

LIMITING CONFIGURATIONS FOR SOLUTIONS OF HITCHIN'S EQUATION

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ABSTRACT. We review recent work on the compactification of the moduli space of Hitchin's self-duality equation. We study the degeneration behavior near the ends of this moduli space in a set of generic directions by showing how limiting configurations can be desingularized. Following ideas of Hitchin, we can relate the top boundary stratum of this space of limiting configurations to a Prym variety. A key rôle is played by the family of rotationally symmetric solutions to the self-duality equation on \mathbb{C} , which we discuss in detail here.

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1. INTRODUCTION

The moduli space of Higgs bundles, introduced by Hitchin [Hi87] and Simpson [Si88], is a well investigated object in algebraic geometry and topology. We wish here to study it from the viewpoint of Riemannian geometry. Hitchin showed that there exists a natural hyperkähler metric on the smooth

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locus of the moduli space; in many cases the moduli space has no singularities and the metric is complete. Its asymptotics are still not well understood, and we survey here some recent approaches to a set of questions about the behavior of this metric near the ends of this moduli space.

There are several reasons to study this metric carefully. The first is to understand the L^2 -cohomology of this space. Hausel proved [Ha99] that the image of the compactly supported cohomology in the ordinary cohomology vanishes, leading him to conjecture that the L^2 -cohomology of the Higgs bundle moduli space must vanish. This was made in analogy with Sen's conjecture about the L^2 -cohomology of the monopole moduli spaces [Se94]. Hitchin proved a rather general result [Hi00] showing that under conditions satisfied in both these cases, the L^2 -cohomology vanishes outside the middle degree. Hausel's conjecture remains open. Following, for example, the approach of [HHM05], an understanding of this middle-degree cohomology relies on some finer knowledge of the metric structure at infinity.

However, this is part of a much broader picture concerning hyperkähler metrics on algebraic completely integrable systems. Indeed, the work of Gaiotto, Moore and Neitzke [GMN10, GMN13] hints at an asymptotic development of this hyperkähler metric g , where the leading term is a so-called *semiflat* metric and the correction terms decay at increasingly fast exponential rates. The exponents and coefficients of these correction terms are described in terms of expressions coming from a wall-crossing formalism, but these are unfortunately a priori divergent. Clarifying this circle of ideas is a high priority.

Our goal here is to review the main result of [MSWW14], which constructs a dense open subset near infinity in the moduli space of Hitchin's self-duality equations. The degeneration behavior of generic solutions is captured by the notion of *limiting configurations*. These constitute a family of singular solutions to the self-duality equations which give a geometric realization of the elements of the top stratum in the compactification of the moduli space. As a second step, we present a desingularization theorem for limiting configurations. We present here an alternate description of these limiting configurations, different than the one one given in [MSWW14]; the approach here was communicated to us by Hitchin [Hi14], and we are grateful to him for allowing us to use it here. We review this desingularization result and conclude with a sketch of what these results indicate about the asymptotic behavior of the hyperkähler metric; complete proofs of this will appear in a subsequent paper.

The most pressing question is to understand the metric asymptotics in all the remaining “non-generic” cases. This parallels the story of the moduli space of $SU(2)$ -monopole metrics of charge k studied by Bielawski. Our genericity condition corresponds to a multi-monopole breaking up into k

monopoles of charge one, cf. [Bi95]. In this “free” region the natural hyperkähler metric is asymptotic to the so-called *Gibbons-Manton metric* [Bi98]. For the general case which involves so-called clusters (where monopoles break up into monopoles of smaller charge but not necessarily equal to one), Bielawski found hyperkähler metrics approximating the natural hyperkähler metric at an exponential rate [Bi08].

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2. HOLOMORPHIC BUNDLES WITH HIGGS FIELDS

In this section we review some relevant background. A more complete introduction can be found, for example, in the appendix in [We08]. For generalities on hermitian holomorphic vector bundles see [Ko87].

2.1. Stable bundles. Let X be a closed Riemann surface, i.e. a compact (orientable) surface endowed with a complex structure. To streamline the discussion we always assume that the genus γ of X is at least 2. We also fix a complex vector bundle $E \rightarrow X$ of rank $r = r(E)$ and degree $d = d(E)$. The pair (r, d) determines E as a smooth bundle. Given the holomorphic structure on X it is natural to classify the holomorphic structures on E . These are characterized in terms of *pseudo-connections*, i.e., \mathbb{C} -linear maps $\bar{\partial}_E : \Omega^0(E) \rightarrow \Omega^{0,1}(E)$ such that $\bar{\partial}_E(fs) = \bar{\partial}f \otimes s + f\bar{\partial}_E s$ for any complex-valued smooth function f and section s of E . Given such a pseudo-connection we can define a complex horizontal subspace of the tangent space of E . Together with the complex structure of the fibres we therefore obtain an almost complex structure on TE . The integrability condition for this is $\bar{\partial}_E \circ \bar{\partial}_E = 0$ (see for instance [Ko87, Proposition I.3.7]) which on a Riemann surface holds trivially for dimensional reasons. Note in passing that any pseudo-connection arises as the $(0,1)$ -part of a full covariant derivative $d_A = \partial_A + \bar{\partial}_A$. A complex automorphism $g \in \mathcal{G}^c := \text{Aut}(E)$ acts on a holomorphic structure by

$$\bar{\partial}_E^g := g^{-1} \circ \bar{\partial}_E \circ g.$$

The associated moduli space

$$\mathcal{N}_{r,d} = \{\text{pseudo-connections } \bar{\partial}_E\} / \mathcal{G}^c$$

is not Hausdorff in general. To obtain a well-behaved quotient one must restrict attention to the subclass of *stable* bundles, following ideas from Mumford's geometric invariant theory [Mu65, Ne78]. For a general complex vector bundle, we define the *slope* of E to be the number $\mu(E) = d(E)/r(E)$. We say that E is (*slope-*)*stable* if and only if

$$\mu(F) < \mu(E).$$

for any proper non-trivial holomorphic subbundle F of E . The *moduli space of stable bundles* $\mathcal{N}_{r,d}^s$ is a smooth quasi-projective variety of (complex) dimension $1 + r^2(\gamma - 1)$; this space is projective if r and d are coprime.

A completely different approach to the moduli space of stable bundles was developed by Atiyah, Bott and Donaldson [AtBo83, Do83]. This builds upon the seminal work of Narasimhan and Seshadri [NaSe65], who proved that a stable holomorphic structure on E is equivalent to a *projectively flat unitary connection*. More concretely, fix a hermitian metric H on E . A unitary connection A is projectively flat if the induced connection on the associated principal $\mathrm{PU}(r)$ -bundle is flat. This may also be described directly in terms of the curvature F_A of A by the condition

$$(1) \quad F_A = -i\mu \cdot \omega.$$

Letting $\mathcal{G} = \{g \in \mathcal{G}^c \mid gg^* = \mathrm{Id}_E\}$ denote the space of *unitary* gauge transformations, then there is a corresponding moduli space

$$(2) \quad \mathbf{N}_{r,d}^{\mathrm{irr}} = \{A \text{ irreducible solution to (1)}\}/\mathcal{G}.$$

When (r, d) are coprime again, this is a smooth manifold of (real) dimension $2 + 2r^2(\gamma - 1)$ and is diffeomorphic to $\mathcal{N}_{r,d}^s$.

