No Tukey reduction of Lebesgue null to Silver null sets

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Abstract

We prove that consistently the Lebesgue null ideal is not Tukey reducible to the Silver null ideal. This contrasts with the situation for the meager ideal which, by a recent result of the author, is Tukey reducible to the Silver ideal.

1 Introduction

Given partial orders \( P \) and \( Q \), a map \( f : P \to Q \) is called a Tukey function, or also a Tukey reduction of \( P \) to \( Q \), provided that for every unbounded \( U \subseteq P \) the pointwise image \( f[U] \) is unbounded in \( Q \). If such a reduction exists we write \( P \leq_T Q \). This notion is of particular interest if applied to ideals \( I, J \subseteq \mathcal{P}(X) \) on a set \( X \), in which case the order on \( I, J \) is inclusion. The existence of a Tukey reduction has an effect on their cardinal coefficients.

Recall the following definitions:

- \( \text{add}(I) = \min \{|F| : F \subseteq I \land \bigcup F \notin I \} \)
- \( \text{cov}(I) = \min \{|F| : F \subseteq I \land \bigcup F = X \} \)
- \( \text{non}(I) = \min \{|Y| : Y \subseteq X \land Y \notin I \} \)
- \( \text{cof}(I) = \min \{|F| : F \subseteq I \land \forall Y \in I \exists Z \in F \ Y \subseteq Z \} \)

These are called the additivity, covering, uniformity and cofinality coefficients of \( I \), respectively. If \( I \) is a \( \sigma \)-ideal, we easily see that \( \aleph_1 \leq \text{add}(I) \leq \text{cov}(I), \text{non}(I) \leq \text{cof}(I) \). By a simple observation (see [2]), if \( I \leq_T J \), then \( \text{add}(I) \geq \text{add}(J) \) and \( \text{cof}(I) \leq \text{cof}(J) \). Among the historically first ideals for which these coefficients have been studied are the (Lebesgue) null ideal \( \mathcal{N} \) and the meager ideal \( \mathcal{M} \). For these, Rothberger has shown

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1 The author is partially supported by DFG grant SP383/4-1.
cov($\mathcal{M}$) ≤ non($\mathcal{N}$) and cov($\mathcal{N}$) ≤ non($\mathcal{M}$). Miller [16] and Truss [25] showed that $\text{add}(\mathcal{M}) = \min\{b, \text{cov}(\mathcal{M})\}$, where $b$ is the bounding number, i.e. the least size of an unbounded family in $(\omega^\omega, \leq^*)$, and Fremlin proved the dual $\text{cof}(\mathcal{M}) = \max\{d, \text{non}(\mathcal{M})\}$, where $d$ is the cofinality of $(\omega^\omega, \leq^*)$. But the most interesting results here are those of (independently) Bartoszynski [1] and Raisonnier, Stern [19], saying that $\text{add}(\mathcal{N}) \leq \text{add}(\mathcal{M})$ and $\text{cof}(\mathcal{M}) \leq \text{cof}(\mathcal{N})$. It was Fremlin [8] who had the intuition that a Tukey reduction of $\mathcal{M}$ to $\mathcal{N}$ might be the deeper reason for these inequalities. Finally Pawlikowski [18] proved $\mathcal{M} \leq_T \mathcal{N}$.

For a while there was a hope that the approach of Tukey reduction would provide a way to study cardinal invariants of the continuum on the basis of ZFC, in particular giving substantial information also under CH. Hence it came as a disappointment when Yiparaki (see [27]) proved that under CH any two partial orders appearing in Cichon’s diagram (see [2]) (especially $\mathcal{M}$ and $\mathcal{N}$) are Tukey equivalent.

In this paper we investigate the ideal of Silver null sets, also called Mycielski ideal (see [17]). This is one of the classical tree ideals defined as follows: Let $P$ be one of the classical tree forcings like Sacks, Silver, Laver or Miller forcing ($Sa, Si, La, Mi$ for short). In the first two cases $I(P)$ is the ideal on $2^\omega$ consisting of those $X \subseteq 2^\omega$ such that
$$\forall p \in P \exists q \in P (q \leq p \land X \cap [q] = \emptyset).$$
Here $[q] = \{x \in 2^\omega : \forall n \; x|_n \in q\}$. For $P \in \{La, Mi\}$, $I(P)$ is defined analogously on $\omega^\omega$. These ideals are not c.c.c. and do not have a Borel base. Note that $\text{non}(I(P)) = 2^{\aleph_0}$ holds in ZFC, as every $p \in P$ has continuum many subtrees in $P$ such that no two of them have a common branch.

In [9] it has been shown that under MA $\text{add}(I(P)) = 2^{\aleph_0}$ for $P \in \{La, Mi\}$. On the other hand, in [11] and independently in [26] it has been shown that $\text{add}(I(Sa)) = \aleph_1$ is consistent with MA + $\neg$CH. The analogous result for $I(Si)$ had been proved prior to [11] by Steprans (see [11]) and was published in [7]. Note that this implies the consistency of $\text{add}(I(P)) < \text{add}(\mathcal{N})$ for $P \in \{Sa, Si\}$, as MA implies $\text{add}(\mathcal{N}) = 2^{\aleph_0}$ (see [13]). Moreover, it implies the consistency of $\text{add}(I(P))) < \text{add}(I(Q))$ for any such $P$ and any $Q \in \{La, Mi\}$.

Let us also mention that while $\text{cof}(\mathcal{N}) \leq 2^{\aleph_0}$ is trivial and hence all coefficients of Cichon’s diagram are at most $2^{\aleph_0}$, this is false for $\text{cof}(I(P))$ for $P \in \{Sa, Si, La, Mi\}$. In [11] it has been shown that even $\text{cf}(\text{cof}(I(Sa))) > 2^{\aleph_0}$ holds. The same proof applies to $I(Si)$. The analogous result for $La$ and for $Mi$ has been proved recently by Brendle, Khomskii and Wohofsky (see [6]). Note that by what we said above, this implies $I(P) \not\leq_T \mathcal{N}$ in ZFC for
\( P \in \{ Sa, Si, La, Mi \} \).

It is natural to try to clarify the relative size of the cardinal coefficients of these tree ideals compared to one another and to the ones of \( \mathcal{M} \) and \( \mathcal{N} \). For the covering coefficients this has been done comprehensively by Brendle in his Thesis for Habilitation (see [5]). The additivities are in a sense more difficult to handle. The natural forcing to increase the additivity of one of the mentioned ideals is an amoeba forcing, i.e. one that adds a positive set (with respect to the ideal) of generic reals (for the forcing in question). E.g. a Silver amoeba is a forcing adding a generic Silver tree such that each branch is Silver-generic. It is not hard to see that the natural amoeba forcings add Cohen reals and hence their iteration will be hard to handle, especially if one tries to keep some other cardinal coefficients small. However, for \( Sa \) (see [14]) and \( La, Mi \) (see [22]) it turned out that amoeba forcings adding neither Cohen nor random reals do exist which preserve this property in a sense more difficult to handle. Hence the consistency of \( \max \{ \text{cov}(\mathcal{M}), \text{cov}(\mathcal{N}) \} < \text{add}(\mathcal{I}(P)) \) was known for \( P \in \{ Sa, La, Mi \} \). For \( Si \) this remained open, but the conjecture was that the only thing missing was the right Silver amoeba to obtain the analogous result.

However, in [23] I showed \( \mathcal{M} \leq_T \mathcal{I}(Si) \), and hence \( \text{add}(\mathcal{I}(Si)) \leq \text{add}(\mathcal{M}) \) holds in ZFC and therefore every Silver amoeba that is proper and can be iterated to produce a model for \( \aleph_1 < \text{add}(\mathcal{I}(Si)) \) must add Cohen as well as dominating reals. Actually, the new element of [23] is the ZFC-inequality \( \text{add}(\mathcal{I}(Si)) \leq \text{cov}(\mathcal{M}), \text{as} \text{add}(\mathcal{I}(Si)) \leq b \) had already been proved in [24]. Let us mention that \( \text{add}(\mathcal{I}(Sa)) \leq b \) is also true. If the continuum is regular this follows by a result of Simon [21] and an observation in [11]. For singular continuum see [10].

Note that by \( \text{add}(\mathcal{I}(Si)) \leq \text{add}(\mathcal{M}) \), in the model obtained by iterating the Sacks amoeba with the Laver property from [14] we have \( \text{add}(\mathcal{I}(Si)) < \text{add}(\mathcal{I}(Sa)) \). Whether the converse inequality is consistent with ZFC is an open problem. We conjecture that it holds in the model constructed in this paper.

The problem to separate the cofinality coefficients of different tree ideals has been attacked only very recently in [20]. The main result of this paper implies that for any \( P, Q \in \{ Sa, Si, La, Mi \} \), if \( \text{add}(\mathcal{I}(P)) < \text{add}(\mathcal{I}(Q)) \) is consistent, then so is \( \text{cof}(\mathcal{I}(Q)) < \text{cof}(\mathcal{I}(P)) \). Hence in particular \( \text{cof}(\mathcal{I}(Sa)) < \text{cof}(\mathcal{I}(Si)) \) is consistent. Again, the converse inequality is open.

In view of Pawlikowski’s result\( \mathcal{M} \leq_T \mathcal{N} \), the natural question that was raised by our result \( \mathcal{M} \leq_T \mathcal{I}(Si) \) is whether even \( \mathcal{N} \leq_T \mathcal{I}(Si) \) is true or, conversely,
there exists a Silver amoeba not adding random reals that could be iterated to construct a model of $\text{cov} (\mathcal{N}) < \text{add} (\mathcal{I} (\mathcal{S}i))$. Here we prove that the second alternative is true. Note that by Yiparaki’s result, CH implies that $\mathcal{M}$ and $\mathcal{N}$ are Tukey equivalent and hence also $\mathcal{N} \leq_T \mathcal{I} (\mathcal{S}i)$. Therefore, $\mathcal{N} \leq_T \mathcal{I} (\mathcal{S}i)$ is independent of ZFC. The Silver amoeba $\mathcal{A} (\mathcal{S}i)$ that we shall construct has the pure decision property, i.e. given a finite disjunction $\varphi = \bigvee_{i<n} \varphi_i$ that is forced to be true by some condition $p \in \mathcal{A} (\mathcal{S}i)$, we can find a stronger condition $q$ with the same finite part as $p$ which decides $\varphi$, i.e. $q \Vdash \varphi_i$ is true for some $i < n$. By $\mathcal{M} \leq_T \mathcal{I} (\mathcal{S}i)$, $\mathcal{A} (\mathcal{S}i)$ adds Cohen reals, hence does not have the Laver property, so a more specific reason must be found why $\mathcal{A} (\mathcal{S}i)$ does not add random reals.

The crucial observation is that virtually every amoeba for some tree forcing has the property that each of its conditions $p$ naturally gives rise to a countable family $\langle R^k : k < \omega \rangle$ where $R^k = \langle r^k_\nu : \nu \in k+1, \omega \rangle$ is a (countable) maximal antichain below $p$ such that $R^{k+1}$ refines $R^k$ according to their indexing (i.e. $r^k_{\nu-j} < r^k_{\nu}$). Actually, $R^k$ consists of all extensions of depth $k+1$ (see Definition 2.4 below) of $p$ that have the same infinite part as $p$. If now the given amoeba additionally has the pure decision property and some basic Axiom-A-like fusion property, then we can decide initial segments of a given name for a real $\dot{x}$ along the $R^k$. If this happens fast enough we produce a null set in the ground model and a condition that forces $\dot{x}$ to belong to it. Substantially more work is needed to show that this specific antichain structure of $\mathcal{A} (\mathcal{S}i)$ is preserved by a countable support iteration of it. For this we apply some elements of the general theory of Axiom-A forcings in [3] and the more specific iteration theory for compact tree forcings in [4]. By our proofs it will become clear that we can prove a more general iteration theorem, saying that if $P$ is a countable support iteration of Axiom-A forcings which have the pure decision property and below each condition a countable system of countable maximal antichains as indicated, then $P$ does not add random reals. We won’t try to make this precise, as it would involve to axiomatize the notion of pure decision property.

2 A Silver amoeba with pure decision

**Definition 2.1** A tree $p \subseteq 2^{<\omega}$ is a **Silver tree** if there exist a coinfinite $a \subseteq \omega$ and $x : a \rightarrow 2$ such that

$$p = \{ \nu \in 2^{<\omega} : \forall i \in a \cap |\nu| \nu(i) = x(i) \}.$$ 

Let $\mathcal{S}i$ denote the set of all Silver trees ordered by inclusion. Given $p \in \mathcal{S}i$, some $\nu \in p$ is called **splitnode** of $p$ if $\nu\downarrow 0$ and $\nu\downarrow 1$ both belong to $p$. Let
split(p) denote the set of all splitnodes of p. By stem(p) we denote the stem of p, i.e. the shortest splitnode of p. By Lev(p) we mean the n-th splitlevel of p, i.e. the set of all splitnodes of p that have precisely n+1 initial segments that are splitnodes of p. Thus Lev(0) = {stem(p)}. Given p,q ∈ Si, we write p ≤ n q if p ≤ q and Lev(p) = Lev(q) (and hence Lev(i) = Lev(q) for every i ≤ n). For ν ∈ p we let pν = {μ ∈ p : μ ⊆ ν ∨ ν ⊆ μ}.

Definition 2.2 (1) If p ∈ Si, σ = stem(p) and τ ∈ 2<ω we let pτ = {σ ∨ ν : σ ∨ ν ∈ p} ∪ {τ ∨ 0, ..., τ ∨ |τ|}. Hence pτ is obtained from p by substituting τ for σ.

(2) Suppose that p,q ∈ Si, q ≤ p, and n < ω. We define glob(n,q,p) ∈ Si, the n-globalization of q in p as follows: Letting m < ω so that Lev(q) ⊆ Lev(p) (hence m ≥ n), we have Lev(glob(n,q,p))(m) = Lev(p,m) and for every σ ∈ Lev(p,m) there is τ ∈ Lev(q) such that glob(n,q,p)σ = qτ (equivalently: for every τ ∈ Lev(q) the equation holds for every σ ∈ Lev(p,m)). Obviously glob(n,q,p) ≤ m p.

(3) Given p ∈ Si, τ ∈ Lev(n,p) and ⟨τ0,...,τk⟩ a properly increasing sequence in split(p) such that τk ⊆ τ, letting τ = ⟨τ0,...,τk,τ⟩ and

R = {ρ ∈ Lev(p) : ∀i ∈ |ρ|(|τi| : j ≤ k} ρ(i) = τ(i)},

we define

glob(p,τ) = ∪{ρp : ρ ∈ R},

and call it the τ-globalization of p in p. Moreover, we call R the τ-globalization of τ in p. If here k = 0, then R has two elements, τ and τ′, such that ∀i ∈ |τ|(|τi|, τ(i) = τ′(i) and τ′(|τ0|) = 1 − τ(|τ0|). In this case we call τ′ the τ0-twin of τ, and we write p(τ,τ′) instead of glob(p,⟨τ0,τ⟩). Clearly glob(p,τ) ∈ Si and glob(p,τ) ≤ p.

(4) Given p ∈ Si and C ⊆ split(p), we call C p-avoidable, iff C ∩ split(q) = ∅ for every q ∈ Si with q ≤ p.

(5) The Hamming weight of any ν ∈ 2<ω is defined as the number of i < |ν| with ν(i) = 1. It is denoted by HW(ν).

Let us give some natural examples of unavoidable sets. A simple but crucial observation in [23] is that for every p ∈ Si and n < ω, the set I(p) := {HW(ν) : ν ∈ Lev(p)} is an interval of length n+1. Moreover, by thinning out p we can arrange that the I(p) are disjoint and min(I(p)) is arbitrarily large. Hence clearly, for every choice of k < ℓ < ω, the set

{ν ∈ split(p) : HW(ν) = k mod ℓ}
is $p$-unavoidable. Clearly the set of all $p$-unavoidable sets is coanalytic. I do not know whether it is Borel. To clarify this, probably an improvement of the following Lemma is needed. It shows that unavoidable sets must satisfy a certain density condition.

