Different cofinalities of tree ideals

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Abstract

We introduce a general framework of generalized tree forcings, GTF for short, that includes the classical tree forcings like Sacks, Silver, Laver or Miller forcing. Using this concept we study the cofinality of the ideal \( I(Q) \) associated with a GTF \( Q \). We show that if for two GTF’s \( Q_0 \) and \( Q_1 \) the consistency of \( \text{add}(I(Q_0)) < \text{add}(I(Q_1)) \) holds, then we can obtain the consistency of \( \text{cof}(I(Q_1)) < \text{cof}(I(Q_0)) \). We also show that \( \text{cof}(I(Q)) \) can consistently be any cardinal of cofinality larger than the continuum.

1 Introduction

The classical tree forcings like Sacks, Silver, Laver or Miller forcing consist of certain subtrees of \( 2^{<\omega} \) or \( \omega^{<\omega} \) (see [2]). They will be denoted by \( Sa, Si, Mi, La \) respectively. As usual, for given \( Q \in \{ Sa, Si, La, Mi \} \) and \( p \in Q \), \([p] \) denotes the set of branches of \( p \), so a subset of \( \mathbb{R} \), where \( \mathbb{R} \) stands for \( 2^{<\omega} \) or \( \omega^{<\omega} \) appropriately. Then the tree ideal \( I(Q) \) consists of all \( X \subseteq \mathbb{R} \) such that for every \( p \in Q \) there exists \( q \in Q \) with \( q \subseteq p \) and \([q] \triangleleft X = \emptyset \). By using standard fusion arguments, it is easily seen that \( I(Q) \) is a \( \sigma \)-ideal. Hence we have \( \aleph_1 \leq \text{add}(I(Q)) \leq 2^{\aleph_0} \), where \( \text{add}(I(Q)) \) denotes the additivity of \( I(Q) \), i.e. the minimal cardinality of some \( X \subseteq I(Q) \) with \( \bigcup X \notin I(Q) \). By \( \text{cof}(I(Q)) \) we denote the minimal cardinality of some \( X \subseteq I(Q) \) that is cofinal in \((I(Q), \subseteq)\). The same definitions make sense for many more tree forcings that are studied in set theory. This is one reason for us to introduce in Section 3 the general concept of generalized tree forcing. However, some knowledge about the antichain structure of the concrete forcing is needed for this framework to be applicable.

The original motivation for this paper was to gain insight into the cofinalities of classical tree ideals, as very little has been known about them. To our

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knowledge, the only papers dealing with this topic are [8] and [3]. In [8] it has been shown that $2^{\aleph_0} < \text{cf} (\text{cof}(\mathcal{I}(\mathcal{S}a)))$ holds in ZFC and that consistently $\text{cof}(\mathcal{I}(\mathcal{S}a))$ can be any cardinal with cofinality $> 2^{\aleph_0}$. The same facts are true for $\mathcal{S}i$ with essentially the same proofs. Similar results for $\mathcal{L}a, \mathcal{M}i$ have been obtained in [3]. Here we attack the question whether we can consistently obtain $\text{cof}(\mathcal{I}(\mathcal{Q}_0)) \neq \text{cof}(\mathcal{I}(\mathcal{Q}_1))$ for different $\mathcal{Q}_0, \mathcal{Q}_1 \in \{\mathcal{S}a, \mathcal{S}i, \mathcal{L}a, \mathcal{M}i\}$. The main result of this paper implies that $\text{cof}(\mathcal{I}(\mathcal{Q}_1)) < \text{add}(\mathcal{I}(\mathcal{Q}_0))$ is consistent for any pair of $\mathcal{Q}_0, \mathcal{Q}_1 \in \{\mathcal{S}a, \mathcal{S}i, \mathcal{L}a, \mathcal{M}i\}$ for which $\text{add}(\mathcal{I}(\mathcal{Q}_0)) < \text{add}(\mathcal{I}(\mathcal{Q}_1))$ is consistent.

Unfortunately, distinguishing the additivities of different tree ideals is also a difficult matter. However there are some cases where this has been achieved, as much more work has been done about additivities of tree ideals. Let us mention [17], [8], [4], [10], [5], [6], [13], [14], [16], [15] (chronological order).

In [8] for $\mathcal{Q} = \mathcal{S}a$ and in [4] for $\mathcal{Q} = \mathcal{S}i$ it has been shown that $\text{MA}$ does not imply $\text{add}(\mathcal{I}(\mathcal{Q})) = 2^{\aleph_0}$, whereas on the other hand, [5] and [6] show that this is true for $\mathcal{Q} = \mathcal{L}a$ or $\mathcal{Q} = \mathcal{M}i$. So we can apply our theorem for any choice of $\mathcal{Q}_0 \in \{\mathcal{S}a, \mathcal{S}i\}$ and $\mathcal{Q}_1 \in \{\mathcal{L}a, \mathcal{M}i\}$. Another such case is when $\mathcal{Q}_0 = \mathcal{S}i$ and $\mathcal{Q}_1 = \mathcal{S}a$. Implicitly in [10], an amoeba for $\mathcal{S}a$ with the Laver property has been constructed. Iterating this with countable supports $\aleph_2$ times one obtains a model for $\text{cov}(\mathcal{M}) = \aleph_1$ and $\text{add}(\mathcal{I}(\mathcal{S}a)) = \aleph_2$. But by [14], $\text{add}(\mathcal{I}(\mathcal{S}i)) \leq \text{cov}(\mathcal{M})$ holds in ZFC. (Here $\text{cov}(\mathcal{M})$ is the minimal number of meager sets needed to cover $\mathbb{R}$.)

All the other cases are open. However, by the work of [13] and [15] soft amoebas for $\mathcal{Q} \in \{\mathcal{M}i, \mathcal{L}a\}$ (with the Laver property) and for $\mathcal{S}i$ (with the pure decision property) exist. We expect that using these for making $\text{add}(\mathcal{I}(\mathcal{Q})) = \aleph_2$ we can produce more models where our main theorem can be applied.

2 $\ast_d$-Iterations

In [11], the first author introduced a general framework to iterate forcings that are $(< \lambda)$-closed and have the $\lambda^+$-c.c. with supports of size $< \lambda$, where $\lambda$ is some regular cardinal with $\lambda^{< \lambda} = \lambda$. The main goal is to guarantee that also the iteration is $\lambda^+$-c.c. For this the $\ast_d$-property is introduced as follows:

\textbf{Definition 2.1} Let $\lambda$ be a regular cardinal with $\lambda^{< \lambda} = \lambda$.

\begin{enumerate}
\item A \textbf{c.c.-parameter} is a quintuple $d = (\lambda, D, \varepsilon, \sigma, S)$ such that
\end{enumerate}
(a) $D$ is a normal filter on $\lambda^+$ containing $S^\lambda_\lambda$ and $\varepsilon < \lambda$ is a limit ordinal,

(b) $\sigma$ is a cardinal with $2 \leq \sigma < \lambda$ and $S \subseteq [S^\lambda_\lambda]^<(1+\sigma)$ has nonempty intersection with $[S]^<(1+\sigma)$ for every stationary set $S \subseteq S^\lambda_\lambda$.

(2) Given a forcing notion $Q$ and a c.c.-parameter $d$ we define the game $\mathcal{G}(Q,d)$ as follows: It lasts for $\varepsilon$ moves. In his $\xi$th move player I plays $(\langle q^\xi_i : i < \lambda^+ \rangle, f_\xi)$ and player II plays $\langle p^\xi_i : i < \lambda^+ \rangle$, where

(a) $\forall \iota < \lambda^+ \forall \zeta < \varepsilon^*(\xi^\iota_i, p^\xi_i \in Q \wedge q^\xi_0 = 1_Q)$,

(b) for every $1 \leq \zeta < \varepsilon f_\xi : \lambda^+ \to \lambda^+$ is regressive, $f_0 : \lambda^+ \to \lambda^+$ is constantly 0, and

(c) $\forall \zeta < \varepsilon^D \iota < \lambda^+ q^\xi_i \leq p^\xi_i$ and $\forall \zeta < \varepsilon^D \iota < \lambda^+ q^\xi_i \leq q^\xi_i$.

(3) Player I wins a play $\langle (\langle q^\xi_i : i < \lambda^+ \rangle, f_\xi), (p^\xi_i : i < \lambda^+ : \zeta < \varepsilon) \rangle$ provided that there exists $E \subseteq D$ such that for every $u \in [E]^{<(1+\sigma)} \cap S$ with the property $\forall i, j : u \{ \zeta < \varepsilon f_\xi(i) = f_\xi(j) \}$ the set $\{ p^\xi_i : \zeta < \varepsilon, i \in u \}$ has a lower bound in $Q$.

(4) Given a c.c.-parameter $d$, we say that forcing $Q$ satisfies property $*_d$ if player I has a winning strategy in the game $\mathcal{G}(Q,d)$.

**Remark 2.1** (1) Let $Q$ be a forcing notion with $*_d$, where $d = (\lambda, D, \varepsilon, \sigma, S)$ is a c.c.-parameter with $D = \text{CLUB}_{\lambda^+}$ and $S = [S^\lambda_{\lambda^+}]^\kappa$ for some cardinal $\kappa$ with $2 \leq \kappa < 1 + \sigma$. Given $(p_i : i < \lambda^+)$, a sequence in $Q$, there exists a club $E \subseteq \lambda^+$ such that for every stationary $S \subseteq E \cap S^\lambda_\lambda$ there is $u \in \mathcal{P}(S) \cap S$ with the property that the set $\{ p_i : i \in u \}$ has a lower bound. Indeed, let $\langle (\langle q^\xi_i : i < \lambda^+ \rangle, f_\xi), (p^\xi_i : i < \lambda^+ : \zeta < \varepsilon) \rangle$ be a play of $\mathcal{G}(Q,d)$ where player I uses his winning strategy and player II plays $\langle p^\xi_i : i < \lambda^+ \rangle = \langle p_i : i < \lambda^+ \rangle$ and afterwards just repeats the moves of player I. By Definition 2.1(3) there exists a club $E$ as there. Given a stationary set $S \subseteq E \cap S^\lambda_\lambda$ for every $i < \lambda^+$ we can find $\alpha_i < i$ such that the sequence $\langle f_\xi(i) : \zeta < \varepsilon \rangle$ is bounded by $\alpha_i$. By the Pressing-down-Lemma there exist a stationary set $S_* \subseteq S$ and $\alpha_*$ such that $\alpha_i = \alpha_*$ for every $i \in S_*$. By our assumption $\lambda^c \lambda^c \lambda$ and there exists $U \subseteq S_*$ of size $\lambda^+$ such that $\langle f_\xi(i) : \zeta < \varepsilon \rangle = \langle f_\xi(j) : \zeta < \varepsilon \rangle$ for any $i, j \in U$. By construction and Definition 2.1(3), every $u \in \mathcal{P}(U) \cap S$ is as desired. By the choice of $S$, such $u$ exist. In particular, $Q$ is $\lambda^+\text{-c.c.}$.

(2) Suppose that $Q$ is $\lambda$-closed and $\lambda$-centered, and $d = (\lambda, D, \varepsilon, \sigma, S)$ is any c.c.-parameter. Then $Q$ satisfies $*_d$. Indeed, if $Q = \bigcup_{\mu < \lambda} Q_\mu$ where each $Q_\mu$ is centered, in his $\xi$th move player I plays $(\langle q^\xi_i : i < \lambda^+ \rangle, f_\xi)$ such that $q^\xi_i$ is a lower bound of player II moves $\langle p^\xi_i : \zeta < \varepsilon \rangle$ and $f_\xi(i) = \mu$ such that $q^\xi_i \in Q_\mu$. Then this is a winning strategy for player I.
In [11], the following preservation theorem is proved:

**Theorem 2.1** Suppose that $\lambda$ is a cardinal with $\lambda^{<\lambda} = \lambda$, $d = (\lambda, D, \varepsilon, \sigma, S)$ is a c.c.-parameter and $\langle P_\alpha, Q_\beta : \alpha \leq \mu, \beta < \mu \rangle$ is a $(<\lambda)$-support iteration such that for every $\beta < \mu$, $\Vdash_{P_\alpha} "Q_\beta \text{ satisfies } *_d"$. Then $P_\alpha$ satisfies $*_d$.