Remark. If $(E, \bar{\partial}_E, H)$ is a hermitian holomorphic vector bundle, there is a unique connection $A = A(H, \bar{\partial}_E)$, called the *Chern connection*, on E which is unitary and satisfies $\bar{\partial}_A = \bar{\partial}_E$. If the underlying holomorphic structure is clear we shall simply write A_H or even A for the Chern connection associated with H . This gives an action of \mathcal{G}^c on H -unitary connections by setting $A^g = A(H, \bar{\partial}_E^g)$ (see [Ko87, Chapter VII]). Donaldson showed that if $(E, \bar{\partial})$ is stable, then we can find in any complex gauge orbit of the Chern connection a projectively flat unitary connection. This connection is not in general the same as the Chern connection for H . It is unique only up to H -unitary gauge transformations. The precise choice of the hermitian background metric H is immaterial. Indeed, complex gauge transformations act transitively on the space of hermitian metrics, so a change in the metric will only affect the representative of the orbit solving Eq. (1), but not the moduli space itself. Equivalently, we can regard the holomorphic structure $\bar{\partial}_E$ as being fixed and then consider Eq. (1) as an equation on H via the assignment $H \mapsto A(H, \bar{\partial})$. Thus, starting with an arbitrary metric H , we find a complex gauge transformation g – unique up to H -unitary transformations – such that the Chern connection $A(H^g, \bar{\partial}_E)$ solves Eq. (1).

2.2. Higgs bundles. The cotangent bundle of a complex manifold carries a natural holomorphic symplectic structure, i.e., a closed non-degenerate holomorphic 2-form. If this is a real-analytic Kähler manifold, then there exists a hyperkähler metric in a neighbourhood of the zero section of the cotangent bundle which generates this holomorphic symplectic structure [Fe01, KaVe99] (see also Section 5 for the definition of hyperkähler metrics). It is straightforward, using the linearization of the equation $\bar{\partial}_E \circ \bar{\partial}_E = 0$ to compute that $T_E \mathcal{N}_{r,d}^s = H^1(X, \text{End}(E))$. By Serre duality, the fibres of this cotangent bundle are

$$T_E^* \mathcal{N}_{r,d} = H^1(X, \text{End}(E))^* \cong H^0(X, \text{End}(E) \otimes K),$$

where $K \rightarrow X$ is the canonical line bundle of X . Sections of $T_E^* \mathcal{N}_{r,d}$, i.e., holomorphic bundle maps $\Phi : E \rightarrow E \otimes K$, are called *Higgs fields*. A *Higgs bundle* is a pair consisting of a holomorphic vector bundle $(E, \bar{\partial}_E)$ and a Higgs field Φ on it.

As for stable bundles, this picture has a gauge theoretic interpretation given by Hitchin's *self-duality equations*. We describe these now. Fix a hermitian metric H on E . We consider pairs (A, Φ) , where A is a unitary connection and $\Phi \in \Omega^{1,0}(\text{End}(E))$ is an (a priori smooth) Higgs field. The equations we require these to satisfy are

$$(3) \quad F_A + [\Phi \wedge \Phi^*] = -i\mu \cdot \omega, \quad \bar{\partial}_A \Phi = 0.$$

These arose as a dimensional reduction of the instanton equation on \mathbb{R}^4 . The term ‘‘Higgs field’’ was coined in analogy with the three-dimensional counterpart of Eq. (3), the so-called *Bogomolny equations*. Solutions are absolute minimizers of a dimensionally reduced Euclidean Yang-Mills-Higgs functional in dimension three [JaTa80].

The second equation of (3) states that Φ is holomorphic with respect to the holomorphic structure defined by $\bar{\partial}_A$. Hence a solution of (3) specifies a Higgs bundle $(E, \bar{\partial}_A, \Phi)$. Omitting the Higgs field, we simply recover Eq. (1). Conversely, any Higgs bundle (E, Φ) where E is stable in the sense of Section 2.1 arises as an irreducible solution of Eq. (3). Here irreducibility means that (E, A, Φ) does not admit a decomposition into a direct sum of two hermitian bundles $(E_1 \oplus E_2, A_1 \oplus A_2, \Phi_1 \oplus \Phi_2)$ with unitary connections A_i and Higgs fields $\Phi_i \in \Omega^{1,0}(\text{End}(E_i))$. On the other hand, not every Higgs bundle which arises from an irreducible solution of Eq. (3) necessarily has an underlying stable holomorphic vector bundle. To capture *all* the irreducible solutions (A, Φ) we therefore generalize the stability condition. Thus we say that (E, Φ) is a *stable Higgs bundle* if for any proper non-trivial holomorphic subbundle F of E with $\Phi(F) \subset F \otimes K$ we have $\mu(F) < \mu(E)$. This reduces to the usual notion of stability when $\Phi = 0$. The moduli space

$$\mathcal{M}_{r,d}^s = \{(E, \Phi) \mid \text{stable Higgs bundle}\} / \mathcal{G}^c$$

is a quasi-projective variety of (complex) dimension $2 + 2r^2(\gamma - 1)$ and contains $T^* \mathcal{N}_{r,d}$ as an open dense subset [Hi87, Ni91, Si88]. In particular, for

a generic Higgs bundle the underlying holomorphic vector bundle is stable in the sense of Section 2.1. We refer to [Hi87, Section 3] for examples of a Higgs bundle (E, Φ) where E is not stable. On the other hand,

$$\mathbf{M}_{r,d}^{\text{irr}} = \{(A, \Phi) \text{ an irreducible solution to (3)}\} / \mathcal{G}$$

is a smooth manifold of (real) dimension $4 + 4r^2(\gamma - 1)$ which is diffeomorphic to $\mathcal{M}_{r,d}^s$.

Remark. From another point of view we can think of the Higgs bundle moduli space as the moduli space of irreducible and projectively flat *complex* connections [Do87, Co88], i.e.

$$\mathcal{M}_{r,d}^s = \{A \text{ irreducible complex solution of (1)}\} / \mathcal{G}^c.$$

We therefore obtain a Narasimhan-Seshadri type theorem for complex connections. This has been generalized from Riemann surfaces to higher-dimensional compact Kähler manifolds by Simpson in his quest to parametrize the flat complex connections which arise in the complex variation of a Hodge structure. For higher dimensions it is necessary to impose the extra condition $\Phi \wedge \Phi = 0$ (which is trivially satisfied on a Riemann surface).

In this paper we consider a variant of the self-duality equations where we work in the *fixed determinant* case. If A is a unitary connection, then its curvature F_A decomposes as

$$F_A = F_A^\perp + \frac{1}{r} \text{Tr}(F_A) \otimes \text{Id}_E,$$

where $F_A^\perp \in \Omega^2(\mathfrak{su}(E))$ is the *trace-free* part of the curvature and $\mathfrak{su}(E) \subseteq \text{End}(E)$ denotes the bundle of traceless skew-hermitian endomorphisms. The *central part* $\text{Tr}(F_A) \in \Omega^2(i\mathbb{R})$ is precisely equal to the curvature of the induced connection on $\det E$. Let us fix a background connection A_0 from now on and consider only those connections A which induce the same connection on $\det E$ as A_0 does, i.e. $A = A_0 + \alpha$ where $\alpha \in \Omega^1(\mathfrak{su}(E))$. In other words, any such A is trace-free “relative” to A_0 . We may now consider the pair of equations

$$(4) \quad \begin{aligned} F_A^\perp + [\Phi \wedge \Phi^*] &= 0, \\ \bar{\partial}_A \Phi &= 0, \end{aligned}$$

where A is trace-free relative to A_0 . Since the trace of a holomorphic Higgs field is constant, we may as well restrict to trace-free Higgs fields $\Phi \in \Omega^{1,0}(\text{End}_0(E))$. There always exists a unitary connection A_0 on E such that $\text{Tr} F_{A_0} = -ir(E)\omega$, and with this as background connection, a solution of (4) provides a solution to (3). Of course we now need to restrict to gauge transformations of unit determinant \mathcal{G}_0^c and \mathcal{G}_0 when building the moduli spaces. The precise choice of A_0 is immaterial in the sense that the moduli spaces corresponding to two such choices are isomorphic.

In the sequel we specialize not only to the fixed determinant case, but also to rank 2 bundles. This is the case originally considered by Hitchin.

We denote by $\Lambda \rightarrow X$ a fixed degree d line bundle; this carries a natural connection induced by A_0 . In particular, it is holomorphic. We consider rank 2 stable Higgs bundles (E, Φ) with $\det E = \Lambda$ (as holomorphic line bundles). The moduli space of all such bundles is denoted \mathcal{M}_Λ , where for simplicity we drop any reference to stability or irreducibility.

