brooks [13] that brings a very important notion from geometric group theory into the picture: amenability. While fractal dimensions of limit sets were not his main motivation, but rather classical considerations in Riemannian geometry concerning the role played by the bottom of the L^2-spectrum of a Riemannian manifold in its large scale geometry, which he first investigated in [11], Brooks was of course aware of the fact that his results from [13] actually generalise earlier observations on critical exponents of Kleinian groups by M. Rees [39],[40]. Brooks [13] essentially shows that as long as $\delta(G) > n/2$ for a convex cocompact Kleinian group G acting on $\mathbb{H}^{n+1}$, we have for any normal subgroup N of G that

$$\delta(N) = \delta(G) \iff G/N \text{ amenable.}$$

Using Bishop and Jones' elegant observation above, this can be easily seen to produce an entire new class of groups with dimension gap [19]. It is worth noting here that there is a purely algebraic analogon [17], [22] of Brooks' theorem which predates his result, but of which he does not seem to have been aware. We discuss all these ideas at some length in Section 2.

Since the connection between amenability and the question of existence of dimension gaps becomes evident through the work of Brooks, it is obvious that any successful attempt at generalising his result will produce new examples of Kleinian groups with dimension gap. Let us first mention the one approach which is closest in method and spirit to the original. S. Tapie [52] takes Brooks' idea further by considering copies of a Riemannian manifold with boundary that can be glued together to a Riemannian manifold without boundary following the pattern of an infinite graph. Under a certain analytic condition, namely that the building block manifold admits an eigenfunction to the bottom of its L^2-spectrum which 'extends well' to a function on the glued manifold, Tapie shows that the amenability of the skeleton graph is equivalent to the bottoms of the L^2-spectra of the building block and the glued manifold coinciding. The one aspect which also goes beyond Brooks' considerations is the fact that Tapie gives an estimate for the size of the gap in terms of the associated isoperimetric constants. For more details on Tapie's rather ingenious and involved work we refer the reader to his paper [52].

Next, we delve into the realm of symbolic dynamics. In an attempt to drop the condition on $\delta(G)$ in the statement of Brooks' theorem, which is in place essentially due to Brooks' method of proof, M. Stadlbauer [47] formulates and solves the problem in terms of symbolic dynamics or, more precisely, topological Markov chains. He shows that under certain circumstances, the Gurevič pressures of a topological Markov chain and a group extension of this chain by some countable group coincide precisely when the group is amenable. This translates to Kleinian groups and their limit sets via the encoding procedure for the dynamics of the geodesic flow given by Bowen and Series [10] and developed further by Stadlbauer and Stratmann [46],[48]. Stadlbauer shows that while the assumption $\delta(G) > n/2$ can be dropped, another condition on G needs to be satisfied, namely, instead of G being convex cocompact, one needs to assume that it is *essentially free* [48], i.e. that all relations of G are due to parabolic elements. It is also interesting to note that Stadlbauer in fact recovers the algebraic version of the dimension gap problem [17], [22] mentioned above. Details on Stadlbauer's work are to be found in Section 3. We also refer the reader to the closely related and very interesting work of J. Jaerisch [24], [25], [26] and [27].

Independently of Tapie's afore-mentioned approach [52] to generalise Brooks' theorem, a similar attempt at a generalisation is given in [19], with more emphasis on the dynamical aspects of the problem. Just as in [52], the convex cocompact group G is eliminated from the picture allowing for the study of hyperbolic surfaces given by infinitely generated Fuchsian groups (i.e. Kleinian groups acting on hyperbolic 2-space), which, in contrast to [52], are *not* made of identical copies of a surface with boundary. In [19] the critical exponent $\delta(G)$ is replaced by an invariant called *convex core entropy* which actually turns out to coincide with the upper Minkowski dimension of the limit set. After formulating the main results which relate the amenability of the graph associated with a so-called *pants decomposition* of the hyperbolic surface of infinite type under scrutiny to the existence of a dimension gap (see Section 4 for more details), we go on to split up the main question into two questions. First, when is the critical exponent strictly smaller than the convex core entropy, and second, when does the Hausdorff dimension of the limit set coincide with the convex core entropy of the surface. At the end of Section 4 we discuss conditions under which the latter question has a positive answer. Finally, in Section 5 we discuss some open problems and possible strategies to attack these two questions.

2. Brooks' Theorem; the role of amenability

In a series of papers, most notably [11] and [13], R. Brooks considers the fundamental question of the behaviour of the bottom λ_0 of the L^2-spectrum under Riemannian coverings, i.e. local isometries $M_2 \to M_1$, where M_1 is a complete Riemannian manifold. He resolved this question first [11] for the case that M_1 is compact, thus making $\lambda_0(M_1) = 0$, and M_2 is its universal cover, and then [13] for the more general case that $\pi_1(M_2)$ is normal in $\pi_1(M_1)$. Here, the notion of amenability plays a crucial role. While it is not too difficult to show that if $\pi_1(M_1)/\pi_1(M_2)$ is amenable, then $\lambda_0(M_1) = \lambda_0(M_2)$ (see also [45] and [42] for different points of view on this implication), the main difficulty consists in showing the converse.