### 3 Amoebas for generalized tree forcings

**Definition 3.1** Let $\lambda = 2^{\aleph_0}$. (1) A $\textit{GTF}_0$ (here GTF stands for generalized tree forcing) is a quintupel $\langle Q, \dot{\zeta}, \text{set}, Q^*, \perp \rangle$ such that

(a) $Q = (Q, <_Q)$ is a forcing notion, $Q \subseteq H(\lambda)$ and $\dot{\zeta}$ is a $Q$-name such that $\Vdash_Q \dot{\zeta} \in \mathbb{R}$;

(b) $Q^*$ is a dense subset of $Q$;

(c) set is a function from $Q^*$ to Borel subsets of $\mathbb{R}$ such that

(α) if $p \leq q$ then $\text{set}(p) \subseteq \text{set}(q)$,

(β) $p \Vdash_Q \dot{\zeta} \in \text{set}(p)$,

(γ) $\Vdash_Q \{\dot{\zeta} : \text{set}(p) \cap \dot{\mathcal{G}}_Q\}$, (where $\dot{\mathcal{G}}_Q$ is the canonical $Q$-name of the generic filter);

(d) for every $A \in [\mathbb{R}]^{<\lambda}$ the set $\{p \in Q^* : \text{set}(p) \cap A = \emptyset\}$ is dense in $Q$;

(e) $\perp$ is a binary, symmetric relation on $Q^*$ such that

(α) if $p \perp q$, then $p$ and $q$ are incompatible in $Q$,

(β) if $p \perp q$, then $\text{set}(p) \cap \text{set}(q) = \emptyset$,

(γ) if $\beta < \lambda$ and $\langle p_\alpha : \alpha < \beta \rangle$ is a sequence in $Q^*$, then there is $q \in Q^*$ such that $\forall \alpha < \beta p_\alpha \perp q$,

(δ) if $\beta < \lambda$, $\langle p_\alpha : \alpha < \beta \rangle$ is a sequence in $Q^*$ and $p \in Q$ is incompatible with every $p_\alpha$, then there is $q \in Q^*$ such that $q \leq p$ and $\forall \alpha < \beta p_\alpha \perp q$.

(2) If $Q = (Q, \dot{\zeta}, \text{set}, Q^*, \perp)$ is as in (1) except that in (e), (γ) and (δ) are replaced by the weaker $(\gamma)_1$ and $(\delta)_1$ which ask the same thing as those, but only for orthogonal sequences $\langle p_\alpha : \alpha < \beta \rangle$, i.e. $p_\alpha \perp p_\alpha'$ for any $\alpha < \alpha' < \beta$, then we call $Q$ a $\textit{GTF}_1$. 

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If $Q = (Q, \dot{\zeta}, \text{set}, Q^*, \bot)$ is a GTF we define

$$\mathcal{I}(Q) = \{X \subseteq \mathbb{R} : \forall p \in Q^* \exists q \in Q^* (q \leq p \land \text{set}(q) \cap X = \emptyset)\},$$

and we let

$$\text{add}(\mathcal{I}(Q)) = \min\{|X| : X \subseteq \mathcal{I}(Q) \land \bigcup X \notin \mathcal{I}(Q)\},$$

$$\text{cof}(\mathcal{I}(Q)) = \min\{|X| : X \subseteq \mathcal{I}(Q) \land \forall X \in \mathcal{I}(Q) \exists Y \in X X \subseteq Y\}.$$

**Remark 3.1** (1) Clearly we have $\text{GTF}_0 \subseteq \text{GTF}_1$.

(2) Clearly $\mathcal{I}(Q)$ is an ideal, in its definition we could replace $Q^*$ by $Q$, and by Definition 3.1(d) we have $[\mathbb{R}]^{<\lambda} \subseteq \mathcal{I}(Q)$.

(3) Given $I \subseteq Q^*$ let

$$X(I) = \mathbb{R} \setminus \bigcup \{\text{set}(p) : p \in I\}.$$

Then clearly the following sets are bases of $\mathcal{I}(Q)$:

$$\{X(I) : I \subseteq Q^* \text{ is predense}\},$$

$$\{X(I) : I \subseteq Q^* \text{ is a maximal antichain}\}.$$

Note that by applying (d) and (e) of Definition 3.1(1) we can obtain the following:

**Claim 1** Let $2^{\aleph_0} = \lambda = \lambda^{<\lambda}$. Given a GTF $Q = (Q, \dot{\zeta}, \text{set}, Q^*, \bot)$ and a dense open subset $D \subseteq Q$, there exists a maximal antichain (with respect to $(Q, <_Q)$) $\langle q_\varepsilon : \varepsilon < \lambda \rangle$ in $Q^* \cap D$ such that

(a) $\forall \varepsilon < \xi < \lambda q_\varepsilon \perp q_\xi$;

(b) $\forall r \in Q^* (\text{set}(r) \not\subseteq \bigcup \{\text{set}(q_\varepsilon) : \varepsilon < \lambda\} \lor \exists B \in [\lambda]^{<\lambda} \text{set}(r) \subseteq \bigcup \{\text{set}(q_\varepsilon) : \varepsilon \in B\})$.

For classical tree forcings this has been proved and applied first in [JMSH] and later was applied often.
Definition 3.2 Let $\lambda = 2^{\aleph_0}$.

(1) Given a GTF$_0$ $Q = (Q, \zeta, \text{set}, Q^*, \bot)$ we define an amoeba forcing for $Q$, denoted by $\mathbb{A}_0(Q)$, as follows:

Elements of $\mathbb{A}_0(Q)$ are pairs $p = (\overline{p}, A) = (\overline{p}_p, A_p)$ such that $\overline{p}$ is a sequence of length $< \lambda$ of members of $Q^*$ and $A \subseteq I(Q)$ is a set of size $< \lambda$. Sometimes we write $\overline{p}_p = (p_{p, \varepsilon} : \varepsilon < \lg (\overline{p}_p))$.

The order on $\mathbb{A}_0(Q)$ is defined by letting $p \leq q$ iff $\overline{p}_q$ is an initial segment of $\overline{p}_p$, $A_q \subseteq A_p$ and for every $B \in A_q$ and $\varepsilon \in [\lg (\overline{p}_q), \lg (\overline{p}_p))$ we have $\text{set}(p_{p, \varepsilon}) \cap B = \emptyset$.

(2) Letting $\hat{G}$ denote the canonical $\mathbb{A}_0(Q)$-name for the generic filter, we let $\hat{p} = \hat{p}_G$ be a name for $\bigcup \{ \overline{p}_p : p \in \hat{G} \}$, which we also denote by $\langle \hat{p}_\varepsilon : \varepsilon < \hat{\mu} \rangle$, where $\hat{\mu} = \hat{\mu}_G = \lg (\hat{p})$, and for $\varepsilon < \hat{\mu}$ we let $\hat{B}_\varepsilon$ be a name for $\mathbb{R} \setminus \bigcup \{ \text{set}(\hat{p}_\zeta) : \zeta \in [\varepsilon, \hat{\mu}) \}$. Finally, $\overline{B} = (\hat{B}_\varepsilon : \varepsilon < \hat{\mu})$.

(3) Given a GTF$_1$ $Q$, we define $\mathbb{A}_1(Q)$ as $\mathbb{A}_0(Q)$ except that for its members $p = (\overline{p}, A)$ we require that $\overline{p}$ is an antichain (in $Q^*$) with respect to $\bot$. If $\hat{G}$ denotes the canonical $\mathbb{A}_1(Q)$-name for the generic filter and $\overline{p} = \hat{p}_G = (\hat{p}_\varepsilon : \varepsilon < \hat{\mu})$ is defined for it as in (2), we define $\hat{B}_0$ as $\mathbb{R} \setminus \bigcup \{ \text{set}(\hat{p}_\zeta) : \varepsilon < \hat{\mu} \}$ and $\hat{B}_\varepsilon = \hat{B}_0$ for every $\varepsilon < \hat{\mu}$.

Lemma 3.1 Suppose that $\lambda = 2^{\aleph_0} = \lambda^\text{<}\lambda$.

(A) Let $Q$ be a GTF$_0$ and $\text{add}(I(Q)) = \lambda$.

(1) $\mathbb{A}_0(Q)$ is $\lambda$-closed, in particular every decreasing sequence of length $< \lambda$ has a largest lower bound (llb for short); moreover $\mathbb{A}_0(Q)$ is $\lambda$-centered. Hence it satisfies $\ast_d$ for every c.c.-parameter $d = (\lambda, D, \varepsilon, \sigma, \mathcal{S})$.

(2) $\Vdash_{\mathbb{A}_0(Q)} " \hat{\mu} = \lambda \land \forall \varepsilon < \zeta < \lambda (\hat{B}_\varepsilon \in I(Q) \land \hat{B}_\zeta \subseteq \hat{B}_\varepsilon) "$, and for every $B \in I(Q) \cap V$, $\Vdash_{\mathbb{A}_0(Q)} \exists \varepsilon < \lambda B \subseteq \hat{B}_\varepsilon$.

(3) $\forall B \in I(Q) \cap V \forall \mathcal{A}(Q), B_0 \nsubseteq B$.

(B) Let $Q$ be a GTF$_1$ and $\text{add}(I(Q)) = \lambda$. Then (A)(1), the first part of (A)(2) and (A)(3) also hold for $\mathbb{A}_1(Q)$, and, as for the second part of (A)(2), for every $\mathcal{A} \in V$ such that $\mathcal{A} \subseteq I(Q)$ and $|\mathcal{A}| < \lambda$ we have $(\emptyset, \mathcal{A}) \Vdash_{\mathbb{A}_1(Q)} \mathcal{A} \subseteq \hat{B}_0$.

Proof: (A)(1) Given $\langle p_{\alpha} : \alpha < \gamma \rangle$ a descending chain in $\mathbb{A}_0(Q)$ with $\gamma < \lambda$, clearly we have that $\bigcup_{\alpha < \gamma} \overline{p}_{p_{\alpha}} \cup \bigcup_{\alpha < \gamma} A_{p_{\alpha}}$ is its largest lower bound in $\mathbb{A}_0(Q)$.
Moreover, given \( p, q \in \mathbb{A}_0(Q) \) with \( 
abla p = \nabla q \); clearly \((\nabla p, \mathcal{A}_p \cup \mathcal{A}_q) \preceq p, q \). By \( \lambda < \lambda = \lambda \) we conclude that \( \mathbb{A}_0(Q) \) is \( \lambda \)-centered. By Remark 2.1(2) we conclude that \( \mathbb{A}_0(Q) \) satisfies \( \ast_d \).

(2) Given \( p \in \mathbb{A}_0(Q) \), \( \gamma < \lambda, p \in Q \cap V \) and \( B \in \mathcal{I}(Q) \cap V \), by assumption we have that \( X := \bigcup \mathcal{A}_p \in \mathcal{I}(Q) \). By Definition 3.1(1)(b) we can find \( \langle p_\varepsilon : \varepsilon < \gamma \rangle \) in \( Q^* \) such that \( p_0 \preceq p \) and \( \forall \varepsilon < \gamma \ X \cap \text{set}(p_\varepsilon) = \emptyset \), and hence, letting \( q := (\nabla p_\varepsilon : \varepsilon < \gamma), \mathcal{A}_p \cup \{B\} \), we have \( q \in \mathbb{A}_0(Q), q \preceq p \) and \( q \vdash \mu \geq \gamma \wedge \forall \varepsilon < \lg(\nabla p_\varepsilon) \exists q \in Q^*(q \preceq p \land B_\varepsilon \cap \text{set}(q) = \emptyset) \wedge \forall \varepsilon \geq \lg(\nabla q) B \subseteq B_\varepsilon^\ast \). Hence by genericity and as \( \mathbb{A}_0(Q) \) does not add new elements to \( H(\lambda) \), we conclude that (2) holds.

(3) Given \( p \in \mathbb{A}_0(Q) \) and \( B \in \mathcal{I}(Q) \cap V \), by Definition 3.1(1)(e) there is \( q \in Q^* \) such that \( \forall \varepsilon < \lg(\nabla p_\varepsilon) p_\varepsilon \perp q \). By Definition 3.1 there exists some nonempty \( X \in \text{set}(q) \subseteq \lambda \) such that \( X \cap B = \emptyset \), and hence \( q := (\nabla p_\varepsilon, \mathcal{A}_p \cup \{X\}) \in \mathbb{A}_0(Q), q \preceq p \) and \( q \vdash B_\varepsilon \subseteq B \) (note that by Definition 3.1(1)(e)(\beta) we have \( \forall \varepsilon < \lg(\nabla p_\varepsilon) \exists q \in Q^*(q \preceq p \land B_\varepsilon \cap \text{set}(q) = \emptyset) \wedge \forall \varepsilon \geq \lg(\nabla q) B \subseteq B_\varepsilon^\ast \)).