2.3. Parabolic Higgs bundles. There is an extension of the definition of stability of bundles to the setting of punctured Riemann surfaces. We recall this briefly since we make auxilliary use of this later. *Parabolic bundles* were first introduced by Seshadri [Se77], [MeSe80]. The corresponding notion of a parabolic Higgs bundle was introduced by Simpson in [Si90] and developed further in [NaSt95, BoYo96].

Definition 2.1. A *parabolic structure* on a holomorphic vector bundle E over X with marked points $\mathfrak{p} = \{p_1, \dots, p_n\} \subset X$ consists of the following data:

- a filtration $E_p = F_p^1 \supseteq \dots \supseteq F_p^{l_p} \supseteq 0$ at $p \in \mathfrak{p}$;
- a system of associated *weights* $0 \leq \alpha_1 < \dots < \alpha_{l_p} < 1$.

The *parabolic degree* of E is defined as

$$\text{pardeg}(E) = d(E) + \sum_{p_i \in \mathfrak{p}} \alpha_i (\dim F_i - \dim F_{i+1}).$$

A *parabolic Higgs bundle* (E, Φ) consists of a holomorphic bundle E with parabolic structure and a Higgs field which is nilpotent with respect to the flags at the marked points $p_i \in \mathfrak{p}$, i.e., $\Phi(F_p^i) \subset F_p^{i+1} \otimes K$. (Alternately, one may require that $\Phi(F_j) \subset F_j \otimes K$.)

To motivate this definition, consider the punctured surface $X^\times = X \setminus \mathfrak{p}$. There exists a discrete subgroup Γ of $\text{PSL}(2, \mathbb{R})$ acting freely on the upper half-plane \mathbb{H}^2 so that $X^\times = \mathbb{H}^2/\Gamma$ and $\Gamma \cong \pi_1(X^\times)$. The elements of Γ corresponding to loops around the punctures are of parabolic type, which means that their extensions to $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ have exactly one fixed point on the boundary $\partial_\infty \mathbb{H}^2$, see for instance [BePe92, Chapter A]. If we denote by $(\mathbb{H}^2)^+$ the union of \mathbb{H}^2 with all fixed points of parabolic elements in Γ , then $X = (\mathbb{H}^2)^+/\Gamma$ is a compact Riemann surface with marked points p_1, \dots, p_n , where each p_j is the ‘endpoint’ of a parabolic cusp, whence the appellation parabolic bundle. The isotropy group of a parabolic cusp under the action of Γ is cyclic. Let a_i denote the loop in $\pi_1(X^\times) = \Gamma$ induced by a generator of the cyclic group corresponding to p_i . Then a unitary representation $\rho: \Gamma \rightarrow \text{U}(r)$ gives rise to matrices

$$(5) \quad \rho(a_i) = \begin{pmatrix} \exp(2\pi i \alpha_1) & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi i \alpha_r) \end{pmatrix}$$

for a suitably chosen unitary basis e_1, \dots, e_r of E near p_i . This gives the weighted flag.

Remark. There exists a natural notion of stability for parabolic (Higgs) bundles, just as in the unmarked case, leading to moduli spaces of stable parabolic Higgs structures.

2.4. Spectral curves. The map

$$\det : \mathcal{M}_\Lambda \rightarrow H^0(X, K^2), \quad [E, \Phi] \mapsto \det \Phi$$

is proper and surjective and defines the so-called *Hitchin fibration* of \mathcal{M}_Λ . If $q \in H^0(X, K^2)$ has simple zeroes, the fibre $\det^{-1}(q)$ can be identified with the Prym variety of the spectral curve associated with q [Hi87, Hi87b].

Since the zeroes of q are assumed to be simple,

$$\hat{X}_q := \{a_x \in T^*X \mid a_x^2 = q(x)\}$$

defines a smooth embedded curve, which is called the *spectral curve* associated with q . We write this simply as \hat{X} if there is no risk of confusion. The projection $\pi(a_x) = x$ is a twofold cover $\pi : \hat{X} \rightarrow X$ ramified at the divisor $q^{-1}(0)$ in X . By the Riemann-Hurwitz formula, the genus of \hat{X} is $4\gamma - 3$. We denote by $\sigma : \hat{X} \rightarrow \hat{X}$ the involution which interchanges the sheets of π . The pull-back bundle π^*K admits a tautological holomorphic section $a_x \in \hat{X} \mapsto a_x \in K_x = (\pi^*K)_x$ which we suggestively denote by \sqrt{q} . Now let

$$\text{Prym}_\Lambda(\hat{X}) = \{\text{holomorphic line bundles } L \rightarrow \hat{X} \mid \det \pi_* L = \Lambda \otimes K^{-1}\}$$

where π_* denotes the *direct image* of the holomorphic line bundle L , cf. [Gu67, §3b] or [HSW99, Chapter 2.4]. Since $\hat{X} \rightarrow X$ has degree two and $L \rightarrow \hat{X}$ is a holomorphic line bundle, $\pi_* L \rightarrow X$ is a rank 2 holomorphic vector bundle. For instance, $\pi_* \mathcal{O}_{\hat{X}} = \mathcal{O}_X \oplus K^{-1}$ ([Ha77, Ex. IV.2.6]). If $\Lambda = \mathcal{O}$ we recover the *Prym variety* of \hat{X} ; by definition, this is the $3\gamma - 3$ -dimensional subvariety of the Jacobian of \hat{X} defined by the property $\sigma^* L = L^*$ [Mu74].

To define Higgs bundles from elements in $\text{Prym}_\Lambda(\hat{X})$ we first choose a holomorphic square root $K^{1/2}$ of K . For $L \in \text{Prym}_\Lambda(\hat{X})$, set $M = L \otimes \pi^* K^{1/2}$ and define $E = \pi_* M$. By the *projection formula* [Gu67, Lemma 10], multiplication by \sqrt{q} induces a holomorphic map $M \rightarrow M \otimes \pi^* K$ which descends to a holomorphic map

$$\Phi : E = \pi_*(M) \rightarrow E \otimes K = \pi_*(M \otimes \pi^* K).$$

Since M is associated with a Prym variety, this Higgs field is trace free, and satisfies $\det \Phi = q$. Note also that

$$\det E = \det(\pi_* L) \otimes K^{-1/2} = \Lambda,$$

whence E has the right determinant. Conversely, let (E, Φ) be a Higgs bundle on X with $\det E = \Lambda$ and $\det \Phi = q$. Pulling back the Higgs bundle to the spectral curve \hat{X}_q we obtain a holomorphic rank 1 subbundle $T \subset \ker(\pi^* \Phi - \sqrt{q})$. Then $E = \pi_* M$ for $M = T \otimes \pi^* K$, see [BNR87, Remark 3.7], so that $L = M \otimes \pi^* K^{-1/2} \in \text{Prym}_\Lambda(\hat{X})$.

3. LIMITING CONFIGURATIONS

As the name suggests, limiting configurations are the structures which arise as limits of solutions of the Hitchin equations. Conversely, it is possible to desingularize such limiting configurations to obtain ‘large’ elements in the Higgs bundle moduli space. We wish here to explain this further and discuss the existence of such limiting configurations.

3.1. Motivation. Following [Hi87] we know that the function

$$\mathcal{M}_\Lambda \rightarrow \mathbb{R}, \quad [A, \Phi] \mapsto \|\Phi\|_{L^2}^2$$

is a proper Morse-Bott function. In other words, if (A_n, Φ_n) is a sequence of solutions to Hitchin’s equations, and if the L^2 norms of the Φ_n are bounded, then these solutions lie in a compact subset of \mathcal{M}_Λ . On the other hand, fixing the Higgs field Φ , then Hitchin’s existence theorem guarantees the existence of a pair $(A_t, t\Phi_t)$ in the complex gauge orbit of $(A_H, t\Phi)$ (in particular, Φ_t is complex gauge equivalent to Φ), satisfying

$$F_{A_t}^\perp + t^2[\Phi_t \wedge \Phi_t^*] = 0, \quad \bar{\partial}_{A_t} \Phi_t = 0.$$

This family of solutions approaches the end of the moduli space as $t \rightarrow \infty$.