**Lemma 2.3** Let $p \in S_i$ and $E \subseteq \text{split}(p)$ $p$-unavoidable.

1. There exists $\varepsilon > 0$, such that for almost all $n < \omega$
   \[
   \frac{|E \cap \text{Lev}_p(n)|}{2^n} \geq \varepsilon;
   \]

2. Suppose $E = E_0 \cup E_1$. There exist $q \in S_i$ and $i < 2$ such that $q \leq p$ and $E_i$ is $q$-unavoidable.

3. If $q \in S_i$ and $q \leq p$ then $E \cap \text{split}(q)$ is $q$-unavoidable.

**Proof:** Let $\sigma = \text{stem}(p)$.

1. Suppose the claim is false. Then for every $\varepsilon > 0$ there are infinitely many $n$ such that
   \[
   \frac{|E \cap \text{Lev}_p(n)|}{2^n} < \varepsilon.
   \]
   Recursively we construct $q \leq^0 p$ such that $\text{split}(q) \cap E = \emptyset$, which will be a contradiction. Clearly for every $n > 0$ there are $2^n - 1$ pairs $(\tau, \tau')$ in $\text{Lev}_p(n)$ such that $\tau'$ is the $\sigma$-twin of $\tau$. As any two such twin pairs are disjoint and they cover $\text{Lev}_p(n)$, if $E$ meets each of them, then
   \[
   \frac{|E \cap \text{Lev}_p(n)|}{2^n} \geq \frac{1}{2}.
   \]
   Hence we can find $n_0 > 0$ and a twin pair $(\tau_0, \tau'_0)$ in $\text{Lev}_p(n_0)$ disjoint from $E$. We stipulate that $\text{Lev}_q(1) = \{\tau_0, \tau'_0\}$. Now suppose that $\text{Lev}_q(n)$ has been determined and $\tau_n$ is the leftmost element of it. Let $m$ be such that $\text{Lev}_q(n) \subseteq \text{Lev}_p(m)$. Let $\tau_i$ be the initial segment of $\tau_n$ belonging to $\text{Lev}_q(i)$ for $i < n$ and $\bar{\tau} = \langle \tau_0, \ldots, \tau_n \rangle$. For every $m' > m$ and $\tau' \in \text{Lev}_p(m')$ with $\tau_n \subseteq \tau'$ and $\tau'([\tau_n]) = 0$ let $R_{\tau'}$ be the $\bar{\tau} \tan \tau'$-globalisation of $\tau'$ in $p$. Clearly $|R_{\tau'}| = 2^{n+1}$, for different $\tau'$ the $R_{\tau'}$ are disjoint and they cover $\text{Lev}_{\text{glob}(p, \bar{\tau})}(m')$. Note that
   \[
   \frac{|\text{glob}(p, \bar{\tau})(m')|}{|p(m')|} = \frac{|\text{Lev}_q(n)|}{|\text{Lev}_p(m)|} =: \varepsilon.
   \]
   Hence if $E$ meets each such $R_{\tau'}$, then
   \[
   \frac{|E \cap \text{Lev}_p(m')|}{2^{m'}} \geq \frac{\varepsilon}{2^{n+1}}.
   \]
Therefore, by assumption we can find $m' > m$ and $\tau' \in \text{Lev}_p(m')$ with $\tau_n \subseteq \tau'$ and $\tau'(|\tau_n|) = 0$ such that $R_{\tau'} \cap E = \emptyset$. Now we let $\text{Lev}_q(n + 1) = R_{\tau'}$. This finishes our construction of $q$.

(2) If there exists $q \in Si$, $q \leq p$ such that $E_0 \cap \text{split}(q) = \emptyset$, then $E_1$ must be $q$-unavoidable. Otherwise $E_0$ is $p$-unavoidable.

(3) This is trivial. \qed

Now we are going to define a special amoeba for Silver forcing.

**Definition 2.4** Elements of $A(Si)$ are triples $p = (p, n, E)$ such that $p \in Si$, $n < \omega$ and if $\sigma$ is the lexicographically least element of $\text{Lev}_p(n)$, for every $i \leq n \sigma_i \in \text{Lev}_p(i)$ is the unique element with $\sigma_i \subseteq \sigma$ (thus $\sigma_n = \sigma$) and $\sigma = \langle \sigma_0, \ldots, \sigma_n \rangle$, then $E$ is a function with $\text{dom}(E)$ a subset of the set

$$T = \{ \sigma^\tau : \exists k < \omega \exists \tau_0, \ldots, \tau_{k-1} \in \text{split}(p_{\sigma^0}) \forall i < k - 1 (\tau_i \subseteq \tau_{i+1} \wedge \tau_{i+1}(|\tau_i|) = 0 \wedge \tau = \langle \tau_0, \ldots, \tau_{k-1} \rangle) \}$$

such that the following recursive property holds:

(1) $\sigma$ is the shortest element of $\text{dom}(E)$, $E(\sigma) \subseteq \text{split}(p_{\sigma^0})$ is $p_{\sigma^0}$-unavoidable, and

(2) for every $k < \omega$, if $\sigma^\tau \in T$ belongs to $\text{dom}(E)$ and $\tau = \langle \tau_0, \ldots, \tau_{k-1} \rangle$, then $\sigma^\tau \tau_k \in \text{dom}(E)$ iff $\tau_k \in E(\sigma^\tau)$, and then $E(\sigma^\tau \tau_k) \subseteq \text{split}(p_{\tau_k^0})$ is $p_{\tau^0_k}$-unavoidable.

Given $p = (p, n^p, E^p)$, $q = (q, n^q, E^q) \in A(Si)$, we let $q \leq p$ iff the following hold:

(3) $q \subseteq p$;

(4) $n^p \leq n^q$;

(5) $\text{Lev}_q(n^p) = \text{Lev}_p(n^p)$ (hence $p$ and $q$ are equal up to their $n^p$-th splitlevel);

(6) if $\sigma$ is the lexicographically least element of $\text{Lev}_q(n^q)$, $\sigma = \langle \sigma_0, \ldots, \sigma_{n^q} \rangle$, $\sigma_{n^q} = \sigma$ as above, then for all $i$, $n^p < i \leq n^q$, we have $\sigma \upharpoonright i \in \text{dom}(E^p)$ and $\sigma_i \in E^p(\sigma \upharpoonright i)$;

(7) $\text{dom}(E^q) \subseteq \text{dom}(E^p)$ and hence (by (1), (2)) for every $\tau$, if $\sigma^\tau \in \text{dom}(E^q)$ (where $\sigma$ is as in (6)), then $E^q(\sigma^\tau) \subseteq E^p(\sigma^\tau)$.
We define \( q \leq^0 p \) by \( q \leq p \) and \( n^p = n^q \). Then we say that \( q \) is a pure extension of \( p \). We call \( n^p \) the depth of \( p \) and denoted it by \( dp(p) \).

Remark 2.5 (1) Note that given \( p \in A(S_i) \) \( E^p \) is determined by \( \text{dom}(E^p) \) and \( \text{dom}(E^p) \) is a tree on \( 2^{<\omega} \) with stem \( \bar{\sigma} = (\sigma_0, \ldots, \sigma_n) \) defined as in Definition 2.4. Therefore we shall occasionally identify \( E^p \) and \( \text{dom}(E^p) \) and use the tree notation for \( E^p \), e.g. we can consider the subtree \( (E^p)_\tau \). Note that if \( \tau = (\tau_0, \ldots, \tau_m) \in E^p, \bar{\sigma} \subseteq \tau \), then \( \bar{\sigma} \cap \nu \) is an immediate successor of \( \tau \) in \( E^p \) iff \( \nu \in E^p(\sigma) \).

(2) Let \( p = (p, n^p, E^p) \in A(S_i) \) and \( q \in S_i, q \leq p \) such that \( \text{Lev}_q(n^p) = \text{Lev}_p(n^p) \). We can define \( E^p \upharpoonright q \), the restriction of \( E^p \) to \( q \) as follows: \( \text{stem}(E^p \upharpoonright q) = \bar{\sigma} = (\sigma_0, \ldots, \sigma_{n^p}) \), where \( \sigma_{n^p} \) is the lexicographically least element of \( \text{Lev}_p(n^p), \sigma_i \subseteq \sigma_{n^p}, \sigma_i \in \text{Lev}_p(i) \) for \( i < n^p \). If \( \bar{\sigma} \cap \tau \in E^p \upharpoonright q \), then \( E^p \upharpoonright q(\sigma \cap \tau) = E^p(\bar{\sigma} \cap \tau) \cap \text{split}(q) \) and hence \( \bar{\sigma} \cap \nu \in E^p \upharpoonright q \) iff \( \nu \in E^p \upharpoonright q \). By Lemma 2.3(3), \( (q, n^p, E^p \upharpoonright q) \in A(S_i) \), and clearly \( (q, n^p, E^p \upharpoonright q) \leq^0 (p, n^p, E^p) \). We call \( E^p \) the tree of possible extensions of \( p \).

(3) Given \( p \in A(S_i) \), every stronger condition can be obtained by performing, one after the other, three specific operations to strengthen a condition: first, say, increasing \( n \) to some \( n' \), choosing a permitted section between the \( n \)-th and the (new) \( n' \)-th splitting level, leaving \( p \) unchanged above \( n' \); then thinning out \( p \) as in (2); then thinning out \( E \) (while remaining unavoidable, of course).

It is easy to see that \( A(S_i) \) is an amoeba for \( S_i \). Indeed, given some \( A(S_i) \)-generic filter \( G \), we let \( p(G) = \bigcap \{ p : p = (p, n^p, E^p) \in G \} \). Equivalently, \( p(G) \) is the Silver tree generated by \( \bigcup \{ \text{Lev}_p(n^p) : p = (p, n^p, E^p) \in G \} \). That \( p(G) \) must contain conditions of every depth follows by an easy density argument. To see that every branch of \( p(G) \) is a Silver real we need the following more general fact:

Lemma 2.6 Let \( p \in A(S_i) \) and \( D \subseteq S_i \) dense open. There exists \( q \in A(S_i) \) such that \( q \leq^0 p \) and, letting \( q = (q, n^q, E^q) \), we have \( q_\tau \in D \) for every \( \tau \in \text{Lev}_q(n^q + 1) \).

Proof: Let \( \bar{\sigma} = (\sigma_0, \ldots, \sigma_{n^p}) = \text{stem}(E^p) \). By building a finite descending chain in \( S_i \), it is easy to find \( r \in S_i \) such that \( r \leq p, \sigma_{n^p} \cap \tau \subseteq \text{stem}(r) \) and

\[
\frac{r \upharpoonright \tau \cap i}{\sigma_{n^p} \cap \tau} \in D
\]
(see Definition 2.2) holds for every \( \tau \in \text{Lev}_{n^p}(p) \) and \( i < 2 \). We let
\[
q = \bigcup \{ r^{\tau \wedge i}_{\sigma_{n^p} \wedge 0} : \tau \in \text{Lev}_{n^p}(p), i < 2 \},
\]
\( n^q = n^p \) and \( E^q = E^p \upharpoonright q \). Then \( q \) is as desired. \( \square \)

By genericity this implies that for every dense open \( D \subseteq S_i \) in the ground model there exists \( n < \omega \) such that for every \( \nu \in p(G) \cap 2^n \), the tree \( p(G)_{\nu} \) is a subtree of a member of \( D \). This is much stronger than what we need.

Now let \( n_i \) be the length of any member of \( \text{Lev}_{p(G)}(i) \). Another genericity argument easily shows that \( \langle n_i : i < \omega \rangle \) is a dominating real. As already mentioned in the introduction, in [24] \( \text{add}(\mathcal{I}(S_i)) \leq b \) has been proved, and hence, as an amoeba for a tree forcing \( P \) naturally increases \( \text{add}(\mathcal{I}(P)) \) (for \( P = S_i \) see the proof of Theorem 5.1 below), every reasonable amoeba for \( S_i \) must add a dominating real.

It is less obvious to see, \( A(S_i) \) also adds a Cohen real, and by [23], again this is necessarily so. Let us describe how to find it. For \( n < \omega \), let \( c(n) \) be the binary expansion of \( n \) in reverse order. Let \( \sigma_i \) be the lexicographically least element of \( \text{Lev}_{p(G)}(i) \), \( \tau_0 = \sigma_0 \) and \( \tau_{i+1} \) of length \( n_{i+1} - n_i \) such that \( \sigma_{i+1} = \sigma_i \wedge \tau_{i+1} \). Let \( \xi_i = c(\text{HW}(\tau_i)) \). By using the Coding Lemma of [23] it can be seen that \( \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \ldots \) is a Cohen real.

**Definition 2.7** Let \( p = (p, n^p, E^p) \in A(S_i) \) and \( \sigma \) as in Definition 2.4. Given \( \tau = \langle \tau_0, \ldots, \tau_m \rangle \) such that \( \sigma \wedge \tau \in E^p \), we call it a **depth-\( m+1 \)-sequence of \( p \)**, and we define
\[
p(\tau) = (\text{glob}(p, \sigma \wedge \tau), n^p + m + 1, (E^p)_{\sigma \wedge \tau}).
\]
The set of all depth-\( m+1 \)-sequences of \( p \) will be denoted by \( DS(p, m+1) \).

Moreover, for every \( m < \omega \) we let
\[
R^{m}(p) = \{ p(\tau) : |\tau| = m + 1 \land \sigma \wedge \tau \in E^p \}.
\]

**Remark 2.8** (1) Given \( p \in A(S_i) \), the set \( DS(p, m+1) \) carries a canonical well-order of type \( \omega^{m+1} \). Indeed, we have the canonical well-order \( \prec \) of \( \omega^{m+1} \) in type \( \omega \) defined by letting \( \nu \prec \mu \) iff \( |\nu| < |\mu| \) or \( |\nu| = |\mu| \) and \( \nu \) precedes \( \mu \) lexicographically. Using \( \prec \) we lexicographically order \( m+1(\omega^{m+1}) \). Then clearly \( DS(p, m+1) \) is a subset of \( m+1(\omega^{m+1}) \) of order type \( \omega^{m+1} \) (as unavoidable sets are always infinite). When we talk about the **canonical listing** of \( DS(p, m+1) \) we mean its increasing enumeration with respect to its canonical well-ordering.
We shall also need to consider the **fast listing** of $DS(p, m+1)$ that has length $\omega$, thus is of the form $\langle \tau_k^{m+1} : k < \omega \rangle$, where $\tau_k^{m+1} = \langle \tau_k^{m+1}(0), \ldots, \tau_k^{m+1}(m) \rangle$. Here we require that for any $k < k'$ we must have $|\tau_k^{m+1}(m)| \leq |\tau_{k'}^{m+1}(m)|$ and if equality holds, then $\tau_k^{m+1}$ precedes $\tau_{k'}^{m+1}$ lexicographically. Hence the canonical and the fast listing of $DS(p, m+1)$ coincide iff $m = 0$. Moreover, for every $n < m$ and $k'$, if $\tau_k^{m+1} \upharpoonright n = \tau_k^{n+1}$, then $k \leq k'$. This fact will be referred to as **coherence** of the fast listings.

(2) Note that $R^n(p)$ is a maximal antichain of $\mathbb{A}(Si)$ below $p$ such that for every $q \leq p$ with $dp(q) \geq n^p + m + 1$ we have $q \leq p(\tau)$ for precisely one $p(\tau) \in R^n(p)$. Moreover $R^{m+1}(p)$ refines $R^m(p)$ such that $$\forall r \in R^m(p) \exists \infty r' \in R^{m+1}(p) r' < r.$$ 

(3) Let $\langle \tau_{(j_0, \ldots, j_m)} \rangle : (j_0, \ldots, j_m) \in m^+1$ canonically list all depth-$m + 1$-sequences of $p$. If $u \in \mathbb{A}(Si)$ with $u \leq^0 p$ there exists a subtree $\Omega_p(u)$ of $\omega^<\omega$ such that for every $m < \omega$ $$\langle \tau_{(j_0, \ldots, j_m)} \rangle : (j_0, \ldots, j_m) \in m^+1 \cap \Omega_p(u)$$ canonically lists all depth-$(m + 1)$-sequences of $u$. Then clearly every node of $\Omega_p(u)$ has infinitely many immediate successors, thus $\Omega_p(u)$ is a **Laver tree** with stem$(\Omega_p(u)) = \emptyset$.

We shall prove that $\mathbb{A}(Si)$ has the **pure decision property**, i.e. given some $\mathbb{A}(Si)$-name $\hat{\mu}$ and $p \in \mathbb{A}(Si)$ such that $p \models_{\mathbb{A}(Si)} \hat{\mu} \in \{0, 1\}$, there exists a pure extension of $p$ which decides $\hat{\mu}$. Our proof fits into a general pattern that has been used frequently in the past. The first instance, where this pattern has been discovered, is Mathias’ proof in [15] that Mathias forcing has the pure decision property. For amoeba forcings it has been successfully applied in [14] and [22].

The first step consists of the following crucial Lemma:

**Lemma 2.9** Suppose $p \models_{\mathbb{A}(Si)} \hat{\mu} \in \{0, 1\}$ and there does not exist a pure extension of $p$ deciding $\hat{\mu}$, then $p$ has a pure extension $q$, such that for every $r \in \mathbb{A}(Si)$, if $r \leq q$ and $dp(r) = dp(q) + 1$, then $r$ does not decide $\hat{\mu}$.

**Proof:** For notational simplicity only we assume $n^p = 0$. The general case is obtained by a straightforward generalization of the arguments given.

Let $\sigma = \text{stem}(p)$. We shall construct a fusion sequence $\langle (p^i, 0, E^i) : i < \omega \rangle$ in $\mathbb{A}(Si)$ such that the following properties hold:
Then clearly $(p^i, 0, E^i) = p_i$.