(B) The proof is almost the same as for \( \mathbb{A}_0(Q) \) in (A). \( \square \)

**Theorem 3.1** Suppose that \( Q \) is a GTF \( _1 \). \( 2^{\aleph_0} = \lambda = \lambda < \lambda < \mu = \text{cf}(\mu) < \chi = \chi^{< \lambda} \) and add \( (\mathcal{I}(Q)) = \lambda \). There exists a forcing \( P \) such that

(a) \( |P| = \chi, P \) is \( \lambda \)-closed and \( \lambda^+-\text{c.c.} \)

(b) \( V^P \vDash 2^\lambda = \chi \wedge \text{cof}(\mathcal{I}(Q)) = \mu. \)

**Proof:** Let us first assume that \( Q \) is even GTF \( \alpha \). Let \( P \) be the limit of a \( (< \lambda) \)-support iteration \( \langle P_\alpha, Q_\alpha : \alpha < \mu, \beta < \mu \rangle \) where \( Q_0 = Fn(\chi, 2, \lambda) \) (which is the standard forcing for adding \( \chi \) Cohen subsets of \( \lambda \) with conditions of size \( < \lambda \)) and \( Q_{1+\beta} \) denotes \( \mathbb{A}_0(Q) \) in \( V^{P_1+\beta} \).

It is easy to check that \( Fn(\chi, 2, \lambda) \) satisfies \( \ast_d \) for every c.c.-parameter \( d = (\lambda, D, \varepsilon, \sigma, S) \). Actually, a simplified version of the proof of Lemma 4.2 below can be used. Hence by Lemma 3.1(A)(1) and Theorem 2.1, \( P \) has \( \ast_d \) and hence, letting \( S = [S^\lambda]_\kappa \) for \( \kappa = 2 \), by Remark 2.1(1) \( P \) is \( \lambda^+-\text{c.c.} \). Clearly, by Lemma 3.1(A)(1) and as we have \( (< \lambda) \)-supports, \( P \) is also \( \lambda \)-closed.

Let \( G \) be a \( P \)-generic filter over \( V \). For \( 1 \leq \beta < \mu \) let \( \langle B_\varepsilon^\beta : \varepsilon < \lambda \rangle \) be the generic sequence in \( \mathcal{I}(Q) \cap V[G_{\beta+1}] \) determined by \( G(\beta) \). Note that by \( \lambda \)-closedness \( P \) does not add new elements to \( H(\lambda) \) and hence we have
\[ \mathcal{I}(Q)^{V[G_\beta]} = \mathcal{I}(Q)^{V[G]} \cap V[G_\beta] \] for every \( \beta < \lambda \). By the \( \lambda^+\)-c.c. of \( P \) and the regularity of \( \mu \), every \( X \in V[G] \) of size \( < \mu \) with \( X \subseteq V \) belongs to \( V[G_\beta] \) for some \( \beta < \mu \). Hence by Lemma 3.1(A)(2) we conclude that \( \{ B_\alpha^\beta : 1 \leq \beta < \mu, \varepsilon < \lambda \} \) is cofinal in \( \mathcal{I}(Q)^{V[G]} \), thus \( \text{cof}(\mathcal{I}(Q)) \leq \mu \) in \( V[G] \).

For the same reason, given \( X \in \mathcal{V}[G] \) such that \( \mathcal{V}[G] \models X \subseteq I(Q) \land |X| < \mu \), there is \( \beta < \mu \) such that \( X \subseteq V[G_\beta] \) (and actually \( X \in V[G_\beta] \)). By Lemma 3.1(A)(3) we conclude that no member of \( X \) contains \( B_0^\beta \). Hence \( \mathcal{V}[G] = \text{cof}(I(Q)) = \mu \).

If \( Q \) is only \( \text{GTF}_1 \) we define the iteration \( P \) as above except that iterand \( \dot{Q}_{1+\beta} \) denotes \( A_1(Q) \) in \( \forall P_{1+\beta} \). The proof is almost the same as in the first case, except that now we argue that \( \{ B_\alpha^\beta : 1 \leq \beta < \mu \} \) is cofinal in \( \mathcal{I}(Q)^{V[G]} \). In fact, given \( X \in \mathcal{I}(Q)^{V[G]} \), as \( X \subseteq V[G_\beta] \) for some \( \beta < \mu \), by genericity we have \( \emptyset, \{ X \} \in G(\gamma) \) for some \( \beta < \gamma < \mu \), and hence \( X \subseteq B_0^\gamma \) by Lemma 3.1(B).

Lemma 3.2 Suppose \( 2^{\aleph_0} = \lambda = \lambda^{<\lambda} \), \( Q \) is a \( \text{GTF}_1 \) and \( P \) is a \( \lambda \)-closed forcing.

(1) If \( \text{add}(\mathcal{I}(Q)) = \lambda \), then \( V^P = \text{add}(\mathcal{I}(Q)) = \lambda \).

(2) If \( \text{add}(\mathcal{I}(Q)) = \mu \) for some cardinal \( \mu < \lambda \), then \( V^P = \text{add}(\mathcal{I}(Q)) = \mu \).

Proof: We only prove (1), as (2) is similar. Let \( Q = (Q, \dot{\zeta}, \text{set}(Q^*), \bot) \). Suppose \( p \in P, \beta < \lambda \) and \( \{ X_\alpha : \alpha < \beta \} \) are \( P \)-names such that

\[ p \Vdash_P \forall \alpha < \beta \exists X_\alpha \in \mathcal{I}(Q). \]

By Claim 1, wlog we may assume that there are \( \dot{X}_\alpha = \{ \dot{q}_\alpha^\varepsilon : \varepsilon < \lambda \} \) for \( \alpha < \beta \) such that the following hold:

(1) \( p \Vdash_P \forall \alpha < \beta \exists X_\alpha \in \mathcal{I}(Q) \)

(2) for every \( \alpha < \beta \), \( p \Vdash \forall r \in Q^*(\exists x \in \text{set}(r)x \notin \bigcup_{\varepsilon < \lambda} \text{set}(\dot{q}_\alpha^\varepsilon)) \lor \exists B \in [\lambda]^{<\lambda} \exists x \in \bigcup_{\varepsilon \in B} \text{set}(\dot{q}_\alpha^\varepsilon)); \)

(3) \( p \Vdash \forall \varepsilon < \xi < \lambda \exists \dot{q}_\xi^\alpha \bot \dot{q}_\varepsilon^\alpha \).

Note that as \( P \) does not add new reals nor elements of \( H(\lambda) \), by absoluteness we have \( \text{set}(r)^V = \text{set}(r)^{V^P} \) for every \( r \in Q^* \). Moreover, for every \( r \in Q^*, \)
by strengthening $p$ in (2) we can decide which alternative holds and also the witness for this (so some $x \in \text{set}(r)$ or $B \in [\lambda]^{<\lambda}$).

Let $\langle r_\varepsilon : \varepsilon < \lambda \rangle$ list $Q^\ast$. By the $\lambda$-closedness of $P$ and the remark just made we can easily construct a decreasing sequence $\langle p_\varepsilon : \varepsilon < \lambda \rangle$ in $P$ and a sequence $\langle \zeta_\varepsilon : \varepsilon < \lambda \rangle$ of ordinals in $\lambda$ such that

1. $p_0 = p, \zeta_\varepsilon \geq \varepsilon$;
2. for all $\alpha < \beta$ and $\varepsilon < \lambda$, $p_{\varepsilon+1}$ decides $\langle q^\alpha_\xi : \xi < \zeta_\varepsilon \rangle$, say as $\langle q^{\alpha,\xi} : \xi < \zeta_\varepsilon \rangle$;
3. for all $\alpha < \beta$ and $\varepsilon < \lambda$ there is $\xi < \zeta_\varepsilon$ such that $r_\varepsilon$ and $q^{\alpha,\xi}$ are compatible (in $Q$);
4. for all $\alpha < \beta$ and $\varepsilon < \lambda$, $p_{\varepsilon+1}$ decides which alternative of (2) for $r = r_\varepsilon$ holds and also in either case the witness for this (so either $x^{\alpha,\varepsilon} \in \text{set}(r_\varepsilon)$ or $B^{\alpha,\varepsilon} \in [\lambda]^{<\lambda}$).

For $\alpha < \beta$ we let $A_\alpha = \langle q^{\alpha,\xi} : \xi < \lambda \rangle$. Then by construction every $A_\alpha$ is a maximal antichain (with respect to $(Q, <_Q)$) in $Q^\ast$ and hence $X(A_\alpha) \in \mathcal{I}(Q)$. By hypothesis, $\bigcup_{\alpha < \beta} X(A_\alpha) \in \mathcal{I}(Q)$. Choose $r \in Q^\ast$ such that $\text{set}(r) \cap \bigcup_{\alpha < \beta} X(A_\alpha) = \emptyset$, thus

1. $\text{set}(r) \subseteq \bigcup \{\text{set}(q^{\alpha,\xi}) : \xi < \lambda\}$ for every $\alpha < \beta$. Let $r = r_\varepsilon$. Note that by (7), we must have
2. $\langle q^{\alpha,\xi}_\zeta : \xi < \lambda \rangle$ for every $\alpha < \beta$, $p_{\varepsilon+1} \Vdash \text{set}(r) \subseteq \bigcup_{\xi \in B^{\alpha,\varepsilon}} \text{set}(q^\alpha_\xi)$.

Indeed, otherwise we had $\alpha < \beta$ and $x^{\alpha,\varepsilon} \in \text{set}(r_\varepsilon)$ such that

$$p_{\varepsilon+1} \Vdash x^{\alpha,\varepsilon} \notin \bigcup_{\xi < \lambda} \text{set}(q^\alpha_\xi).$$

By (8) there is $\xi_0 < \lambda$ such that $x^{\alpha,\varepsilon} \in \text{set}(q^{\alpha,\xi_0})$. Letting $\mu > \max\{\varepsilon, \xi_0\}$ we have $p_\mu \leq p_{\varepsilon+1}$ and $p_\mu \Vdash q^{\alpha}_\xi = q^{\alpha,\xi_0}$, thus $p_\mu \Vdash x^{\alpha,\varepsilon} \in \text{set}(q^{\alpha,\xi_0})$, which is a contradiction.

As (9) holds for a dense set of $r \in Q^\ast$, we conclude that $p \Vdash \bigcup_{\alpha < \beta} \hat{X}_\alpha \in \mathcal{I}(Q)$. \qed
4 Small additivity and large cofinality - the antiamoeba

Definition 4.1 Let $2^{\aleph_0} = \lambda = \lambda^{<\lambda} = \kappa^+$ and $Q = (\mathbb{Q}, \dot{\mathbb{Q}}, \text{set}, Q^*, \perp)$ be a GTF$_1$. We say that $Q$ has a strong witness $W$ for $\text{add}(\mathcal{I}(Q)) \leq \kappa$ if there exists an orthogonal maximal antichain (w.r.t. $(Q, \leq)$) $\mathcal{Q}^* = \langle q^*_\varepsilon : \varepsilon < \lambda \rangle$ in $Q^*$ and for every $\iota < \kappa$ and $\varepsilon < \lambda$ some family $\mathcal{Q}^*_{\iota, \varepsilon} = \langle q^*_\iota, \varepsilon, \zeta : \zeta < \lambda \rangle$ in $Q^*$ below $q^*_\varepsilon$ such that $\mathcal{Q}^*_{\iota, \varepsilon}$ is predense (w.r.t. $(Q, \leq)$) below $q^*_\varepsilon$, hence

$$X_{\iota, \varepsilon} := \text{set}(q^*_\varepsilon) \setminus \bigcup \{\text{set}(q^*_\iota, \varepsilon, \zeta) : \zeta < \lambda\}$$

belongs to $\mathcal{I}(Q)$, but $Y_{\iota} := \bigcup_{\varepsilon < \lambda} X_{\iota, \varepsilon} \notin \mathcal{I}(Q)$. Then $W = (\mathcal{Q}^*, (\mathcal{Q}^*_{\iota, \varepsilon} : \iota < \kappa, \varepsilon < \lambda))$.

Definition 4.2 (1) Let $\chi > 2^{\aleph_0} = \lambda = \lambda^{<\lambda} = \kappa^+$ and $Q = (\mathbb{Q}, \dot{\mathbb{Q}}, \text{set}, Q^*, \perp)$ a GTF$_1$ with a strong witness $W$ for $\text{add}(\mathcal{I}(Q)) \leq \kappa$, and let $W = (\mathcal{Q}^*, (\mathcal{Q}^*_{\iota, \varepsilon} : \iota < \kappa, \varepsilon < \lambda))$ be as in Definition 4.1. We define a forcing notion $\mathbb{A}(Q, W, \chi)$ as follows ("$\mathbb{A}$" stands for "anti-amoebe"):

(A) (a) Conditions $p \in \mathbb{A}(Q, W, \chi)$ have the form $p = (u, \zeta, \tau, S, f) = (u_p, \zeta_p, \tau_p, S_p, f_p)$ where

(b) $u \in [\chi]^{<\kappa}$ and $\zeta < \lambda$,

(c) $\tau = \langle r_{\alpha, \varepsilon} : \alpha \in u, \varepsilon < \zeta \rangle$ and $\tau^{[\alpha]} := \langle r_{\alpha, \varepsilon} : \varepsilon < \zeta \rangle$ (for $\alpha \in u$) are such that every $r_{\alpha, \varepsilon}$ is a member of $Q^*$ below some $q^*_\varepsilon$ (from the strong witness)

(d) $S \subseteq \{\overline{\alpha} : \alpha \in u \text{ is increasing} \}$ and $|S| \leq \kappa$,

(e) $f : S \to \lambda$ is such that for every $\overline{\alpha}_1, \overline{\alpha}_2 \in S$

(a) if $f(\overline{\alpha}_1) = f(\overline{\alpha}_2)$ then $(\overline{\alpha}_1, \overline{\alpha}_2)$ is a $\Delta$-system pair, i.e. $\forall i, j < \kappa(\overline{\alpha}_1(i) = \overline{\alpha}_2(j) \Rightarrow i = j)$ and

(b) if $f(\overline{\alpha}_1) \neq f(\overline{\alpha}_2)$, then $|\text{ran}(\overline{\alpha}_1) \cap \text{ran}(\overline{\alpha}_2)| \leq 1$.