Let us give a brief reminder on the notion of amenability of (countably generated) discrete groups. J. von Neumann defined such a group G to be *amenable*, if there exists an *left-invariant mean* on $L^\infty(G,\mathbb{R})$. i.e. a non-negative bounded linear functional $\mu : L^\infty(G,\mathbb{R}) \to \mathbb{R}$ so that $\mu(1) = 1$ and $\mu(\tau_g f) = \mu(f)$ for all $f \in L^\infty(G,\mathbb{R})$, where, for each $g \in G$, $\tau_g : L^\infty(G,\mathbb{R}) \to L^\infty(G,\mathbb{R})$ is defined by $\tau_g f(h) := f(g^{-1}h)$. Later, Følner [21] gave the following very useful characterisation in the finitely generated case. A finitely generated group G is amenable if and only if there exists a sequence (F_n) of finite sets in G so that $\bigcup_n F_n = G$ and, for all $g \in G$,

$$\lim_{n\to\infty} \frac{\#\, gF_n \triangle F_n}{\# F_n} = 0,$$

where $\triangle$ refers to the symmetric difference. The condition which is most intuitive for our purposes is the following (see discussion of amenability in [11]). A finitely generated group G is amenable if and only if there exists a sequence (F_n) of finite sets in G so that

$$\lim_{n\to\infty} \frac{\#\partial F_n}{\# F_n} = 0,$$

where ∂F_n is the boundary of F_n in the Cayley graph of G.

Theorem 2.1 ([13]). *Let M_1 be a $(n+1)$-dimensional, $n \in \mathbb{N}$, hyperbolic manifold given by a convex cocompact Kleinian group which satisfies $\lambda_0(M_1) < n^2/4$, and consider some normal cover M_2 of M_1. Then,*

$$\lambda_0(M_1) = \lambda_0(M_2) \iff \pi_1(M_1)/\pi_1(M_2) \text{ amenable.}$$

Note that this generalises previous results by M. Rees [39] [40] who considered the special case where $\pi_1(M_1)/\pi_1(M_2)$ is isomorphic to $\mathbb{Z}^d$ for some $d \in \mathbb{N}$, i.e. is abelian. For the proof of the more involved implication, Brooks employs a method of Sullivan [51] and Agmon [1] to shift the bottom of the L^2-spectrum of a Riemannian manifold M to 0; one uses a positive harmonic function ϕ corresponding to the eigenvalue λ of the Laplace-Beltrami operator Δ_M on M to define the new operator

$$P := \frac{1}{\phi}\,(\Delta_M - \lambda) \circ [\phi],$$

where $[\phi]$ denotes the multiplication operator by ϕ. P has the same spectrum as Δ_M, just shifted by λ, and is self-adjoint w.r.t. the volume form of M multiplied by ϕ^2. Then Brooks follows the line of thought from his earlier work [11],[12] where he showed that, for a compact Riemannian manifold M, the amenability of $\pi_1(M)$ is equivalent to the bottom of the L^2-spectrum of the universal cover $\widetilde{M}$ being 0. Assuming $\lambda_0(\widetilde{M}) = 0$, one considers a sequence (f_k) of smooth functions with compact support on $\widetilde{M}$ whose Rayleigh quotients satisfy

$$\frac{\int_{\widetilde{M}} |\nabla f_k|^2}{\int_{\widetilde{M}} |f_k|^2} \longrightarrow 0.$$

Then one uses Cheeger's method [16] to produce hypersurfaces S_k separating $\widetilde{M}$ into a bounded component $\mathrm{int}(S_k)$ and an unbounded one, so that the corresponding isoperimetric ratios satisfy

$$\frac{\mathrm{area}(S_k)}{\mathrm{vol}(\mathrm{int}(S_k))} \longrightarrow 0.$$

After replacing the hypersurfaces S_k by integral currents T_k, $k \in \mathbb{N}$, of uniformly bounded mean curvature with isoperimetric ratio less or equal to that of the corresponding S_k, one is then able to apply the following characterisation of amenability [11, Proposition 2]. The fundamental group of M is amenable if and only if for any $\varepsilon > 0$ there is a union H_ε of fundamental domains of the action of $\pi_1(M)$ on $\widetilde{M}$ which satisfies the isoperimetric inequality

$$\frac{\text{area}(\partial H_\varepsilon)}{\text{vol}(H_\varepsilon)} < \varepsilon.$$

Of course these ideas from [11] need adjustment in order to prove Theorem 2.1 and we refer the reader to [13] for further details.

It is of course clear that while Brooks formulates and proves his theorem in terms of the bottom of the L^2-spectrum, the statement can be reinterpreted for our purposes in terms of critical exponents of Kleinian groups via the following well-known correspondence.