To get a feeling for how degenerations occur, *assume* that $(A_t, \Phi_t) \rightarrow (A_\infty, \Phi_\infty)$ as $t \nearrow \infty$ in $\mathcal{C}_{\text{loc}}^\infty$ (note that we have normalized the Higgs field $t\Phi_t$ in the solution by dividing by t). Then it necessarily holds that

$$[\Phi_\infty \wedge \Phi_\infty^*] = 0, \quad \bar{\partial}_{A_\infty} \Phi_\infty = 0.$$

In particular, Φ_∞ is normal, and hence unitarily diagonalizable. Therefore, at a point $p \in X$ where $\det \Phi_\infty \in H^0(X, K^2)$ vanishes, this order of vanishing must be at least two; order one vanishing occurs if and only if Φ is nilpotent at p . However, for generic Higgs fields, $\det \Phi$ has *simple zeroes* (we also say that Φ is *simple*). This means that Φ_∞ must be singular at these zeroes. Depending on the point of view one takes, cf. Section 2.1, either the hermitian metric degenerates (so that normality is no longer defined at p) or else the holomorphic structure breaks down (so $\det \Phi_\infty$ is not holomorphic on all of X). Either way, it seems reasonable to expect that F_{A_t} concentrates near $(\det \Phi_\infty)^{-1}(0)$ as $t \nearrow \infty$. This is consistent with recent results by Taubes [Ta13a, Ta13b], who investigated the analogous degeneration behavior for a three-dimensional analogue of the self-duality equation.

3.2. The fiducial solution. We next study a class of solutions to Hitchin’s equation on \mathbb{C} which are rotationally symmetric in an appropriate sense. We learned about this family of fiducial solutions from Andy Neitzke, who in joint work with Gaiotto and Moore [GMN13] described its basic properties. We are grateful to him for explaining this to us in detail. The symmetry reduces Hitchin’s equation to a Painlevé type III ordinary differential equation, and from that perspective this solution can be traced back to work of Mason and Woodhouse [MaWo93].

Our paper [MSWW14] presents a derivation of these solutions and their basic properties, but we do so again here from a different and simpler perspective. Indeed, as we explain here, the fiducial solutions are obtained rather easily if one interprets Hitchin's equations as equations for the hermitian metric when restricted to the complex gauge orbit of a configuration (A, Φ) with $\bar{\partial}_A \Phi = 0$. In general, fixing a background hermitian metric H_0 on the vector bundle E , we may identify an arbitrary hermitian metric with a hermitian endomorphism field H ; in the fixed determinant case, this field also satisfies $\det H = 1$. Let $*_H$ denote the adjoint taken with respect to the hermitian metric H . Then

$$(6) \quad F_A^\perp + \bar{\partial}_A(H^{-1}\partial_A H) + [\Phi \wedge \Phi^{*H}] = 0$$

if and only if $(A_H, \Phi)^g$ solves Hitchin's equation, where g is a complex gauge transformation which is determined by solving $H^{-1} = gg^*$. Note that g is uniquely determined up to right multiplication by a unitary gauge transformation.

Fix the holomorphic normal form for a rank-2 Higgs bundle over \mathbb{C} whose determinant has a simple zero in $z = 0$. More precisely, consider the holomorphically trivial rank-2 vector bundle over \mathbb{C} and the holomorphic Higgs field

$$\Phi = \varphi dz, \quad \varphi(z) = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix};$$

this has determinant $-zdz^2$. Note that if H_0 is the standard constant hermitian metric, then the Chern connection is the trivial flat connection, denoted here by A .

We ask first if there exists a configuration (A_∞, Φ_∞) in the complex gauge orbit of (A, Φ) satisfying the purely algebraic equation

$$(7) \quad [\Phi_\infty \wedge \Phi_\infty^*] = 0 \iff [\Phi \wedge \Phi^{*H_\infty}] = 0;$$

here H_∞ is a hermitian metric and $*_{H_\infty}$ is the adjoint with respect to H_∞ .

We say that a hermitian metric is rotationally symmetric if its representation H relative to the fixed metric h_0 is rotationally symmetric, i.e.,

$$H(r) = \begin{pmatrix} \alpha(r) & b(r) \\ \bar{b}(r) & \beta(r) \end{pmatrix},$$

where α, β are real valued, with $\alpha > 0$ and $\alpha\beta - |b|^2 = 1$. A straightforward calculation shows that

$$(8) \quad [\Phi \wedge \Phi^{*H}] = \begin{pmatrix} \alpha^2 - |z|^2\beta^2 + 2i \operatorname{Im} b^2 z & 2\alpha b - 2\beta\bar{b}\bar{z} \\ -2\alpha b z + 2|z|^2\beta\bar{b} & |z|^2\beta^2 - \alpha^2 - 2i \operatorname{Im} b^2 z \end{pmatrix} dz \wedge d\bar{z}.$$

We wish to choose $H = H_\infty$ so that this vanishes. Since $\alpha, \beta \neq 0$, setting all entries here equal to 0 implies that $b \equiv 0$, and then that $\alpha = r^{1/2}$, $\beta = r^{-1/2}$.

In other words,

$$[\Phi \wedge \Phi^{*H_\infty}] = 0 \iff H_\infty = \begin{pmatrix} r^{1/2} & 0 \\ 0 & r^{-1/2} \end{pmatrix}$$

This solution is singular at $z = 0$. Recalling that the curvature of any Hermitian metric H is given by

$$(9) \quad \bar{\partial}(H^{-1}\partial H) = - \begin{pmatrix} \partial_{\bar{z}}(\beta\partial_z\alpha - b\partial_z\bar{b}) & \partial_{\bar{z}}(\beta\partial_z b - b\partial_z\beta) \\ \partial_{\bar{z}}(\alpha\partial_z\bar{b} - \bar{b}\partial_z\alpha) & \partial_{\bar{z}}(\alpha\partial_z\beta - \bar{b}\partial_z b) \end{pmatrix} dz \wedge d\bar{z},$$

then in particular,

$$\bar{\partial}(H_\infty^{-1}\partial H_\infty) = - \begin{pmatrix} \partial_{\bar{z}}(r^{-1/2}\partial_z r^{1/2}) & 0 \\ 0 & \partial_{\bar{z}}(r^{1/2}\partial_z r^{-1/2}) \end{pmatrix} dz \wedge d\bar{z} = 0.$$

This proves the following surprising fact:

Lemma 3.1. *The unique hermitian metric H_∞ satisfying $[\Phi \wedge \Phi^{*H_\infty}] = 0$ on \mathbb{C}^\times is flat and therefore solves the decoupled version of Hitchin's equation*

$$\bar{\partial}(H_\infty^{-1}\partial H_\infty) = 0, \quad [\Phi \wedge \Phi^{*H_\infty}] = 0.$$

Setting

$$g_\infty = \begin{pmatrix} r^{-1/4} & 0 \\ 0 & r^{1/4} \end{pmatrix},$$

then one has $H_\infty^{-1} = g_\infty g_\infty^*$, so from the fact that $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}}) = (A, \Phi)^{g_\infty}$, we obtain the expression for the limiting fiducial solution

$$A_\infty^{\text{fid}} = \frac{1}{8} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right), \quad \Phi_\infty^{\text{fid}} = \begin{pmatrix} 0 & r^{1/2} \\ zr^{-1/2} & 0 \end{pmatrix} dz.$$

Now let us look for nonsingular rotationally symmetric solutions H_t of Hitchin's equation

$$(10) \quad \bar{\partial}(H_t^{-1}\partial H_t) + t^2[\Phi \wedge \Phi^{*H_t}] = 0, \quad 0 < t < \infty.$$