(2) $p^{i+1} \leq p^i$ and $\text{Lev}_{p^{i+1}}(i + 1) = \text{Lev}_{p^i}(i + 1)$;

(3) $E^{i+1}$ equals $E^i \upharpoonright p^{i+1}$ except that for $\tau \in \text{Lev}_{p^{i+1}}(i + 1)$ with $(\sigma, \tau) \in E^i$ we only require $(E^{i+1})_{(\sigma, \tau)} \subseteq (E^i \upharpoonright p^{i+1})_{(\sigma, \tau)}$.

Then clearly $(p^{i+1}, 0, E^{i+1}) \leq (p^i, 0, E^i)$ holds, and letting $p^\infty = \bigcap_{i<\omega} p^i$ and $E^\infty = \bigcap_{i<\omega} E^i$ we have $(p^\infty, 0, E^\infty) \in \mathbb{A}(Si)$ and $(p^\infty, 0, E^\infty) \leq (p^i, 0, E^i)$ for all $i < \omega$.

The crucial property that will also hold is the following:

(4) For every $\tau \in \text{Lev}_{p^{i+1}}(i + 1)$, if $(\sigma, \tau) \in E^0$ (thus $\sigma^0 \subseteq \tau$), $\tau'$ is the $\sigma$-twin of $\tau$ and in $\mathbb{A}(Si)$ there exists a pure extension of $(p^{i+1}(\tau, \tau'), 1, (E^{i+1})_{(\sigma, \tau)})$ deciding $\mu$, then $(p^{i+1}(\tau, \tau'), 1, (E^{i+1})_{(\sigma, \tau)})$ already decides $\mu$.

For the construction, given $(p^i, 0, E^i)$, let $\langle \tau_l : l < k \rangle$ list all $\tau \in \text{Lev}_{p^i}(i + 1)$ with $(\sigma, \tau) \in E^i$ (by (3) equivalently: $(\sigma, \tau) \in E^0$) and let $\tau'_l$ be the $\sigma$-twin of $\tau_l$. Recursively we construct a descending sequence $\langle (p^j, 0, E^j) : l < k \rangle$ such that $(p^0, 0, E^0) = (p^i, 0, E^i)$. Suppose that $(p^j, 0, E^j)$ has been constructed and there exists $(\tau, 1, E_\tau) \leq (p^j, 0, E^j)$ deciding $\mu$. We let $p_{i+1} = \text{glob}(1, r, p^j)$ and $E^{i+1} = E^{i+1} \upharpoonright p_{i+1}$ except that $(E^{i+1})_{(\sigma, \tau)} = E_\tau$. In the end we let $(p^{i+1}, 0, E^{i+1}) = (p^{i+k}, 0, E^{i+k})$. It is straightforward to check that (1) - (4) hold. Let $(p^\infty, 0, E^\infty) \in \mathbb{A}(Si)$ be as above. By construction we have (4) for $(p^\infty, 0, E^\infty)$, i.e.:

(5) For every $i < \omega$ and $\tau \in \text{Lev}_{p^\infty}(i)$, if $(\sigma, \tau) \in E^\infty$, $\tau'$ is the $\sigma$-twin of $\tau$ and in $\mathbb{A}(Si)$ there exists a pure extension of $(p^\infty(\tau, \tau'), 1, (E^\infty)_{(\sigma, \tau)})$ deciding $\mu$, then $(p^\infty(\tau, \tau'), 1, (E^\infty)_{(\sigma, \tau)})$ already decides $\mu$.

Now comes the core of the argument illustrating why we added the unavoidable sets to our amoeba conditions. We colour $E^\infty(\langle \sigma \rangle)$ by 3 colours $i < 3$ according to whether $(p^\infty(\tau, \tau'), 1, (E^\infty)_{(\sigma, \tau)})$ decides $\mu$ as 0, as 1, or does not decide $\mu$. As $E^\infty(\langle \sigma \rangle)$ is unavoidable in $p^\infty_{r^0}$, by Lemma 2.3(2) there exist $r \in Si$, $r \leq (p^\infty)_{r^0}, C \subseteq E^\infty(\langle \sigma \rangle) \cap \text{split}(r)$ and $i < 3$ such that $C$ is $r$-unavoidable and every $\tau \in C$ has colour $i$. Letting $\rho = \text{stem}(r)$ and $\rho'$ the $\sigma$-twin of $\rho$ in $p^\infty$, we define $q = r \cup r^{\rho'}$. Hence $q \leq p^\infty$ and stem$(q) = \sigma$. Moreover $E^q$ is defined so that $E^q(\langle \sigma \rangle) = C$ and if $(\sigma, \tau_0, \ldots, \tau_k) \in E^q$, then $(E^q)_{(\sigma, \tau_0, \ldots, \tau_k)} = (E^\infty)_{(\sigma, \tau_0, \ldots, \tau_k)} \upharpoonright q_k$. 

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Now we claim that $i = 2$. By way of contradiction suppose that, say, $i = 0$. As every $(r, n^r, E^r) \leq (q, 0, E^q)$ with $n^r > 0$ extends $(p^\infty(\tau, \tau'), 1, (E^\infty)_{(\sigma, \tau)})$ for some $\tau \in C$ we conclude $(q, 0, E^q) \Vdash \dot{\mu} = 0$. But this is a contradiction to our hypothesis, as $(q, 0, E^q) \not\leq (p^\infty(\tau, \tau'), 1, (E^\infty)_{(\sigma, \tau)})$ for some $\tau \in C$ we conclude $(q, 0, E^q) \Vdash \dot{\mu} = 0$. But this is a contradiction to our hypothesis, as $(q, 0, E^q) \not\leq (p^\infty, 0, E^\infty) \leq (p, 0, E^p)$. But if $i = 2$, the conclusion of Lemma 2.9 holds. \hfill \Box

Remark 2.10 The type of fusion used in the proof of Lemma 2.9 will be used repeatedly in this paper, and it seems to be essentially the only reasonable one for $A(S_i)$. At first glance, property (3) which says that each of the countably many unavoidable sets occurring in the side condition $E$ is shrunk in a non-trivial manner only finitely often seems to be too restrictive. One is tempted to imagine that each of them, i.e. the sets $E((\sigma^{\sim} \tau))$, could be shrunk infinitely often so that the intersection is an unavoidable set. However, there is no natural fusion strategy for this and, moreover, by Lemma 2.3(1) a positive density must be preserved and hence can be determined in one step already.

Theorem 2.11 The forcing notion $A(S_i)$ has the pure decision property.

Proof: We work by contradiction and assume that the assumptions of Lemma 2.9 hold. Again, for the purpose of notational simplicity we assume that $n^p = 0$. Applying Lemma 2.9 iteratively we construct another fusion sequence $\langle (p^i, 0, E^i) : i < \omega \rangle$ such that the following hold:

1. $(p^0, 0, E^0) = p$;
2. $p^{i+1} \leq p^i$ and $\text{Lev}_{p^{i+1}}(i) = \text{Lev}_p(i)$;
3. $E^{i+1}$ equals $E^i \upharpoonright p^{i+1}$ except that for any $\tau = (\tau_0, \ldots, \tau_{i-1})$ such that $\langle \sigma \rangle^{\upharpoonright (\tau_0, \ldots, \tau_{i-1})} \in \text{dom}(E^i)$ and $\tau_{i-1} \in p^{i+1}$ we only require $(E^{i+1})_{(\sigma^{\sim})^{\upharpoonright (\tau_0, \ldots, \tau_{i-1})}} \subseteq (E_i \upharpoonright p^{i+1})_{(\sigma^{\sim})^{\upharpoonright (\tau_0, \ldots, \tau_{i-1})}}$;
4. for every $(r, n^r, E^r) \in A(S_i)$, if $(r, n^r, E^r) \leq (p^{i+1}, 0, E^{i+1})$ and $n^r = i + 1$, then $(r, n^r, E^r)$ does not decide $\dot{\mu}$.

If this works, then by (2) and (3) it is clear that, letting $q = \bigcap_{i < \omega} p^i$ and $E^q = \bigcap_{i < \omega} E^i$, we have $(q, 0, E^q) \in A(S_i)$.

Moreover $(q, 0, E^q) \leq (p^i, 0, E^i) \leq (p, 0, E^p)$ for every $i$, and by (4) we have that no $(r, n^r, E^r) \in A(S_i)$ with $(r, n^r, E^r) \leq (q, 0, E^q)$ decides $\dot{\mu}$. This is a contradiction. Hence we have proved the pure decision property of $A(S_i)$.
The construction of \( \langle(p^i, 0, E^i) : i < \omega \rangle \) is quite clear. Suppose we have gotten \((p^i, 0, E^i)\) as desired. In order to obtain \((p^{i+1}, 0, E^i)\) we perform another fusion construction, building \(\langle(r^k, 0, F^k) : k < \omega \rangle\) such that the following are true:

1. \((r^0, 0, F^0) = (p^i, 0, E^i)\);
2. \(r^{k+1} \subseteq r^k\) and \(\text{Lev}_{r \uparrow^{k+1}}(i + k) = \text{Lev}_{r \uparrow^k}(i + k)\);
3. \(F^{k+1} \equiv F^k \upharpoonright r^{k+1}\) except that for any \(\tau = \langle \tau_0, \ldots, \tau_{i-1} \rangle\) such that \(\langle \sigma \rangle \downarrow \tau \in \text{dom}(F^k)\) and \(\tau_{i-1} \in \text{Lev}_{r \uparrow^{k+1}}(i + k)\) we only require \((F^{k+1})_{\langle \sigma \rangle \downarrow \tau} \subseteq (F^k \upharpoonright r^{k+1})_{\langle \sigma \rangle \downarrow \tau} \);
4. \(\text{dom}(\langle \sigma \rangle \downarrow \tau) = \text{dom}(F^k)\) except that for any \(\tau = \langle \tau_0, \ldots, \tau_{i-1} \rangle\) such that \(\langle \sigma \rangle \downarrow \tau \in \text{dom}(F^k)\) and \(\tau_{i-1} \in \text{Lev}_{r \uparrow^{k+1}}(i + k)\) we only require \((F^{k+1})_{\langle \sigma \rangle \downarrow \tau} \subseteq (F^k \upharpoonright r^{k+1})_{\langle \sigma \rangle \downarrow \tau} \);
5. For every \(\tau = \langle \tau_0, \ldots, \tau_{i-1} \rangle\) as in (4), for every \((r, n^r, E^r) \in \mathfrak{A}(Si)\), if \((r, n^r, E^r) \le (\text{glob}(r^{k+1}, \tau), i, (F^{k+1})_{\langle \sigma \rangle \downarrow \tau})\) and \(n^r = i + 1\), then \((r, n^r, E^r)\) does not decide \(\dot{\mu}\).

Suppose we have already obtained \((r^k, 0, F^k)\). In order to obtain \((r^{k+1}, 0, F^{k+1})\) we list the finitely many \(\tau\) as in (7). Suppose we have \(h\) of them. We are building a descending length-\(h\) chain of conditions of the form \((r^i, 0, F^i)\) below \((r^k, 0, F^k)\) such that \(\text{Lev}_{r \uparrow^i}(i + k) = \text{Lev}_{r \uparrow^k}(i + k)\), at each step taking care of another \(\tau\). The last condition then will be \((r^{k+1}, 0, F^{k+1})\). Suppose we have already gotten \((r^i, 0, F^i)\) and have to take care of \(\tau = \langle \tau_0, \ldots, \tau_{i-1} \rangle\). As (4) holds for \((p^i, 0, E^i)\), we know that no pure extension of \((\text{glob}(r^i, \langle \sigma \rangle \downarrow \tau), i, (F^i)_{\langle \sigma \rangle \downarrow \tau})\) decides \(\dot{\mu}\). We apply Lemma 2.9 and obtain \((u, i, E^u) \le (\text{glob}(r^i, \langle \sigma \rangle \downarrow \tau), i, (F^i)_{\langle \sigma \rangle \downarrow \tau})\) such that no extension of it of depth \(i + 1\) decides \(\dot{\mu}\).

We let \(r'' = \text{glob}(i, u, r')\) and \(F'' = F'\) except that \((F'')_{\langle \sigma \rangle \downarrow \tau} = E^u\). Then \((r'', 0, F'')\) is the next element of the length-\(h\) chain we are about to build. In this way we obtain \((r^{k+1}, 0, F^{k+1})\). This finishes the construction of \(\langle(r^k, 0, F^k) : k < \omega \rangle\). All requirements can be verified easily. Now we let

\[
\begin{align*}
p_{i+1} &= \bigcap_{k < \omega} r_k \\
E^{i+1} &= \bigcap_{k < \omega} F^k
\end{align*}
\]

Then \((p^{i+1}, 0, E^{i+1})\) is as desired. This finishes the construction of \(\langle(p^i, 0, E^i) : i < \omega \rangle\). \(\square\)
3 \( A(Si) \) does not add random reals

An immediate consequence of the pure decision property of \( A(Si) \) (Theorem 2.11) is the following:

Corollary 3.1 Suppose that \( p \in A(Si) \), \( n < \omega \) and \( \dot{x} \) is an \( A(Si) \)-name such that \( p \models \dot{x} \in 2^\omega \). There exist \( q \in A(Si) \) and \( \xi \in 2^n \) such that \( q \leq^0 p \) and

\[
q \models \dot{x} \upharpoonright n = \xi.\]

Below, by \( \mu \) we will denote the standard measure on the Cantor space, i.e. the product measure of the equidistributive measure on \( \{0, 1\} \). Given \( \xi \in 2^n \), by \( [\xi] \) we denote the basic open set in \( 2^\omega \) determined by \( \xi \). Thus we have \( \mu([\xi]) = 1/(n+1) \).

Theorem 3.2 The forcing notion \( A(Si) \) does not add a random real.

Proof: Suppose that \( p = (p, n^p, E^p) \in A(Si) \) and \( \dot{x} \) is an \( A(Si) \)-name such that

\[
p \models \dot{x} \in 2^\omega.\]

Let \( \sigma = \langle \sigma_0, \ldots, \sigma_{n^p} \rangle \) such that \( \sigma_i \) is the leftmost element of \( \text{Lev}_{p}(i) \) for every \( i \leq n^p \). Let \( \langle \varepsilon_i : i < \omega \rangle \) be a sequence of positive reals with \( \lim_{i \to \infty} \varepsilon_i = 0 \).