Theorem 2.2 ([18], [34], [51])**.** *For any Kleinian group G acting on $\mathbb{H}^{n+1}$, $n \geq 1$, with critical exponent $\delta = \delta(G)$ we have*

$$\lambda_0(M) = \begin{cases} \dfrac{n^2}{4}, & \text{if } \delta \leq \dfrac{n}{2} \\ \delta(n-\delta), & \text{if } \delta \geq \dfrac{n}{2} \end{cases}$$

Thus, Brooks' theorem can be formulated as follows, using non-amenability instead of amenability in order to emphasize the existence of a dimension gap. Let G be convex cocompact acting on $\mathbb{H}^{n+1}$ so that $\delta(G) > n/2$ and let $N \triangleleft G$ be a non-trivial normal subgroup. Then,

$$\delta(N) < \delta(G) \iff G/N \text{ not amenable.}$$

Now consider a normal subgroup N of a convex cocompact Kleinian group G so that $\delta(G) > n/2$ and G/N is non-amenable; apply Brooks' Theorem, together with Bishop and Jones' observation [8] described in the first section and the fact that G is convex cocompact, to obtain

$$\dim_{\mathrm{H}}(L_r(N)) = \delta(N) < \delta(G) = \dim_{\mathrm{H}}(L(G)) = \dim_{\mathrm{H}}(L(N)).$$

This means that, for the hyperbolic manifold $\mathbb{H}^{n+1}/N$, we do have a dimension gap, i.e. 'recurrent dynamics has strictly smaller Hausdorff dimension than transient dynamics'. Here is a class of examples of such normal subgroups N.

Example 2.3. ([20]) Let G_0 and G_1 be Schottky groups with fundamental domains F_0 resp. F_1, so that $F_0^{\mathbf{c}} \cap F_1^{\mathbf{c}} = \emptyset$. Assume G_0 is freely generated by hyperbolic isometries $g_1, \ldots, g_k$, and that G_1 is freely generated by more than one hyperbolic isometry. Put $N := ker(\varphi)$, where $\varphi : G \to G_1$ is the canonical group homomorphism. Thus, $0 \to N \to G \to G_1 \to 0$ is a short exact sequence and

$$N = \langle h g_i h^{-1} : i = 1, ..., k, h \in G_1 \rangle$$

Furthermore, N is the normal subgroup of G generated by G_0 in G, and $G/N \cong G_1$. Clearly, G_1 is not amenable since it is freely generated by at least two generators.

As mentioned in the first section, there are also other classes of (infinitely generated) Kleinian groups with critical exponent strictly smaller than the Hausdorff dimension of their limit set, e.g. [35], [36] (see also [20]). The examples in [36] are constructed so that the limit set has full Hausdorff dimension but the critical exponent can be chosen to be arbitrarily close to 0. In the normal subgroup case, however, the gap cannot get 'too large':

Theorem 2.4 ([20]). *If N is a non-trivial normal subgroup of a Kleinian group G, then*

$$\delta(N) \geq \frac{\delta(G)}{2}.$$

This result has been refined using different techniques [42], [27], where it is shown in addition to the above that if G is of divergence type, then the inequality is strict. The inequality also turns out to be sharp, as shown in [9].

We finish this section by discussing an algebraic analogon of Brooks' theorem and Theorem 2.4 due to Cohen [17] and Grigorchuk [22]. Let G be a finitely generated group and let $|\cdot|$ denote the word length w.r.t. some finite generating set. The *exponential growth rate* of G is defined as

$$\gamma(G) := \limsup_{n\to\infty} \sqrt[n]{\#\{g \in G : |g| \leq n\}} = \lim_{n\to\infty} \sqrt[n]{\#\{g \in G : |g| \leq n\}}.$$

The fact that the limit exists is a theorem [17]. In order to emphasize the analogy to the geometric results, we modify slightly the definition of *cogrowth* or *entropic dimension* as in [17], [22], by saying that the *entropy* of G is given by

$$h(G) := \log \gamma(G) = \lim_{n\to\infty} \frac{\log \#\{g \in G : |g| \leq n\}}{n}.$$

Now, if G is the free group with k generators, then it is not difficult to see that

$$\gamma(G) = 2k-1 \quad \text{and} \quad h(G) = \log(2k-1),$$

and we have the following result which predates the other theorems described in this section.

Theorem 2.5 ([17], [22]). *If G is the free group in k generators, and $N \triangleleft G$ some non-trivial normal subgroup, then*

$$\frac{h(G)}{2} < h(N) \leq h(G).$$

Equality in the second inequalitiy occurs precisely when G/N is amenable.

3. Group extensions of topological Markov chains

In this section we discuss the generalisation of Brooks' Theorem 2.1 given by M. Stadlbauer [47] who extends Brooks' ideas to the realm of symbolic dynamics or, more precisely, topological Markov chains, and then employs the thermodynamical formalism and methods from the theory of random walks on infinite graphs to obtain his main result.

We shall introduce some key notions in symbolic dynamics and then formultate the main theorems of the section. For more details we refer the reader to the original article [47]. The one-sided *topological Markov chain* (Σ_A, θ) with countable *alphabet* I consists of the symbolic space

$$\Sigma_A := \{(w_k)_{k\in\mathbb{N}} : w_k \in I \text{ and } a_{w_k w_{k+1}} = 1\ \forall k \in \mathbb{N}\}$$

where $A := (a_{ij})_{i,j\in I}$ is an adjacency matrix so that $a_{ij} \in \{0,1\}$ and $\sum_j a_{ij} > 0$ for all $i \in I$, and the *shift map*

$$\theta : \Sigma_A \to \Sigma_A,\ (w_1 w_2 \ldots) \mapsto (w_2 w_3 \ldots).$$

This is a topological dynamical system with topology given by *cylinder sets* $[w]$, where w is a finite admissible word, for which the notions of *topological transitivity* and *mixing* make sense. Also, one can define the so-called *big images and preimages (b.i.p.) property* [44] which in fact coincides with the notion of *finite irreducibility* as introduced in [32]. Given a strictly positive, continuous *potential function* $\varphi : \Sigma_A \to \mathbb{R}^+$ of bounded variation, one can define the *Gurevič pressure* [43] of $(\Sigma_A, \theta, \varphi)$