We see that $[\Phi \wedge \Phi^{*H_t}]$ is rotationally symmetric if and only if

$$H_t = \begin{pmatrix} \alpha_t & 0 \\ 0 & \alpha_t^{-1} \end{pmatrix}$$

for some function $\alpha_t = \alpha_t(r) > 0$. With this reduction, Hitchin's equation reduces to the ODE

$$(11) \quad \partial_{\bar{z}}(\alpha_t^{-1}\partial_z\alpha_t) - t^2(\alpha_t^2 - |z|^2\alpha_t^{-2}) = 0.$$

We let

$$\alpha_t = e^{h_t + \frac{1}{2}\log(r)}$$

where $h_t = h_t(r)$ is real-valued. Then (11) is equivalent to

$$\begin{aligned} 0 &= \partial_{\bar{z}} \partial_z (h_t + \frac{1}{2} \log(r)) - t^2 (r e^{2h_t} - r e^{-2h_t}) \\ &= \frac{1}{4} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) (h_t + \frac{1}{2} \log(r)) - t^2 (r e^{2h_t} - r e^{-2h_t}) \\ &= \frac{1}{4} (h_t'' + r^{-1} h_t') - t^2 (r e^{2h_t} - r e^{-2h_t}), \end{aligned}$$

and hence finally to

$$(12) \quad h_t'' + r^{-1} h_t' = 8t^2 r \sinh(2h_t).$$

Now substitute $h_t(r) = \psi(\rho)$ with $\rho = \frac{8}{3} t r^{3/2}$; this transforms (12) to the Painlevé type III equation

$$(13) \quad \psi'' + \frac{\psi'}{\rho} = \frac{1}{2} \sinh(2\psi).$$

The properties of solutions to (13) are well known, see [MSWW14] and the references therein. There is a unique solution h_t to (12) satisfying $h_t(r) + \frac{1}{2} \log(r) \rightarrow 0$ as $r \searrow 0$ and $h_t(r) \rightarrow 0$ as $r \nearrow \infty$.

The hermitian metric H_t then equals

$$H_t = \begin{pmatrix} r^{1/2} e^{h_t(r)} & 0 \\ 0 & r^{-1/2} e^{-h_t(r)} \end{pmatrix},$$

and

$$H_t^{-1} = g_t g_t^* \quad \text{for} \quad g_t = \begin{pmatrix} r^{-1/4} e^{-h_t(r)/2} & 0 \\ 0 & r^{1/4} e^{h_t(r)/2} \end{pmatrix}.$$

This yields, finally, the entire family of desingularized fiducial solutions by the formula $(A_t^{\text{fid}}, \Phi_t^{\text{fid}}) = (A, \Phi)^{g_t}$, so

$$A_t^{\text{fid}} = f_t(r) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right), \quad \Phi_t^{\text{fid}} = \begin{pmatrix} 0 & r^{1/2} e^{h_t(r)} \\ z r^{-1/2} e^{-h_t(r)} & 0 \end{pmatrix} dz$$

where $f_t(r) = \frac{1}{8} + \frac{1}{4} r \partial_r h_t$.

We have therefore arrived at the following result:

Theorem 3.2. *For each $t \in (0, \infty)$ there exists a unique hermitian metric H_t solving Hitchin's equation (10) such that*

- (i) H_t is rotationally symmetric, and
- (ii) $H_t \sim H_\infty = \begin{pmatrix} r^{1/2} & 0 \\ 0 & r^{-1/2} \end{pmatrix}$ as $r \nearrow \infty$.

These metrics correspond to the fiducial solutions $(A_t^{\text{fid}}, \Phi_t^{\text{fid}})$.

A byproduct of this approach is that we may now easily derive fiducial solutions corresponding to Higgs fields Φ , the determinants of which have zeroes of order greater than one. This touches on the ongoing thesis work of Laura Fredrickson [F], who is describing the behavior of families of solutions on \mathbb{C} near this degenerate case, i.e., families where the Higgs field is simple but which limit to these degenerate solutions. She also constructs fiducial

solutions in the higher rank case. More precisely, for $k \geq 1$, consider the Higgs field

$$\Phi = \begin{pmatrix} 0 & 1 \\ z^k & 0 \end{pmatrix} dz,$$

which has determinant $-z^k dz^2$. Setting

$$H_t = \begin{pmatrix} \alpha_t & 0 \\ 0 & \alpha_t^{-1} \end{pmatrix}$$

we obtain that

$$[\Phi \wedge \Phi^{*H_t}] = \begin{pmatrix} \alpha_t^2 - |z|^{2k} \alpha_t^{-2} & 0 \\ 0 & |z|^{2k} \alpha_t^{-2} - \alpha_t^2 \end{pmatrix} dz \wedge d\bar{z}.$$

If we let $\alpha_t = e^{h_t + \frac{k}{2} \log(r)}$, then the same calculation as above leads now to the ODE

$$(14) \quad h_t'' + r^{-1} h_t' = 8t^2 r^k \sinh(2h_t).$$

With the substitution

$$\rho = 4tr^{\frac{k}{2}+1}/(\frac{k}{2}+1)$$

this is equivalent to the same Painlevé III equation (13) as in the case where $\det \Phi$ has a simple zero. The Higgs pair (A_t, Φ_t) corresponding to this rotationally symmetric hermitian metric H_t is now calculated to be

$$A_t = f_t(r) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right), \quad \Phi_t = \begin{pmatrix} 0 & r^{k/2} e^{h_t(r)} \\ z^k r^{-k/2} e^{-h_t(r)} & 0 \end{pmatrix} dz$$

where $f_t(r) = \frac{1}{8} + \frac{1}{4} r \partial_r h_t$ and h_t is a solution of (14). The limiting solution is given by

$$A_\infty = \frac{k}{8} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right), \quad \Phi_\infty = \begin{pmatrix} 0 & r^{k/2} \\ z^k r^{-k/2} & 0 \end{pmatrix} dz.$$

The gluing result from [MSWW14] may be implemented using this higher order fiducial solution. Details will appear elsewhere.

3.3. Construction of limiting configurations. Motivated by the preceding discussion we consider the following model for degeneration at infinity [MSWW14].

Definition 3.1. Let (E, H, Φ) be a hermitian Higgs bundle, where Φ is simple. A *limiting configuration* is a Higgs pair (A_∞, Φ_∞) which satisfies the decoupled self-duality equations

$$(15) \quad F_{A_\infty}^1 = 0, \quad [\Phi_\infty \wedge \Phi_\infty^*] = 0, \quad \bar{\partial}_{A_\infty} \Phi_\infty = 0$$

on X^\times , and which agrees with $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$ near each point of $(\det \Phi)^{-1}(0)$ with respect to some unitary frame for E and holomorphic coordinate system such that $\det \Phi = -z dz^2$ near the zeroes of $\det \Phi$.

The main result for limiting configurations is this.

Theorem 3.3. *Let (E, H, Φ) be a Hermitian Higgs bundle with simple Higgs field. Then in the complex gauge orbit of (A_H, Φ) over $X \setminus (\det \Phi)^{-1}(0)$ there exists a limiting configuration (A_∞, Φ_∞) unique up to unitary gauge transformation. Up to a smooth complex gauge transformation on X the limiting (singular) complex gauge transformation g_∞ takes the form*

$$g_\infty = \begin{pmatrix} |z|^{-\frac{1}{4}} & 0 \\ 0 & |z|^{\frac{1}{4}} \end{pmatrix}$$

near $(\det \Phi)^{-1}(0)$.

This was proved in [MSWW14] analytically, in particular relying on the Fredholm theory of conic elliptic operators. We present here an alternative approach using spectral curves which was explained to us by Nigel Hitchin [Hi14], and we are grateful for his permission to present it here.