We shall construct a sequence \( \langle (p^i, n^p, E^i) : i < \omega \rangle \) in \( A(Si) \) and a sequence \( \langle A_i : n^p < i < \omega \rangle \) of open subsets \( A_i \subseteq 2^\omega \) such that the following properties hold:

1. \( (p^0, n^p, E^0) = p; \)
2. \( p^{i+1} \leq p^i \) and \( \text{Lev}_{p^{i+1}}(n^p + i) = \text{Lev}_{p^i}(n^p + i); \)
3. \( E^{i+1} \) equals \( E^i \upharpoonright p^{i+1} \) except that for every \( \tau = \langle \tau_0, \ldots, \tau_{i-1} \rangle \) such that \( \sigma \upharpoonright \tau \in \text{dom}(E^i) \) and \( \tau_{i-1} \in p^{i+1} \) we only require

\[
(E^{i+1})_{\sigma \upharpoonright \tau} \subseteq (E^i \upharpoonright p^{i+1})_{\sigma \upharpoonright \tau};\]
4. \( \mu(A_i) < \varepsilon_i; \)
5. \( (p^{i+1}, n^p, E^{i+1}) \models \dot{x} \in A_{n^p+i+1}. \)
If this construction can be performed, we let \( q = \bigcap_{i<\omega} p^i, n^q = n^p, E^q = \bigcap_{i<\omega} E^i, q = (q, n^q, E^q) \) and \( A = \bigcap_{n^p<i<\omega} A_i \). We conclude that \( q \in A(S_i), q \triangleq (p^{i+1}, n^p, E^{i+1}) \leq p \) for every \( i \) and \( \mu(A) = 0 \). By (5) we conclude
\[
q \models \dot{x} \in A.
\]

For the construction, we assume that for \( i < \omega \) we have gotten \( \langle (p^j, n^p, E^j) : j \leq i \rangle \) and \( \langle A_j : n^p < j \leq i \rangle \) as desired. Let \( \langle \delta_k : k < \omega \rangle \) be a sequence of positive reals such that \( \sum_{k=0}^{\infty} \delta_k \leq \varepsilon_{i+1} \). In order to obtain \( (p^{i+1}, n^p, E^{i+1}) \) and \( A_{i+1} \) as desired we perform another fusion construction, by which we build sequences \( \langle (r^k, n^p, F^k) : k < \omega \rangle \) and \( \langle B_k : k < \omega \rangle \) such that the following requirements are met:

\[
\begin{align*}
(6) \quad & (r^0, n^p, F^0) = (p^i, n^p, E^i); \\
(7) \quad & r^{k+1} \leq r^k \text{ and } \text{Lev}_{r,k+1}(n^p + i + k) = \text{Lev}_{r,k}(n^p + i + k); \\
(8) \quad & F^{k+1} \text{ equals } F^k | r^{k+1} \text{ except that for every } \bar{\rho} = \langle \rho_0, \ldots, \rho_{i-1} \rangle \text{ such that } \bar{\sigma}^{-1} \rho \in \text{dom}(F^k) \text{ and } \rho_{i-1} \in \text{Lev}_{r,k}(n^p + i + k) \text{ (hence } \rho_{i-1} \in r^{k+1} \text{ by (7)) we only require } \\
& (F^{k+1})_{\bar{\sigma}^{-1} \bar{\rho}} \subseteq (F^k)_{\bar{\sigma}^{-1} \bar{\rho}}; \\
(9) \quad & B_k \subseteq 2^\omega \text{ is open and } \mu(B_k) \leq \delta_k; \\
(10) \quad & \text{for every } \bar{\rho} \text{ as in (8) we have } \\
& (\text{glob}(r^{k+1}, \bar{\sigma}^{-1} \bar{\rho}), n^p + i, (F^{k+1})_{\bar{\sigma}^{-1} \bar{\rho}}) \models \dot{x} \in B_{k+1}.
\end{align*}
\]

Suppose we have obtained \( (r^k, n^p, F^k) \) and \( B_k \) as desired. (Let \( B_0 = \emptyset \).) We list all the finitely many \( \bar{\rho} \) as in (8). Suppose we have \( h \) of them. We are building a descending length-\( h \) chain of conditions of the form \( (r', n^p, F') \) below \( (r^k, n^p, F^k) \) such that \( \text{Lev}_{r'}(n^p+i+k) = \text{Lev}_{r,k}(n^p+i+k) \), at each step taking care of another \( \bar{\rho} \). The last condition then will be \( (r^{k+1}, n^p, F^{k+1}) \). Let \( \zeta_{k+1} = \frac{\delta_{k+1}}{h} \). Suppose we have already gotten \( (r', n^p, F') \) and have to take care of \( \bar{\rho} = \langle \rho_0, \ldots, \rho_{i-1} \rangle \). We apply Corollary 3.1 to \( (\text{glob}(r', \bar{\sigma}^{-1} \bar{\rho}), n^p+i, (F')_{\bar{\sigma}^{-1} \bar{\rho}}) \), \( n := 1/\zeta_{k+1} \) and \( \dot{x} \), and find \( (u, n^p+i, E^u) \triangleq (\text{glob}(r', \bar{\sigma}^{-1} \bar{\rho}), n^p+i, (F')_{\bar{\sigma}^{-1} \bar{\rho}}) \) and some open set \( B' \subseteq 2^\omega \) with \( \mu(B') \leq \zeta_{k+1} \), such that
\[
(u, n^p+i, E^u) \models \dot{x} \in B'.
\]
Now we let $r'' = \text{glob}(n^p + i, u, r')$ and $F'' = F'$ except that $(F'')_{\sigma''} = E^u$. Then $(r'', n^p, F'')$ is the next element of the length-$h$ chain we are about to build.

In this way we obtain $(r^{k+1}, n^p, F^{k+1})$ and we let $B_{k+1}$ be the union of all sets $B'$ we got during the $h$ many steps. This finishes the construction of $\langle (r^k, n^p, F^k) : k < \omega \rangle$ and $\langle B_k : k < \omega \rangle$. All requirements can be verified easily. Now we let

$$p^{i+1} = \bigcap_{k<\omega} r_k,$$

$$E^{i+1} = \bigcap_{k<\omega} F^k,$$

$$A_{n^p+i+1} = \bigcup_{k<\omega} B_k.$$

Then (1) - (5) hold. This finishes the construction of $\langle (p^i, n^p, E^i) : i < \omega \rangle$ and $\langle A_i : n^p < i < \omega \rangle$ as desired, and hence the proof is complete. □

**Remark 3.3** Note that our proof of Theorem 3.2 actually gives the following general scheme: Suppose that $p \in \mathbb{A}(S_i)$, $\langle \sigma_\nu : \nu \in <\omega \omega \rangle$ and $\langle M(\nu) : \nu \in <\omega \omega \rangle$ are such that $\forall \nu M(\nu) < \omega$ and $p \models \forall \nu \sigma_\nu \in M(\nu)$. Let $\langle \tau_{(j_0, \ldots, j_m)} : (j_0, \ldots, j_m) \in m+1 \omega \rangle$ canonically list all depth-$m + 1$-sequences of $p$ (see Remark 2.8(3)). Then there exists $q \leq^0 p$ such that for every $m < \omega$ and every $\langle j_0, \ldots, j_{m-1} \rangle \in \Omega_p(q)$ (see 2.8(3)) we have that $q(\tau_{(j_0, \ldots, j_{m-1})})$ decides $\sigma_{(j_0, \ldots, j_{m-1})}$.

## 4 Iterating $\mathbb{A}(S_i)$

In this section we show that iterating $\mathbb{A}(S_i)$ with countable supports does not add random reals. By standard arguments this implies that we can obtain a $\text{ZFC}$-model where $\text{cov}(\mathcal{N}) = \aleph_1$ and $\text{add}(\mathcal{I}(S_i)) = \aleph_2$. In order to perform fusions in iterations of $\mathbb{A}(S_i)$ we need a more effective tool than recursion along the length of depth-sequences (see Definition 2.7) used in the proofs of Theorems 2.11 and 3.2 above. For this reason we shall introduce the concept of level-sequence in the next definition. Given $p = (p, n^p, E^p) \in \mathbb{A}(S_i)$, a level-$k + 1$-sequence of $p$ is simply a depth-sequence $\bar{\tau}$ of $p$ such that its last member is in $\text{Lev}_p(n^p + k + 1)$. Then clearly, the depth of $\bar{\tau}$ (i.e. its length) is at most $k + 1$. Moreover, every depth-$m + 1$-sequence of $p$ is a level-$k + 1$-sequence of $p$ for some $k \geq m$, and for fixed $m$, for almost every $k$ there are level-$k + 1$-sequences of depth $m + 1$. For this reason, even though the proofs
of Theorems 2.11 and 3.2 are the natural ones, they needed infinitely many infinite fusions. The reason for introducing level-sequences is that now we can perform just one fusion that incorporates all the old ones by recursion on the levels, at each level taking care of all the finitely many level-sequences at that level.

**Definition 4.1** (1) Let \( p = (p, n^p, E^p) \in \mathbb{A}(S_i) \) and let \( \sigma = \text{stem}(E^p), \tau = \langle \sigma_0, \ldots, \sigma_{n^p} \rangle \) (see Def. 2.4). Let \( \tau \in \text{Lev}_p(n^p + k + 1) \) such that \( \sigma_{n^p} \subseteq \tau \) and \( \tau(|\sigma_{n^p}|) = 0 \). Let \( \langle \tau_0, \ldots, \tau_k \rangle \) be such that \( \tau_i \) is the initial segment of \( \tau \) in \( \text{Lev}_p(n^p + i + 1) \), thus \( \tau_k = \tau \). Letting \( \rho \in k^2 \) be defined by setting \( p(i) = \tau(|\tau_i|) \) for \( i < k \), we call \( \rho \) the \textbf{p-code of} \( \tau \), denoted by \( \text{code}_p(\tau) \). Moreover, letting \( \langle k(j) : j \leq m \rangle \) be any subsequence of \( \langle 0, \ldots, k \rangle \) with \( k(m) = k \) and \( \tau := \langle \tau_{k(j)} : j \leq m \rangle \), we call \( \tau \) a \textbf{level-} \( k \)-\textbf{sequence of} \( p \) provided that \( \tau \) is a depth-\( m \)-1-sequence of \( p \). In this case we call \( \langle k(j) : j < m \rangle \) the \textbf{type of} \( \tau \) in \( p \), denoted by \( \text{tp}_p(\tau) \). Note that \( \tau \) is a depth-\( m \) \( 1 \)-sequence of \( p \).

(2) We let
\[
\text{TP}(p, k + 1) = \{ (\rho, M) \in k^2 \times \mathcal{P}(k) : \exists \tau, \tau(\tau \in \text{Lev}_p(n^p + k + 1) \land \\
\rho = \text{code}_p(\tau) \land |\tau| = \tau \land \\
\tau \text{ is a level-} k \text{-} 1 \text{-sequence of } p \land M = \text{tp}_p(\tau) \} \}.
\]

Given a level-\( k \)-1-sequence \( \tau \) of \( p \), in Definition 2.7 we have defined \( p(\tau) \in \mathbb{A}(S_i) \) as
\[
(\text{glob}(p, \tau), n^p + |\text{tp}_p(\tau)| + 1, (E^p)\tau, \tau).
\]

Let \( (\rho, M) \in k^2 \times \mathcal{P}(k) \). If there are a level-\( k \)-1-sequence \( \tau \) of \( p \) and \( \tau = \tau(|\tau| - 1) \) with \( \rho = \text{code}_p(\tau) \) and \( M = \text{tp}_p(\tau) \) we write \( \tau = \tau(\rho, p, \tau) = \tau(\rho, M, p) \). Finally, we call \( \emptyset \) the \textbf{level-0-sequence of} \( p \). We let \( p(\emptyset) = p, TP(p, 0) = \{ \emptyset \} \) and \( TP(\emptyset, p) = \emptyset \).

(3) We let \( TP_0 = \{ \emptyset \} \) and \( TP_{k+1} = k^2 \times \mathcal{P}(k) \) the set of all \textbf{potential type pairs} of level-\( k \)-1-sequences of members of \( \mathbb{A}(S_i) \), and we fix a bijection \( TP_{N_{k+1}} : TP_{k+1} \rightarrow 2^{k} \) assigning to each type pair \( (\rho, M) \) its \textbf{type pair number} \( TP_{N_{k+1}}(\rho, M) \). We let \( TP_{N_0}(\emptyset) = 0 \). Finally let \( TP = \bigcup_{k<\omega} TP_{k+1} \).

There exists a natural partial ordering on \( TP \) defined as follows: Given \( (\rho_0, M_0), (\rho_1, M_1) \in TP \) we let \( (\rho_0, M_0) < (\rho_1, M_1) \) iff \( (\rho_0, M_1) \in TP_{k+1} \) for some \( k_0 < k_1, \rho_0 \subseteq \rho_1 \) and \( M_1 \cap (k_0 + 1) = M_0 \cup \{ k_0 \} \). Note that given \( (\rho, M) \in TP_{k+1} \) there exists precisely one maximal chain in \( TP \) with \( (\rho, M) \) as its maximum: It has length \( |M| + 1 \) and if \( \langle k(j) : j < m \rangle \) increasingly enumerates \( M \) then it is \( \langle (\rho \upharpoonright k(j)), \{ k(i) : i < j \} : j < m \rangle \). Therefore
we define \( \text{depth}(\rho, M) = |M| + 1 \). Note that if \((\rho, M) \in TP(p, k + 1)\) for some \( p \in A(S_i) \), then \((\rho \upharpoonright k(j), \{ k(i) : i < j \}) \in TP(p, k(j) + 1)\) for every \( j < m \). Moreover, note that in this case \( P(\rho, M, p) \) is a depth-\( M + 1 \)-sequence of \( p \).

(4) We define a push-down function as a one-to-one function \( P : \omega \rightarrow \omega \) such that the following hold:

(i) \( P(\varnothing) = \varnothing \) and \( P(\nu) = \nu \) for every \( \nu \in \omega \);

(ii) \( P(\nu) \subseteq P(\nu') \) whenever \( \nu \subseteq \nu' \) and \( |\nu| = 1 \);

(iii) if \( |\nu| = k + 1, \nu(k) = j \cdot 2^k + TPN_{k+1}(t) \) for some \( t \in TP_{k+1} \) and \( \text{depth}(t) = m + 1 \), then \( P(\nu^<i) \in m+2\omega \);

(iv) given \( \nu, \nu', k < k', i, i', j, j' \) such that \( |\nu| = k + 1, |\nu'| = k' + 1, \nu \subseteq \nu' \), \( \nu'(k + 1) = j, i = j \cdot 2^k + TPN_{k+1}(t) \) and \( i' = j' \cdot 2^{k'} + TPN_{k'+1}(t') \) for some \( t \in TP_{k+1} \) and \( t' \in TP_{k'+1} \) such that and \( t < t' \), we have \( P(\nu^<i) \subseteq P(\nu'^<i') \).

**Remark 4.2**

(1) The reason for defining \( TP(p, k) \) is that if \( p \) is only a name for a member of \( A(S_i) \), then while the set of potential level-\( k \)-sequences of \( p \) may be infinite the set \( TP(p, k) \) can attain only finitely many values, hence can be purely decided.

Note that there are potential type pairs \( t \) which can never be equal to some \( t' \in TP(p, k) \) for any \( p, k \), as any such \( t' = (\rho, M) \) always satisfies \( \rho(j) = 0 \) for \( j \in M \).

(2) Note that \( TP(p, 1) \) is either empty or contains one element which then is uniquely determined. Moreover the set of \( p \in A(S_i) \) for which the second alternative holds is dense. Therefore, wlog we may assume that this is true for every \( p \in A(S_i) \), hence we will never have to decide \( TP(p, 1) \).

(3) Note that push-down functions trivially exist. The definition is closely linked to the proof of the crucial fact that the peculiar antichain structure of \( A(S_i) \) already described in the introduction is preserved by the iteration. (See the proof of Lemma 4.6 below where it appears in its simplest form and a push-down function is used.)

Before we shall show how these concepts get into action, we check that \( A(S_i) \) satisfies Baumgartner’s Axiom A (see [3]), i.e. a property that guarantees that the iteration does not collapse \( \aleph_1 \). It is well-known that Axiom A is strictly stronger than properness.
Definition 4.3 Given \( p, q \in A(S_i) \) and \( n < \omega \), we define \( p \leq^n q \) to hold iff the following are satisfied:

1. \( p \leq^0 q \), and if \( n > 0 \) then in addition
2. \( \text{Lev}_p(n^p + n) = \text{Lev}_q(n^p + n) \) and
3. letting \( \sigma = \text{stem}(E^q) \), then \( E^p = E^q \upharpoonright p \) except that for every \( i < \omega \) and every level-\( n+i \)-sequence \( \tau \) of \( p \) we only require

\[
(E^p)_{\sigma \triangleright \tau} \subseteq (E^q \upharpoonright p)_{\sigma \triangleright \tau}.
\]

Lemma 4.4 Forcing \( A(S_i) \) satisfies Axiom A.

Proof: First it is trivial to see that \( p \leq^{n+1} q \) implies \( p \leq^n q \). Secondly, we need to check that given \( \langle p_i : i < \omega \rangle \) in \( A(S_i) \) such that \( p_{i+1} \leq^i p_i \) for every \( i \), there exists \( q \in A(S_i) \) such that \( q \leq^i p_i \) for all \( i \). But this is very much analogous to the argument we used in the proof of Lemma 2.11. If \( p_i = (p_i, n^{p_i}, E^{p_i}) \), we can let \( q = \bigcap_{i<\omega} p_i, n^q = n^{p_0} \) and \( E^q = \bigcap_{i<\omega} E^{p_i} \). By (2) \( q \) is a Silver tree and by (3) \( E^q \) is a tree of possible extensions for \( q \). Thus by construction \( (q, n^q, E^q) \) is as desired. In the following, such sequence \( \langle p_i : i < \omega \rangle \) will be called a fusion sequence and \( q \) its fusion. Sometimes we shall denote it by \( \bigwedge \{ p_i : i < \omega \} \).

Thirdly, given \( p \in A(S_i), m < \omega \) and an \( A(S_i) \)-name \( \dot{a} \) such that \( p \Vdash \dot{a} \in V \), we must find \( q \leq^m p \) and a countable set \( x \in V \) such that \( q \Vdash \dot{a} \in x \). Let \( D \) be the set of all extensions of \( p \) that decide \( \dot{a} \). Hence \( D \) is dense open below \( p \). Very similarly as in the proof of Lemma 2.11 we construct \( \langle p_i : i < \omega \rangle \) in \( A(S_i) \) such that \( p_0 = p, p_{i+1} = p_i \leq^{m+i} p_i \) for every \( i \) and for every \( \tau \) a level-\( m+i \)-sequence of \( p_{i+1} \), if some pure extension of \( p_{i+1}(\tau) \) belongs to \( D \), then \( p_{i+1}(\tau) \) already does. Letting \( q \) be the fusion of \( \langle p_i : i < \omega \rangle \), we have that \( q \leq^m p \) and every \( r \leq q \) is compatible with \( q(\tau) \) for some level-\( m+i \)-sequence \( \tau \), for some \( i \). Hence we can let \( x \) be the set of all decisions made about the value of \( \dot{a} \) by any \( q(\tau) \) like this that belongs to \( D \). \( \square \)

Remark 4.5 Alternatively we could define the orderings \( \leq^n \) witnessing that \( A(S_i) \) satisfies Axiom A by replacing (3) in Definition 4.3 by

\[
(3') \text{ letting } \sigma = \text{stem}(E^q), \text{ then } E^p = E^q \upharpoonright p \text{ except that for every depth-}n-\text{sequence } \tau \text{ of } p \text{ we only require}
\[
(E^p)_{\sigma \triangleright \tau} \subseteq (E^q \upharpoonright p)_{\sigma \triangleright \tau}.
\]

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The fusion sequences with respect to these orderings are then precisely the 
one we constructed in the proofs of Theorems 2.11 and 3.2. However, as we 
explained above, we do not want to use these for the iteration.