$$P(\theta, \varphi) := \limsup_{n\to\infty} \frac{1}{n} \log Z_n^a$$

as the exponential growth rate of certain *partition functions* Z_a^n, $a \in I$, w.r.t. the shift map and the potential function. If $\log \varphi$ is Hölder continuous and $(\Sigma_A, \theta, \varphi)$ topologically mixing, then a variational principle holds [43]. For a finite admissible word v of length $n \in \mathbb{N}$ and $f : \Sigma_A \to \mathbb{C}$, let $f \circ \tau_v : \Sigma_A \to \mathbb{C}$ be given by $x \mapsto \mathbf{1}_{\varphi^n([v])}(x) \cdot f(\tau_v(x))$, where $\mathbf{1}.$ denotes the characteristic function of a set. Using this, define the *Ruelle* or *transfer operator* by

$$L_\varphi f := \sum_v (\varphi \circ \tau_v) \cdot (f \circ \tau_v),$$

where the sum is taken over all admissible words of length 1 and f is in some appropriate complex function space such that the possibly infinite sum is well defined.

We now come to the analogue of normal covers in the context of topological Markov chains, namely their group extensions. Let (Σ_A, θ) be a topological Markov chain, G a countable discrete group and $\psi : \Sigma_A \to G$ so that ψ is constant on the cylinder sets in (Σ_A, θ) associated to admissible words of length 1. With $X := \Sigma_A \times G$ endowed with the product topology, we define the *group extension* (X, T) of (Σ_A, θ) by

$$T : X \to X, \;\; (x, g) \mapsto (\theta x, g\,\psi(x)).$$

As it turns out, (X, T) is also a topological Markov chain. Furthermore, any potential $\varphi : \Sigma_A \to \mathbb{R}^+$ lifts to a potential $\hat{\varphi} : X \to \mathbb{R}$ by simply setting $\hat{\varphi}(x, g) := \varphi(x)$, inverse branches of T are given by $\hat{\tau}_v(x, g) := (\tau_v(x), g\,\psi(v)^{-1})$ for all admissible finite words v, the Gurevič pressure $P(T, \hat{\varphi})$ is defined in the same way as $P(\theta, \varphi)$ and, finally, the Ruelle operator for (X, T) is defined by

$$\hat{L}_{\hat{\varphi}}(\xi, g) := \sum_v \hat{\varphi} \circ \hat{\tau}_v(\xi, \mathrm{id}_G) \;\cdot\; f \circ \hat{\tau}_v(\xi, g),$$

where $\xi \in [v]$ with v admissible of length 1, and $g \in G$.

The core idea in Stadlbauer's work is to generalise H. Kesten's [29] characterisation of amenability of groups to group extensions of topological Markov chains. Kesten shows that a probability measure on a countable group which satisfies a certain symmetry condition defines an operator on the ℓ^2-space of the group whose spectral radius is equal to 1 if and only if the group is amenable. While this theorem holds 'fibre-wise' for group extensions of topological Markov chains, it becomes necessary for the Markov chain to be *(weakly) symmetric* in a certain sense in order to formulate and prove an analogous result for its group extensions. (Again, we refer the interested reader to [47] for further details.)

Theorem 3.1 ([47]). *Consider the topologically transitive and symmetric group extension (X, T) of the topologically mixing topological Markov chain (Σ_A, θ) by a discrete group G, and a weakly symmetric potential φ so that $P(\theta, \varphi) < \infty$. Then,*

$$G \text{ amenable} \implies P(T, \hat{\varphi}) = P(\theta, \varphi).$$

While this result can still be seen as a more or less direct consequence of Kesten's theorem mentioned above, the proof of the converse is way more involved and relies on a very nifty and careful analysis of the action of the Ruelle operator on the embedding of $\ell^2(G)$ into a certain subspace of the space of continuous functions on X.

Theorem 3.2 ([47])**.** *Consider a topological Markov chain (Σ_A, θ) with the b.i.p.-property, a Hölder continuous potential φ with $\|L_\varphi(1)\| < \infty$, and a topologically transitive group extension (X, T) by the countable group G. Then,*

$$P(T, \hat{\varphi}) = P(\theta, \varphi) \implies G \text{ amenable.}$$

Now how does this relate to conformal dynamics? In other words, how do the above theorems generalise Brooks' Theorem 2.1 about the critical exponents of a Kleinian group G and a normal subgroup N of G? The idea is to use the so-called *Bowen-Series map* [10], generalised by Stadlbauer and Stratmann [46],[48] to fit the present situation, in order to encode the dynamics of the geodesic flow on $\mathbb{H}^{n+1}/G$ and $\mathbb{H}^{n+1}/N$ into a group extension of a topological Markov chain. More precisely, under certain circumstances it is possible to give a coding of a radial limit point ξ of G by keeping count not only of which fundamental domains a geodesic ray towards ξ intersects, but also by which side of that fundamental domain the ray enters it. The alphabet of the Markov chain is given by the collection of sides of a fundamental domain. The condition needed on G is for it to be *essentially free* [48], meaning that all relations in any presentation of G are due to parabolic elements of G. One then applies Theorem 3.1 and Theorem 3.2 to the group extension induced by N, relates the Gurevič pressures to the critical exponent of the underlying Kleinian groups, and obtains the following.