Following the notation from Section 2.4, let \hat{X} be the spectral curve associated with $q = -\det \Phi$. Start with $L \in \text{Prym}_\Lambda(\hat{X})$, but now consider the holomorphic rank 2 bundle $\hat{E} = L \oplus \sigma^* L$ over \hat{X} . Near the ramification locus (which is the zero locus of \sqrt{q}) we fix a holomorphic trivialization of L and declare it to be unitary. This determines a hermitian metric \hat{h} near these points, which we extend to all of \hat{X} and set $\hat{H} = \hat{h} \oplus \sigma^* \hat{h}$. Finally, with respect to the decomposition of \hat{E} we define a Higgs field by $\hat{\Phi} = \text{diag}(\sqrt{q}, -\sqrt{q})$. This determines a hermitian Higgs bundle $(\hat{E}, \hat{H}, \hat{\Phi})$ where $\hat{\Phi}$ is normal on \hat{X} and \hat{H} is flat near the ramification locus.

The \mathbb{Z}_2 -action on \hat{X} generated by σ is covered by a representation

$$\tau : \mathbb{Z}_2 \rightarrow \text{GL}_r \mathbb{C}, \quad \text{where} \quad \tau(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

in a suitable basis (e, e') . Writing the action of σ on an eigenvector as $e^{i\pi x}$, we obtain the *isotropy weights* $x = 0$ and $x' = 1$ for this action. Since the Higgs bundle $(\hat{E}, \hat{H}, \hat{\Phi})$ is σ -invariant, it descends to X and defines there an orbifold hermitian Higgs bundle (E, H, Φ) , see [NaSt95] and also [Bo91, FuSt92]. In particular, the underlying orbifold bundle has trivializations of the form $\hat{D} \times \mathbb{C}^2 / \sigma \times \tau$ on neighbourhoods $D = \hat{D} / \sigma$ of points in $q^{-1}(0)$. In addition, if w is a coordinate on \hat{D} , then $z = w^2$ is a coordinate on D .

Orbifold Higgs bundles can be desingularized in a natural way by parabolic Higgs bundles with cusps at $\mathfrak{p} = q^{-1}(0)$. Namely, at a marked point p with isotropy weights $x = 0$ and $x' = 1$ we consider the bundle \mathcal{E} defined by

$$E|_{X \setminus \{p\}} \cup_\Psi D \times \mathbb{C}^2$$

with clutching function Ψ given in local coordinates by its \mathbb{Z}_2 -equivariant lifting

$$\begin{aligned} \hat{\Psi} : (\hat{D} \setminus \{0\}) \times \mathbb{C}^2 &\rightarrow D \times \mathbb{C}^2 \\ (w, (v_1, v_2)) &\mapsto (w^2, (w^{-x}v_1, w^{-x'}v_2)) = (w^2, (v_1, w^{-1}v_2)). \end{aligned}$$

Thus \mathcal{E} carries a natural parabolic structure which at a marked point $p \in q^{-1}(0)$ is given by

$$(16) \quad \begin{array}{ccccccc} \mathcal{E}_p & = & \mathbb{C}^2 & \supset & \mathbb{C}e' & \supset & \{0\} \\ 0 & \leq & 0 & < & \frac{1}{2} & < & 1 \end{array} .$$

To describe the holomorphic sections of \mathcal{E} near p we note that the invariant holomorphic sections of $(\hat{D}^2 \times \mathbb{C}^2)/(\sigma \times \tau)$ with respect to (e, e') take the form $(s_1(w), s_2(w))$ with $s_1(w) = \tilde{s}_1(w^2)$ and $s_2(w) = w\tilde{s}_2(w^2)$. Under the clutching map Ψ this gives a bounded holomorphic section $(\tilde{s}_1(z), \tilde{s}_2(z))$ of \mathcal{E} which extends over 0. In particular, $f = e$ and $f' = e/w$ induce a holomorphic frame for \mathcal{E} on the punctured disk $D \setminus \{0\}$. It follows that $\pi^*\Lambda^2\mathcal{E} = \pi^*(\Lambda \otimes K^{-1})$, hence $\Lambda^2\mathcal{E} = \Lambda \otimes K^{-1}$ for the map $\pi^* : \text{Jac}(X) \rightarrow \text{Jac}(\hat{X})$ between the respective Jacobians is injective [Mu74]. Furthermore, with respect to the frame (f, f') , the induced hermitian metric \mathcal{H} is given by

$$\mathcal{H} = \begin{pmatrix} 1 & 0 \\ 0 & |z| \end{pmatrix}.$$

As for the Higgs field we find

$$\Phi(z) = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} dz$$

by [NaSt95, Section 5A]. Note that Φ is indeed parabolic at $z = 0$.

To get a holomorphic vector bundle with determinant Λ we must as in Section 2.4 twist with the square root $K^{1/2}$ to get $E_\infty = \mathcal{E} \otimes K^{1/2}$ with corresponding Higgs field $\Phi_\infty : E_\infty \rightarrow E_\infty \otimes K^{1/2}$. Restricted to X^\times any line bundle is holomorphically trivial, and a hermitian metric is just a nowhere vanishing function h with respect to this trivialization. We trivialize K^2 by q and define $h = |q\bar{q}|^{-2}$. Taking $\sqrt[4]{h}$ for a trivialization of $K^{1/2}$ yields a hermitian metric which is given by $1/|z|$ near the marked points and is locally of the form $f\bar{f}$ for a holomorphic function f . In particular, this metric is flat so that the product metric H_∞ on $E_\infty|_{X^\times}$ has the same curvature as \mathcal{H} . Near a marked point,

$$H_\infty = \begin{pmatrix} \frac{1}{\sqrt{|z|}} & 0 \\ 0 & \sqrt{|z|} \end{pmatrix}$$

so the associated Chern connection A_∞ is precisely equal to A_∞^{fid} . Similarly, writing Φ_∞ with respect to the rescaled unitary frame gives the fiducial Higgs field Φ_∞^{fid} .

To show that (A_∞, Φ_∞) is complex gauge equivalent to (A_H, Φ) on X^\times we may assume (cf. Section 2.1) that modulo a smooth complex gauge transformation over X our initial hermitian metric H is given by $\text{diag}(1, 1)$ with respect to the frame (f, f') . Then $H_\infty = g_\infty^* H g_\infty$ on X^\times where

$$g_\infty = \begin{pmatrix} |z|^{-\frac{1}{4}} & 0 \\ 0 & |z|^{\frac{1}{4}} \end{pmatrix}$$

near the marked points.

Remark. This viewpoint is somehow dual to the one in [MSWW14]. Indeed, here we look for a limiting hermitian metric, which then determines a holomorphic bundle E_∞ endowed with a singular hermitian metric. By contrast, in [MSWW14] we look for a limiting unitary connection; this gives a smooth hermitian metric but a singular complex structure.

4. DESINGULARIZATION BY GLUING

We now give a brief sketch of the main theorem in [MSWW14], which globalizes the phenomenon that the family of smooth fiducial solutions $(A_t^{\text{fid}}, \Phi_t^{\text{fid}})$ desingularize the limiting fiducial solution $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$.

Theorem 4.1 ([MSWW14]). *Suppose that (A_∞, Φ_∞) is a simple limiting configuration, i.e., so that $\det \Phi_\infty$ has simple zeroes. Then there exists a family of solutions to the Hitchin equations $(A_t, t\Phi_t)$ with $A_t \rightarrow A_\infty$ and $\Phi_t \rightarrow \Phi_\infty$ in $\mathcal{C}_{\text{loc}}^\infty$ at exponential rate in t on the complement of $\det \Phi_\infty^{-1}(0)$.*

These solutions are obtained by gluing (A_∞, Φ_∞) to the elements of the family $(A_t^{\text{fid}}, \Phi_t^{\text{fid}})$. In fact, it is possible to prove this theorem either by performing the gluing as just indicated, or else by constructing a desingularizing family of hermitian metrics H_t as solutions to (6). These two approaches are quite close, and there seems to be nothing to recommend one over the other, but we explain the former.