There exists a preservation theorem for not adding random reals due to Judah 
and Repicky (see [12]) which says that if \( \langle P_\alpha, \dot{Q}_\alpha : \alpha < \beta \rangle \) is an iteration 
of proper forcings such that no \( P_\alpha \) adds a random real, then its countable 
support limit does neither. On the other hand it is known that there exists 
a two step iteration of universally Baire proper real forcing such that neither 
step adds a random real while the iteration does (see [28], p. 114).

We first take the time to prove that the two-step-iteration \( A(S_i) * \dot{A}(S_i) \) does 
not add random reals. The following key lemma shows that \( A(S_i) * \dot{A}(S_i) \) shares an analog of the crucial antichain structure of \( A(S_i) \) exhibited in 
Remark 3.3, hence does not add random reals. We could then go on and 
similarly show that every finite iteration of universally Baire proper real forcing such that neither 
step adds a random real while the iteration does (see Lemma 4.10 below). For this we will 
elaborate on the ideas that are presented in a simple form in the following 
Lemma.

**Lemma 4.6** Suppose \((p, \dot{q}) \in A(S_i) * \dot{A}(S_i), \langle \dot{\sigma}_\nu : \nu \in \omega \rangle \) is a family of 
\( A(S_i) * \dot{A}(S_i) \)-names and \( \langle M(\nu) : \nu \in \omega \rangle \) is a family in \( \omega \) such that 
\((p, \dot{q}) \Vdash_{A(S_i) * \dot{A}(S_i)} \forall \nu \in \omega \ \dot{\sigma}_\nu \in M(\nu)^2 \).

There exist \((u, \dot{v}) \in A(S_i) * \dot{A}(S_i), \) a Laver tree \( \Omega \) with empty stem, family 
\( R = \langle R_k : k < \omega \rangle \) where \( R_k = \langle r_\nu : \nu \in \Omega \cap k^\omega \rangle \) and family \( \langle a_\nu : \nu \in \Omega \rangle \) in 
\( \omega^2 \) such that

1. \((u, \dot{v}) \leq^0 (p, \dot{q}), \)

2. every \( R_\nu \) is a maximal antichain in \( A(S_i) * \dot{A}(S_i) \) below \((u, \dot{v})\) such 
   that \( R_0 = \{(u, \dot{v})\}, R_{k+1} \) refines \( R_k \) according to their indexing (i.e. 
   \( r_{\nu <^* \nu} \), \( R_{k+1} \) refines \( R_k \) according to their indexing (i.e. 
   \( r_{\nu <^* \nu} \), \)

3. \( \forall \nu \in \Omega \ r_\nu \Vdash_{A(S_i) * \dot{A}(S_i)} \dot{\sigma}_\nu = a_\nu. \)
**Proof:** By the pure decision property of $A(Si)$ applied twice, wlog we may assume that $(p, \dot{q})$ decides $\dot{\sigma}_q$. Let $a_0$ be its value. We fix a push-down function $P$. Let $G$ be an $A(Si)$-generic filter containing $p$. Let $q = q[G]$, thus $q = (q, n^q, E^q)$, and let $\sigma = (\sigma_0, \ldots, \sigma_{n^q})$ be stem($E^q$).

In $V[G]$ we construct a fusion tree $\langle s_\nu : \nu \in \omega^\omega \rangle$ in $A(Si)$ below $q$, i.e. we demand that the following hold:

1. $s_\emptyset = q$;
2. $s_{\nu \cdot j} \leq^{[\nu]} s_\nu$;
3. if $|\nu| = k$, then for every $t \in TP(s_\nu, k)$ and for all $j$ we have
   $$s_{\nu \cdot j}(\tau(t, s_\nu)) \models_{A(Si)} \dot{\sigma}_{P(\mu(\nu, t, j))} = a(\mu(\nu, t, j)),$$
   where $\mu(\nu, t, j) = \nu \cdot j = \langle j \rangle$ if $k = 0$,
   $$\mu(\nu, t, j) = \nu \cdot j \cdot 2^{(k-1)} + TPN_k(t))$$
   if $k > 0$ and $a(\mu(\nu, t, j)) \in \omega^2$.

For the definition of $\tau(t, s_\nu)$ see Definition 4.1(2). Note that by Definition 4.1(4)(iii) we have $|P(\mu(\nu, t, j))| = \text{depth}(t) + 1$. Moreover let us make the trivial remark that $\mu(\nu, t, j) \neq \mu(\nu', t', j')$ for $(\nu, t, j) \neq (\nu', t', j')$. As the function $P$ is supposed to be one-to-one, we conclude $P(\mu(\nu, t, j)) \neq P(\mu(\nu', t', j'))$.

This construction can be achieved by repeatedly applying the pure decision property of $A(Si)$ and modifying its fusion technique that we established in §§1,2,3 in a straightforward manner. Therefore we only give a brief sketch: If we have obtained $s_\nu$ where $|\nu| = k$, we can determine the finite set $TP(s_\nu, k)$. Suppose that it has $h$ many members. For given $j < \omega$, we construct a length-$h$-chain below $s_\nu$ that descends with respect to $\leq_k$, at each step taking care of another $t \in TP(s_\nu, k)$, i.e. if $s'$ is the last element of that chain constructed so far, then its next element $s''$ equals $s'$ except that $s'((\tau(t, s_\nu)))$ is replaced by some pure extension that decides $\dot{\sigma}_{P(\mu(\nu, t, j))}$. As $\tau(t, s_\nu)$ is a level-$k$-sequence we have $s'' \leq^k s'$. In the end, $s_{\nu \cdot j}$ is the last member of the length-$h$-chain we just described.

In $V$ we can find $A(Si)$-names $\langle \dot{s}_\nu : \nu \in \omega^\omega \rangle$ and $\langle \dot{a}(\langle j \rangle) : j < \omega \rangle$ and $\langle \dot{a}(\mu(\nu, t, j)) : t \in TP(s_\nu, k), j < \omega \rangle$ if $|\nu| = k > 0$ such that (4), (5), (6) are forced by $p$ to hold for these.
We let \( \dot{\sigma}'(j) = \dot{a}(j) \) for \( j < \omega \) and

\[
\dot{\sigma}'_{\nu \leftarrow j} = \langle \dot{a}(\mu(\nu, t, j)) : t \in TP(\dot{s}_{\nu, k}) \rangle \cap TP(\dot{s}_{\nu \leftarrow j}, k + 1) \]

for every \( k > 0, \nu \in k^\omega \) and \( j < \omega \). By the proof of Theorem 3.2 (see Remark 3.3 for the notations below) we obtain \( u \leq 0^p \) and families

\[
C^k = \langle c^k_\nu TPV(\nu, k + 1) : \nu \in k^+ \cap \Omega_p(u) \rangle
\]

where TPV abbreviates ”type pair values” and

\[
TPV : \bigcup_{k<\omega} k^1 \times \{k + 1\} \rightarrow TP
\]

such that, letting \( \langle \tau_{(j_0, \ldots, j_m)} : (j_0, \ldots, j_m) \in m^+ \rangle \) canonically list all depth-\( m + 1 \)-sequences of \( p \) (see Remark 2.8(3)),

\[
u(\tau_{(j)}) \models_{\mathbb{A}(\mathcal{S}i)} \dot{\sigma}'_{(j)} = c^0_{(j)}
\]

for every \( (j) \in 1^1 \cap \Omega_p(u) \) and

\[
u(\tau_{\nu \leftarrow j}) \models_{\mathbb{A}(\mathcal{S}i)} \dot{\sigma}'_{\nu \leftarrow j} = c^k_{\nu \leftarrow j} \cap TPV(\nu \leftarrow j, k + 1)
\]

for every \( k > 0, \nu \in k^\omega \) and \( j < \omega \) such that \( \nu \leftarrow j \in \Omega_p(u) \). Note that \( u(\tau_{\nu \leftarrow j}) \leq u(\tau_{\nu}) \) and consequently we have

\[
u(\tau_{\nu \leftarrow j}) \models_{\mathbb{A}(\mathcal{S}i)} TP(\dot{s}_{\nu}, k) = TPV(\nu, k).
\]

Hence there exists \( \langle b^k(\mu(\nu, t, j)) : t \in TPV(\nu, k) \rangle \) which equals \( c^k_{\nu \leftarrow j} \). We let \( b^0(\langle j \rangle) = c^0_{\langle j \rangle} \). Now combining these decisions with (6), for every \( k > 0, \nu \in k^\omega, t \in TPV(\nu, k) \) and \( j < \omega \) with \( \nu \leftarrow j \in \Omega_p(u) \) we obtain

\[
u(\tau_{\nu \leftarrow j}) \models_{\mathbb{A}(\mathcal{S}i)} \dot{\sigma}_{\nu \leftarrow j} TP(\dot{s}_{\nu}, k) = TPV(\nu, k)
\]

In order to define \( \dot{v} \) let \( G \) be an \( \mathbb{A}(\mathcal{S}i) \)-generic filter containing \( u \). There exists \( x_G \in [\Omega_p(u)]^1 \) such that \( u(\tau_{x_G[k+1]}^G) \in G \) for every \( k \). By (5), \( \langle s_{x_G[k+1]}^G : k < \omega \rangle \) is a fusion sequence, and hence \( \bigwedge \{s_{x_G[k+1]}^G : k < \omega \} \) exists in \( \mathbb{A}(\mathcal{S}i) \). Let \( \dot{v} \) be the canonical \( \mathbb{A}(\mathcal{S}i) \)-name denoting it. Clearly, for every \( \nu \in k^\omega \cap \Omega_p(u) \) we have

\[
u(\tau_{\nu}) \models_{\mathbb{A}(\mathcal{S}i)} TP(\dot{v}, k) = TP(\dot{s}_{\nu}, k) = TPV(\nu, k)
\]
and for every \( t \in TPV(\nu, k) \)
\[
\mathbf{u}(\tau_\nu) \Vdash_{\mathcal{A}(Si)} \dot{\tau}(t, \dot{v}) = \dot{\tau}(t, \dot{s}_\nu).
\]
Moreover by (7), if in addition \( j < \omega \) is such that \( \nu^j \in \Omega_p(\mathbf{u}) \), then
\[
(8) \ (\mathbf{u}(\tau_\nu^{j-1}), \dot{v}(\tau(t, \dot{v}))) \Vdash_{\mathcal{A}(Si) + \mathcal{A}(Si)} \dot{a}(\mu(\nu, t, j)) = b^k(\mu(\nu, t, j)).
\]

Now that everything has been set up nicely, we are ready to enter into the core of the matter. We define \( \langle R_m : m < \omega \rangle \) as follows:
\[
R_0 = \{(\mathbf{u}, \dot{v})\},
\]
\[
R_1 = \{(\mathbf{u}(\tau_\nu), \dot{v}) : \nu \in 1^\omega \cap \Omega_p(\mathbf{u})\},
\]
\[
R_{m+2} = \{(\mathbf{u}(\tau_\nu^{j-1}), \dot{v}(\tau(t, \dot{v}))) : k \geq m + 1 \wedge \nu \in 1^\omega \wedge j < \omega \wedge
\nu^j \in \Omega_p(\mathbf{u}) \wedge t \in TPV(\nu, k) \wedge \text{depth}(t) = m + 1\}.
\]

We claim that every \( R_m \) is a maximal antichain below \( (\mathbf{u}, \dot{v}) \). For \( R_1 \) this is clear as \( \{\mathbf{u}(\tau_\nu) : \nu \in 1^\omega \cap \Omega_p(\mathbf{u})\} = R^1(\mathbf{u}) \) (see Definition 2.7) is a maximal antichain below \( \mathbf{u} \). The basic idea in defining \( R_{m+2} \) is that it is a modification of the set of all \( (\mathbf{u}, \dot{v}(\tau)) \) where \( \tau \) denotes a depth-\( m+1 \)-sequence of \( \dot{v} \). As we need to have decisions taken by each member of it and we cannot shrink infinitely often at the same place, if \( \tau \) is a level-\( k \)-sequence, then we replace \( (\mathbf{u}, \dot{v}(\tau)) \) by the set of all \( (\mathbf{u}(\bar{\tau}), \dot{v}(\bar{\tau})) \) where \( \mathbf{u}(\bar{\tau}) \in R^{k+1}(\mathbf{u}) \).

Let us check maximality of \( R_{m+2} \): Let \( (\mathbf{u}', \dot{v}') \leq (\mathbf{u}, \dot{v}) \). We choose \( G \) an \( \mathcal{A}(Si) \)-generic filter such that \( \mathbf{u}' \in G \). As \( \mathbf{u}' \leq \mathbf{u} \), there exists a unique branch \( x_G \in [\Omega_p(\mathbf{u})] \) such that
\[
\forall k < \omega \mathbf{u}(\pi_{x_G} | k) \in G.
\]
By construction of \( \dot{v} \) we have
\[
\dot{v}[G] = \bigwedge_{k < \omega} \dot{s}_{x_G | k}[G].
\]
Let us write \( \dot{v}'[G] = \mathbf{v}' = (\nu', n', E^{\nu'}) \) and \( \dot{v}[G] = \mathbf{v} = (\nu, n^\nu, E^n) \). Thus \( \mathbf{v}' \leq \mathbf{v} \). Wlog we may assume that \( n' \geq n^\nu + m + 1 \). Hence there exists a depth-\( m+1 \)-sequence \( \pi \) of \( \mathbf{v} \) such that \( \mathbf{v}' \leq \mathbf{v}(\pi) \). Let \( k \) such that \( \pi \) is a level-\( k \)-sequence of \( \mathbf{v} \). Clearly we have \( k \geq m+1 \). There exists \( t \in TPV(x_G \upharpoonright k, k) \) such that \( \pi = \bar{\tau}(t, \mathbf{v}) \). Wlog we may assume
\[
\mathbf{u}' \Vdash_{\mathcal{A}(Si)} \mathbf{v}' \leq \dot{v}(\bar{\tau}(t, \dot{v})).
\]
As \( u(\tau_{xG|k+1}) \) and \( u' \) both belong to \( G \) they are compatible, and we conclude that \((u', v')\) and \((u(\tau_{xG|k+1}), \hat{v}(\tau(t, \hat{v})))\) are compatible. But \((u(\tau_{xG|k+1}), \hat{v}(\tau(t, \hat{v}))) \in R_{m+2}\).

Now let us prove that \( R_{m+2} \) is an antichain. Let \( r = (u(\tau_{\nu^{-j}}), \hat{v}(\tau(t, \hat{v}))) \) and \( r' = (u(\tau_{\nu'^{-j'}}), \hat{v}(\tau(t', \hat{v}))) \) be different elements and \( k = |\nu|, k' = |\nu'| \). If \( \nu^\sim j, \nu'^\sim j' \) are such that none is an initial segment of the other one, \( u(\tau_{\nu^{-j}}) \) and \( u(\tau_{\nu'^{-j'}}) \) extend different elements of \( R^l(u) \) for some \( l \leq k \) and hence \( r \) and \( r' \) are incompatible. If \( \nu'^\sim j = \nu^\sim j \) then \( k = k' \) and hence \( t, t' \) must be different members of \( TPV(\nu, k) \). Consequently,

\[
u(\tau_{\nu}) \models_{\mathbb{A}(S_i)} \tau(t, \hat{v})) \text{ and } \tau(t', \hat{v})) \text{ are different level-} k \text{-sequences of } \hat{v}
\]

and hence

\[
u(\tau_{\nu}) \models_{\mathbb{A}(S_i)} \hat{v}(\tau(t, \hat{v})) \text{ and } \hat{v}(\tau(t', \hat{v})) \text{ are incompatible.}
\]

This clearly implies that \( r \) and \( r' \) are incompatible.