Theorem 3.3 ([47])**.** *Let G be an essentially free Kleinian group acting on $\mathbb{H}^{n+1}$ and let $N \lhd G$ be a non-trivial normal subgroup. Then,*

$$\delta(N) = \delta(G) \iff G/N \text{ amenable.}$$

Note that while the condition on G to be essentially free is more restrictive than for it to be convex cocompact, the condition on the critical exponent of G from Brooks' Theorem 2.1 has disappeared. It would seem that with some work and careful consideration the essentially free condition can in fact be replaced with convex cocompactness in Theorem 3.3.

4. The convex core entropy

This section deals mainly with yet another partial generalisation of Brooks' Theorem 2.1, as given in [19]. As in [52], the idea is to consider Kleinian groups which are *not* normal subgroups of some other 'well-behaved' (e.g. convex cocompact or even geometrically finite) group G. While Brooks' result identifies the (non-)amenability of G/N as the criterion for the existence of a dimension gap between $\delta(N)$ and $\dim_H(L(N))$, where N is some normal subgroup of G, we remove G from the picture and replace G/N by a 'combinatorial skeleton' of the hyperbolic manifold given by N. Thus, the main question becomes what to replace $\delta(G)$ with, as dealing directly with $\dim_H(L(N))$, which, as already seen, coincides with $\delta(G)$ in the case where $N \lhd G$ and G is convex cocompact, seems rather difficult.

We shall look at two ways to find the desired replacement for $\delta(G)$; first, by generalising a classical notion from dynamics, the volume entropy, and second, by

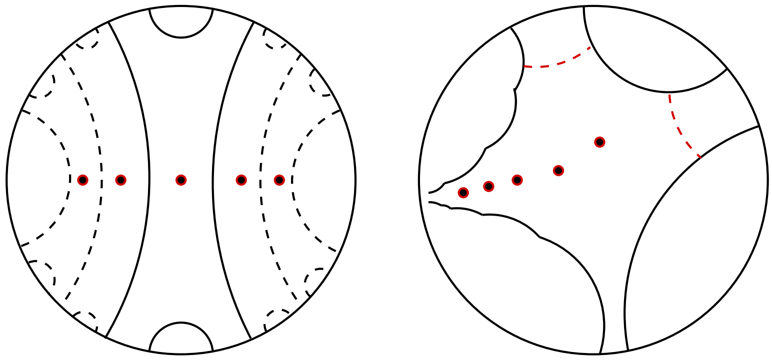

FIGURE 2. Real and 'fake' orbit points.

considering an 'extended Poincaré series' of N together with its critical exponent. We shall then see in Theorem 4.1 that the two approaches in fact lead to the same invariant.

Recall that for a compact Riemannian manifold (X, g) the volume entropy is defined as

$$\lim_{R\to\infty} \frac{\log \mathrm{vol}_g B_R(z)}{R},$$

where $B_R(z)$ is the ball of radius R centred at some $z \in \widetilde{X}$, with $\widetilde{X}$ the universal cover of X, and vol_g is the volume element induced by the Riemannian metric g; for any (not necessarily compact) hyperbolic manifold $\mathbb{H}^{n+1}/G$, with G some Kleinian group, the volume entropy just equals n.

For a closed set Λ in $\partial\mathbb{H}$, we define the *convex core entropy* [19] as

$$h_{\mathrm{c}}(\Lambda) := \limsup_{R\to\infty} \frac{\log \mathrm{vol}\,(B_R(z) \cap H_\varepsilon(\Lambda))}{R},$$

where $B_R(z)$ is the ball of centre z and radius R in $\mathbb{H}^{n+1}$, and $H_\varepsilon(\Lambda)$ is the ε-neighbourhood (for some $\varepsilon > 0$) of the convex hull $H(\Lambda)$ of Λ in $\mathbb{H}^{n+1}$. For a hyperbolic manifold $M_G = \mathbb{H}^{n+1}/G$, we define the convex core entropy $h_{\mathrm{c}}(M_G)$ to be $h_{\mathrm{c}}(L(G))$.

Note that the definition is independent of the choice of $z \in \mathbb{H}^{n+1}$ and of a positive constant $\varepsilon > 0$. The latter is seen below in Theorem 4.1. The simplest reason why we need to take the ε-neighbourhood is that, in the degenerate case such as a Fuchsian group viewed as a Kleinian group, the volume of the convex hull is zero and there is no use considering its entropy.

Let us now turn to the idea of an 'extended Poincaré series' and its critical exponent. We shall look at certain discrete sets which are also known under other names like 'ε-nets', 'uniformly discrete sets' or 'Delone sets'. Roughly speaking, one 'fills up' the convex hull of some closed set in the boundary of hyperbolic space with uniformly distributed 'fake' orbit points and then considers the resulting series. We call a discrete set $X = \{x_i\}_{i=1}^{\infty}$ in the convex hull $H(\Lambda) \subset \mathbb{H}^{n+1}$ *uniformly distributed* if the following two conditions are satisfied:

(i) there exists a constant $M < \infty$ such that, for every point $z \in H(\Lambda)$, there is some $x_i \in X$ such that $d(x_i, z) \le M$;

(ii) there exists a constant $m > 0$ such that any distinct points x_i and x_j in X satisfy $d(x_i, x_j) \geq m$.