The steps in the proof follow a familiar pattern. We first construct a family of approximate solutions A_t^{app} and Φ_t^{app} as follows. Since the limiting configurations, by definition, agree with the limiting fiducial solution in a neighbourhood of each marked point, and noting that $(A_t^{\text{fid}}, \Phi_t^{\text{fid}})$ converges exponentially in t to $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$ on any annulus $\{\frac{1}{2}\epsilon \leq |z| \leq \epsilon\}$, we can use a partition of unity to patch $(A_t^{\text{fid}}, \Phi_t^{\text{fid}})$ to (A_∞, Φ_∞) to obtain $(A_t^{\text{app}}, \Phi_t^{\text{app}})$. The error term, i.e., the deviation which measures the extent to which these do not satisfy Hitchin's equations, is supported in this annulus and is exponentially small in t . For the second step we seek a small complex gauge transformation $g = \exp \gamma$ which has the property that $(A_t^{\text{app}}, t\Phi_t^{\text{app}})^g$ is an exact solution. Note that we may assume that γ is a section of the bundle $i\mathfrak{su}(E)$ of hermitian endomorphisms of E since this is transverse to the infinitesimal real gauge transformations.

The second step requires more work. To set things up, fix a background connection A_0 and consider the Hitchin operator

$$\mathcal{H}_t(A, \Phi) = (\mathcal{H}_t^{(1)}, \mathcal{H}_t^{(2)}) = (F_A^\perp + t^2[\Phi \wedge \Phi^*], \bar{\partial}_A \Phi)$$

for connections A which are traceless relative to A_0 and traceless Higgs fields Φ . Consider also the orbit map

$$(17) \quad \mathcal{O}_{(A, \Phi)}(\gamma) = (A^g, \Phi^g), \quad g = \exp(\gamma).$$

The goal is to find a pair (A, Φ) which lies in the nullspace of $\mathcal{H}_t \circ \mathcal{O}_{(A, \Phi)}$. Since the condition $\bar{\partial}_A \Phi = 0$ is complex gauge invariant, we may disregard the second component of \mathcal{H}_t , so it suffices to find a solution of

$$(18) \quad \mathcal{F}_t(\gamma) := \mathcal{H}_t^{(1)} \circ \mathcal{O}_{(A, \Phi)}(\exp(\gamma)) = 0,$$

or more explicitly,

$$F_{A^g}^\perp + t^2[\Phi^g \wedge (\Phi^g)^*] = 0, \quad g = \exp(\gamma).$$

Since we may assume that $\gamma \in \Omega^0(\mathfrak{isu}(E))$, we calculate first that

$$D\mathcal{O}_{(A, \Phi)}|_{g=\text{Id}}(\gamma) = (\bar{\partial}_A \gamma - \partial_A \gamma, [\Phi, \gamma])$$

and then that

$$D\mathcal{H}_t^{(1)}(\dot{A}, \dot{\Phi}) = d_A(\dot{A}) + t^2([\Phi \wedge \dot{\Phi}^*] + [\Phi^* \wedge \dot{\Phi}])$$

whence

$$D\mathcal{F}_t(\gamma) = (\partial_A \bar{\partial}_A - \bar{\partial}_A \partial_A) \gamma + t^2([\Phi \wedge [\Phi, \gamma]^*] + [\Phi^* \wedge [\Phi, \gamma]]).$$

Using the standard identities

$$2\bar{\partial}_A \partial_A = F_A - i * \Delta_A, \quad 2\partial_A \bar{\partial}_A = F_A + i * \Delta_A,$$

as well as the fact that

$$[\Phi \wedge [\Phi, \gamma]^*] = -[\Phi \wedge [\Phi^*, \gamma]],$$

we can write

$$(19) \quad D\mathcal{F}_t(\gamma) = i * \Delta_A \gamma + t^2 M_\Phi \gamma,$$

where

$$M_\Phi \gamma := [\Phi^* \wedge [\Phi, \gamma]] - [\Phi \wedge [\Phi^*, \gamma]].$$

We study instead the operator on $\Omega^2(\mathfrak{su}(E))$ given by

$$L_t(\gamma) = -i * D\mathcal{F}_t(\gamma) = \Delta_A \gamma - i * t^2 M_\Phi \gamma.$$

A brief calculation, see [MSWW14], shows that if $\gamma \in \Omega^0(\mathfrak{isu}(E))$, then

$$(20) \quad \langle L_t \gamma, \gamma \rangle = \|d_A \gamma\|^2 + 4 \|[\Phi, \gamma]\|^2 \geq 0.$$

In particular, $L_t \gamma = 0$ if and only if $[\Phi, \gamma] = 0$ and $d_A \gamma = 0$.

The main analytic part of this gluing argument is to determine the mapping properties of the inverse of L_t , and to keep track of this behavior uniformly in t . As part of this, we must also show that L_t is invertible. To do all of this, decompose X into a union of disks $B_\epsilon(p_j)$ around each

p_j and the complementary (or exterior) region $X \setminus \sqcup B_\epsilon(p_j)$. Assume that the underlying metric on X is Euclidean in each $B_\epsilon(p_j)$. We then analyze the restriction of this operator to each disk, which we denote by L_t^{int} , with Neumann boundary conditions. This is done using separation of variables in polar coordinates, and is an explicit but lengthy computation. We see from this that L_t^{int} is invertible and satisfies

$$\|(L_t^{\text{int}})^{-1}\|_{\mathcal{L}(L^2, L^2)} \leq C, \quad \|(L_t^{\text{int}})^{-1}\|_{\mathcal{L}(L^2, H^2)} \leq Ct^2.$$

On the other hand, L_t^{ext} , the restriction of L_t to the exterior region, is independent of t , so it suffices only to check that it has no nullspace (with Neumann boundary conditions). Integrating by parts as in (20) shows that if γ is in the nullspace of L_t^{ext} (with Neumann boundary conditions), then γ is a parallel section (with respect to A_∞ of the line bundle $\mathcal{L}_{\Phi_\infty}$ of elements γ which satisfy $[\Phi_\infty, \gamma] = 0$, i.e., which commute with the limiting Higgs field Φ_∞). However, this line bundle is nontrivial around each p_j and thus has no parallel sections, so we conclude that $\gamma = 0$ and this nullspace is trivial. Finally, a standard domain decomposition lemma shows that the operator L_t is invertible on the entire surface X and satisfies

$$\|L_t^{-1}\|_{\mathcal{L}(L^2, L^2)} \leq C, \quad \|L_t^{-1}\|_{\mathcal{L}(L^2, H^2)} \leq Ct^2.$$

The rest of the proof of the gluing theorem proceeds by expanding \mathcal{F}_t into its first order Taylor approximation plus remainder:

$$\mathcal{F}_t(\exp(\gamma)) = \mathcal{H}_t^{(1)}(A_t^{\text{app}}, \Phi_t^{\text{app}}) + L_t\gamma + Q_t(\gamma).$$

Since L_t is invertible, the solutions of $\mathcal{F}_t(\exp(\gamma)) = 0$ are the same as solutions of

$$\gamma = -L_t^{-1}(\mathcal{H}_t^{(1)}(A_t^{\text{app}}, \Phi_t^{\text{app}}) + Q_t(\gamma)),$$

or in other words, as fixed points of the mapping

$$T: B_\rho \rightarrow H^2(i\mathfrak{su}(E)), \quad \gamma \mapsto -L_t^{-1}(\mathcal{H}_t^{(1)}(A_t^{\text{app}}, \Phi_t^{\text{app}}) + Q_t(\gamma)),$$

where B_ρ is a ball in H^2 .

To show that T really does have fixed points, we show that it is a contraction when ρ is sufficiently small. The key points here are that the norm of L_t^{-1} grows polynomially in t while the error term $\mathcal{H}_t^{(1)}(A_t^{\text{app}}, \Phi_t^{\text{app}})$ decays exponentially in t .

We remark that the exact same proof can be used to desingularize limiting configurations (A_∞, Φ_∞) where $\det \Phi_\infty$ has multiple zeroes. Indeed, we simply use the corresponding family of fiducial solutions introduced earlier. The more serious issue, however, is to understand families of solutions near to these ones with multiple zeroes. As mentioned earlier, this is likely possible using the forthcoming work of Fredrickson [F].