(B) The order on $\mathbb{A}(Q, W, \chi)$ is defined as follows: For $p_1, p_2 \in \mathbb{A}(Q, W, \chi)$ we declare $p_2 \leq p_1$ iff

(a) $u_{p_1} \subseteq u_{p_2}, \zeta_{p_1} \subseteq \zeta_{p_2}, \tau_{p_1} = \tau_{p_2} \setminus u_{p_1} \times \zeta_{p_1}, S_{p_1} \subseteq S_{p_2}, f_{p_1} \subseteq f_{p_2}$ and
(b) if \((\alpha, \varepsilon) \in (u_{p_2} \times \zeta_{p_2}) \setminus (u_{p_1} \times \zeta_{p_1}), \xi(p_2, \alpha, \varepsilon)\) is the unique \(\xi\) such that \(r^{p_2}_{\alpha, \varepsilon} \leq q_\varepsilon^*\) and \(\beta \in S_{p_1}, \iota < \kappa\) are such that \(f_{p_1}(\beta) = \xi(p_2, \alpha, \varepsilon)\) and \(\beta_\iota = \alpha\) (note that this implies \(\alpha \in u_{p_1}\) by \((A)(d)\), and by \((A)(e)(\alpha)\) \(\iota\) does not depend on \(\beta\)), then \(r^{p_2}_{\alpha, \varepsilon} \leq q_{r_{f_{p_1}(\beta)}^*}\) for some \(\zeta < \lambda\).

(2) Letting \(G_{AA}(Q, W, \chi)\) the canonical name for the \(AA(Q, W, \chi)\)-generic filter, for \(\alpha < \chi\) we let \(\dot{p}_\alpha = \{\dot{r}_{\alpha, \varepsilon} : \varepsilon < \lambda\}\) be the \(AA(Q, W, \chi)\)-name \(\bigcup \{r^\alpha_p : p \in G_{AA}(Q, W, \chi) \wedge \alpha \in u_p\}\) and \(X_\alpha = \mathbb{R} \setminus \{\text{ran}(\dot{r}_{\alpha, \varepsilon}) : \varepsilon < \lambda\}\).

Lemma 4.1 With the notation of Definition 4.2 the following statements are true:

(1) Every descending sequence in \(AA(Q, W, \chi)\) of length < \(\lambda\) has a largest lower bound.

(2) \(AA(Q, W, \chi)\) is not empty and for every \(r_* \in Q, \alpha_* < \chi\) and \(p_1 \in \AA(Q, W, \chi)\) there exists \(p_2 \in \AA(Q, \chi)\) such that

(a) \(p_2 \leq p_1\),
(b) \(\zeta_{p_1} < \zeta_{p_2}\) and \(\alpha_* \in u_{p_2}\),
(c) for some \(\varepsilon < \zeta_{p_2}\) we have that \(r^{p_2}_{\alpha_*, \varepsilon}\) and \(r_*\) are compatible;
(d) \(\forall \alpha < \chi : \models_{\AA(AA(Q, W, \chi))} \dot{p}_\alpha\) lists a predense subset of \(Q\).

(3) Suppose that \(p \in \AA(Q, W, \chi), \xi < \lambda, \beta \in ^\kappa \chi\) are such that \(\xi \notin \{\nu < \lambda : \exists (\alpha, \varepsilon) \in u_p \times \zeta_p, r^\alpha_{\alpha, \varepsilon} \leq q_\nu^* \} \cup \text{ran}(f_p)\) and, letting

\[ q := (u_p, \zeta_p, \tau_p, S_p \cup \{\beta\}, f_p \cup \{\langle \beta, \xi \rangle\}) \]

we have \(q \in \AA(Q, W, \chi)\) and hence \(q \leq p\), then

\[ q \models_{\AA(AA(Q, W, \chi))} \bigcup_{\iota < \kappa} \{\text{set}(q^*_{\iota}) \setminus \{\text{set}(\dot{r}_{\beta, \varepsilon}) : \varepsilon < \lambda\} \} \notin I(Q) \]

and hence \(q \models_{\AA(AA(Q, W, \chi))} \bigcup_{\iota < \kappa} X_{\beta_\iota} \notin I(Q)\).

Remark 4.1 Note that in (3), for \(q \in \AA(Q, W, \chi)\) to hold we only need that \(\beta\) is increasing and for every \(\overline{\pi} \in S_p\) we have \(|\text{ran}(\overline{\pi}) \cap \text{ran}(\beta)| \leq 1\).
Proof: (1) Given a descending chain \( \langle p_\alpha : \alpha < \mu < \lambda \rangle \) in \( \mathcal{A}(Q,W) \) we define \( \xi, \eta \) such that for every \( \beta < \mu \), \( \xi < \eta < \lambda \) and \( \eta \) is the largest lower bound.

There is no pair of antichain, and fix \( \xi = \xi, \alpha, \eta = \eta, \beta < \mu < \lambda \) such that \( \eta < \lambda \) is the largest lower bound.

Next we construct \( p_{\alpha, \beta} \) such that \( (\xi, \eta) \) is a maximal antichain below \( q_{\xi, \eta} \) and \( r_{\alpha, \beta} \) is compatible for every \( \varepsilon < \zeta_p \). We can easily define \( r_{\alpha, \beta} \) such that \( t_{\alpha, \beta} = q_{\xi, \eta} \) such that \( f_{\alpha, \beta} = f_{\alpha, \beta} \), as otherwise we let \( p_p = f_{\alpha, \beta} \).

We can easily define \( r_{\alpha, \beta} \) such that \( t_{\alpha, \beta} = q_{\xi, \eta} \) such that \( f_{\alpha, \beta} = f_{\alpha, \beta} \), as otherwise we let \( p_p = f_{\alpha, \beta} \).

(2) \( \mathcal{A}(Q,W) \) is not empty as \( (\emptyset, \emptyset, \emptyset, \emptyset) \) is an element. Let us check density. We do it in two steps. First we find \( p_{\alpha, \beta} \) such that \( \zeta_p < \zeta_p \). We can choose \( \xi < \zeta_p < \zeta_p \) such that \( \eta < \lambda \) is the largest lower bound. We also assume that \( r_{\alpha, \beta} \) and \( s_{\alpha, \beta} \) are compatible for every \( \varepsilon < \zeta_p \), as otherwise we let \( p_p = f_{\alpha, \beta} \).

Finally we construct \( p_{\alpha, \beta} \) such that \( (\xi, \eta) \) is a maximal antichain below \( q_{\xi, \eta} \). We also assume that \( r_{\alpha, \beta} \) and \( r_{\alpha, \beta} \) are compatible for every \( \varepsilon < \zeta_p \), as otherwise we let \( p_p = f_{\alpha, \beta} \).

We can easily define \( r_{\alpha, \beta} \) such that \( t_{\alpha, \beta} = q_{\xi, \eta} \) such that \( f_{\alpha, \beta} = f_{\alpha, \beta} \), as otherwise we let \( p_p = f_{\alpha, \beta} \).

Case 1 There exist \( \beta, \varepsilon \) such that \( f_{\alpha, \beta} = \xi \) and \( f_{\alpha, \beta} = \alpha \).

Note that by Definition 4.2 (A)(e) \( \varepsilon \) is uniquely determined. As \( t_{\alpha, \beta} \) is a maximal antichain below \( q_{\xi, \eta} \), there exists \( \zeta < \lambda \) such that \( q_{\xi, \eta} \) and \( r \) are compatible. We define \( p_p \) such that \( t_{\alpha, \beta} = q_{\xi, \eta} \) such that \( f_{\alpha, \beta} = f_{\alpha, \beta} \), as otherwise we let \( p_p = f_{\alpha, \beta} \).

We construct \( p_p \) as in Case 1 except that \( \varepsilon < \zeta_p \) can be chosen randomly.

(3) Given \( q' \leq q \) and \( (\alpha, \varepsilon) \in u_{\alpha, \varepsilon} \times \zeta_p \) such that \( \alpha = \beta \), for some \( \varepsilon < \zeta_p \) and \( r_{\alpha, \beta} \leq q_{\xi, \eta} \), then, by Definition 4.2(1)(B)(b), for some \( \zeta < \lambda \) we have \( r_{\alpha, \beta} \leq q_{\xi, \eta} \). By (2)(c) we conclude

\[ q \models \mathcal{A}(Q,W) \forall \varepsilon < \lambda (\forall \beta, \varepsilon \leq q_{\xi, \eta} \rightarrow \exists \zeta < \lambda (\forall \beta, \varepsilon \leq q_{\xi, \eta}) \land \forall \zeta < \lambda (\exists \varepsilon < \lambda \forall \beta, \varepsilon \leq q_{\xi, \eta} \rightarrow q_{\xi, \eta} \leq q_{\xi, \eta}).]
As \( \iota < \kappa \) was arbitrary, by Definition 4.1 we conclude that (3) is true. \( \Box \)

**Lemma 4.2** Suppose \( \chi > 2^{\aleph_0} = \lambda = \lambda^{\kappa} = \kappa^+ \), \( Q \), strong witness \( W \) for add(\( I(Q) \)) \( \leq \kappa \) and \( \mathbb{A}(Q,W,\chi) \) are as in Definition 4.2. If \( \langle p_\alpha : \alpha < \kappa^+ \rangle \) is a family of conditions in \( \mathbb{A}(Q,W,\chi) \) there exist a club \( E \subseteq \lambda^+ \) and a regressive function \( h : E \cap S_\lambda^+ \to \lambda^+ \) such that for every \( w \subseteq E \cap S_\lambda^+ \) of cardinality at most \( \kappa \), if \( h \upharpoonright w \) is constant then \( \langle p_\alpha : \alpha < w \rangle \) has a largest lower bound in \( \mathbb{A}(Q,W,\chi) \).

**Proof:** Let \( \langle p_\alpha : \alpha < \kappa^+ \rangle \) be given. We write \( p_\alpha = (u_\alpha, \zeta_\alpha, r_\alpha, S_\alpha, f_\alpha), r_\alpha = \langle r_\alpha^{\gamma, \varepsilon} : \gamma < \zeta_\alpha, \varepsilon < \zeta_\alpha \rangle, r_\alpha^{\gamma, \varepsilon} = \langle r_\alpha^{\gamma, \varepsilon} : \varepsilon < \zeta_\alpha \rangle \). For every \( \alpha < \kappa^+ \) let \( g_\alpha : otp(u_\alpha) \to u_\alpha \) be the unique increasing surjection. We define a binary relation \( R_\cdot \) on \( \chi_1^+ \) by letting \( \alpha R_\cdot \beta \iff \)

1. \( otp(u_\alpha) = otp(u_\beta), otp(\alpha \cap u_\alpha) = otp(\beta \cap u_\beta), \zeta_\alpha = \zeta_\beta, \) and
2. \( g_\beta \circ g_\alpha^{-1} \) is an isomorphism from \( p_\alpha \) onto \( p_\beta \), i.e. \( \gamma_1 \)
   - \( (\alpha) \) if \( g_\beta \circ g_\alpha^{-1}(\gamma_1) = \gamma_2 \), then \( r_\beta^{\gamma_1} = r_\alpha^{\gamma_2} \) and
   - \( (\beta) \) if \( \gamma_1 : \iota < \kappa \in f^*(\lambda^+) \), then \( \gamma_1 \in S_\alpha \) iff \( g_\beta \circ g_\alpha^{-1}(\gamma_1) := \langle g_\beta \circ g_\alpha^{-1}(\gamma_i) : \iota < \kappa \rangle \in S_\beta \) and \( f_\alpha(\gamma_1) = f_\beta(g_\beta \circ g_\alpha^{-1}(\gamma_1)) \).