Finally, if \( \nu^\sim j \subsetneq \nu'^\sim j' \) we have \( k < k' \), \( u(\tau_{\nu'^{-j'}}) \subset u(\tau_{\nu^{-j}}) \) and by \( r, r' \in R_{m+2}\),

\[
u(\tau_{\nu'^{-j'}}) \models_{\mathbb{A}(S_i)} \tau(t, \hat{v})) \text{ is a level-} k \text{-sequence of } \hat{v} \text{ and } \tau(t', \hat{v})) \text{ is a level-} k' \text{-sequence of } \hat{v} \text{ and depth}(t) = \text{ depth}(t') = m + 1.
\]

Clearly this implies

\[
u(\tau_{\nu'^{-j'}}) \models_{\mathbb{A}(S_i)} \hat{v}(\tau(t, \hat{v})) \text{ and } \hat{v}(\tau(t', \hat{v})) \text{ are incompatible.}
\]

Then clearly \( r \) and \( r' \) are incompatible. Note that for this last argument the fact that \( u(\tau_{\nu'^{-j'}}) < u(\tau_{\nu^{-j}}) \) is not really needed, as for every \( v \in \mathbb{A}(S_i) \) and level-sequences \( \tau \) and \( \tau' \) from different levels of \( v \) but with the same depth (i.e. \( |\tau| = |\tau'| \)) we have that \( v(\tau) \) and \( v(\tau') \) are incompatible.

It is not hard to see that every \( R_{m+1} \) refines \( R_m \). For \( m \leq 1 \) this is obvious. If \( m > 1 \) let \((u(\tau_{\nu^{-j}}), \hat{v}(\tau(t, \hat{v}))) \in R_{m+1}\). By definition we have that \( \nu \in k^\omega \) for some \( k \geq m + 1 \), \( \nu^\sim j \in \Omega_p(u) \), \( t \in TPV(\nu, k) \) and \( \text{depth}(t) = m + 1 \). Let \( t = (\rho, M) \in k^{-12} \times P(k-1) \) (see Definition 4.1). Let \( k_0 = \max(M), \rho_0 = \rho | k_0, M_0 = M \cap k_0 \) and \( t_0 = (\rho_0, M_0) \). Note that \( t_0 \in TPV(\nu | k_0 + 1, k_0 + 1) \) and \( \text{depth}(t_0) = m \). We conclude that

\[
u(\tau_{\nu|k_0+2}), \hat{v}(\tau(t_0, \hat{v}))) \in R_m
\]

and

\[
u(\tau_{\nu^{-j}}), \hat{v}(\tau(t, \hat{v}))) \leq (u(\tau_{\nu|k_0+2}), \hat{v}(\tau(t_0, \hat{v}))).
\]

As for every depth-\( m \)-sequence there are infinitely many depth-\( m+1 \)-sequences extending it, it is now clear that the \( R_m \) can be indexed as promised in Lemma

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4.6. Alternatively we can use our push-down function $P$ to produce such an indexing. For this purpose we define

$$
\Omega = \{ P(\mu(\nu,j,t)) : 0 < k < \omega \land \nu \in k^\omega \land \nu \upharpoonright j \in \Omega_p(u) \land t \in TPV(\nu,k) \}
\cup \{ \emptyset \} \cup (\Omega_p(u) \cap 1^\omega).
$$

By the properties of $P$ (see Definition 4.1(4)), we can, by recursion on levels, check that $\Omega$ is a Laver tree with stem($\Omega$) = $\emptyset$ and at the same time define how its level $m$ indexes antichain $R_m$. Clearly, $\Omega_p(u) \cap 1^\omega$ is infinite. For $\mu \in \Omega_p(u) \cap 1^\omega$, we let $r_\mu = (u(\tau_\mu), \dot{v})$, hence

$$
R_1 = \{ r_\mu : \mu \in \Omega \cap 1^\omega \}.
$$

Given $\nu \in \Omega_p(u) \cap 1^\omega$, there are infinitely many $j$ such that $\nu \upharpoonright j \in \Omega_p(u)$. Moreover, by our convention (see Remark 4.2(2)) $TPV(\nu,1)$ contains precisely one element, say $t$, we have $TPN_1(t) = 0$ and hence

$$
\mu(\nu,j,t) = \nu \upharpoonright j
$$

(see Lemma 4.6(6)) and clearly depth$(t) = 1$, by the definition of a push-down function (see Definition 4.1(4)) we have that $\nu \subseteq P(\mu(\nu,j,t)) \in \Omega \cap 2^\omega$ for all $j$ as above. Hence if for such $\nu, j$ and $t$ we let

$$
r_P(\mu(\nu,j,t)) = (u(\tau_\nu \upharpoonright j), \dot{v}(\tau(t), \dot{v})),
$$

we have $r_P(\mu(\nu,j,t)) \in R_2, r_P(\mu(\nu,j,t)) < r_\nu$. But now note that not every member of $R_2$ is obtained in this way or equivalently, $\Omega \cap 2^\omega$ contains much more than the $P(\mu(\nu,j,t))$ of this particular shape. By the denseness of unavoidable sets, for almost every $k > 1$ there are $\nu \in k^\omega \cap \Omega_p(u)$ such that $TPV(\nu,k)$ contains some $t$ of depth 1. Let us fix such $\nu, t$ and also $j$ such that $\nu \upharpoonright j \in \Omega_p(u)$. Now we have

$$
\mu(\nu,t,j) = \nu \upharpoonright (j \cdot 2^{2(k-1)} + TPN_k(t)),
$$

but as depth$(t) = 1$, by Definition 4.1(4)(ii) and (iii) we have

$$
\nu \upharpoonright 1 \subseteq P(\mu(\nu,j,t)) \in \Omega \cap 2^\omega.
$$

Again we define

$$
r_P(\mu(\nu,j,t)) = (u(\tau_\nu \upharpoonright j), \dot{v}(\tau(t), \dot{v})),
$$

and we have $r_P(\mu(\nu,j,t)) < r_{\nu \upharpoonright 1}$. But now clearly we have $R_2 = \{ r_\mu : \mu \in \Omega \cap 2^\omega \}$.

Let us also look at the third level of $\Omega$, as here we see how property (iv) of the push-down function $P$ comes into action. We fix $\nu, j$ and $t$ such that $P(\mu(\nu,j,t)) \in \Omega \cap 2^\omega$. Hence $|\nu| = k + 1$ for some $k < \omega$ such that
Now we can easily check that \( r_t \) and \( t \in TPV(\nu, k+1) \) and \( \mu(\nu, j, t) = \nu^{\land j}2^k + TPN_k(t) \). Moreover, we have \( \text{depth}(t) = 1, \nu^{\land j} \in \Omega_p(u) \) and \( r_{P(\mu(\nu, j, t))} = (u(\tau^\nu_j), v(\tau(t, \nu)). \)

Again for almost every \( k' > k \) there are \( \nu' \in k'+\omega \cap \Omega_p(u), j' \) and \( t' \in TPV(\nu', k'+1) \) such that \( \nu^{\land j'} = \nu' \), \( \nu'^{\land j'} \in \Omega_p(u), \) depth(\( t' \)) = 2 and \( t < t' \).

Fix such \( \nu', j' \) and \( t' \). By Definition 4.1(iv) we have

\[
P(\mu(\nu, j, t)) \subseteq P(\mu(\nu', j', t')),
\]

and by (iii), \( P(\mu(\nu', j', t')) \in \Omega \cap 3\omega \). We define

\[
r_{P(\mu(\nu', j', t'))} = (u(\tau^\nu_{j'}), v(\tau(t', \nu)).
\]

Now we can easily check that \( r_{P(\mu(\nu', j', t'))} \in R_3 \) and \( r_{P(\mu(\nu', j', t'))} < r_{P(\mu(\nu, j, t))}. \)

Note that for this last fact we need the requirement that \( \nu'(k+1) = j \) (which might seem unnatural in Definition 4.1(iv)). By similar arguments we gave for \( R_2 \) it can now be shown that \( R_3 = \{r_\mu : \mu \in \Omega \cap 3\omega\}. \)

The proof of this last equality for arbitrary \( m \) is a straightforward generalization of the arguments above.

Finally, for any \( \mu \in \Omega \) of length \( m+1 > 1 \) we find \( m+1 \leq k < \omega, \nu \in k^\omega, j < \omega \) and \( t \in TPV(\nu, k) \) such that \( \nu^{\land j} \in \Omega_p(u), \) depth(\( t \)) = \( m+1 \) and \( P^{-1}(\mu) = \mu(\nu, j, t). \) We define \( a_\mu := b^k(\mu(\nu, j, t)). \) For \( \mu \in \Omega \cap 1^\omega \) let \( a_\mu = b^0(\langle \mu(0) \rangle). \) Then by (7) and (9), for every \( \mu \in \Omega \) we conclude

\[
r_\mu \Vdash \check{a}_\mu = a_\mu.
\]

This finishes the proof of Lemma 4.6. \( \square \)

**Corollary 4.7** The forcing notion \( \mathbb{A}(Si)*\hat{\mathbb{A}}(Si) \) does not add a random real.

**Proof:** Suppose that \( (p, \check{q}) \in \mathbb{A}(Si)*\hat{\mathbb{A}}(Si) \) and \( \check{x} \) is an \( \mathbb{A}(Si)*\hat{\mathbb{A}}(Si) \)-name such that \( (p, \check{q}) \Vdash_{\mathbb{A}(Si)*\hat{\mathbb{A}}(Si)} \check{x} \in \omega^2. \) Let \( \langle M(\nu) : \nu \in <^\omega \omega \rangle \in (\omega^\omega)^{\omega^\omega} \) be one-to-one and let \( \langle \check{\sigma}_\nu : \nu \in <^\omega \omega \rangle \) be defined by \( \check{\sigma}_\nu = \check{x} \upharpoonright M(\nu). \) By Lemma 4.6 we can find \( (u, \check{v}) \in \mathbb{A}(Si)*\hat{\mathbb{A}}(Si), \) a Laver tree \( \Omega \) with empty stem and families \( R = \langle R_k : k < \omega \rangle \) where \( R_k = \langle r_\nu : \nu \in \Omega \cap k^\omega \rangle \) and \( \langle a_\nu : \nu \in \Omega \rangle \) such that (1),(2),(3) hold. Define open sets \( \langle A_k : k < \omega \rangle \) by letting

\[
A_k = \bigcup \{[a_\nu] : \nu \in k^\omega \cap \Omega\}.
\]

By (2) and (3) we conclude

\[
(u, \check{v}) \Vdash_{\mathbb{A}(Si)*\hat{\mathbb{A}}(Si)} \check{x} \in \bigcap \{A_k : k < \omega\}.
\]

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Clearly $\bigcap\{A_k : k < \omega\}$ is a null set. □

Now let $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$ be the countable-support iteration of $A(Si)$, i.e. $\forall \alpha < \omega_2 \parallel_\alpha Q_\alpha = A(Si)$, and $P = P_{\omega_2}$ its limit. Recall that given some finite $F \subseteq \alpha$, $n < \omega$ and $p, q \in P_\alpha$, the ordering $p \leq_{r,n} q$ is defined by $p \leq q$ and

$$\forall \beta \in F \; p \upharpoonright \beta \parallel_\beta p(\beta) \leq^n q(\beta).$$

Moreover, given $p \in P$ and $\alpha < \omega_2$ we let $p_\alpha, p^\alpha$ denote the restriction $p \upharpoonright \alpha$, the remainder $p \upharpoonright [\alpha, \omega_2)$, respectively.

The crucial property of the forcings $A(Si)$ and $A(Si) * A(Si)$ that was used in the proofs of Theorem 3.2 and Corollary 4.7 is their peculiar antichain structure, i.e. given any condition $p$, name of a real $\dot{x}$ and function $h : <\omega \omega \to \omega$, there exist $q \leq p$, a pruned tree $\Omega \subseteq <\omega \omega$ and countably many countable antichains $R_k = \{r_\nu : \nu \in \Omega \cap \upsilon \omega \}$ for $k < \omega \setminus \{0\}$ such that each $R_k$ is maximal below $q$, $R_{k+1}$ refines $R_k$ according to their indexing and every $r_\nu$ decides the value of $\dot{x} \upharpoonright h(\nu)$. Below we shall show that this property is preserved by a countable support iteration of $A(Si)$. Let us first mention two elementary facts about $A(Si)$ which will have their analogs for the iteration.

**Lemma 4.8** (a) Let $p, q, r \in A(Si)$ such that $q \leq^0 p$. If $r$ is of the form $r = p(\bar{r}) \in R^\alpha(p)$ for some $\bar{r}$ and $m < \omega$ (see Definition 2.7) and $q$ and $r$ are compatible, then $q \wedge r$ exists in $A(Si)$.

(b) Let $m, n < \omega$ and $p, q, q' \in A(Si)$ such that $q' \leq^m q \leq p$ and if $\text{Lev}_q(m) \subseteq \text{Lev}_p(i)$, then $n \leq i$ (clearly this holds if $m \geq n$). There exists $p' \in A(Si)$ such that $p' \leq^n p$ and every $u \in A(Si)$, $u \leq p'$ that is compatible with $q$ is compatible with $q'$.

**Proof:** (a) If $q$ and $r$ are compatible, then $q \wedge r = q(\bar{r})$.

(b) Pick $\sigma \in q' \cap \text{Lev}_p(n)$ and let $p' = \text{glob}(m, q'_\sigma, p)$. Let $E^{p'} = E^p \upharpoonright p'$ except for $E^{p'} \upharpoonright q' = E^{q'}$. Now let $p'' = (p', n^p, E^{p'})$. □

**Definition 4.9** Suppose $p, q \in P_\alpha$ are such that $q \leq p$. We call $q$ **meeting below** $p$ if for every $r \leq p, r \in P_\alpha$, either $q$ and $r$ are incompatible or else their infimum $q \wedge r$ exists in $P_\alpha$ (in the first case we write $q \wedge r = 0$).

E.g., given $p \in P_\alpha$ and $\bar{r}$ any level-sequence of $p(0)$, then Lemma 4.8(a) implies that the condition $\langle p(0)(\bar{r}) \rangle^{p^1}$ is meeting below $p$. Indeed, given $r \leq p$ in $P_\alpha$, we have $q \wedge r = (p(0)(\bar{r}) \wedge r(0))^{r^1}$.

The crucial lemma is the following:
Lemma 4.10 Let $\alpha \leq \omega_2$, $p \in P_\alpha$, $F \subseteq \alpha$ finite, $k, n < \omega$, $\langle \dot{\sigma}_\nu : \nu \in \omega \omega \rangle$ a family of $P_\alpha$-names, $\langle M(\nu) : \nu \in \omega \omega \rangle$ a family in $\omega$ such that $p \forces_\alpha \forall \nu \dot{\sigma}_\nu \in M(\nu)2$.

Moreover let $S = \langle s_\nu : \nu \in \Omega_S \rangle$, where $\Omega_S \subseteq k \omega$, be a maximal antichain in $P_\alpha$ below $p$ such that every $s_\nu$ is meeting below $p$.

There exist $q \in P_\alpha$, a nonempty pruned tree $\Omega$, antichains $R_k = \langle r_\nu : \nu \in k \omega \cap \Omega \rangle$ for $0 < k < \omega$ such that for every $\nu \in k \omega$ and $j < \omega$ such that $\nu\upharpoonright j \in \Omega$ we have $r_\nu \upharpoonright j < s_\nu$.

Let $\langle a_\nu : \nu \in \Omega \rangle$ be a family in $\omega$ such that

1. $q \leq F, n p$,
2. every $R_i$ is a maximal antichain in $P_\alpha$ below $q$ such that $R_{i+1}$ refines $R_i$ according to their indexing (i.e. $r_\nu \upharpoonright j < r_\nu$), and every member of $R_i$ is meeting below $q$,
3. $\Omega \cap k \omega \subseteq \Omega_S$ and $R_{k+1}$ refines $S$ such that for every $\nu \in k \omega$ and $j < \omega$ such that $\nu \upharpoonright j \in \Omega$ we have $r_\nu \upharpoonright j < s_\nu$,
4. $\forall \nu \in \Omega \setminus k \omega (a_\nu \in M(\nu)2$ and $r_\nu \forces_\alpha \dot{\sigma}_\nu = a_\nu$).

Clearly this implies our main theorem:

Theorem 4.11 For every $\alpha \leq \omega_2$, the forcing notion $P_\alpha$ does not add a random real.

Proof of 4.11: Let $\dot{x}$ be a $P_\alpha$-name and $p \in P_\alpha$ such that $p \forces_\alpha \dot{x} \in 2^\omega$.