5. THE HYPERKÄHLER METRIC

Now let us turn to a consideration of the natural L^2 , or Weil-Petersson, metric on \mathbf{M}_Λ , the moduli space of solutions to the self-duality equation (4). As shown initially by Hitchin, this metric is of a very special type.

Definition 5.1. A manifold M^{4k} endowed with a Riemannian metric g and three integrable complex structures I_1, I_2 and I_3 is called *hyperkähler* if

- g is Kähler with respect to I_j , $j = 1, 2, 3$;
- these complex structures satisfy the quaternion relations $I_1 I_2 = I_3$ etc.

As Hitchin shows, \mathbf{M}_Λ carries a natural hyperkähler metric on its smooth locus; the smooth locus is the entire moduli space if d is odd [Hi87], and this metric is then complete by Uhlenbeck compactness. We recall next how this metric arises through a moment map interpretation of the self-duality equations.

5.1. Moment maps. An action of a Lie group G on a symplectic manifold (M, ω) is called *hamiltonian* if there exists a G -equivariant *moment map* $\mu : M \rightarrow \mathfrak{g}^*$ from M to the dual of the Lie algebra \mathfrak{g} of G . This map is defined by the property that for $v \in \mathfrak{g}$, the smooth map $\mu_v(x) = \mu(x)(v)$ satisfies $d_x \mu_v = \omega(v^\sharp(x), \cdot)$, where $v^\sharp(x)$ denotes the induced fundamental vector field at x . If G acts freely and $\lambda \in \mathfrak{g}^*$ is central, the quotient $\mu^{-1}(\lambda)/G$ carries a natural symplectic structure inherited from M .

This symplectic quotient construction can be adapted to hyperkähler manifolds as follows. Assume that there is a hamiltonian action by G with respect to any of the three Kähler forms $\omega_j = g(I_j \cdot, \cdot)$, $j = 1, 2, 3$. We then assemble the individual symplectic moment maps μ_j into $\mu = (\mu_1, \mu_2, \mu_3)$, which takes values in $\mathbb{R}^3 \otimes \mathfrak{g}^*$. It turns out that for a central element $\lambda \in \mathbb{R}^3 \otimes \mathfrak{g}^*$, the quotient $\mu^{-1}(\lambda)/G$ inherits a natural hyperkähler metric [HKLR87].

This construction applies to our setting as follows [Hi87]. The choice of a base connection A_0 identifies the space of unitary connections of fixed determinant with $\Omega^1(\mathfrak{su}(E))$. This, in turn, can be identified with the complex vector space $\mathcal{A} := \Omega^{0,1}(\text{End}_0(E))$, the complex conjugate of the space of traceless Higgs fields $\Omega^{1,0}(\text{End}_0(E))$. The L^2 -inner product

$$\langle (\alpha, \Phi), (\alpha, \Phi) \rangle = 2i \int_X \text{Tr}(\alpha^* \alpha + \Phi \Phi^*)$$

gives the quaternionic vector space $\mathcal{A} \times \bar{\mathcal{A}}$ the structure of a flat hyperkähler manifold with Kähler forms ω_j induced by the complex structures

$$I_1(\alpha, \Phi) = (i\alpha, i\Phi), \quad I_2(\alpha, \Phi) = (i\Phi^*, -i\alpha^*), \quad I_3(\alpha, \Phi) = (-\Phi^*, \alpha^*).$$

Formally, the gauge group \mathcal{G} acts in a hamiltonian fashion with moment maps

$$\mu_1(A, \Phi) = F_A + [\Phi \wedge \Phi^*], \quad \mu_2(A, \Phi) = \text{Re}(\bar{\partial}_A \Phi), \quad \mu_3(A, \Phi) = \text{Im}(\bar{\partial}_A \Phi).$$

The L^2 -metric on $\mathcal{A} \times \bar{\mathcal{A}}$ induces a hyperkähler metric on the quotient $\mu^{-1}(-\mu i\omega \otimes \text{Id}, 0, 0)/\mathcal{G} = \mathbf{M}_\Lambda$.

Remark. Note that for any $(a, b, c) \in S^2 \subset \mathbb{R}^3$, the complex structure $I_{(a,b,c)} := aI_1 + bI_2 + cI_3$ is an isometry for g . However, the corresponding holomorphic structures on \mathbf{M}_Λ are not equivalent. In fact, $(\mathbf{M}_\Lambda, \pm I_1)$ is biholomorphic to \mathcal{M}_Λ , while for any other complex structure defined by $S^2 \setminus \{(\pm 1, 0, 0)\}$, $(\mathbf{M}_\Lambda, I_{(a,b,c)})$ is biholomorphic to the moduli space of irreducible projectively flat complex connections together with its natural complex structure as discussed in the Remark of Section 2.2.

5.2. The semi-flat metric. One of the most interesting open questions is to determine the asymptotic structure of the hyperkähler metric g on \mathcal{M}_Λ which we identify with \mathbf{M}_Λ using the complex structure $\pm I_1$. Guiding any investigation into this problem is the far-reaching conjectural picture by Gaiotto-Moore-Neitzke [GMN10, GMN13] which describes this metric as a perturbation series off a so-called semi-flat metric g_{sf} . More specifically, g_{sf} is a metric on the open dense set of the moduli space where the Higgs field Φ is simple, i.e., such that $\det \Phi$ has only simple zeroes. To understand the name ‘semiflat’, we recall that \mathcal{M}_Λ is the total space of a singular fibration $\det : \mathcal{M}_\Lambda \rightarrow \mathcal{H}\mathcal{Q}$, where the base is the space of holomorphic quadratic differentials on X ; this is called the Hitchin fibration. The regular fibre of \det is the Prym variety associated with q , which is a torus of real dimension $6\gamma - 6$, where γ is the genus of X , cf. Section 2.4.

Restricting to the regular part of the fibration this data constitute an algebraic completely integrable system and therefore the base carries a special Kähler structure [Fr99]. A special Kähler manifold is a Kähler manifold (M, g, J, ω) together with an additional flat torsion-free connection ∇ on TM satisfying $\nabla\omega = 0$ and $d_\nabla J = 0$, where J is viewed as a TM -valued 1-form. As shown in [Fr99] one may now use the horizontal distribution on T^*M to lift the metric g to the horizontal part of a hyperkähler metric g_{sf} on T^*M . The restriction of g_{sf} to each cotangent fibre is then a flat metric, which explains the terminology. Moreover, since locally the total space of the integrable system is represented as the quotient of T^*M by a family of lattices (parallel with respect to ∇), the semiflat metric descends to a metric on the regular part of the Hitchin fibration, again denoted by g_{sf} . This metric is incomplete, however, and needs to be corrected in order to extend over the singular fibres. A description of this correction process using a wall-crossing formalism is one of the main achievements of the work of Gaiotto, Moore and Neitzke.

The first application of our new construction of ‘large’ solutions in \mathcal{M}_Λ shows that the hyperkähler metric g is indeed well approximated by the semi-flat metric g_{sf} far out in the ends of the moduli space. The way we seek to approach this is as follows: As shown in [MSWW14], the space of limiting configurations associated with $q \in \mathcal{H}\mathcal{Q}$ with simple zeroes is a torus of real dimension $6\gamma - 6$, just like the Prym variety is. A family of limiting

configurations $(A_\infty(s), \Phi_\infty(s))$ associated with a curve of holomorphic quadratic differentials $q(s)$ may now, using Theorem 4.1, be perturbed into a family of solutions $(A(s), t\Phi(s))$ of Hitchin's equation for sufficiently large t . After a suitable gauge choice, the derivative of this family with respect to s is a tangent vector to \mathcal{M}_Λ which is vertical with respect to the Hitchin fibration. The L^2 -metric may now be directly evaluated on this tangent vector, and is easily seen to be an exponentially small (in t) correction to the vertical part of the semiflat metric as $t \rightarrow \infty$. Horizontal tangent vectors may be obtained by varying $q \in \mathcal{H}\mathcal{Q}$, leading likewise to a comparison between the horizontal parts of the metrics.

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