It is easy to check that \( R_\cdot \) is an equivalence relation and that by our assumption \( \lambda = \lambda^{\kappa} E_\cdot \) has \( \lambda \) many equivalence classes.

For every \( \alpha < \lambda^+ \) let \( U_{\alpha} = \bigcup \{ u_\beta : \beta < \alpha \}, v_\alpha = \{ \iota < otp(u_\alpha) : g_\alpha(\iota) \in U_{\alpha} \} \).

We define the function \( h_1 : \lambda^+ \to \lambda^+ \) by \( h_1(\alpha) = \min \{ \beta \in \text{Ord} : \beta \geq \alpha \land \forall \gamma_1 < \lambda^+ (\text{ran}(g_{\gamma_1} \upharpoonright v_{\gamma_1}) \subseteq U_{\alpha} \Rightarrow \exists \gamma_2 < \beta (\gamma_1 R_\cdot \gamma_2 \land g_{\gamma_2} \upharpoonright v_{\gamma_2} = g_{\gamma_1} \upharpoonright v_{\gamma_1})) \} \).

Note that as \( R_\cdot \) has \( \lambda \) equivalence classes and by our assumption \( \lambda^{< \lambda} = \lambda \) and \( |U_{< \alpha}|^{< \lambda} \leq \lambda^{< \lambda} \), the function \( h_1 \) maps indeed into \( \lambda^+ \).

Let \( E = \{ \gamma < \lambda^+ : \gamma \) is a limit ordinal and \( \forall \alpha < \gamma h_1(\alpha) < \gamma \} \). Thus \( E \) is a club on \( \lambda^+ \).

Finally we define our desired function \( h : E \cap S_\lambda^+ \to \lambda^+ \) by letting \( h(\gamma) = \min \{ \delta < \lambda^+ : g_\delta \upharpoonright v_{\delta} = g_\gamma \upharpoonright v_{\gamma} \land R_\cdot \delta \} \). Then \( h(\gamma) \leq \gamma \) holds trivially. But by construction even \( h(\gamma) < \gamma \), hence \( h \) is regressive. Indeed, by definition \( \text{ran}(g_\gamma \upharpoonright v_{\gamma}) \subseteq U_{< \gamma} \). As \( |\text{ran}(g_\gamma \upharpoonright v_{\gamma})| < \lambda \) and \( \text{cf}(\gamma) = \lambda \) we can find \( \delta_1 < \gamma \) such that \( \text{ran}(g_{\delta_1} \upharpoonright v_{\delta_1}) \subseteq U_{< \delta_1} \). Since \( h_1(\delta_1) < \gamma \) there exists \( \delta_2 < \gamma \) such that \( g_{\delta_2} \upharpoonright v_{\delta_2} = g_{\gamma} \upharpoonright v_{\gamma} \) and \( \delta_2 R_\cdot \gamma \), and hence \( h(\gamma) \leq \delta_2 < \gamma \).
Suppose now that \( w \subseteq E \cap S^1_\chi \), \( |w| \leq \kappa \) and \( h \upharpoonright w \) is constant. By definition of \( h \), \( g_\alpha \upharpoonright v_\alpha = g_\beta \upharpoonright v_\beta =: g^* \) for any \( \alpha, \beta \in w \). By definition of \( v_\alpha \) we conclude that \( \langle u_\alpha : \alpha \in w \rangle \) is a \( \Delta \)-system with root \( \text{ran}(g^*) \) and \( g_\beta \circ g_\alpha^{-1} \) is the identity on \( \text{ran}(g^*) \) for any \( \alpha, \beta \in w \).

Moreover, by definition of \( R_* \) we have \( \zeta_\alpha = \zeta_\beta =: \zeta_w \), and if \( \gamma \in u_\alpha \cap u_\beta \), hence \( \gamma \in \text{ran}(g^*) \) and \( g_\beta \circ g_\alpha^{-1}(\gamma) = \gamma \), then \( f^\gamma_{\zeta_w} = f^\gamma_{\zeta_w} \).

Now we define \( q = q_w \in \mathbb{A}_w(Q, W, \chi) \) as follows:

(a) \[ u_q = \bigcup \{ u_\alpha : \alpha \in w \}. \]

Note that \( |u_q| \leq \kappa \) as required, as \( |w| \leq \kappa \).

(b) \[ \zeta_q = \zeta_w. \]

(c) \[ \tau_q = \{ f^\alpha_{\gamma, \varepsilon} : \alpha \in w, \gamma \in u_\alpha, \varepsilon < \zeta_w \}. \]

Note that by the remark above this is well defined (i.e. \( f^\alpha_{\gamma, \varepsilon} = f^\alpha_{\gamma, \varepsilon} \) if \( \gamma \in u_\alpha \cap u_\beta \)).

(d) \[ S_q = \bigcup \{ S_\alpha : \alpha \in w \}. \]

Again \( |S_q| \leq \kappa \) as required, by \( |w| \leq \kappa \).

(e) \[ f_q = \bigcup \{ f_\alpha : \alpha \in w \}. \]

Note that \( f_q \) is a function. Indeed, if \( \tau \in S_\alpha \cap S_\beta \) for \( \alpha, \beta \in w \) then \( \text{ran}(\tau) \subseteq u_\alpha \cap u_\beta = \text{ran}(g^*) \). Since \( g_\beta \circ g_\alpha^{-1} \upharpoonright \text{ran}(g^*) \) is the identity, by (b)/(\( \beta \)) in the definition of \( R_* \) we have \( f_\alpha(\tau) = f_\beta(\tau) \).

Let us check (A)(e) from Definition 4.2(1). Let \( \alpha, \beta \in w, \alpha \neq \beta \), and \( \tau^1 \in S_\alpha, \tau^2 \in S_\beta \). Let \( \tau^3 := g_\alpha \circ g_\alpha^{-1}(\tau^1) \), thus \( \tau^3 \in S_\beta, f_\alpha(\tau^1) = f_\beta(\tau^3) \), and \( (\tau^1, \tau^3) \) is a \( \Delta \)-system pair (see Definition 4.2(1)(e)(\( \alpha \))). If \( \tau^2 = \tau^3 \), hence \( f_q(\tau^1) = f_q(\tau^2) \), we are done. Now suppose \( \tau^2 \neq \tau^3 \). Note that \( \{ (\tau, \nu) \in \kappa^2 : \gamma^1 = \gamma^3 \} \subseteq \{ (\tau, \nu) \in \kappa^2 : \gamma^3 = \gamma^3 \} \). If \( f_\alpha(\tau^1) = f_\beta(\tau^3) \), hence \( f_\beta(\tau^2) = f_\beta(\tau^3) \) and thus \( (\tau^2, \tau^3) \) is a \( \Delta \)-system pair, we are done. Otherwise \( f_\alpha(\tau^1) \neq f_\beta(\tau^3), \) hence \( f_\beta(\tau^2) \neq f_\beta(\tau^3) \) and thus \( |\text{ran}(\tau^2) \cap \text{ran}(\tau^3)| \leq 1 \). But this implies \( |\text{ran}(\tau^1) \cap (\tau^2, \tau^3)| \leq 1 \).

Finally, it is straightforward to verify \( q \leq p_\alpha \) for every \( \alpha \in w \). That \( q \) actually is the largest lower bound is also clear. \( \square \)
5 Different cofinalities if amoeba and anti-amoeba interact

Lemma 5.1 Suppose $\chi > 2^\aleph_0 = \lambda = \lambda^\leq \kappa^+$. $Q$, strong witness $W$ for $\text{add}(\mathcal{I}(Q))$ $\leq \kappa$ and $\mathbb{AA}(Q, W, \chi)$ are as in Definition 4.2. Moreover let $d = (\lambda, \text{CLUB}_{\lambda^+}, \varepsilon, \kappa, [S^\lambda_\alpha]^\kappa_\beta)$ where $\varepsilon < \lambda$ (so $d$ is a c.c.-parameter). If $\dot{P}$ is an $\mathbb{AA}(Q, W, \chi)$-name for a forcing such that $\Vdash_{\mathbb{AA}(Q, W, \chi)} \dot{P}$ satisfies $\ast_d$, then

$$\Vdash_{\mathbb{AA}(Q, W, \chi)} \text{cof}(\mathcal{I}(Q)) \geq \chi.$$ 

Proof: Towards a contradiction we assume that there are $p_s \in \mathbb{AA}(Q, W, \chi) \ast \dot{P}$, cardinal $\alpha_s < \chi$ and a family $\langle \dot{B}_\alpha : \alpha < \alpha_s \rangle$ of $\mathbb{AA}(Q, W, \chi) \ast \dot{P}$-names such that

$$p_s \Vdash_{\mathbb{AA}(Q, W, \chi) \ast \dot{P}} " \langle \dot{B}_\alpha : \alpha < \alpha_s \rangle \text{ is a cofinal sequence in } \mathcal{I}(Q)."$$

We must have $\alpha_s > \lambda$. For $\alpha < \chi$ we can find $p_\alpha \in \mathbb{AA}(Q, W, \chi) \ast \dot{P}$ below $p_s$ and $\gamma(\alpha) < \alpha_s$ such that

$$(a) \ p_\alpha \Vdash_{\mathbb{AA}(Q, W, \chi) \ast \dot{P}} \dot{X}_\alpha \subseteq \dot{B}_{\gamma(\alpha)},$$

where $\dot{X}_\alpha$ is the $\mathbb{AA}(Q, W, \chi)$-name as in Definition 4.2(2). We can find some unbounded $U \subseteq \alpha_3^+$ and $\gamma_3 < \alpha_s$ such that $\gamma(\alpha) = \gamma_3$ for every $\alpha \in U$. By renumbering we may assume $U = \alpha_3^+$. In the sequel we only make use of $\langle p_\alpha : \alpha < \lambda^+ \rangle$. Let $p_\alpha = (p_\alpha^1, p_\alpha^2)$ where $p_\alpha^1 \in \mathbb{AA}(Q, W, \chi)$ and $\Vdash_{\mathbb{AA}(Q, W, \chi)} p_\alpha^2 \in \dot{P}$.

In $V^{\mathbb{AA}(Q, W, \chi)}$ we consider the game $G(\dot{P}, d)$ (see Definition 2.1), for which, by assumption, player I has a winning strategy. Let $((\langle \dot{f}_i^\lambda : i < \lambda^+ \rangle, \dot{f}_\zeta), \langle \dot{s}_i^\lambda : i < \lambda^+ \rangle, \zeta < \varepsilon)$ be the play described in Remark 2.1(1) with $\langle \dot{s}_i^0 : i < \lambda^+ \rangle = \langle \dot{p}_i^0 : i < \lambda^+ \rangle$. As player I wins this play, there exists a $\mathbb{AA}(Q, W, \chi)$-name $E_2$ for a club of $\lambda^+$ as in the winning rule for $G(\dot{P}, d)$. As by Lemma 4.2 $\mathbb{AA}(Q, W, \chi)$ has the $\lambda^+$-c.c., wlog we may assume $E_2 = E_2 \in V$.

By $\lambda$-closedness of $\mathbb{AA}(Q, W, \chi)$, for every $\alpha < \lambda^+$ we can find $p_\alpha^3 \in \mathbb{AA}(Q, W, \chi)$ below $p_\alpha^1$ and $g_\alpha : \varepsilon \to \lambda^+$ in $V$ such that

$$(b) \ p_\alpha^3 \Vdash_{\mathbb{AA}(Q, W, \chi)} " \langle \dot{f}_\zeta(\alpha) : \zeta < \varepsilon \rangle = g_\alpha."$$

By Lemma 4.1(2)(b), we may assume that $\alpha \in u_{p_\alpha^3}$ (see Definition 4.2(1)(A)(a)). Applying Lemma 4.2 to $\langle p_\alpha^3 : \alpha < \lambda^+ \rangle$ we can find a club $E_1 \subseteq \lambda^+$ and a regressive function $f_1 : E_1 \cap S^\lambda_\alpha \to \lambda^+$ as there.
As in Remark 2.1(1) we have a regressive function \( f_2 : S_\lambda^+ \to \lambda^+ \) such that \( \text{ran}(g_\alpha) \) is bounded by \( f_2(\alpha) \) for every \( \alpha \in S_\lambda^+ \).

We shall use notation and proof of Lemma 4.2 below. As \( E_1 \cap E_2 \cap S_\lambda^+ \) is a stationary subset of \( \lambda^+ \), there are ordinals \( \gamma_1, \gamma_2 \) such that the set

\[
S := \{ \alpha < \lambda^+ : \alpha \in E_1 \cap E_2 \cap S_\lambda^+ \wedge f_1(\alpha) = \gamma_1 \wedge f_2(\alpha) = \gamma_2 \}
\]

is stationary. By \( \lambda = \lambda^\chi \) we can find some unbounded set \( V \subseteq S \) and \( g_* \) such that \( g_\alpha = g_* \) for every \( \alpha \in V \).