Let $\langle M(\nu) : \nu \in \omega \omega \rangle$ be a one-to-one familiy in $\omega$, $\dot{\sigma}_\nu = \dot{x} \mid M(\nu)$ and $S = \{p\}$. We apply Lemma 4.10 and obtain $q \leq p$, a nonempty pruned tree $\Omega$, antichains $R_k = \langle r_\nu : \nu \in k \omega \cap \Omega \rangle$ for $0 < k < \omega$ and familiy $\langle a_\nu : \nu \in \Omega \rangle$ as there. If we let

$$A_k = \bigcup \{[a_\nu] : \nu \in k \omega \cap \Omega \}$$

then clearly by (2) and (4) we have

$$q \forces_\alpha \dot{x} \in \bigcap \{A_k : k < \omega \}.$$  

As $\langle M(\nu) : \nu \in \omega \omega \rangle$ is one-to-one, $\bigcap \{A_k : k < \omega \}$ is a null set. \hfill \Box

Proof of 4.10: The proof is by induction on $\alpha$. As $P_\alpha$ satisfies Axiom A, we may assume that every $\dot{\sigma}_\nu$ and every $p(\delta)$, $\delta \in \dom(p)$, is a nice name that is hereditarily countable. Hence the set $C \subseteq \alpha$ consisting of all coordinates
needed to evaluate any $\dot{\sigma}_\nu$ or $p(\delta)$ is countable. Hence, wlog we may assume that $C = \text{dom}(p)$ and either $\alpha = \beta + 1$ for $\beta = \max(C)$ or $\alpha$ is a limit ordinal of countable cofinality and $\alpha = \sup(C)$. We present our proof only for the case that $n = 0$. The general case adds only notational complexity. Moreover, at first we assume $k = 0$ and $S = \{p\}$. At the end of the proof we shall indicate how we can reduce the general case to this special situation.

Now suppose first that $\alpha$ is a limit. Hence $\text{cf}(\alpha) = \omega$, and we can find families $\langle \delta_k : k < \omega \rangle$ and $\langle F_k : k < \omega \rangle$ such that

\begin{align*}
\text{(5)} & \quad \forall k \delta_k(k) < \delta(k + 1) \text{ and } \sup\{\delta(k) : k < \omega\} = \alpha, \\
\text{(6)} & \quad F_0 = F, \bigcup\{F_k : k < \omega\} = \text{dom}(p) \text{ and } \forall k (|F_k| < \aleph_0 \land F_k \subseteq F_{k+1} \land \max(F_k) = \delta(k)).
\end{align*}

Recursively we shall construct a family $\langle q_k : k < \omega \rangle$ in $P_\alpha$, slices of trees $\langle \Omega_k : 0 < k < \omega \rangle$ where $\Omega_k \subseteq k^\omega$, and families $\langle R_k : k < \omega \rangle$ where $R_k = \langle r_\nu : \nu \in \Omega_k \rangle$ and $\langle a_\nu : \nu \in \Omega_k \rangle$ such that

\begin{align*}
\text{(7)} & \quad q_0 = p, q_k^{k+1} \leq_{F_k, \iota_{k+1}} q^k, \text{dom}(q^k) = \text{dom}(p) \\
\text{(8)} & \quad \forall k > 0 \forall \nu \in \Omega_k \nu \in \Omega_k, \\
\text{(9)} & \quad \text{every } R_k \text{ is a maximal antichain in } P_\alpha \text{ below } q^k \text{ consisting of elements that are meeting below } q^k \text{ such that } R_0 = S = \{p\}, R_{k+1} \text{ refines } R_k \text{ according to their indexing (i.e. } r_\nu \gamma_j < r_\nu\text{), and } R_{k+1} \upharpoonright \delta(k) + 1 = \langle r_\nu \upharpoonright \delta(k) + 1 : \nu \in \Omega_{k+1} \rangle \text{ is an antichain in } P_{\delta(k)+1} \text{ (and hence a maximal one below } q^{k+1} \upharpoonright \delta(k) + 1), \\
\text{(10)} & \quad \forall k > 0 \forall \nu \in \Omega_k r_\nu \upharpoonright \alpha \dot{\sigma}_\nu = a_\nu.
\end{align*}

Suppose we have gotten $q^i$ and $R_i$ for $i \leq k$ and $\langle a_\nu : \nu \in \Omega_i \rangle$ for $0 < i \leq k$. In order to know which $\dot{\sigma}_\nu$ should be decided by which condition during step $k + 1$ we have to fix what we call a local push-down function, i.e. a function $P(\xi, \cdot)$ depending on $\xi \in \omega^k$ such that $P(\xi, \cdot)$ is one-to-one and maps sequences of the form

$$
\mu(\nu, t, j) = \nu^\gamma (j \cdot 2^{(k+1) - 1} + TPN_{k+1}(t))
$$

where $\nu \in \omega^k$, $j < \omega$ and $t \in TP_{k+1}$ with depth($t$) = $k$ to length-$k+1$-sequences that extend $\xi$. If here $k + l = 0$, then in addition we require $P(\emptyset, (j)) = (j)$. 

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Now let $G_{\delta(k)}$ be a $P_{\delta(k)}$-generic filter such that $(q^k)_{\delta(k)} \in G_{\delta(k)}$. By (9) and (5) there exists a unique $\xi = \xi(G_{\delta(k)}) \in \kappa^\omega$ such that $r_{\xi[i]} \in G_{\delta(k)}$ for every $i \leq k$. Note that by (9) again, we must have that

$$(q^k)^{\delta(k)}[G_{\delta(k)}] \leq (r_{\xi[i]})^{\delta(k)}[G_{\delta(k)}].$$

Among the properties of the following construction this will be listed explicitly (see (12)(c) below). It will be used later to show that the $R_{k+1}$ we are going to construct refines $R_k$. In $V[G_{\delta(k)}]$ we construct a fusion tree $\langle (s_\nu,p_\nu) : \nu \in \kappa^\omega \rangle$ in $P_{\delta(k),\alpha} = \dot{A}(S_i) * P_{\delta(k)+1,\alpha}$ below $(q^k)^{\delta(k)}$ such that the following hold:

11) $(s_\emptyset,p_\emptyset) = (q^k)^{\delta(k)}$;

12) (a) $\forall \nu < k^\omega \exists j (s_{\nu \setminus j},p_{\nu \setminus j}) = (s_\nu,p_\nu)$;

(b) $\forall \nu < k^\omega \exists j (s_{\nu \setminus j},p_{\nu \setminus j}) \leq \{\delta(k)\},|\nu| (s_\nu,p_\nu)$;

(c) $\forall \nu < k^\omega (s_\nu,p_\nu) \leq (r_{\xi})^{\delta(k)}[G_{\delta(k)}]$

13) if $|\nu| = k + l$, then for every $t \in TP(s_\nu,k+l)$ such that $\text{depth}(t) = k$ we have

$$(s_{\nu \setminus j}(\tau(t,s_\nu)),p_{\nu \setminus j}) \models_{\mathcal{K}(S_i)} \dot{\sigma}_{P(\xi,\mu(\nu,t,j))} = a(\mu(\nu,t,j)),$$

where as in Lemma 4.6(6), $\mu(\nu,t,j) = \nu^\omega j = (j)$ if $k + l = 0$, and

$$\mu(\nu,t,j) = \nu^\omega j \cdot 2^{k+l-1} + TP_{n+1}(t)$$

if $k + l > 0$, and $a(\mu(\nu,t,j)) \in <\omega 2$.

By our assumption about $p$ and the $\dot{\sigma}_\nu$, we may assume that $\text{dom}(p_\nu) \subseteq \text{dom}(p)$ for every $\nu$. The construction is completely analogous to the one given in Lemma 4.6(4),(5),(6). In $V$ we find families of $P_{\delta(k)}$-names $\langle \dot{s}_\nu : \nu \in \kappa^\omega \rangle$, $\langle \dot{a}(\mu(\nu,t,j)) : j < \omega \rangle$ and

$$\langle \dot{a}_{\mu(\nu,t,j)} : t \in TP(\dot{s}_\nu,k+l) \land \text{depth}(t) = k, j < \omega \rangle$$

in the case $|\nu| = k + l$ and $k + l > 0$, and a family $\langle \dot{p}_\nu : \nu \in \kappa^\omega \rangle$ of $P_{\delta(k)+1}$-names such that (11), (12), (13) are forced by $q^k$ to hold about these. By [3](7.3(c)), actually we may assume that $\dot{p}_\nu = p_\nu \in P_{\delta(k)+1,\alpha}$ is not only a name for a member of $P_{\delta(k)+1,\alpha}$.

We let $\dot{\sigma}'_{(j)} = \dot{a}(\mu(\nu,t,j))$ for $j < \omega$ and

$$\dot{\sigma}'_{\nu \setminus j} = \langle \dot{a}(\mu(\nu,t,j)) : t \in TP(\dot{s}_\nu,k+l) \land \text{depth}(t) = k \rangle^TP(\dot{s}_{\nu \setminus j},k+l+1)$$
in the case $|\nu| = k + l$ and $k + l > 0$ and $j < \omega$.

We apply the inductive hypothesis for $\delta(k)$, $q^k \upharpoonright \delta(k)$, $F_k \cap \delta(k)$, $k$, $n$, $\langle \sigma'_{\nu-j} : \nu \in \omega^\omega, j < \omega \rangle$ and $S = R_k \upharpoonright \delta(k)$. Note that by (9) we have that $S$ is a maximal antichain below $q^k \upharpoonright \delta(k)$ consisting of elements that are meeting below $q^k \upharpoonright \delta(k)$. We obtain

$$q' \leq_F \mathcal{F}_k \cap \delta(k), (q^k)_{\delta(k)};$$

a nonempty pruned tree $\Omega'$ extending part of $\Omega \cap k \omega$, family $\mathcal{R}' = \langle R'_i : k < i < \omega \rangle$ where $R'_i = \langle r'_\nu : \nu \in \Omega' \cap i \omega \rangle$, and families

$$C'_{k+l+1} = \langle c'_{\nu} \mathcal{TPV}(\nu, k + l + 1) : \nu \in \Omega' \cap k^{l+1} \omega \rangle$$

as stated in the Theorem such that

$$r'_{\nu-j} \upharpoonright \delta(k) \sigma'_{\nu-j} = c'_{\nu-j} \mathcal{TPV}(\nu \upharpoonright j, k + l + 1),$$

for every $k, j < \omega$ and $\nu$ such that $\nu \upharpoonright j \in \Omega' \cap k^{l+1} \omega$. Hence by (13), for all such $k, j, \nu$ there exists a family

$$(b'(\mu(\nu, j, t)) : t \in \mathcal{TPV}(\nu \upharpoonright j, k + l + 1) \land \text{depth}(t) = k)$$

which equals $c'_{\nu-j}$, and for every $t \in \mathcal{TPV}(\nu \upharpoonright j, k + l + 1)$ such that depth$(t) = k$ we have

$$(14) \quad r'_{\nu-j} \upharpoonright \langle \dot{s}_{\nu-j}(\dot{\tau}(t, \dot{s}_{\nu})) \rangle \mathcal{TPV}(\nu \upharpoonright j, k + l + 1) = \dot{a}(\mu(\nu, j, t)) = b'(\mu(\nu, j, t)).$$

We define $q^{k+1} = (q^{k+1})_{\delta(k)} \upharpoonright \langle \dot{\nu} \rangle \mathcal{TPV}(q^{k+1})_{\delta(k)+1} \leq F_{k\omega}$ $p$ as follows: We let $q^{k+1} = q'$. Choose $G_{\delta(k)}$ $P_{\delta(k)}$-generic such that $(q^{k+1})_{\delta(k)} \in G_{\delta(k)}$. As $R_k \upharpoonright \delta(k)$ and every $R'_i$ for $i > k$ is a maximal antichain below $(q^{k+1})_{\delta(k)}$, there exists a unique $x = x_{G_{\delta(k)}} \in \omega \omega$ such that

$$r_{x|k} \upharpoonright \delta(k) \in G_{\delta(k)} \land \forall k < i < \omega \ r'_{x|i} \in G_{\delta(k)}.$$ 

By (12) we have that $\langle \dot{s}_{x|i}[G_{\delta(k)}] : i < \omega \rangle$ is a fusion sequence in $\mathcal{A}(Si)$. We let

$$\nu = \bigwedge \{ \dot{s}_{x|i}[G_{\delta(k)}] : i < \omega \}.$$ 

Now we recall that $\{ \nu(\bar{t}) : \bar{t} \text{ is a depth-}k\text{-sequence of } \nu \}$ is a maximal antichain below $\nu$. Moreover, by construction we have that the depth-$k$-sequences of $\nu$ consist of all $\bar{t}(t, s_{x|i-1})$ where $s_{\nu} = \dot{s}_{\nu}[G_{\delta(k)}]$, $i > k$, and $t \in TP(s_{x|i-1}, i - 1)$ is of depth $k$. We define $(q^{k+1})_{\delta(k)+1}$ by stipulating
that given any depth-k-sequence \( \tau \) of \( v \), thus we have unique \( i > k \) and \( t \in TP(s_{x|i-1}, i-1) \) such that \( \tau = \tau(t, s_{x|i-1}) \),
\[
v(\bar{\tau}(t, s_{x|i-1})) \models_{A(S_i)} (q^{k+1})^{\delta(k)+1} = p_{x|i}.
\]

Note that by Lemma 4.8(a) and the example after Definition 4.9, we have that any condition in \( P_{\delta(k), a} \) of the form
\[
(v(\tau(t, s_{x|i-1})), (q^{k+1})^{\delta(k)+1})
\]
is meeting below \( (v, (q^{k+1})^{\delta(k)+1}) \). Since \( G_{\delta(k)} \) containing \( (q^{k+1})^{\delta(k)} \) was arbitrary, by the forcing theorem we can find a name \( \dot{v} \) for \( v \) such this is forced by \( (q^{k+1})^{\delta(k)} \). Note that again by [3](7.3(c)) there is no need to introduce a name denoting \( (q^{k+1})^{\delta(k)+1} \). We define
\[
q^{k+1}(\delta(k)) = \dot{v}.
\]

Note that now (14) becomes
\[
(15) \quad r'_{\nu \rightarrow j} \prec \langle \dot{v}(\tau(t, \dot{v})) \rangle \prec (q^{k+1})^{\delta(k)+1} \models_{A} \delta(P(\nu[k,\mu(\nu, j, t))) = \bar{a}(\mu(\nu, j, t)) = b'(\mu(\nu, j, t)).
\]

for every \( \nu \rightarrow j \in k^{l+l} \omega \cap \Omega' \) and \( \tau = \tau(t, \dot{v}) \) where \( t \in TP(\dot{v}, k + l) \) and depth\( (t) = k \). We define antichain \( R_{k+1} \) as follows:
\[
R_{k+1} = \{ r'_{\nu \rightarrow j} \prec \langle \dot{v}(\hat{\tau}) \rangle \prec (q^{k+1})^{\delta(k)+1} : r'_{\nu \rightarrow j} \models_{\delta(k)} \tau = \tau(t, \dot{v}) \}.
\]

Before we determine the right indexing of the \( R_{k+1} \) let us check that it is a maximal antichain of \( P_{\alpha} \) below \( q^{k+1} \), consists of elements that are meeting below \( q^{k+1} \) and refines \( R_k \). That elements of \( R_{k+1} \) are meeting below \( q^{k+1} \) easily follows from the inductive assumption and what we said above before defining \( \dot{v} \). To check incompatibility, let
\[
s = r'_{\nu \rightarrow j} \prec \langle \dot{v}(\bar{\tau}) \rangle \prec (q^{k+1})^{\delta(k)+1}, s' = r'_{\nu' \rightarrow j'} \prec \langle \dot{v}(\bar{\tau}') \rangle \prec (q^{k+1})^{\delta(k)+1}
\]
be different elements of \( R_{k+1} \). If \( \nu \rightarrow j, \nu' \rightarrow j' \) are such that none is an initial segment of the other, incompleteness of \( s \) and \( s' \) follows from the refining property of the \( R_i \) (see (2)). So in fact, even incompleteness of \( s \upharpoonright \delta(k) \) and \( s' \upharpoonright \delta(k) \) is true. If \( \nu \rightarrow j' = \nu' \rightarrow j \), let \( k + l \) be the common length of \( \nu \) and \( \nu' \). There must exist different members \( t, t' \) of \( TPV(\nu[k,\mu(\nu, j, t))) \) such that \( \tau = \tau(t, \dot{v}) \) and \( \tau' = \tau(t', \dot{v}) \). By the ordering of \( A(S_i) \) it follows that
\[
(16) \quad r'_{\nu \rightarrow j} \models_{\delta(k)} \dot{v}(\bar{\tau}), \dot{v}(\bar{\tau}') \text{ are incompatible.}
\]
Note that in this case we have shown that \( s \upharpoonright \delta(k) + 1 \) and \( s' \upharpoonright \delta(k) + 1 \) are incompatible in \( P_{\delta(k) + 1} \).

Finally, if \( \nu^j \leq \nu'^j \) we have \( l < l' \) where \( |\nu| = k + l \) and \( |\nu'| = k + l' \) and \( r'_{\nu^j} < r'_{\nu'^j} \). Moreover there are \( t \in TPV(\nu, k + l) \) and \( t' \in TPV(\nu', k + l') \) such that

\[
r_{\nu^j}^k \models_{\delta(k)} \tau = \tau(t, \dot{v}) \text{ is a level}-k \text{-sequence of depth } k \wedge \tau' = \tau(t', \dot{v}) \text{ is a level}-k \text{-sequence of depth } k.
\]

Again we conclude that (16) is true and also that \( s \upharpoonright \delta(k) + 1 \) and \( s' \upharpoonright \delta(k) + 1 \) are incompatible in \( P_{\delta(k) + 1} \).