Note that Λ is thus the limit set of X in the sense that $\Lambda = \overline{X} \setminus X$. The first natural example of such a uniformly distributed set in an convex hull is of course any orbit of some (convex) cocompact Kleinian group inside the convex hull of its limit set. Now, for a uniformly distributed set X, we define [19] the *Poincaré series* of X with exponent $s > 0$ and reference point $z \in \mathbb{H}^{n+1}$ by

$$P^s(X, z) := \sum_{x \in X} e^{-s\, d(x,z)}.$$

The *critical exponent* of X is

$$\Delta = \Delta(X) := \inf\{s > 0 \mid P^s(X, z) < \infty\} = \limsup_{R \to \infty} \frac{\log \#(B_R(z) \cap X)}{R}.$$

The Poincaré series of X or, more simply, X itself is said to be of *convergence type* if $P^\Delta(X, z) < \infty$, and of *divergence type* otherwise.

As it turns out, not only do the convex core entropy $h_c(\Lambda)$ and the critical exponent $\Delta(X)$ of some uniformly distributed set X in the convex hull of Λ coincide, but they are also equal to the upper Minkowski dimension $\overline{\dim}_{\mathrm{M}}(\Lambda)$ of Λ, since both $\Delta(X)$ and $\overline{\dim}_{\mathrm{M}}(\Lambda)$ are exponential growth rates of the number of balls of decreasing size necessary to cover Λ.

Theorem 4.1 ([19])**.** *Given a closed set $\Lambda \subset \mathbb{S}^n$, we have for any uniformly distributed set X in the convex hull $H(\Lambda)$*

$$\Delta(X) = h_c(\Lambda) = \overline{\dim}_{\mathrm{M}}(\Lambda).$$

When $\Lambda = L(G)$ is the limit set of some non-elementary Kleinian group G, we thus have

$$\delta(G) \leq \dim_H(L(G)) \leq h_c(L(G)) = \overline{\dim}_{\mathrm{M}}(L(G)).$$

In light of this result we have to revisit the initial question of when $\delta(G) < \dim_H(L(G))$ holds for a Kleinian group G, as we now have two gaps to tend to. We will need some preparations in order to adress these new questions. For now we shall restrict to the 2-dimensional case, i.e. to $\mathbb{H}^2$, since this simplifies the arguments relating the 'combinatorial' isoperimetric constant to the classical one, which are necessary in [11], [12] and [13]. This will allow us to give the desired generalisation of Brooks' Theorem 2.1 to a situation where G is dropped altogether and one considers the amenability of some other object than G/N.

Every hyperbolic surface S has a not necessarily uniquely determined *pants decomposition* $\mathcal{P} = \mathcal{P}(S)$, in the sense that there is a (possibly infinite) collection of disjoint simple closed geodesics of S such that the complement consists of pieces homeomorphic to a disk with two holes called *pairs of pants*. Metrically, pants could be relatively compact in S, or could have punctures or funnels, i.e. boundary at infinity. If $\mathcal{P}$ is a pants decomposition for S, then the associated graph $\mathcal{G} = \mathcal{G}(\mathcal{P})$ is defined as the set of pants with edges between pants whenever these are adjacent via some boundary simple closed geodesic.

A hyperbolic surface S has *bounded geometry* if the set of all lengths of closed geodesics in S is uniformly bounded away from 0. We say that S has *strongly bounded geometry* if the convex core $C(S)$ admits a pants decomposition possibly including degenerate pairs of pants such that there are constants $C \geq c > 0$ with

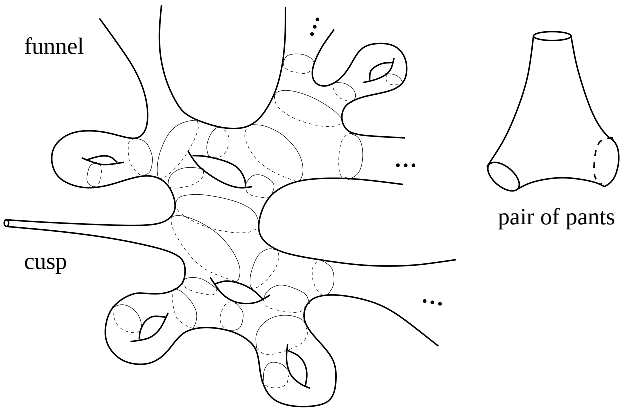

FIGURE 3. The pants decomposition.

the property that the lengths of geodesic boundary components of each pair of pants are bounded from above by C and from below by c.

The notion of amenability generalises to graphs in the following way. A connected graph $\mathcal{G} = (V, E)$ is *amenable* if

$$\inf_K \frac{\#\partial K}{\#K} = 0,$$

where the infimum is taken over all finite connected subgraphs K of $\mathcal{G}$. $\mathcal{G}$ is *strongly amenable* if, for some fixed vertex $v \in V$,

$$\lim_{m\to\infty} \inf_{K^{(m)}} \frac{\#\partial K^{(m)}}{\#K^{(m)}} = 0,$$

where the infimum is taken over all finite connected subgraphs $K^{(m)}$ with $\#K^{(m)} \leq m$ and $K^{(m)} \ni v$. $\mathcal{G}$ is *uniformly strongly amenable* if the above convergence as $m \to \infty$ is uniform independently of the choice of $v \in V$. A connected graph is *transitive* if its autorphism group acts transitively on the set of edges. Note that for transitive graphs the above notions of amenability coincide. Cayley graphs of finitely generated groups are transitive, as such a group always acts transitively on its own Cayley graph.