By the proof of Lemma 4.2 we have that

\[
\begin{align*}
&\text{(c)} \text{ all } \alpha \in S \text{ are } R_\ast\text{-equivalent,} \\
&\text{(d)} \langle u_{\beta_\alpha} : \alpha \in S \rangle \text{ is a } \Delta\text{-system (hence } \alpha \in u_{\beta_\alpha} \setminus \bigcup\{u_{\beta_\alpha} : \beta \in S \wedge \beta \neq \alpha\} \text{ for all } \alpha \in S \rangle.
\end{align*}
\]

We choose \( w \subseteq V \) such that \( \text{otp}(w) = \kappa \) and let \( \bar{\alpha}^\ast \) list \( w \) in increasing order. By the proof of Lemma 4.2 we know that \( \{p_\alpha^\ast : \alpha \in w\} \) has a largest lower bound \( p^w \). We define \( p^1 \leq p^w \) in \( \text{AA}(Q, W, \chi) \) as follows: \( u_{p^1} = u_{p^w}, \) \( \zeta_{p^1} = \zeta_{p^w}, \) \( \bar{r}_{p^1} = \bar{r}_{p^w}, \) \( S_{p^1} = S_{p^w} \cup \{\bar{\alpha}^\ast\}, \) and \( f_{p^1} = f_{p^w} \cup \{(\bar{\alpha}^\ast, \xi)\}, \) where \( \xi < \lambda \) is chosen such that no member of \( \bar{r}_{p^w} \) is below \( q_{p^w}^\xi, \) and hence \( \forall \beta \in u_{p^w} \forall \varepsilon < \zeta_{p^w} \) \( \text{set}(r_{p^w}^\beta) \cap \text{set}(q_{p^w}^\xi) = \emptyset, \) and moreover \( \xi \notin \text{ran}(f_{p^w}). \) By construction (see (c)) we have \( |\text{ran}(\bar{\alpha}^\ast) \cap \text{ran}(\bar{\beta})| \leq 1 \) for every \( \bar{\beta} \in S_{p^w} \) and hence \( p^1 \) is as desired (see Remark 4.1).

Let \( \bar{\alpha}_\xi = \langle \alpha_\xi : \xi < \kappa \rangle. \) By Lemma 4.1(3) we conclude

\[
p^1 \not\Vdash_{\text{AA}(Q, W, \chi)} \bigcup_{\xi < \kappa} \bar{X}_\alpha, \notin I(Q).
\]

By construction (especially the definition of \( p_\alpha^3 \) and \( g_\alpha \) in (b)), there exists some \( \text{AA}(Q, W, \chi)\)-name \( p^2 \) such that \( \Vdash_{\text{AA}(Q, W, \chi)} p^2 \in \bar{P} \) and \( p^1 \Vdash_{\text{AA}(Q, W, \chi)} "p^2 \text{ is a lower bound of } \{p_\alpha^2 : \alpha \in w\}". \) But now we have a contradiction, as \( (p^1, p^2) \leq p_* \) and by (a)

\[
(p^1, p^2) \not\Vdash_{\text{AA}(Q, W, \chi)} \bigcup_{\xi < \kappa} \bar{X}_\alpha, \subseteq \bar{B}_{\gamma_*}.
\]

\[\square\]

As a conclusion of what we proved so far we obtain the following:

**Theorem 5.1** Suppose that \( 2^{\aleph_0} = \lambda = \lambda^\chi = \kappa^+ < \mu = \text{cf}(\mu) < \chi = \chi^\chi. \) Moreover we assume the following:
(1) $Q_0 = (Q_0, \dot{\zeta}_0, \text{set}_0, Q^*_0, \bot_0)$ is a GTF$_1$ such that $Q_0$ has a strong witness $W$ for $\text{add}(\mathcal{I}(Q_0)) \leq \kappa$.

(2) $Q_1 = (Q_1, \dot{\zeta}_1, \text{set}_1, Q^*_1, \bot_1)$ is a GTF$_1$ such that $\text{add}(\mathcal{I}(Q_1)) = \lambda$.

(3) Let $\dot{P}$ be the $\mathcal{A}(\mathcal{Q}_0, W, \chi)$-name of the limit of a $(< \lambda)$-support iteration $(\dot{P}_\alpha, \dot{Q}_\beta : \alpha < \mu, \beta < \mu)$ in $\mathcal{V}^{\mathcal{A}(\mathcal{Q}_0, W, \chi)}$, where $\dot{Q}_\beta$ denotes $\mathcal{A}_1(\mathcal{Q}_1)$ in $\mathcal{V}^{\mathcal{A}(\mathcal{Q}_0, W, \chi)}$.

Then the following hold:

(4) $\mathcal{A}(\mathcal{Q}_0, W, \chi) * \dot{P}$ is $\lambda$-closed and $\lambda^+$.c.c.

(5) $\mathcal{V}^{\mathcal{A}(\mathcal{Q}_0, W, \chi)} * \dot{P} \models 2^{\aleph_0} = \lambda \land \text{cof}(\mathcal{I}(Q_0)) = 2^\lambda = \chi \land \text{cof}(\mathcal{I}(Q_1)) = \mu$.

### 6 Application to classical tree forcings

Here we study the well-known classical tree forcings Sacks, Silver, Laver and Miller. We abreviate them by $Sa$, $Si$, $La$ and $Mi$, respectively. We shall show that under certain assumptions they are GTF$_1$ in the sense of Definition 3.1. Then we shall explain for which pairs $(Q_0, Q_1)$ of these the assumptions of Theorem 5.1 are known to be consistent, hence we can get the consistency of $\text{cof}(\mathcal{I}(Q_0)) > \text{cof}(\mathcal{I}(Q_1))$.

**Theorem 6.1** (1) Suppose $d = 2^{\aleph_0}$. Then both, Sacks and Silver forcing, can be considered as GTF$_1$'s.

(2) Suppose $b = 2^{\aleph_0}$. Then both, Laver and Miller forcing, can be considered as GTF$_0$'s.

**Proof:** It is well-known that for every $Q \in \{Sa, Si, La, Mi\}$, every $p \in Q$ has continuum many extensions such that any two of them have no common infinite branch.

(1) Let $Q \in \{Sa, Si\}$. Let $\dot{G}_Q$ be the canonical $Q$-name for the generic filter, let $\dot{\zeta}_Q = \bigcap \dot{G}_Q$, i.e. $\dot{\zeta}_Q$ denotes the Sacks, Silver real, respectively. Let set$_Q(p) = [p]$, let $Q^*$ be the set of all $p \in Q$ such that $[p]$ is nowhere dense, and let $p \bot_Q q$ mean $[p] \cap [q] = \emptyset$. We claim that $(Q, \dot{\zeta}_Q, \text{set}_Q, Q^*, \bot_Q)$ is GTF$_1$. In fact, (1)(a), (b), (c) and (e)(a), (b) are obvious, for (c)(γ) we use the well-known fact that a Sacks or Silver real determines its generic filter.
(1)(d) follows from the remark at the beginning of this proof. Nontrivial are (e)(γ)_1 and (δ)_1. For these we apply the results in [7] (for Sa) and [16] (for Si) that every maximal antichain in Sa or Si that consists of nowhere dense trees must have size at least 9. Then (e)(γ)_1 and (δ)_1 follow easily from our assumption, the remark at the beginning of this proof and the fact that if p, q are incompatible Sacks or Silver trees, then [p] ∩ [q] is countable.

(2) For Q ∈ {La, Mi} we apply the base matrix tree from [1]. This is a family \( \langle A_\alpha : \alpha < b \rangle \) such that every \( A_\alpha \) is a mad family in \( [\omega]^\omega \) of size continuum, \( A_\beta \) refines \( A_\alpha \) (i.e. \( \forall b \in A_\beta \exists a \in A_\alpha \ b \subseteq^* a \)) for every \( \alpha < \beta < b \), and \( \bigcup_{\alpha < b} A_\alpha \) is dense in \( ([\omega]^\omega, \subseteq) \). Actually, by an easy modification of its construction we can achieve the following:

\((*)\) for every sequence \( \langle a_n : n < \omega \rangle \) in \( [\omega]^\omega \) there is \( \alpha < b \) and a sequence \( \langle b_n : n < \omega \rangle \) in \( A_\alpha \) such that \( \forall n b_n \subseteq a_n \).

Now we let \( La^* \) consist of all \( p \in La \) with the property that there exists \( \alpha < b \) such that for every \( \sigma \in p \) extending \( \text{stem}(p) \) we have \( \text{succ}_p(\sigma) \in A_\alpha \), where \( \text{succ}_p(\sigma) = \{ n < \omega : \sigma \upharpoonright n \in p \} \). If \( \sigma \not\in p \) we define \( \text{succ}_p(\sigma) = \emptyset \). As for (1) we let \( \zeta_{La} \) denote the Laver real, set \( \text{set}_{La}(p) = [p] \), and \( p \perp_{La} q \) mean \( [p] \cap [q] = \emptyset \). We claim that \( (La, \zeta_{La}, \text{set}_{La}, La^*, \perp_{La}) \) is \( \text{GTF}_0 \). Let us check Definition 3.1(1): (b) follows easily from property (\(*)\) of the base tree matrix. (c) is clear or well-known. (d) holds by the remark at the beginning of this proof. Nontrivial are (e)(γ) and (δ). Let \( \beta < 2^{80} \) and \( \langle p_\alpha : \alpha < \beta \rangle \) a sequence in \( La^* \). The set

\[ S = \{ \text{succ}_{p_\alpha}(\sigma) : \text{stem}_{p_\alpha} \subseteq \sigma \in p_\alpha \land \alpha < \beta \} \]

has cardinality \( < 2^{80} \) and is contained in the base matrix tree. As \( A_0 \) has size \( 2^{80} \) and the base matrix is a tree with respect to \( \supseteq^* \), there exists \( a \in A_0 \) such that \( a \cap b \) is finite for every \( b \in S \). Let \( p \in La^* \) be the tree with empty stem and \( \text{succ}_p(\sigma) = a \) for every \( \sigma \in p \). Then clearly \( p \) is incompatible with every \( p_\alpha \). We need the following claim which is folklore wisdom:

**Claim 2** Let \( \langle p_\alpha : \alpha < \beta < b \rangle \) be a sequence in \( La \). If \( p \in La \) is such that \( p \) is incompatible (w.r.t. \( (La, \leq) \)) with \( p_\alpha \) for every \( \alpha < \beta \), then there exists \( q \leq p \), \( q \in La \), such that \( \text{stem}(p) = \text{stem}(q) \) and \( [p_\alpha] \cap [q] = \emptyset \) for every \( \alpha < \beta \).

**Proof:** Fix \( \alpha < \beta \). We define a rank function \( \text{rk}_\alpha \) on \( p^- := \{ \sigma \in p : \text{stem}(p) \subseteq \sigma \} \) as follows:

\[ \text{rk}_\alpha(\sigma) = 0 \text{ iff } \text{succ}_p(\sigma) \cap \text{succ}_{p_\alpha}(\sigma) \text{ is finite,} \]
\( \text{rk}_\alpha(\sigma) = \nu \) if \( \nu \in \text{Ord} \) is minimal such that for all except finitely many \( n \in \text{succ}_p(\sigma) \cap \text{succ}_p(\sigma) \text{rk}_\alpha(\sigma \setminus n) < \nu \).

If \( \sigma \) gets no ordinal rank we define \( \text{rk}_\alpha(\sigma) = \infty \).

It is clear that as \( p \) and \( p_\alpha \) are incompatible, every \( \sigma \in p^- \) has an ordinal rank. We define \( f_\alpha : p^- \to \omega \) as follows: If \( \text{rk}_\alpha(\sigma) = 0 \) let \( n = \sup(\text{succ}_p(\sigma) \cap \text{succ}_p(\alpha) : \text{rk}_\alpha(\sigma \setminus m) \geq \text{rk}_\alpha(\sigma)) \) otherwise. Now let \( f_\alpha(\sigma) = n + 1 \). It can easily be checked that if \( g(\sigma) \geq f_\alpha(\sigma) \) for almost all \( \sigma \in p^- \), then, if we prune \( p \) using \( g \), i.e. for every \( \sigma \in p^- \) deleting everything above \( \sigma \setminus m \) for \( m < g(\sigma) \), we obtain a Laver tree \( q \leq p \) with \( [p_\alpha] \cap [q] = \emptyset \). But by \( \beta < b \) we can get \( g \) like this for every \( \alpha < \beta \).

Continuing with the proof of (e)(\( \gamma \)), by the claim and as \( \text{La}^* \) is dense we can find \( q \in \text{La}^* \) with \( q \leq p \) and \( [p_\alpha] \cap [q] = \emptyset \) for every \( \alpha < \beta \), as desired. These arguments also prove (e)(\( \delta \)).