Now let us check that \( R_{k+1} \) is maximal below \( q^{k+1} \). This amounts essentially to checking that \( q^{k+1} \) is a well-defined condition in \( P_\alpha \), which we did already. Let \( r \leq q^{k+1} \). Let \( G_{\delta(k)} \) be a \( P_{\delta(k)} \)-generic filter containing \( r_{\delta(k)} \), hence \( (q^{k+1})_{\delta(k)} \in G_{\delta(k)} \). As we noticed earlier, by our assumptions there exists a unique \( x = x_{G_{\delta(k)}} \in \omega \omega \) such that

\[
(r_{x|k})_{\delta(k)} \in G_{\delta(k)} \wedge \forall k < \omega \ r'_{x|k} \in G_{\delta(k)}.
\]

Hence by construction of \( q^{k+1} \),

\[
V[G_{\delta(k)}] \models q^{k+1}(\delta(k))[G_{\delta(k)}] = \bigwedge \{ s_{x|\iota}[G_{\delta(k)}] : \iota < \omega \}.
\]

Let \( r = r(\delta(k))[G_{\delta(k)}] \) and \( v = q^{k+1}(\delta(k))[G_{\delta(k)}] \). Wlog we may assume that depth(\( r \)) > \( k \), and hence there exists a depth-\( k \)-sequence \( \tau \) of \( v \) such that \( r \leq \nu(\tau) \). There must exist \( l < \omega \) and \( t \in TP(\nu, k + l) \) such that \( \tau = \bar{\tau}(t, \nu) \). Then in \( V[G_{\delta(k)}] \),

\[
\langle r \rangle_{\nu}^{\delta(k) + 1} \leq \langle \nu(\bar{\tau}) \rangle_{\nu}^{(q^{k+1})_{\delta(k) + 1}}.
\]

This must be forced by some condition in \( G_{\delta(k)} \) that, wlog, extends both, \( r_{\delta(k)} \) and \( r'_{x|k+l+1} \). This shows that \( r \) and

\[
r'_{x|k+l+1} \models \langle \nu(\bar{\tau}) \rangle_{\nu}^{(q^{k+1})_{\delta(k) + 1}}
\]

are compatible. But this last condition is a member of \( R_{k+1} \). This proves maximality of \( R_{k+1} \).

The proof that \( R_{k+1} \) refines \( R_k \) follows easily from the construction. Given

\[
r_{\nu^j} \models \langle \nu(\bar{\tau}) \rangle_{\nu}^{(q^{k+1})_{\delta(k) + 1}} \in R_{k+1},
\]

we have that \( r_{\nu^j} \leq (r_{\nu|k})_{\delta(k)} \). Moreover by (12)(c),

\[
(r_{\nu|k})_{\delta(k)} \models (q^{k+1})_{\delta(k)} \leq (r_{\nu|k})_{\delta(k)}.
\]
Hence we have proved
\[ r'_{\nu_\tau} \preceq (\hat{\psi}(t))^{(q^{k+1})^{\delta(k)+1}} \leq r_{\nu|k}. \]

Now let us fix the indexing of \( R_{k+1} \). Given \( r'_{\nu_\tau} \preceq (\hat{\psi}(t))^{(q^{k+1})^{\delta(k)+1}} \in R_{k+1} \) and \( t \) as in its definition, we index it as \( r_{P(\nu|k,\mu_{\nu_\tau},\ell)} \). Moreover we let
\[ a_{P(\nu|k,\mu_{\nu_\tau},\ell)} = b'(\mu_{\nu_\tau},t). \]

Note that by the definition of the local push-down function \( P \) we have \( P(\nu \upharpoonright k,\mu_{\nu_\tau},\ell) \in k+1.\omega \). Finally we define \( \Omega_{k+1} \) as the set of all these values \( P(\nu \upharpoonright k,\mu_{\nu_\tau},\ell) \). Then by (15), everything is fine.

This finishes the recursive step. In the end we let \( q = \bigwedge \{ q^k : k < \omega \} \), \( \Omega \) the tree generated by all \( \nu \in \bigcup \{ \Omega_{k+1} : k < \omega \} \) such that \( r_{\nu} \cap q \neq 0 \), and we replace \( R_k \) by \( \{ r \cap q : r \in R_k \} \setminus \{ 0 \} \). It is easy to check that \( \Omega \) is a nonempty Laver tree. Actually, it is not hard to see that \( \Omega \) is a Laver tree. Therefore, everything is as desired. But recall that we worked in the special situation that the antichain \( S \) we started with is a singleton.

So let us indicate how we can reduce the general case of an infinite antichain \( S \) to the case \( S = \{ p \} \). We enumerate \( S = \{ s(i) : i < \omega \} \). The first step, getting \( q^{k+1} \) and \( R_{k+1} \), is now the result of a fusion sequence \( \langle v(i) : 0 \leq i < \omega \rangle \) where \( v(0) = p \). We use the following Claim that can be easily proved inductively, using Lemma 4.8(b):

**Claim 4.12** Let \( \alpha \leq \omega_2, p,q,q' \in P_\alpha, F \subseteq \alpha \) finite and \( n < \omega \) such that \( q \leq p \). If \( q' \leq F_n q \) there exists \( p' \in P_\alpha, p' \leq F_n p \) such that every extension of \( p' \) that extends \( q \) also extends \( q' \).

Suppose we have already constructed \( v(i) \leq_{F_n} p \). In the next step we we consider \( s(i) \) in case \( v(i) \) and \( s(i) \) are compatible. In this case we apply the special case to \( v(i) \cap s(i) \) (recall that \( s(i) \) is meeting below \( p \)) and obtain \( s'(i) \leq_{F_i,v(i)} s(i) \), below \( s'(i) \) a maximal antichain \( X(i+1) \) etc. as in Lemma 4.10. By Claim 4.12 we obtain \( v(i+1) \leq_{F_i, v(i)} s'(i) \) such that every extension of \( v(i+1) \) that extends \( v(i) \cap s(i) \) also extends \( s'(i) \). In the end we let \( q^{k+1} = \bigwedge \{ v(i) : i < \omega \} \) and
\[ R_{k+1} = \{ r \cap q^{k+1} : r \in \bigcup \{ X(i+1) : i < \omega \} \} \setminus \{ 0 \}. \]

Hence the proof of the limit case is complete.

Now let us assume that \( \alpha = \beta + 1 \). If \( \beta \notin F \) we can argue in very much the same way as in the limit case. Hence we assume \( \beta \in F \). Let \( F' = F \cap \beta \).
This case is similar to the proof of Lemma 4.6 that implies that $A(S_i) \ast A(S_i)$ does not add random reals. We give the proof for the case that $S = \{p\}$. The general case can be handled as in the limit case. Moreover we assume $n = 0$.

We step into the model $V[G_\beta]$ where $G_\beta$ is a $P_\beta$-generic filter such that $p \upharpoonright \beta \in G_\beta$. Let $r = p(\beta)[G_\beta]$, thus $r = (r, n^r, E^r)$, and let $\bar{\sigma} = \langle \sigma_0, \ldots, \sigma_n^r \rangle$ be stem($E^r$).

Precisely as in the proof of Lemma 4.6, we construct a fusion tree $\langle s_\nu : \nu \in \omega^{<\omega} \rangle$ in $A(S_i)$ below $r$. We fix a push-down function $P$. We demand that the following hold:

\begin{align*}
(17) & \quad s_\emptyset = r; \\
(18) & \quad s_\nu \upharpoonright j \leq \nu \leq s_\nu; \\
(19) & \quad \text{if } |\nu| = k, \text{ then for every } t \in TP(s_\nu, k), \text{ letting } \tau = \tilde{\tau}(t, s_\nu), \text{ we have that} \\
& \quad s_\nu \upharpoonright j(\tilde{\tau}) \models_{A(S_i)} \hat{\sigma}_P(\nu') = a(\nu'), \\
& \quad \text{where } \nu' = \nu \upharpoonright j = \langle j \rangle \text{ if } k = 0 \text{ and } \nu' = \nu \upharpoonright (j \cdot 2^{2(k-1)} + TPN_k(t)) \text{ if } k > 0.
\end{align*}

In $V$ we can find $P_\beta$-names $\langle \dot{s}_\nu : \nu \in \omega^{<\omega} \rangle$ and $\langle \dot{a}(\langle j \rangle) : j < \omega \rangle$ and

$$
\langle \dot{a}(\nu \upharpoonright (j \cdot 2^{2(k-1)} + TPN_k(t))) : t \in TP(\dot{s}_\nu, k), j < \omega \rangle
$$

if $|\nu| = k > 0$ such that $(17)$, $(18)$, $(19)$ are forced by $p_\beta$ to hold for these.

We let $\dot{\sigma}'_{\langle j \rangle} = \hat{\sigma}(\langle j \rangle)$ for $j < \omega$ and

$$
\sigma'_{\nu \upharpoonright j} = \langle \dot{a}(\nu \upharpoonright (j \cdot 2^{2(k-1)} + TPN_k(t))) : t \in TP(\dot{s}_\nu, k) \rangle \cap TP(\dot{s}_\nu, k + 1)
$$

for every $k > 0, \nu \in k^{<\omega}$ and $j < \omega$. By the inductive hypothesis for $\beta$ we obtain $q' \leq_{F', n} p_\beta$, a pruned tree $\Omega'$, families

$$
R_{k+1} = \langle r_\nu : \nu \in \Omega' \cap k^{+1}, \nu \rangle, \quad C_{k+1} = \langle c_\nu \cap TPV(\nu, k + 1) : \nu \in \Omega' \cap k^{+1}, \nu \rangle,
$$

the $R_{k+1}$ being maximal antichains of $P_\beta$ below $q'$ as stated in the Theorem, such that

$$
r_{\langle j \rangle} \models_{P_\beta} \dot{\sigma}'_{\langle j \rangle} = c_{\langle j \rangle}
$$

for every $\langle j \rangle \in \Omega'$ and

$$
r_{\nu \upharpoonright j} \models_{P_\beta} \dot{\sigma}'_{\nu \upharpoonright j} = c_{\nu \upharpoonright j} \cap TPV(\nu \upharpoonright j, k + 1)
$$
for every $k > 0$, $\nu \in k\omega$ and $j < \omega$ such that $\nu \cdot j \in \Omega'$. Hence there exists
\[ (b(\nu^\frown (j \cdot 2^{2(k-1)} + \text{TPN}_k(t))) : t \in \text{TPV}(\nu,k)) \]
which equals $c_{\nu \cdot j}$. We let $b((j)) = c_j$. Now for every $k > 0$, $\nu \in k\omega$, $t \in \text{TPV}(\nu,k)$ and $j < \omega$ we have
\[
(20) \quad r_{\nu \cdot j} \cdot \hat{s}_{\nu \cdot j}(\hat{\tau}(t, \hat{s}_\nu)) \models_{\beta} \hat{\sigma}_{P(\nu^\frown j \cdot 2^{2(k-1)} + \text{TPN}_k(t))} = \hat{a}(\nu^\frown j \cdot 2^{2(k-1)} + \text{TPN}_k(t)) = b(\nu^\frown (j \cdot 2^{2(k-1)} + \text{TPN}_k(t))).
\]

We define $q \leq_{F,n} p$ by letting $q_{\beta} = q'$; moreover we let $\hat{x}$ be the canonical $P_\beta$-name such that
\[ q_{\beta} \models_{\beta} \forall n \ r_{\hat{x}|n} \in \hat{G}_{\beta}, \]

and we stipulate that
\[ q_{\beta} \models_{\beta} q(\beta) = \bigwedge \{ \hat{s}_{\hat{x}|n} : n < \omega \}. \]

We define maximal antichains $S_m$ as follows:
\[
S_1 = \{ r_{(j)}^\frown q(\beta) : j < \omega \}, \quad \text{and if } m > 0, \]
\[
S_{m+1} = \{ r_{\nu \cdot j}^\frown q(\beta)(\hat{\tau}) : \nu \in k\omega \land j < \omega \land t \in \text{TPV}(\nu,k) \land \text{depth}(t) = m + 1 \land r_{\nu \cdot j}^\frown q(\beta) \models_{\beta} \hat{\tau} = \tau(t, q(\beta)) \}. \]

The proof that every $S^m$ is a maximal antichain below $q$ and that $S_m$ refines $S_m$ is similar as in Lemma 4.6. That its elements are all meeting below $q$ follows from the inductive hypothesis and Lemma 4.8(a). Finally we have to define the proper indexing of the $S_m$. We define
\[ \Omega = \{ P(\nu^\frown (j \cdot 2^{2(k-1)} + \text{TPN}_k(t))) : 0 < k < \omega, \nu \in k\omega, j < \omega, t \in \text{TPV}(\nu,k) \} \cup \{ \emptyset \} \cup (\Omega' \cap 1\omega). \]

By the properties of a pushing-down function it easily follows that $\Omega$ is a pruned tree with stem($\Omega$) = $\emptyset$. Let $\mu \in ^{<\omega}\omega \setminus \{ \emptyset \}$. If $|\mu| = 1$ we let $s_\mu = r_\mu^\frown q(\beta)$. If $|\mu| = m + 1 > 1$, then
\[
(21) \quad P^{-1}(\mu) = \nu^\frown (j \cdot 2^{2(k-1)} + \text{TPN}_k(t))
\]
for some $m + 1 \leq k < \omega, \nu \in k\omega, j < \omega$ and $t \in \text{TPV}(\nu,k)$ such that depth($t$) = $m + 1$. Then we let
\[ s_\mu = r_{\nu \cdot j}^\frown q(\beta)(\hat{\tau}), \]
where $\hat{\tau}$ is such that $r_{\nu \cdot j} \models_{\beta} \hat{\tau} = \tau(t, q(\beta))$. By construction we have $S_{m+1} = \{ s_\mu : \mu \in ^{m+1}\omega \}$. Finally, defining $b_\mu := b(\nu^\frown (j \cdot 2^{2(k-1)} + \text{TPN}_k(t)))$, by (20) and (21) we conclude
\[ s_\mu \models_{\beta} \hat{\sigma}_\mu = b_\mu. \]

This finishes the proof of Lemma 4.10. \qed
5 Conclusion

We are now able to prove our main theorem as outlined in the introduction.

**Theorem 5.1** Let \( \langle P_\alpha, Q_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle \) be a countable-support iteration of \( \mathcal{A}(Si) \) and \( P = P_{\omega_2} \). If \( V \models \text{GCH} \) and \( G \) is \( P \)-generic over \( V \) then
\[
V[G] \models \text{cov}(\mathcal{N}) < \text{add}(\mathcal{I}(Si)),
\]
and hence
\[
V[G] \models \mathcal{N} \not\subseteq T(\mathcal{I}(Si)).
\]

**Proof:** By Theorem 4.11 we have \( V[G] \models \text{cov}(\mathcal{N}) = \aleph_1 \). We have to show \( V[G] \models \text{add}(\mathcal{I}(Si)) = \aleph_2 \). This is a standard argument. In \( V[G] \), let \( \langle X_\nu : \nu < \omega_1 \rangle \) be a family of sets in \( \mathcal{I}(Si) \). Let \( X = \bigcup \{X_\nu : \nu < \omega_1\} \). We have to show \( X \in \mathcal{I}(Si) \). For every \( \nu < \omega_2 \), the set
\[
D_\nu = \{ q \in Si : [q] \cap X_\nu = \emptyset \}
\]
is open dense in \( Si \). Applying a typical Löwenheim-Skolem argument together with the \( \aleph_2 \)-chain condition of \( P \), we can see that the set
\[
C_\nu = \{ \alpha < \omega_2 : D_\nu \cap Si^{V[G_\alpha]} \in V[G_\alpha] \land D_\nu \cap Si^{V[G_\alpha]} \text{ is dense in } Si^{V[G_\alpha]} \}
\]
is an \( \omega_1 \)-club, i.e. closed under taking suprema of increasing sequences of length \( \omega_1 \).

Now let \( p \in Si \) be arbitrary. We have to find \( q \in Si \) such that \( q \leq p \) and
\[
V[G] \models \forall \nu < \omega_1 [q] \subseteq \bigcup \{ [r] : r \in D_\nu \}.
\]

We can find \( \alpha \in \bigcap \{C_\nu : \nu < \omega_1\} \) large enough so that \( p \in V[G_\alpha] \) and, moreover, \( (p, n^p, E^p) \in G(\alpha) \) for some \( n^p \) and \( E^p \). If \( q = p(G(\alpha)) \) is the generic Silver tree determined by \( G(\alpha) \), by Lemma 2.6 and genericity we conclude that for every \( \nu < \omega_1 \), \( [q] \) is covered by finitely many trees from \( D_\nu \) in \( V[G_\alpha] \). Clearly this is absolute, hence holds also in \( V[G] \). As \( q \leq p \), \( q \) is as desired. \( \square \)
References


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