Next, we consider the notion of an isoperimetric constant which exists both in hyperbolic geometry and for graphs. As a direct consequence of the definition, a graph is non-amenable if and only if

$$\sup_K \frac{\#K}{\#\partial K} < \infty,$$

where the supremum is taken over all finite connected subgraphs K of $\mathcal{G}$. We shall refer to $\sup_K(\#K/\#\partial K)$ as the *isoperimetric constant* of $\mathcal{G}$. For convenience we define the *isoperimetric constant* or *Cheeger constant* of a hyperbolic surface S as

$$h(S) := \sup_W \frac{A(W)}{\ell(\partial W)},$$

where the supremum is taken over all relatively compact domains $W \subset S$ with smooth boundary ∂W. Here, $A(W)$ denotes the hyperbolic area of A and $\ell(\partial W)$ the hyperbolic length of the boundary. Note that this is the reciprocal of the 'usual' isoperimetric constant, as defined and used for instance in [13]. The next classical result establishes a relationship between $h(S)$ and the bottom of the L^2-spectrum of S, denoted as before by λ_0. The first inequality is also employed by Brooks in [11] and [13].

Theorem 4.2 ([16], [14])**.** *For every hyperbolic surface S we have*

$$\frac{1}{4\,h(S)^2} \leq \lambda_0(S) \leq \frac{B}{h(S)}$$

for some universal constant $B > 0$.

In order to obtain the desired partial generalisation of Brooks' theorem, we now only need to put the puzzle pieces together; relate the isoperimetric constants

$$h(S) = \sup_W \frac{A(W)}{\ell(\partial W)} \qquad \text{and} \qquad \sup_K \frac{\#K}{\#\partial K}$$

of S and $\mathcal{G}$, respectively, to each other, thus showing that the non-amenability of $\mathcal{G}$ implies that the Cheeger constant of S is finite. Theorem 4.2 then implies that $\lambda_0(S) > 0$ and thus $\delta(G) < 1$. This is the core of the argument yielding the following result, and is in fact closely related to the methods used in [11] and [13].

Theorem 4.3 ([19])**.** *Consider a hyperbolic surface of infinite type S with strongly bounded geometry, given by the Fuchsian group G; further, consider the graph $\mathcal{G}$ associated to some arbitrary pants decomposition of S. Then we have the following implications:*

- (i) *If $\mathcal{G}$ is uniformly strongly amenable, then either both $\delta(G)$ and $\dim_{\mathrm{H}}(L(G))$ are striclty less than 1, or they are both equal to 1.*
- (ii) *If $\mathcal{G}$ is non-amenable, then $\delta(G) < 1$.*

In particular, if $S = \mathbb{H}^2/G$ and $\mathcal{G}$ are as in Theorem 4.3, and if we assume that $\mathcal{G}$ is transitive and that $\dim_{\mathrm{H}}(L(G)) = h_{\mathrm{c}}(S) = 1$, then $\mathcal{G}$ is non-amenable if and only if $\delta(G) < 1$.This is reason enough to give the following not too surprising conjecture.

Conjecture 4.4. *Let $S = \mathbb{H}^2/G$ be a hyperbolic surface of infinite type with strongly bounded geometry, and $\mathcal{G}$ be the graph associated to some arbitrary pants decomposition of S. Then, $\mathcal{G}$ is non-amenable if and only if $\delta(G) < h_{\mathrm{c}}(S)$.*

Note that we formulate the conjecture in terms of the convex core entropy h_{c}, and *not* the Hausdorff dimension of the limit set, since this seems to be a more approachable problem. It does, however, leave the question open of when $\dim_{\mathrm{H}}(L(G))$ and $h_{\mathrm{c}}(S)$ coincide. In order to address it, we first introduce a certain type of 'homogeneity condition'. For the remainder of this section we return to the general case of $(n+1)$-dimensional hyperbolic space.

We say that a uniformly distributed set X is of *bounded type* [19] if there exists a constant $\rho \geq 1$ such that

$$\frac{\#(X \cap B_R(x))}{\#(X \cap B_R(o))} \leq \rho$$

for every $x \in X$ and for every $R > 0$. Here we assume that the origin $o \in \mathbb{H}^{n+1}$ belongs to X. Clearly, any orbit of a convex cocompact Kleinian group is of bounded type.

We shall now see that if the bounded type condition is satisfied by a uniformly distributed set, then the Hausdorff and box dimensions of the corresponding limit set coincide.