For \( Mi \), analogous arguments work.

**Theorem 6.2**

(1) Suppose \( Q \in \{ Sa, Si \} \). Then \( \text{add}(\mathcal{I}(Q)) \leq b \) holds.

(2) Suppose \( 2^{\aleph_0} = b \) and \( Q \in \{ La, Mi \} \). Then \( \text{add}(\mathcal{I}(Q)) \leq h \).

**Proof:** Let \( \kappa(Q) \) the least cardinal \( \kappa \) such that forcing with \( Q \) changes the cofinality of \( (2^{\aleph_0})^V \) to \( \kappa \).

(1) Simon [12] has proved \( \kappa(Sa) \leq b \). In [8], \( \text{add}(\mathcal{I}(Sa)) \leq \kappa(Sa) \) is proved under the assumption that \( 2^{\aleph_0} \) is regular. In [7] it is proved that this assumption is not needed.

In [16], \( \text{add}(\mathcal{I}(Si)) \leq b \) is proved directly.

(2) In [6], \( \kappa(Q) \leq b \) has been shown for \( Q \in \{ La, Mi \} \). Similarly as in [8] for \( Sa \), one can prove \( \text{add}(\mathcal{I}(Q)) \leq \kappa(Q) \) for \( Q \in \{ La, Mi \} \), provided that \( 2^{\aleph_0} = b \) holds. Actually, for \( Q = Mi, d = 2^{\aleph_0} \) suffices (see [9], Corollary 13).

**Corollary 6.1**

Suppose \( Q_0 \in \{ Sa, Si, La, Mi \} \) is such that \( \text{add}(\mathcal{I}(Q_0)) = 2^{\aleph_0} \). Then the following are true:

(1) Every \( Q \in \{ Sa, Si, La, Mi \} \) is GTF\(_1\) (La and Mi are even GTF\(_0\)).

(2) If \( Q_1 \in \{ Sa, Si, La, Mi \} \) is such that \( \text{add}(\mathcal{I}(Q_1)) \leq \kappa < 2^{\aleph_0} \), then there exists a strong witness for this (see Definition 4.1).
Proof: (1) follows from Theorems 6.1 and 6.2. (2) follows from (1) and the homogeneity of the classical tree forcings.

The following theorem collects all the cases for which the consistency of add($I(Q_0)$) < add($I(Q_1)$) is known, where $Q_0, Q_1 \in \{Sa, Si, La, Mi\}$.

**Theorem 6.3** If ZF is consistent, then the following statements are consistent with ZFC + $2^{\aleph_0} = \aleph_2 = \aleph_2^{\aleph_1}$:

1. $\text{add}(I(Si)) < \text{add}(I(Sa))$,
2. $\forall Q \in \{La, Mi\} \text{add}(I(Sa)) < \text{add}(I(Q))$,
3. $\forall Q \in \{La, Mi\} \text{add}(I(Si)) < \text{add}(I(Q))$.

Proof: (1) In [10], an amoeba forcing for $Sa$ with the Laver property has been constructed. If this forcing is iterated $\aleph_2$ times with countable supports, a model for $\text{cov}(\mathcal{M}) < \text{add}(I(Sa))$ is obtained (where $\mathcal{M}$ is the meager ideal). In [14], $\text{add}(I(Si)) \leq \text{cov}(\mathcal{M})$ has been proved in ZFC.

(2) In [6], it has been shown that MA implies $\text{add}(I(Q)) = 2^{\aleph_0}$ for both $Q \in \{La, Mi\}$. In [8], and independently in [17], it has been shown that MA does not imply $\text{add}(I(Sa)) = 2^{\aleph_0}$, i.e. a model for MA + $\text{add}(I(Sa)) = \aleph_1 < 2^{\aleph_0} = \aleph_2$ is constructed.

(3) In [4] it has been shown that MA does not imply $\text{add}(I(Si)) = 2^{\aleph_0}$, i.e. a model for MA + $\text{add}(I(Si)) = \aleph_1 < 2^{\aleph_0} = \aleph_2$ is constructed.

Alternatively one can use the models in [13], where amoebas for $La$ and $Mi$ with the Laver property have been constructed. In these, $\text{add}(I(Si)) = \aleph_1$ holds by [14] as in (1).

As an immediate consequence of Theorems 5.1, 6.1, 6.2 and 6.3 we obtain the following:

**Theorem 6.4** If ZF is consistent, then the following statements are consistent with ZFC:

1. $\text{cof}(I(Sa)) < \text{cof}(I(Si))$,
2. $\text{cof}(I(Q_1)) < \text{cof}(I(Q_0))$, where $Q_0 \in \{Sa, Si\}$ and $Q_1 \in \{La, Mi\}$.
7 Singular cofinality

In this section we shall show that consistently we can have \( \text{cof}(\mathcal{I}(Q)) \) singular, where \( Q \) is a GTF\(_1\). For this we apply the amoeba from Section 3, but we have to use a more elaborate iteration.

**Theorem 7.1** Suppose that \( Q = (Q, \dot{\zeta}, \text{set, } Q^*, \bot) \) is a GTF\(_1\), \( 2^{\aleph_0} = \lambda = \lambda^{<\lambda} < \theta = \text{cf}(\mu) < \mu < \chi = \chi^{<\chi} \) and \( \text{add}(\mathcal{I}(Q)) = \lambda \). Moreover we assume \( \forall \alpha < \mu \ | \alpha |^\lambda < \mu \). There exists a forcing \( P \) such that

(a) \( P \) is \( \lambda \)-closed and \( \lambda^+-\text{c.c.} \).

(b) \( V^P \models 2^\lambda = \chi \land \text{cof}(\mathcal{I}(Q)) = \mu \).

**Proof:** We fix an increasing sequence \( \langle \lambda_i : i < \theta \rangle \) of regular cardinals \( \lambda_i < \mu \) with \( \lambda < \lambda_0 \) and \( \sup \{ \lambda_i : i < \theta \} = \lambda \). Let

\[ \mathcal{F} = \{ f \in \prod_{i < \theta} \lambda_i : |\{ i < \theta : f(i) \neq 0 \}| < \lambda \}. \]

For \( f \in \mathcal{F} \) let \( \text{supp}(f) = \{ i < \theta : f(i) \neq 0 \} \). Let \( \leq_{\mathcal{F}} \) denote the natural partial order on \( \mathcal{F} \) defined by \( f \leq_{\mathcal{F}} g \) iff \( \text{supp}(f) \subseteq \text{supp}(g) \) and \( \forall i \in \text{supp}(f) \ f(i) < g(i) \). By our assumptions, clearly \( |\mathcal{F}| = \mu \) and \( \mathcal{F} \) is \( (< \lambda^+) \)-directed. Let \( \langle f^*_β : \beta < \mu \rangle \) list \( \mathcal{F} \) such that \( f^*_0 \) is the constantly 0 function.

**Definition 7.1** Let the assumptions of Theorem 7.1 hold.

(1) We call a family \( q = q(Q) = \langle P_\alpha, \dot{Q}_\beta, u_\beta, \dot{\eta}_\beta, f_\beta : \alpha \leq \alpha_q, \beta < \alpha_q \rangle \) a \( (< \lambda) \)-support iteration of \( Q \) with memory if

(a) \( \chi < \alpha_q \) is a limit ordinal, and \( \langle P_\alpha, \dot{Q}_\beta : \alpha \leq \alpha_q, \beta < \alpha_q \rangle \) is a \( (< \lambda) \)-support iteration such that for every \( \beta < \alpha_q \), \( \Vdash_{P_\beta} " \dot{Q}_\beta \) has has a subset of \( \mathcal{P}(H(\lambda)) \) as its set of elements and \( \dot{\eta}_\beta \subseteq \dot{Q}_\beta \) is the generic filter".

(b) \( u_\beta \subseteq \beta \) such that \( \forall \gamma \in u_\beta u_\gamma \subseteq u_\beta \) (transitivity of the memory \( \langle u_\beta : \beta < \alpha_q \rangle \)).

(c) \( \forall \beta < \chi \ (u_\beta = \emptyset \land \Vdash_{P_\beta} " \dot{Q}_\beta = (\langle \lambda, \geq \rangle)" ) \).

(d) \( \forall \beta \in [\chi, \alpha_q) \Vdash_{P_\beta} " \dot{Q}_\beta = A_1(Q)V^{[\dot{\eta}[u_\beta]]}, \) where \( \dot{\eta}[u] \) denotes \( \langle \dot{\eta}_\nu : \nu \in u \rangle \) for \( u \subseteq \beta \).
(e) (α) \( f_\beta \in \mathcal{F} \) and if \( \beta < \mu \) then \( f_\beta = f_0^* \).

(β) If \( \beta \in u_\gamma \) then \( f_\beta \leq \mathcal{F} f_\gamma \).

(γ) If \( \beta \in u_\gamma \) and \( \beta < \mu \) then \( \sup \{ \lambda_\varepsilon : \varepsilon < \iota \} \leq \beta < \lambda_\iota \) implies \( \beta < f_\gamma(\iota) \).

(2) Let \( q \) be as in (1) and \( \bar{u} = \langle u_\beta : \beta < \alpha_q \rangle \). A subset \( U \subseteq \alpha_q \) is called \( \bar{u} \)-closed if \( \forall \beta \in U \) \( u_\beta \subseteq U \).

Claim 3 Let \( q = q(Q) \) be as in Definition 7.1 and \( U \subseteq [\chi, \alpha_q) \) such that

(1) \( \forall u \in [\alpha_q]^{\leq \lambda} \exists \beta \in U \) \( u \subseteq u_\beta \).

Let \( \hat{p}_\beta = \langle \hat{p}_\epsilon^\beta : \epsilon < \lambda \rangle \) denote the generic maximal antichain in \( Q \) added by \( Q_\beta \) and \( \hat{X}_\beta = X(\hat{p}_\beta) \) the associated set in \( I(Q)^{V[\beta, \beta]} \).

Then \( V^{P_{\alpha_q}} \models " \langle \hat{X}_\beta : \beta \in U \rangle \) is cofinal in \( I(Q) \), hence cof \( (I(Q)) \leq |U| " \).

Proof: Note that (1) implies cf\( (\alpha_q) > \lambda \) and hence

(1)' \( \forall u \in [\alpha_q]^{\leq \lambda} \exists \beta \in U \) \( u \subseteq u_\beta \).

Now suppose \( p \models P_{\alpha_q} \). Wlog we may assume that there exists a family of \( P_{\alpha_q} \)-names \( \langle \hat{q}_\epsilon : \epsilon < \lambda \rangle \) such that

\( p \models P_{\alpha_q} \langle \hat{q}_\epsilon : \epsilon < \lambda \rangle \) is a maximal antichain of \( Q \) and \( \hat{\tau} = X(\langle \hat{q}_\epsilon : \epsilon < \lambda \rangle) \).

Each \( \hat{q}_\epsilon \) can be viewed as a pair \( (A_\epsilon, h_\epsilon) \) where \( A_\epsilon \) is a maximal antichain in \( P_{\alpha_q} \) and \( h_\epsilon : A_\epsilon \to Q \). As \( P_{\alpha_q} \) has the \( \lambda^+ \)-c.c., \( |A_\epsilon| \leq \lambda \). Note that by the definition of the \( Q_\beta \), if \( p \models P_\beta " \hat{\sigma} \in Q_\beta " \) then \( \hat{\sigma} \) can be coded in essentially the same way as \( \hat{\tau} \), i.e. by \( \lambda \) may maximal antichains of \( P_\beta \). As \( q \) is a \( (\lambda) \)-support iteration, doing this for every \( p(\beta) \) where \( p \in A_\epsilon \) and \( \beta \in \text{dom}(p) \) and then proceeding similarly, we obtain a wellfounded tree \( T \) on \( (\alpha_q, >) \) such that every node has at most \( \lambda \) many immediate successors, \( T \) has no infinite branch, and \( \hat{\tau} \) can be evaluated from \( \langle \hat{q}_\nu : \nu \in T \rangle \). As \( |T| \leq \lambda \), by (1)' there are \( \lambda \) many \( \alpha_\iota \in U \) such that \( T \subseteq u_{\alpha_\iota} \). By Lemma 3.1(B) we conclude \( p \models P_{\alpha_q} \exists \iota < \lambda \hat{\tau} \subseteq X_{\alpha_\iota} \). Note that for this argument no memory is needed.

\( \square \)

Definition 7.2 Let \( q = q(Q) \) and \( \bar{u} \) be as in Definition 7.1. By induction on \( \alpha \leq \alpha_q \), for all \( \bar{u} \)-closed \( U \subseteq \alpha \), we define \( P_U' \subseteq P_\alpha \) and prove
(a) $P'_U$ consists of all $p \in P_\alpha$ such that $\text{dom}(p) \subseteq U$ and for every $\beta \in \text{dom}(p)$, $p(\beta)$ is a $P'_{u_\beta}$-name for a subset of $H(\lambda)$ (so either for an element of $<\lambda\lambda$ or of $A_1(Q)V^{[\beta][u_\beta]}$).