Theorem 4.5 ([19]). *Let Λ be a closed subset in $\partial\mathbb{H}^{n+1}$ and assume that there is a uniformly distributed set X of bounded type in the convex hull $H(\Lambda)$. Then the Δ-dimensional Hausdorff measure of Λ is positive. In particular,* $\dim_{\mathrm{H}}(\Lambda) \geq \Delta(X)$, *and hence we have that*

$$\dim_{\mathrm{H}}(\Lambda) = \Delta(X) = h_{\mathrm{c}}(\Lambda) = \underline{\dim}_{\mathrm{M}}(\Lambda) = \overline{\dim}_{\mathrm{M}}(\Lambda).$$

Furthermore, the Poincaré series of X is of divergence type, i.e. $P^{\Delta}(X,z) = \infty$.

A moment's consideration shows that the bounded type condition for uniformly distributed sets in fact prevents orbits of geometrically finite Kleinian groups with parabolic elements from being examples of such sets. This is clearly not a satisfactory state of affairs, so we need to adapt the definition slightly. We say that a uniformly distributed set X is of *weakly bounded type* [19] if there exist a constant $\rho \geq 1$ and a family of mutually disjoint horoballs $\{D_p\}_{p\in\Phi}$ in $\mathbb{H}^{n+1}$ with set of tangency points $\Phi \subset \partial\mathbb{H}^{n+1}$ such that

$$\frac{\#(X \cap B_R(x))}{\#(X \cap B_R(o))} \leq \rho$$

for every $x \in X'$ and for every $R > 0$. Here $X' = X \cap (\mathbb{H}^{n+1} \setminus \bigcup_{p\in\Phi} D_p)$ and we assume that the origin $o \in \mathbb{H}^{n+1}$ belongs to X'. It is not difficult to see that any orbit of a geometrically finite Kleinian groups with parabolic elements is an elementary example of such a X'.

In this situation one can show the following [19]. Suppose that a uniformly distributed set X in the convex hull of $\Lambda \subset \mathbb{S}^n$ is of weakly bounded type and the *Patterson measure* μ_z, $z \in \mathbb{H}^{n=1}$, *associated with* X has no atom on the tangency points $p \in \Phi \subset \partial\mathbb{H}^{n+1}$ of the horoballs $\{D_p\}_{p\in\Phi}$ appearing in the definition of weak boundedness. (For more details on μ_z see [19].) Then the packing dimension $\dim_{\mathrm{P}}(\Lambda)$ of Λ coincides with $\Delta(X) = h_c(\Lambda) = \overline{\dim}_{\mathrm{M}}(\Lambda)$. Moreover, the Poincaré series of X is of divergence type, i.e. $P^{\Delta}(X,z) = \infty$. In particular, let G be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$ all of whose parabolic fixed points are bounded. If the limit set $L(G)$ of G is of weakly bounded type for the set Φ of all parabolic fixed points of G, i.e. its convex hull contains a uniformly distributed set X of weakly bounded type, then

$$\dim_{\mathrm{P}}(L(G)) = h_c(L(G)) = \Delta(X) \quad \text{and} \quad P^{\Delta}(X,z) = \infty.$$

5. An outlook; open problems

Given some (non-elementary) Kleinian group G, it makes sense to split up the question whether a gap $\delta(G) < \dim_H(L(G))$ exists into the following two questions:

(i) When is $\dim_{\mathrm{H}}(L(G)) = h_{\mathrm{c}}(L(G))$? More generally, as we have seen that $h_{\mathrm{c}}(L(G)) = \dim_{\mathrm{M}}(L(G))$, when is $\dim_H(L(G)) = \dim_{\mathrm{M}}(L(G))$?
(ii) When is $\delta(G) < h_{\mathrm{c}}(L(G))$?

About (ii): So far there exist only sufficient conditions for the general equality

$$\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{M}}(\Lambda), \quad \Lambda \text{ closed},$$

which usually amount to some sort of suitable ‘homogeneity’ of Λ. Our bounded type condition on uniformly distributed sets is such a ‘homogeneity condition’, not directly on Λ but on its convex hull inside hyperbolic space. However, there is no useful characterisation for this equality, albeit many concrete examples of $\dim_{\mathrm{H}}(\Lambda) < \dim_{\mathrm{M}}(\Lambda)$, which employ some sort of ‘inhomogeneity’. This should not exist in conformal dynamical systems (or at least in some large class of such systems), which means that conjecturing

$$\dim_{\mathrm{H}}(L(G)) = h_{\mathrm{c}}(L(G))$$

for all (or ‘most’) non-elementary Kleinian groups is not completely unrealistic.

About (i): This appears to be more likely to be answered than the direct question of whether $\delta(G) < \dim_{\mathrm{H}}(L(G))$, since $h_{\mathrm{c}}(L(G)) = \Delta(X)$ for some uniformly distributed X within $H(L(G))$ and $\delta(G)$ are ‘similarly defined’ invariants and may thus be easier to put in relation to each other. There are at least two possible approaches. The first would be the attempt to generalise Brooks’ approach, as exercised by Tapie [52] or in [19]. The problem with the method in [19] is that it appears difficult to generalise the pants decomposition of surfaces along totally geodesic pieces, i.e. simple closed geodesics, to higher dimensions, where there is no analogous decomposition method with totally geodesic separating submanifolds of codimension 1.

The other approach would be attempting to generalise Stadlbauer’s method [47] of proof for Brooks’ theorem . The problem here is that translating the geometric problem from hyperbolic manifolds to the setting of Markov chains works via the Bowen-Series map, which has only been meaningfully defined for rather restricted classes of Kleinian groups.

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