(b) If $\alpha_1 < \alpha$ then $P'_{U \cap \alpha_1} \subseteq P'_U$ (clearly $U \cap \alpha_1$ is $\bar{u}$-closed).

(c) $P'_\alpha$ is dense in $P_\alpha$.

(d) $P'_U$ is a dense subset of the limit of the $<\lambda$)-support iteration of the form $(P^*_\beta, \dot{Q}^*_\beta : \beta \in U)$ such that for every $\beta \in U \cap \alpha$, $\Vdash_{P^*_\beta} " \dot{Q}^*_\beta = (\langle \lambda, \subseteq \rangle", \text{and for every } \beta \in U \cap [\lambda, \alpha_q), \Vdash_{P^*_\beta} \dot{Q}^*_\beta = A_1(Q)V^{[\beta][u_\beta]}"
\text{(Here } \eta_3 \text{ and } \eta[u_\beta] \text{ are defined as in Definition 7.1. Note again that}
\text{if } u_\beta \subseteq U \text{ as } U \text{ is } \bar{u}\text{-closed.) Hence, letting } U = \alpha, \text{ we have (c).}

(e) $P'_U$ is a complete suborder of $P_\alpha$.

(f) For every $q \in P'_\alpha$, $q \upharpoonright U \in P'_U$ and $q \leq P'_{\alpha_1}$, $q \upharpoonright U$.

(g) For every $q \in P'_\alpha$ and $p \in P'_U$, if $p \leq P'_U$, $q \upharpoonright U$, then $p$ and $q$ are compatible in $P'_U$, in fact, $p \cup q \upharpoonright (\text{dom}(q) \setminus U)$ is a lower bound of $p$ and $q$.

(h) $p \in P'_U$ iff $p \in P'_\alpha$ and $\text{dom}(p) \subseteq U$.

**Proof:** We won’t use (d), hence we omit its proof. The main point is (c), as (f), (g), and (h) are clear, and hence (e) follows from (c). So let us prove (c) by induction on $\alpha$. The case $\alpha = 0$ is trivial.

Let $\alpha = \beta + 1$ and $p \in P_\alpha$. Wlog we may assume that $\beta \in \text{dom}(p)$, as otherwise we can apply the induction hypothesis. For the same reason we know that $P'_{u_\beta}$ is a complete subforcing of $P_\beta$ and $P'_{u_\beta}$ is dense in $P_\beta$. Clearly we have $P'_{u_\beta} \subseteq P'_{\beta}$. Hence by definition we have

\[\Vdash_{P'_{u_\beta}} "p(\beta) \in V[[\dot{\eta}_\gamma : \gamma \in u_\beta]]".\]

As $\langle \dot{\eta}_\gamma : \gamma \in u_\beta \rangle$ is (forced to be) $P'_{u_\beta}$-generic, there exist a $P'_{u_\beta}$-name $\dot{\tau}$ and $p_1 \leq P_{u_\beta} p \upharpoonright \beta$ in $P'_{u_\beta}$ such that $p_1 \Vdash_{P'_{u_\beta}} p(\beta) = \dot{\tau}$. Let $q = (p_1, \dot{\tau})$. Then $q \in P'_{\alpha}$ and $q \leq p$.

Now suppose that $\alpha$ is a limit ordinal and $p \in P_\alpha$. As $|\text{dom}(p)| < \lambda$ we may assume that $\text{cf}(\alpha) < \lambda$. Let $\langle \alpha^*_\iota : \iota < \text{cf}(\alpha) \rangle$ be increasing and cofinal in $\alpha$. We choose $\langle q_\iota : \iota \leq \text{cf}(\alpha) \rangle$ such that $q_\iota \in P'_{\alpha^*_\iota}$, $q_\iota \leq P'_{\alpha^*_\iota}$, $p \upharpoonright \alpha^*_\iota$ and if $\iota < \nu < \text{cf}(\alpha)$ then $q_\nu \leq P'_{\alpha^*_\nu} q_\iota$. For the successor step we apply the inductive hypothesis. Suppose that $\nu \leq \text{cf}(\alpha)$ is a limit ordinal and $\langle q_\iota : \iota < \nu \rangle$ have been chosen as desired. Let $\gamma \in \bigcup_{\iota < \nu} \text{dom}(q_\iota)$. Choose $\iota(\gamma)$ such that
\(\gamma \in q_{\ell(\gamma)}.\) Then in \(V, \langle q_\ell(\gamma) : \ell \in [s(\gamma), \nu) \rangle\) is a sequence of \(P_{u_\gamma}\)-names for members of \(\dot{Q}_\gamma\) such that this sequence is forced to be decreasing. But this forcing is forced to be \(< \lambda\)-complete and can be evaluated in \(V^{P_{u_\gamma}}\). Hence we can choose \(q_\ell(\gamma)\) as a \(P_{u_\gamma}\)-name that is forced to be a lower bound of it. Hence we have \(q_{\ell(\alpha)} \in P_{\alpha}\) and \(q_{\ell(\alpha)} \leq p\). □

In order to get \(V^{P_{\alpha q}} \models \text{cof}(\mathcal{I}(Q)) \geq \mu\) we must make \(q\) more concrete as follows: We let

\[(2) \alpha_q = \chi + \mu \cdot \lambda^+,
\]

\[(3) \text{if } \beta < \chi, \text{ then } u_\beta = \emptyset \text{ and } f_\beta = f_\beta^*,
\]

\[(4) \text{if } \beta = \chi + \mu \cdot \ell + \nu \text{ for } \ell < \lambda^+ \text{ and } \nu < \mu, \text{ then } f_\beta = f_\beta^* \text{ and } u_\beta = \{\alpha < \mu : \sup\{\lambda_\nu : \nu < \ell\} \leq \alpha < \lambda \} \cup \{\alpha \in [\mu, \beta) : f_\alpha \leq_f f_\beta\}.
\]

Note that \(\langle u_\beta : \beta < \alpha_q \rangle\) is transitive: Let \(\beta \in u_\alpha \) and \(\alpha \in u_\beta\). We must have \(\chi \leq \beta < \gamma\) and hence \(f_\beta \leq f_\gamma\). If \(\alpha < \mu\), hence \(\sup\{\lambda_\nu : \nu < \ell\} \leq \alpha < \lambda\) for some \(\ell < \theta\), we have \(\alpha < f_\beta(\ell) \leq f_\gamma(\ell)\). If \(\mu \leq \alpha\) we have \(f_\alpha \leq f_\beta \leq f_\gamma\) and we are done.

Also note that \((1)'\) holds for \(U = [\chi, \alpha_q]\): Let \(u \subseteq \alpha_q\) have size \(\lambda\). As \(\mathcal{F}\) is \((< \lambda^+\)-directed, we can easily find \(f \in \mathcal{F}\) such that

\[(5) u \cap [\sup\{\lambda_\nu : \nu < \ell\}, \lambda]\) \text{ is bounded by } f(\ell) \text{ for every } \ell < \theta, \text{ and}
\]

\[(6) f_\beta \leq_f f \text{ holds for every } \beta \in u \cap [\mu, \alpha_q].
\]

It follows that for every \(\gamma \in [\sup(u) + 1, \alpha_q]\) such that \(f_\gamma = f\), we have \(u \subseteq u_\gamma\). As by construction there are at least \(\lambda^+\) such \(\gamma\), we are done.

Now let us prove \(V^{P_{\alpha q}} \models \text{cof}(\mathcal{I}(Q)) \geq \mu\), where \(q\) is the iteration just defined. By Definition 7.2(e) we have \(V^{P_{\alpha q}} = V^{P_{\alpha q}}\). By contradiction suppose we had \(\nu(\ast) < \theta\), \(p \in P_{\alpha q}\) and a family \(\langle \dot{Y}_\alpha : \alpha < \lambda_{i(\ast)} \rangle\) of \(P_{\alpha q}\)-names such that

\[p \models_{P_{\alpha q}} \langle \dot{Y}_\alpha : \alpha < \lambda_{i(\ast)} \rangle \text{ is cofinal in } \mathcal{I}(Q).
\]

Wlog we may assume that every \(\dot{Y}_\alpha\) is forced to be of the form \(X(\langle \dot{q}_{\alpha, \varepsilon} : \varepsilon < \lambda \rangle)\) (see Remark 3.1(3)), where \(\langle \dot{q}_{\alpha, \varepsilon} : \varepsilon < \lambda \rangle\) is forced to be a maximal antichain of \(Q\). Since \(Q \subseteq \mathbb{R}\) and \(P_{\alpha q}\) does not add reals, wlog we may assume that every \(\dot{q}_{\alpha, \varepsilon}\) is a nice \(P_{\alpha q}\)-name, i.e. has the form \((A_{\alpha, \varepsilon}, f_{\alpha, \varepsilon})\) where \(A_{\alpha, \varepsilon}\) is a maximal antichain of \(P_{\alpha q}^{\prime}\) and \(f_{\alpha, \varepsilon} : A_{\alpha, \varepsilon} \rightarrow Q\). Let \(\alpha_q = \bigcup\{\text{dom} : (p) : p \in A_{\alpha, \varepsilon}\}\), thus \(v_\alpha \in [\alpha_q]^{\leq \lambda}\) and hence, by \((1)'\) for our memory \(\bar{u}\), we find \(\gamma_\alpha < \alpha_q\) such that \(v_\alpha \subseteq u_{\gamma_\alpha}\).
Let $\beta^* = \sup \{ f_\alpha(\iota(\ast) + 1) + 1 : \alpha < \iota(\ast) \}$ and $u^* = \bigcup \{ u_\alpha : \alpha < \iota(\ast) \}$. Then clearly $\beta^* < \lambda_{\iota(\ast) + 1}$, $u^*$ is $\bar{u}$-closed and $u^* \cap [\lambda_{\iota(\ast)}, \lambda_{\iota(\ast) + 1}) = [\lambda_{\iota(\ast)}, \beta^*)$. By Definition 7.2(e) we have that $P_{u^*}$ is a complete subforcing of $P_{\alpha_q}$, and hence every $\eta_\beta$ for $\beta \in [\beta^*, \lambda_{\iota(\ast) + 1})$ is $\lambda$-Cohen, i.e. generic for $(\prec \lambda, \supseteq)$, over $V^{P_{u^*}}$. As $\langle \hat{Y}_\alpha : \alpha < \lambda_{\iota(\ast)} \rangle$ is forced to belong to $V^{P_{u^*}}$, the following claim will complete the proof of Theorem 7.1:

**Claim 4** If $Q$ is $\text{GTF}_1$, $2^{< \delta_0} = \lambda$ and $\eta : \lambda \to \lambda$ is $\lambda$-Cohen, i.e. generic for $(\prec \lambda, \supseteq)$, over $V$, then in $V[\eta]$ there exists $X \in I(Q)$ that is not contained in any member of $I(Q)^V$.

**Proof:** Let $\langle r_\varepsilon : \varepsilon < \lambda \rangle$, $\langle p_\varepsilon : \varepsilon < \lambda \rangle$ list $\mathbb{R}$, $Q$ respectively. In $V[\eta]$ we define families $\langle s_\varepsilon : \varepsilon < \lambda \rangle$ in $\mathbb{R}$ and $\langle q_\varepsilon : \varepsilon < \lambda \rangle$ in $Q$ as follows: Let $s_0 = r_{\eta(0)}$ and let $q_0$ be the $\eta(1)$th $p_\varepsilon$ that satisfies $p_\varepsilon \leq p_0$ and $s_0 \not\subseteq [p_\varepsilon]$. If $\langle s_\varepsilon : \varepsilon < \nu \rangle$ and $\langle q_\varepsilon : \varepsilon < \nu \rangle$ have been determined for some $\nu < \lambda$, let $s_\nu$ be the $\eta(\nu \cdot 2)$th $r_\varepsilon$ such that $r_\varepsilon \not\subseteq \bigcup_{\varepsilon < \nu} [q_\varepsilon]$. To define $q_\nu$ we distinguish two cases. If $p_\nu$ is compatible with some $q_\varepsilon$ for $\varepsilon < \nu$ we let $q_\nu = q_0$. Otherwise, let $q_\nu$ the $\eta(\nu \cdot 2 + 1)$th $p_\varepsilon$ such that $p_\varepsilon \leq p_\nu$ and $[p_\varepsilon] \cap \{ s_\xi : \xi < \nu \} = \emptyset$. As $Q$ is $\text{GTF}_1$, this construction is possible. Now $X = \{ s_\xi : \xi < \lambda \}$ is as desired. □

□Theorem 7.1

**References**


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